Shifted sheaves for space-time

On the occasion of the retirement of Catriona Byrne, with my sincere friendship

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March 2022

Abstract

After some historical considerations, we describe a strange phenomenon extracted from [GKS12], which shows that if one considers the sheaf associated with a closed ball living in a Euclidean vector space whose radius grows linearly with time t, then its natural prolongation for $t < 0$ is the open ball *shifted* (in the langage of sheaves) by the dimension of the space. There is a similar result for a sphere, the shift increasing at each passage of the poles.

Books are of fundamental importance in Science, for the future, for the readers of course and also for the authors: what better than a book to make your ideas spread? I have a long history of publishing books with Springer, starting with LNM 126 in 1970! But my most signifiant experience was the publication of the book GL192, with Masaki Kashiwara, published in 1990. Catriona Byrne played a prominent role in the process, making suggestions and corrections, offering criticism and, more importantly, providing psychological support. Thank you Catriona!

The book mentioned above is about the microlocal theory of sheaves. Here, three names have to be quoted: Jean Leray, who invented sheaf theory in the 40s, when he was a prisoner of war; Alexander Grothendieck, who gave to this theory its full strength by developing it in the framework of categories[1] and emphasizing the functorial point of view through "the six operations"; and Mikio Sato, who introduced the microlocal point of view, showing that what we think of as "local" on a manifold is in some sense global, it is the projection on the manifold of phenomena occurring in the cotangent bundle. The book with Kashiwara mentioned above is a formulation of this fundamental idea in the langage of sheaves.

But what is sheaf theory? It is the mathematical treatment of the dichotomy local/global. Some objects, or properties, are completely different when viewed locally or globally. Some have no local existence, although they globally exist, and others only exist locally. The Möbius strip is a popular mathematical illustration of this fact: this

[1]We refer to [Krö07] for a philosophical look at this theory.

strip may be locally oriented but traversing around it once, the orientation is reversed. This dichotomy is present in many everyday phenomena, especially in politics, where it is the source of strong conflicts.

And what does microlocal[2] mean? If you are a point x on a real manifold M, to be local means to observe everything in a small ball around you. But the manifold admits a tangent bundle $\tau_M : TM \to M$ and its dual, the cotangent bundle $\pi_M : T^*M \to M$. At x, roughly speaking, the tangent bundle T_xM is the vector space of all light rays passing through x and its dual T^*_xM is the space of all walls passing through x that block the light. The tangent space is more intuitive but the cotangent space, a symplectic manifold, is more important. It is the "phase space" of the physicists. In a word, to be microlocal does not mean to replace M with T^*M but to work on M with T^*M in mind. For example, the micro-support of a sheaf F on M is the set of co-directions in which the sheaf F does not propagate. The micro-support is co-isotropic for the symplectic structure of T^*M, similarly as the characteristic variety of a coherent D-module in the complex case[3]. Indeed, the micro-support of the sheaf of holomorphic solutions of a coherent D-module is nothing but the characteristic variety of the D-module.

The smallest non-empty co-isotropic sets (assuming some weak regularity condition such as, for example, being subanalytic) are the Lagrangian subvarieties. The sheaves whose micro-support are Lagrangian are exactly the constructible sheaves. A sheaf is constructible if there exists a stratification of M along which it is locally constant. Constructible sheaves are of fundamental importance in various branches of mathematics and also in physics. Generically, when it is smooth and the rank of the projection π_M is constant, a Lagrangian submanifold Λ is the conormal bundle to its projection, a submanifold of M. But when the rank of the projection is no longer constant, the projection becomes singular and caustics appear. In order to calculate asymptotic expansions in a neighborhood of a caustic, Viktor Maslov [Mas65] introduced the index of a closed curve in a Lagrangian submanifold. The so-called Maslov index was studied and reformulated by several authors, including [Arn67, Ler76], until Masaki Kashiwara gave a very simple and elegant description of it (see [KS90, Appendix]).

When a "pure sheaf" F (a notion that we shall not explain here, referring to [KS90, § 7.5]) is microlocally supported by a smooth Lagrangian manifold Λ, at each point p of Λ one can attach to F a half integer, called its shift at p, and this shift jumps when the rank of the projection does so. This should be related to what is called "phase transition" in physics.

Let us illustrate this point with space-time and the expansion of the universe (the Big Bang). Of course, what will be written now is purely mathematical, is very rough and does not have the pretension of corresponding to any physical reality. Let us represent the universe as a ball of dimension n (of course, for us $n = 3$) whose radius R grows linearly with the time t so that we represent space-time as a closed cone in $\mathbb{R}^4$ with vertex at $t = 0$, similarly as a light cone in a Minkowski space. One asks: what happens for $t < 0$? If one replaces the space-time with the constant sheaf supported

[2] See [Sch21] for a detailed survey

[3] This is an important and difficult theorem of [SKK73] later reformulated and proved purely algebraically in [Gab81].

by it, the sheaf $\mathbf{k}_{\{|x|\le t\}}$ (for a given field $\mathbf{k}$), defined on $t \ge 0$, we need to extend it naturally for $t < 0$. The micro-support of this sheaf at the boundary is the interior conormal. If we extend it naturally for $t < 0$ we get the exterior conormal which is the micro-support of the constant sheaf on the open cone. In [GKS12, Exa. 3.10, 3.11] we construct a "distinguished triangle"

$$\mathbf{k}_{\{|x|<-t\}}[n] \to K \to \mathbf{k}_{\{|x|\le t\}} \xrightarrow[\psi]{+1}$$

and the micro-support of K outside the zero-section is the smooth Lagrangian manifold associated with the Hamiltonian isotopy $(x;\xi) \mapsto (x - t\xi/|\xi|;\xi)$. Hence, we get a sheaf

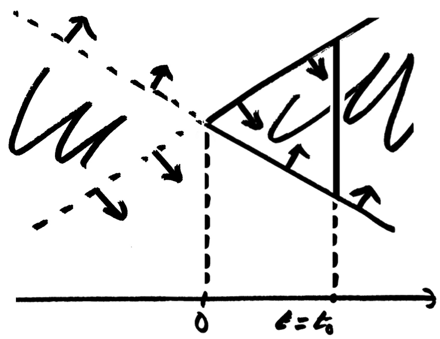

Figure 1: Before the Big Bang

K which corresponds to our intuition for $t \ge 0$, and which is the open cone *shifted by the dimension* for $t < 0$, the space at time t_0 being more or less the dual of the space at time $-t_0$.

One can modify the Lorentzian case encountered above and consider a similar situation on the n-dimensional unit sphere $M = \mathbb{S}^n$ ($n \ge 2$) endowed with the canonical Riemannian metric. In this case, the sheaf obtained has a shift which jumps by the dimension minus one when $t \in \pi\mathbb{Z}$.

What does it mean to be shifted? For sheaves this is a very common notion. What is called "a sheaf" is indeed an object of the derived category, represented by a complex of sheaves, and the shift of a complex is something familiar and elementary. However, one would like the space at each time t to have a Riemannian structure and the notion of shifted Riemannian structure, or that of shifted Lorentzian structure, has not appeared in the literature, in contrast to what happens with symplectic geometry. Shifted symplectic geometry is a part of what is called derived geometry, based on the new langage of ∞-categories.

Conclusion.

These two examples of a shift appearing in sheaf theory could suggest another point of view on the Big Bang, a topic which should be treated with lot of care since it attracts non-scientists and is the occasion of much nonsense. Nevertheless we can mention the paper [MM14], and there is a vast physics literature on this subject, notably under the impulse of Roger Penrose (see for example [Pen12]).

Acknowledgment.

It is a pleasure to thank Stéphane Guillermou for his advice.

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