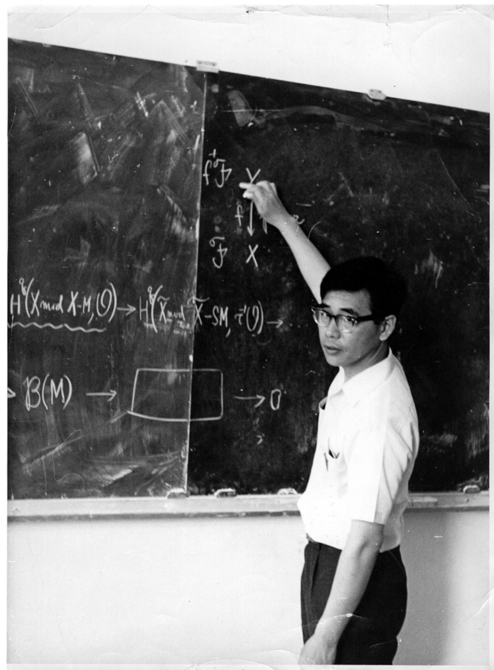

Figure 1: Mikio Sato, Nice (France) 1970

Mikio Sato, a visionary of mathematics

Pierre Schapira[*]

Mikio Sato passed away on January 9, 2023 and it was very sad news for all of us who had the chance to meet him and share his vision of mathematics. A vision radically new and revolutionary, too revolutionary to be immediately understood by the community of mathematicians, in particular by the analysts, despite the fact that initially, Sato's aim was to develop "algebraic analysis", that is, to treat problem of analysis with the tools of algebraic geometry. And Sato never did any effort nor spend a great deal of time or energy to make his ideas propagate.

[*]This paper is a modified version of a text that already appeared in the "Notices of the AMS", Feb. 2007 after a first publication in French, in "La Gazette des Mathématiciens" **97** (2003) on the occasion of Sato's reception of the 2002/2003 Wolf prize. The part concerned with number theory of these publications had benefitted from the scientific comments of Jean-Benoît Bost and Antoine Chambert-Loir, and also Pierre Colmez for the last version. I warmly thank all of them.

Sato did not write a lot, did not communicate easily and attended very few meetings. But he invented a new way of doing analysis, "Algebraic Analysis", and he opened a new horizon in mathematics, the microlocal approach. Thanks to Sato, we understand that many phenomena which appear on a manifold are in fact the projection on the manifold of things living in the cotangent bundle to the manifold. Being "local" becomes, in some sense, global with respect to this projection.

Mikio Sato also created a school, "the Kyoto school", among whom Masaki Kashiwara, Takahiro Kawai, Tetsuji Miwa and Michio Jimbo should be mentioned.

Born in 1928,[1] Sato became known in mathematics only in 1959–60 with his theory of hyperfunctions. Indeed, his studies had been seriously disrupted by the war, particularly by the bombing of Tokyo. After his family home was burned down, he had to work as a coal delivery man and later as a school teacher. At age 29 he became an assistant at the University of Tokyo. He studied mathematics and physics, on his own.

To understand the originality of Sato's theory of hyperfunctions, one has to place it in the mathematical landscape of the time. Mathematical analysis from the 1950s to the 1970s was under the domination of functional analysis, marked by the success of the theory of distributions. People were essentially looking for existence theorems for linear partial differential equations (LPDE) and most of the proofs were reduced to finding "the right functional space", to obtain some a priori estimate and apply the Hahn–Banach theorem.

It was in this environment that Sato defined hyperfunctions [Sat59] in 1959–1960 as boundary values of holomorphic functions, a discovery which allowed him to obtain a position at the University of Tokyo thanks to the clever patronage of Professor Iyanaga, an exceptionally open-minded person and a great friend of French culture. Next, Sato spent two years in the USA, in Princeton, where he unsuccessfully tried to convince André Weil of the relevance of his cohomological approach to analysis.

Sato's method was radically new, in no way using the notion of limit. His hyperfunctions are not limits of functions in any sense of the word, and the space of hyperfunctions has no natural topology other than the trivial one. For his construction, Sato invented local cohomology in parallel with Alexander Grothendieck. This was truly a revolutionary vision of analysis. And, besides its evident originality, Sato's approach had deep implications since it naturally led to microlocal analysis.

The theory of LPDE with variable coefficients was at its early beginnings in the years 1965–1970 and under the shock of Hans Lewy's example which showed that a very simple first order linear equation $(-\sqrt{-1}\partial_1 + \partial_2 - 2(x_1 + \sqrt{-1}x_2)\partial_3)u = v$ had no solution, even a local solution, in the space of distributions [2]. The fact that an equation had no solution was quite disturbing at that time. People thought it was a defect of the theory, that the spaces one had considered were too small to admit solutions. Of course, often just the opposite is true and one finds that the occurrence of a cohomological obstruction heralds interesting phenomena: the lack of a solution is the demonstration of some deep and hidden geometrical phenomena. In the case

[1] See [And07] for more details about Sato's life.

[2] The slightly simpler equation $(\partial_1 + \sqrt{-1}x_1\partial_2)u = v$ does not have any solution in the space of germs at the origin of distributions in $\mathbb{R}^2$ either, nor even in the space of germs of hyperfunctions.

Figure 2: Mikio Sato and the author, ICM, Nice 1970

of the Hans Lewy equation, the hidden geometry is "microlocal" and this equation is microlocally equivalent to an induced Cauchy–Riemann equation on a real hypersurface of the complex space.

In mathematics, as in physics, in order to treat phenomena in a given (affine) space, one is naturally led to compute in the dual space. One way to pass from a vector space to his dual, the most commonly used in analysis, is the Fourier transform. This transform, being not of local nature, is not easily adapted to calculus on manifolds. By contrast, Sato's method is perfectly suited to manifolds. If M is a real analytic manifold, X a complexification of M, what plays the role of the dual space is now the conormal bundle $T^*_M X$ to M in the cotangent bundle T^*X, something local on M. (Note that $T^*_M X$ is isomorphic to $\sqrt{-1}T^*M$.) In order to pass from M to $T^*_M X$, Sato constructed a key tool of sheaf theory, the microlocalization functor $\mu_M(\bullet)$, the "Fourier–Sato" transform of the specialization functor $\nu_M(\bullet)$[3]. This is how Sato defines in [Sat70] the analytic wave front set of hyperfunctions (in particular, of distributions), a closed conic subset of the cotangent bundle, and he shows that if a hyperfunction u is a solution of the equation $Pu = 0$, then its wave front set is contained in the intersection with $T^*_M X$ of the characteristic variety of the operator P. It is then clear (but it was not so clear at this time) that if you want to understand what happens on a real manifold, you better look at what happens on a complex neighborhood of the manifold.

Of course, at this time other mathematicians (especially Lars Hörmander) and physicists (*e.g.,* Daniel Iagolnitzer) had the intuition that the cotangent bundle was the natural space for analysis, and in fact this intuition arose much earlier, in particular in the work of Jacques Hadamard, Fritz John and Jean Leray. Indeed, pseudo-differential operators did exist before the wave front set.

In 1973, Sato and his two students, M. Kashiwara and T. Kawai, published a treatise [SKK73] on the microlocal analysis of LPDE. Certainly this work had a considerable impact, although most analysts didn't understand a single word. Hörmander and his

[3]A presentation of these functors for non specialists may be found in [Sch21].

school then adapted the classical Fourier transform to these new ideas, leading to the now popular theory of Fourier-integral operators (see for example [Hör83, Sjö82]).

The microlocalization functor is the starting point of microlocal analysis but it is also at the origin of the microlocal theory of sheaves, due to Kashiwara and the author [KS82, KS90]. This theory associates to a sheaf on a real manifold M its micro-support, a closed conic subset of the cotangent bundle T^*M and allows one to treat sheaves "microlocally" in T^*M. The theory of systems of LPDE becomes essentially sheaf theory, the only analytic ingredient being the Cauchy-Kowalevsky theorem. One of the deepest result of [SKK73] was the involutivity theorem which asserts that the characteristic variety of a microdifferential system (in particular, of a $\mathcal{D}$-module, see below) is co-isotropic. Similarly the micro-support of a sheaf is proved to be co-isotropic, which makes a link, discovered by Dmitry Tamarkin [Tam12] (see also David Nadler and Eric Zaslow [NZ09] for closely related results) between sheaf theory and symplectic topology, a link at the origin of numerous important results (see for example [Gui23]).

Also note that by a fair return of things, microlocal analysis, through microlocal sheaf theory, appeared quite recently in algebraic geometry, under the impulse of Sacha Beilinson [Bei16] (see [Sai17] for developments).

Already in the 1960s, Sato had the intuition of $\mathcal{D}$-module theory, of holonomic systems and of the b-function (the so-called Bernstein–Sato b-function). He gave a series of talks on these topics at Tokyo University but had to stop for lack of combatants. His ideas were reconsidered and systematically developed by Masaki Kashiwara in his 1969 thesis (see [Kas70], [Kas03]). See [Sch18] for an overview of Kashiwara's work, a part of which was deeply influenced by Sato's ideas. As its name indicates, a $\mathcal{D}$-module is a module over the sheaf of rings $\mathcal{D}$ of differential operators, and a module over a ring essentially means "a system of linear equations"[4] with coefficients in this ring. The task is now to treat (general) systems of LPDE. This theory, which also simultaneously appeared in the more algebraic framework developed by Joseph Bernstein, a student of Israel Gelfand, quickly had considerable success in several branches of mathematics. In 1970–1980, Kashiwara obtained almost all the fundamental results of the theory, in particular those concerned with holonomic modules, such as his constructibility theorem [Kas75], his index theorem for holomorphic solutions of holonomic modules [Kas73], the proof of the rationality of the zeroes of the b-function [Kas76] and his proof of the (regular) Riemann-Hilbert correspondence [Kas80, Kas84].

The mathematical landscape of 1970–1980 had thus considerably changed. Not only did one treat equations with variable coefficients, but one treated systems of such equations and moreover one worked microlocally, that is, in the cotangent bundle, the phase space of the physicists. But there were two schools in the world: the C^∞ school, in the continuation of classical analysis and headed by Hörmander who developed the calculus of Fourier integral operators[5], and the analytic school that Sato established, which was almost nonexistent outside Japan and France.

France was a strategic place to receive Sato's ideas since they are based on, or

[4] According to Mikio Sato (personal communication), at the origin of this idea is the mathematician and philosopher of the 17th century, E. W von Tschirnhaus.

[5] Many names should be quoted at this point, in particular those of Viktor Maslov and Vladimir Arnold.

parallel to, those of both Jean Leray and Alexander Grothendieck. Like Leray, Sato understood that singularities have to be sought in the complex domain, even for the understanding of real phenomena. Sato's algebraic analysis is based on sheaf theory, a theory invented by Leray in 1944 when he was a prisoner of war, clarified by Henri Cartan and made extraordinarily efficient by Grothendieck and his formalism of derived categories and the *six opérations*.

Sato, motivated by physics as usual, then tackled the analysis of the S-matrix in light of microlocal analysis. With his two new students, M. Jimbo and T. Miwa [SMJ78], he explicitly constructed the solution of the n-points function of the Ising model in dimension 2 using Schlesinger classical theory of isomonodromic deformations of ordinary differential equations. This naturally led him to the study of KdV-type non-linear equations. In 1981, with his wife Yasuko Sato (see [SS82] and also [Sat89b]), he interpreted the solutions of the KP-hierarchies as points of an infinite Grasmannian manifold and introduced his famous τ-function. These results would be applied to other classes of equations and would have a great impact in mathematical physics in the study of integrable systems and field theory in dimension 2.

In parallel with his work in analysis and in mathematical physics, Sato obtained remarkable results in group theory and in number theory.

He introduced the theory of prehomogeneous vector spaces, that is, of linear representations of complex reductive groups with a dense orbit. The important case where the complement of this orbit is a hypersurface gives good examples of b-functions (see [SS74, SK77]).

In 1962, Sato also discovered how to deduce, using a construction of auxiliary (Kuga-Sato) varieties, the Ramanujan conjecture on the coefficients of the modular form Δ from Weil's conjectures concerning the number of solutions of polynomial equations on finite fields. His ideas allowed Michio Kuga and Goro Shimura to treat the case of compact quotients of the Poincaré half-space and one had to wait another 10 years for Pierre Deligne to definitely prove that Weil's conjectures imply Ramanujan and Petersson's conjecture.

Mikio Sato shared the Wolf prize with John Tate in 2002/03. They also share a famous conjecture[6] in number theory concerning the repartition of Frobenius angles. Let P be a degree 3 polynomial with integer coefficients and simple roots. Hasse has shown that for any prime p which does not divide the discriminant of P, the number of solutions of the congruence $y^2 = P(x) \pmod p$ is like $p - a_p$, with $|a_p| \leq 2\sqrt{p}$. When writing $a_p = 2\sqrt{p}\cos\theta_p$ with $0 \leq \theta_p \leq \pi$, the Sato–Tate conjecture predicts that these angles θ_p follow the law $(2/\pi)\sin^2\theta$ (in absence of complex multiplication). Note that Tate was led to this conjecture by the study of algebraic cycles and Sato by computing numerical data.

Sato's most recent works are essentially unpublished (see however [Sat89a]) and have been presented in seminars attended only by a small group of people. They treat an algebraic approach of non-linear systems of PDE, in particular of holonomic systems, of which theta functions are examples of solutions !

[6] For developments on this conjecture, see [Maz06]. See also [Col17] for a fascinating presentation of the Serre-Tate correspondence and some links with Sato and this conjecture.

Looking back, 50 years later, we realize that Sato's approach to mathematics is not so different from that of Grothendieck, that Sato did have the incredible temerity to treat analysis as algebraic geometry and was also able to build the algebraic and geometric tools adapted to his problems.

His influence on mathematics is, and will remain, considerable.

References

[And07] Emmanuel Andronikof, *Interview with Mikio Sato*, Journ. of the American Mathematical Society **54, n 2** (2007).

[Bei16] Alexander Beilinson, *Constructible sheaves are holonomic*, Selecta Mathematica **22** (2016), 1797–181, available at `arXiv:1505.06768`.

[Col17] Pierre Colmez, *Tate's work and the Serre-tate correspondence*, Bull. AMS **54** (2017), 559–573.

[Gui23] Stéphane Guillermou, *Sheaves and symplectic geometry of cotangent bundles*, Astérisque **440** (2023), available at `arXiv:1905.07341`.

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators I, II, III, IV*, Grundlehren der Math. Wiss, vol. 256, 257, 274, 275, Springer-Verlag, 1983, 1985.

[Kas70] Masaki Kashiwara, *Algebraic study of systems of partial differential equations, Master's thesis, Tokyo university 1970*, Mémoires SMF, vol. 63, Soc. Math. France, 1995. xiii+72 pp.

[Kas73] ———, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. **49** (1973), 803–804.

[Kas75] ———, *On the maximally overdetermined systems of linear differential equations I*, Publ. RIMS, Kyoto Univ. **10** (1975), 563–579.

[Kas76] ———, *B-functions and holonomic systems. Rationality of roots of B-functions*, Invent. Math. **38** (1976/77), 33–53.

[Kas80] ———, *Faisceaux constructibles et systèmes holonômes d'équations aux dérivées partielles linéaires à points singuliers réguliers*, Séminaire Goulaouic-Schwartz, exp 19 (1980).

[Kas84] ———, *The Riemann-Hilbert problem for holonomic systems*, Publ. RIMS, Kyoto Univ. **20** (1984), 319–365.

[Kas03] ———, *$\mathcal{D}$-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003. xvi+254 pp.

[KS82] Masaki Kashiwara and Pierre Schapira, *Microsupport des faisceaux: applications aux modules différentiels*, C. R. Acad. Sci. Paris **295, 8** (1982), 487–490.

[KS90] ———, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1990. x+512.

[Maz06] Barry Mazur, *Controlling our errors*, Nature **443** (2006).

[NZ09] David Nadler and Eric Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.

[Sai17] Takeshi Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Invent. Math. **207** (2017), 597–695.

[Sat59] Mikio Sato, *Theory of hyperfunctions, I & II*, Journ. Fac. Sci. Univ. Tokyo **8** (1959, 1960), 139–193, 487–436.

[Sat70] ———, *Regularity of hyperfunctions solutions of partial differential equations*, Actes ICM 1970, vol. 2, Gauthier-Villars, 1970.

[Sat89a] ———, *$\mathcal{D}$-modules and nonlinear systems*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, 1989.

[Sat89b] ———, *The KP hierarchy and infinite-dimensional Grassmann manifolds*, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), 51–66, Proc. Sympos. Pure Math., vol. 49, Part 1, Amer. Math. Soc., 1989.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Springer Berlin, 1973, pp. 265–529, Lecture Notes in Math. 287.

[SK77] Mikio Sato and Tatsuo Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

[SMJ78] Mikio Sato, Tetsuji Miwa, and Michio Jimbo, *Holonomic quantum fields I-V*, Publ. Res. Inst. Math. Sci. **14, 15, 15, 15, 16** (1978, 79, 79, 79, 80), 223–267, 201–278, 577–629, 871–972, 531–584.

[SS82] Mikio Sato and Yasuko Sato, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold*, Nonlinear partial differential equations in applied science, Tokyo, North-Holland, 1982, pp. 259–271. Lecture Notes Numer. Appl. Anal., 5.

[SS74] Mikio Sato and Takuro Shintani, *On zeta functions associated with prehomogeneous vector spaces*, Ann. of Math. **100** (1974), 131–170.

[Sch18] Pierre Schapira, *Fifty years of Mathematics with Masaki Kashiwara*, Proc. Int. Cong. Math. Rio de Janeiro, Plenary Lectures, vol. 1, World Sci, 2018.

[Sch21] ———, *Microlocal analysis and beyond*, New spaces in Mathematics, Mathieu Anel and Gabriel Catren eds, Cambridge University Press, 2021, pp. 117–152.

[Sjö82] Johannes Sjöstrand, *Singularités analytiques microlocales*, Astérisque, vol. 95, Soc. Math. France, 1982.

[Tam12] Dmitry Tamarkin, *Algebraic and analytic microlocal analysis*, Microlocal conditions for non-displaceability, Vol. 269, Springer, proceedings in Mathematics & Statistics, 2012&2013, pp. 99–223, available at `arXiv:0809.1584`.

Pierre Schapira

Sorbonne Université, CNRS IMJ-PRG

4 place Jussieu, 75252 Paris Cedex 05 France

e-mail: pierre.schapira@imj-prg.fr

http://webusers.imj-prg.fr/˜pierre.schapira/