Fifty years of Mathematics with Masaki Kashiwara

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Before Kashiwara

Masaki Kashiwara was a student of Mikio Sato and the story begins long ago, in the late fifties, when Sato created a new branch of mathematics, now called "Algebraic Analysis" by publishing his papers on hyperfunction theory and developed his vision of analysis and linear partial differential equations (LPDE) in a series of lectures at Tokyo University.

Hyperfunctions on a real analytic manifold M are cohomology classes supported by M of the sheaf $\mathcal{O}_X$ of holomorphic functions on a complexification X of M.

In these old times, trying to understand real phenomena by complexifying a real manifold and looking at what happens in the complex domain was a totally new idea. The use of cohomology of sheaves in analysis was definitely revolutionary.

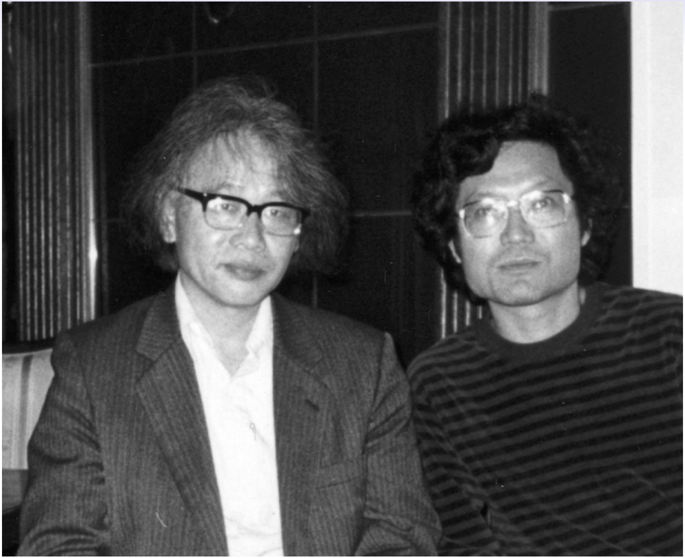

Figure : 佐藤幹夫 and 柏原正樹

Kashiwara's thesis and $\mathcal{D}$-module theory

Then came Masaki Kashiwara. In his master's thesis, Tokyo University, 1970, Kashiwara establishes the foundations of analytic $\mathcal{D}$-module theory, a theory which is now a fundamental tool in many branches of mathematics, from number theory to mathematical physics. (Soon after and independently, Joseph Bernstein developed a similar theory in the algebraic setting.)

On a complex manifold X, one has the non commutative sheaf of rings $\mathcal{D}_X$ of holomorphic partial differential operators. A left coherent $\mathcal{D}_X$-module $\mathcal{M}$ is locally represented (non uniquely) by an exact sequence

$$\mathcal{D}_X^{N_1} \xrightarrow{\cdot A_0} \mathcal{D}_X^{N_0} \to \mathcal{M} \to 0.$$

In other words, $\mathcal{M}$ is locally the cokernel of a matrix of differential operators acting on the right on $\mathcal{D}_X$.

This is the intrinsic way to formulate what is a finite system of LPDE with finitely many unknowns.

In his thesis, Kashiwara gives the main algebraic properties of the sheaf of rings $\mathcal{D}_X$ (coherency, homological dimension, etc.).

He defines the characteristic variety $\mathrm{char}(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M}$, a closed conic complex analytic subset of the cotangent bundle T^*X.

He defines the operations on $\mathcal{D}$-modules and extends the classical Cauchy-Kowalevski theorem to general systems of LPDE.

The SKK paper

The seventies are, for the analysts, the time of microlocal analysis, initiated by Mikio Sato with the analytic wave front set. In the famous paper, known as [SKK], by Mikio Sato, Takahiro Kawai and Masaki Kashiwara, the sheaf of rings of microdifferential operators is constructed on the cotangent bundle T^*X and two fundamental results are obtained.

(i) The involutivity of characteristics of (micro)-differential systems. This fundamental question had, around 1970, only a partial answer (Quillen, Guillemin and Sternberg). Later, in 1981, Gabber will give a purely algebraic proof.

(ii) The microlocal classification at generic points of any system of microdifferential equations on a real manifold. These are proved to be equivalent, after a so-called quantized contact transform, to a combination of a partial De Rham system, a partial Dolbeault system and a Hans Lewy's type system.

The SKK paper had an enormous influence on the analysis of partial differential equations (see in particular the work of Hörmander and Sjöstrand).

The Riemann-Hilbert correspondence

Since the characteristic variety of a coherent $\mathcal{D}$-module is involutive, it is natural to look at the extreme case, when this variety is Lagrangian. One calls such systems "holonomic".

They are the higher dimensional version of classical ordinary differential equations (ODE). Among ODE, there is a class of particular interest, called the class of Fuchsian equations or equations with regular singularities.

Roughly speaking, the classical Riemann-Hilbert (R-H) correspondence is based on the following question: given a finite set of points on the Riemann sphere and, at each point, an invertible matrix of complex numbers, does it exist a unique Fuchsian ODE whose singularities are the given points and such that the monodromy of its holomorphic solutions are the given matrices.
A partial answer is given by Pierre Deligne in 1970 who treats regular connections.

From 1975 to 1980, Masaki Kashiwara gives a precise formulation of the R-H correspondence in any dimension and proves it.

In 1975 he proves that the contravariant functor $\mathcal{S}ol$, which to a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ associates the complex of its holomorphic solutions, takes its values in the derived category of $\mathbb{C}$-constructible sheaves.

At the same time, he proves that the complex one obtains satisfies what will be called five years later by Beilinson, Bernstein, Deligne and Gabber, the perversity conditions.

Notice that the idea of perverse sheaves comes from LPDE!

Then Kashiwara introduced (around 1978, with Oshima and Kawai) the notion of regular holonomic module and proved in 1980 the R-H correspondence, namely an equivalence of categories between the derived category of regular holonomic $\mathcal{D}$-modules and the derived category of $\mathbb{C}$-constructible sheaves.
For that purpose, he introduced the functor Thom of tempered cohomology.

Microlocal sheaf theory

From 1982 to 1990, Kashiwara and myself introduced and developed the microlocal theory of sheaves.

This theory emerged from a joint paper in which we solved the Cauchy problem for hyperfunction solutions of hyperbolic $\mathcal{D}$-modules. Indeed, the basic idea is that of microsupport of sheaves which gives a precise meaning to the concept of propagation for sheaves on real manifolds.

The microsupport of sheaves is, in some sense, a real analogue of the characteristic variety of coherent $\mathcal{D}$-modules on complex manifolds and the functorial properties of the microsupport are very similar to those of the characteristic variety of $\mathcal{D}$-modules.

Moreover, the microsupport of the complex $\mathcal{S}ol(\mathcal{M})$ of holomorphic solutions of a coherent $\mathcal{D}$-module $\mathcal{M}$ is nothing but the characteristic variety of $\mathcal{M}$. This is essentially a corollary of the Cauchy-Kowalewski theorem, in its precise form due to Petrowski-Leray.

Moreover, and this is one of the main results of the theory, the microsupport is involutive.

Microlocal sheaf theory has found applications in other fields of Mathematics.

- First, in representation theory, as we shall see soon.
- In symplectic topology and knot theory, essentially with Tamarkin, Nadler, Zaslow, and also Guillermou, Shende, Treumann and many others.
- Recently, Alexander Beilinson adapted the definition of the microsupport of sheaves to arithmetic geometry and the theory is now developing, in particular by Takeshi Saito.

Other major contributions

In this presentation, I have preferred to concentrate on one aspect of Kashiwara's work, namely Algebraic Analysis, rather than to compile a catalog of all his work, and even in that case 20 minutes would not have been enough.

Let's remember, however, that Masaki Kashiwara obtained fundamental results in many other fields of Mathematics.

Representation theory

- Kashiwara proves with Michèle Vergne a fundamental result on the Campbell-Hausdorff formula (1978). There is currently a lot of activity stemming from this paper.
- With Jean-Luc Brylinski, he solves the Kazhdan-Lusztig conjecture on infinite-dimensional representations of simple Lie algebras (1981) (a result obtained simultaneously by Beilinson-Bernstein).
- These results were generalized to Kac-Moody algebras by Kashiwara with Toshiyuki Tanisaki (1990).
- He reinterprets the Harish-Chandra theory in terms of $\mathcal{D}$-module theory and obtains by this method important theorems on semi-simple Lie algebras with Ryoshi Hotta in 1984 and on real reductive groups with Wilfried Schmid (1994).

Quantum groups

In 1990, inspired by solvable lattice models, Kashiwara introduces the groundbreaking notion of crystal bases of quantum groups.
He developed this idea with collaborators, in particular Seok-Jin Kang, Myungho Kim and Se-jin Oh.

Mathematical Physics

Masaki Kashiwara has also obtained important results in Mathematical Physics.

- In collaboration with T. Kawai and H. Stapp, he applied the theory of holonomic $\mathcal{D}$-modules to the study of Feynman integrals (1977).
- With Etsuro Date, Michio Jimbo and Tetsuji Miwa (1981–82), he found links between hierarchies of soliton equations and representations of infinite dimensional Lie algebras, a subject initiated by Mikio Sato and Yasuko Sato.

Recent work

- Kashiwara and myself have developed the theory of Deformation Quantization (DQ) modules, obtaining in particular an analogue of the Grauert direct image theorem and a theorem of perversity for holonomic DQ-modules (2006–2012).
- Kashiwara and Raphaël Rouquier apply the theory of DQ-modules to construct a microlocalization of rational Cherednik algebras (2008).

- Kashiwara and Kari Vilonen have solved the so-called codimension 3 conjecture (2015), a holonomic version of the classical Siu-Trautmann theorem.

- Kashiwara and myself have introduced and developed the theory of indsheaves and, as a particular case, sheaves for the subanalytic topology (2001). This allows one to define the indsheaf of tempered holomorphic functions on a complex manifold and we showed that the complex $\mathcal{S}ol^t$ of tempered holomorphic solutions of an irregular holonomic $\mathcal{D}$-module contains much more informations than the usual complex $\mathcal{S}ol$.

- On the other hand, around 2010, Takuro Mochizuki made a breakthrough in the theory of irregular holonomic $\mathcal{D}$-modules (with contributions of C. Sabbah and K-S Kedlaya). Based on this work, and with the tool of $\mathcal{S}ol^t$, Kashiwara, in collaboration with A. D'Agnolo, obtained a kind of irregular R-H correspondence (2016).

Conclusion

Kashiwara's contribution to mathematics is really astonishing and it should be mentioned that his influence is not only due to his published work, but also to many informal talks. Important subjects such as second microlocalization, complex quantized contact transformations, the famous "watermelon theorem", etc. were initiated by him, although not published. Kashiwara is an invaluable source of inspiration for many people.

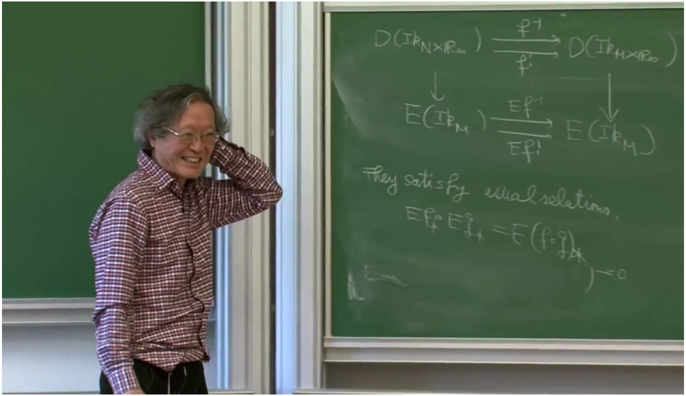

Figure : Masaki Kashiwara at IHES, 2015