Unusual functorialities for weakly constructible sheaves

Andreas Hohl* and Pierre Schapira

Abstract

We prove that various morphisms related to the six Grothendieck operations on sheaves become isomorphisms when restricted to (weakly) constructible sheaves.

1 Introduction

Let $\mathbf{k}$ be a commutative unital ring of finite global dimension and denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves on a good topological space X. There are some classical morphisms which are, in general, not isomorphisms, such as

$$\begin{aligned} &\mathrm{R}\mathscr{H}om\,(F_1, F_2)\overset{\mathrm{L}}{\otimes} F_3 \to \mathrm{R}\mathscr{H}om\,(F_1, F_2\overset{\mathrm{L}}{\otimes} F_3), \\ &f^! G_1 \overset{\mathrm{L}}{\otimes} f^{-1} G_2 \to f^!(G_1 \overset{\mathrm{L}}{\otimes} G_2), \end{aligned}$$

for $F_1, F_2, F_3 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ and $f\colon X \to Y$ a continuous map. We will prove here that, under suitable hypotheses of (weak) constructibility, these morphisms become isomorphisms.

Assuming that the duality functor is conservative in $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ (see (3.3)), we prove the following results in the categories of real analytic manifolds and weakly $\mathbb{R}$-constructible sheaves. (See Theorems 4.2, 4.4 and 4.7.)

Let $f\colon X \to Y$ be a morphism of real analytic manifolds, and let $F_1, F_2, L_X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, $G, L_Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_Y)$ with moreover F_1 being $\mathbb{R}$-constructible, and L_X, L_Y locally constant. Then we have the isomorphisms

$$\begin{aligned} &f^! G \otimes f^{-1} L_Y \xrightarrow{\sim} f^!(G \otimes L_Y), \\ &f^{-1}\mathrm{R}\mathscr{H}om\,(L_Y, G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(f^{-1}L_Y, f^{-1}G), \\ &\mathrm{R}\mathscr{H}om\,(F_1, F_2) \otimes L_X \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(F_1, F_2 \otimes L_X), \\ &\mathrm{R}\mathscr{H}om\,(L_X, F_2) \otimes F_1 \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(L_X, F_2 \otimes F_1). \end{aligned}$$

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For direct images we need slightly stronger assumptions: consider a morphism $f\colon X_\infty \to Y_\infty$ of b-analytic manifolds (see [Sch23] for this notion) and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ be weakly $\mathbb{R}$-constructible up to infinity and L_Y be locally constant. We prove the isomorphisms

$$\begin{aligned}&\mathrm{R}f_*F \otimes L_Y \xrightarrow{\sim} \mathrm{R}f_*(F \otimes f^{-1}L_Y),\\&\mathrm{R}f_!\mathrm{R}\mathscr{H}om\,(f^{-1}L_Y, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(L_Y, \mathrm{R}f_!F).\end{aligned}$$

We start by introducing the notion (implicitly already defined in [KS90, § 3.4]) of weakly cohomologically constructible sheaves (wcc-sheaves, for short) and the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of such sheaves. On a real analytic manifold, the category of $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ contains the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ of weakly $\mathbb{R}$-constructible sheaves.

We prove first that $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ is triangulated. Then our main tool is that for an object F of this category, for $x \in X$ and $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$, one has functorial isomorphisms

$$\mathrm{R}\mathscr{H}om\,(L_X, F)_x \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(L, F_x), \quad \mathrm{R}\Gamma_x F \otimes L \xrightarrow{\sim} \mathrm{R}\Gamma_x(F \otimes L_X),$$

where L_X denotes the constant sheaf associated with L.

The motivation for this note came through the preprint [Hoh23], where field extensions for sheaves are considered, and many of the desired functorialities of loc. cit. indeed follow from the more general set-up developed here.

We make the conjecture that the above results hold without Hypothesis (3.3).

2 Preliminaries

In all this paper, we work in a given universe $\mathcal{U}$. All limits and colimits (in particular, products and direct sums) are assumed to be small. Recall that $\mathbf{k}$ is a a commutative unital ring of finite global dimension. We assume that all topological spaces are "good", that is, Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

For a topological space X as above, one denotes by $\mathrm{Mod}(\mathbf{k}_X)$ the Grothendieck abelian category of sheaves of $\mathbf{k}$-modules and by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ its bounded derived category. We need a slight modification of the notion of cohomologically constructible sheaves (see [KS90, Def. 3.4.1]).

We mainly follow the notations of [KS90]. In particular,

- ω_X denotes the dualizing complex and D_X the duality functor $\mathrm{R}\mathscr{H}om\,(\,\bullet\,, \omega_X)$,
- D denotes the duality functor on $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$,
- $a_X\colon X \to \{\mathrm{pt}\}$ denotes by the unique map from X to a one-point space. Hence, for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, $\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}a_{X*}F$.
- For $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$, L_X denotes the constant sheaf on X with stalk L. More generally, for Z locally closed in X, one denotes by L_{XZ} the constant sheaf L_Z on Z extended by 0 on $X \setminus Z$. When Z is closed, we shall simply denote by L_Z the sheaf L_{XZ}.
- For $x \in X$, denoting by $i_x\colon \{x\} \hookrightarrow X$ the embedding, and for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one denotes as usual by $F_x = i_x^{-1}F$ its stalk at x. One also sets $\mathrm{R}\Gamma_x F = i_x^! F$.

Ind-objects

We shall make use of ind-objects. For a short exposition see [KS90, § 1.11]. For a more detailed study, including new results that we shall use here, see [KS06, Ch. 6, § 8.6, Ch. 15]. Let us recall a few facts that we need, skipping some delicate questions of universes.

If $\mathscr{C}$ is a category, one denotes by $\mathrm{Ind}(\mathscr{C})$ the category of ind-objects of $\mathscr{C}$, a full subcategory of the category $\mathscr{C}^\wedge$ of functors from $\mathscr{C}^{\mathrm{op}}$ to **Set**. Recall [KS06, § 6.1]

- The natural functor $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is fully faithful.
- The category $\mathrm{Ind}(\mathscr{C})$ admits small filtrant colimits, denoted "colim".
- Let $\mathscr{I}$ be a small and filtrant category and $\alpha\colon \mathscr{I} \to \mathscr{C}$ a functor. Let $T\colon \mathscr{C} \to \mathscr{C}'$ be a functor. Then $T(\text{“colim”}\,\alpha) \simeq \text{“colim”}\,(T \circ \alpha)$.

Now we assume that $\mathscr{C}$ is abelian. Recall [KS06, Th. 8.6.5] that

- The category $\mathrm{Ind}(\mathscr{C})$ is abelian and the fully faithful functor $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is exact.
- Small filtrant colimits are exact in $\mathrm{Ind}(\mathscr{C})$.

One should be aware that even if $\mathscr{C}$ is a Grothendieck category, $\mathrm{Ind}(\mathscr{C})$ does not admit enough injectives in general.

Let $\mathscr{I}$ be a small category and $\alpha\colon \mathscr{I} \to \mathscr{C}$ be a functor. As already mentioned, one denotes by "colim" α its colimit in $\mathrm{Ind}(\mathscr{C})$. Note that if $\mathscr{C}$ admits colimits, denoted colim, there is a natural morphism in $\mathrm{Ind}(\mathscr{C})$:

$$\text{“colim”}\,\alpha \to \operatorname{colim}\alpha$$

but this morphism is not an isomorphism in general. However:

- If "colim" α belongs to $\mathscr{C}$, then $\text{“colim”}\,\alpha \xrightarrow{\sim} \operatorname{colim}\alpha$ (see [KS90, Cor. 1.11.7]). In this case, if $T\colon \mathscr{C} \to \mathscr{C}'$ is a functor, then $\text{“colim”}\,(T \circ \alpha) \xrightarrow{\sim} T(\operatorname{colim}\alpha)$. Therefore, "colim" $(T \circ \alpha)$ belongs to $\mathscr{C}$ and hence is isomorphic to $\operatorname{colim}(T \circ \alpha)$.

3 Weakly cohomologically constructible sheaves

Definition 3.1. Let $F \in \mathrm{D^b}(\mathbf{k}_X)$. We say that F is *weakly cohomologically constructible* (wcc for short) if for all $x \in X$, one has the isomorphisms

$$\underset{x\in U}{\text{“colim”}}\,\mathrm{R}\Gamma(U;F) \xrightarrow{\sim} F_x, \quad \mathrm{R}\Gamma_x F \xrightarrow{\sim} \underset{x\in U}{\text{“lim”}}\,\mathrm{R}\Gamma_c(U;F).$$

We denote by $\mathrm{D^b_{wcc}}(\mathbf{k}_X)$ the full subcategory of $\mathrm{D^b}(\mathbf{k}_X)$ consisting of weakly cohomologically constructible sheaves,

Remark 3.2. As explained in [KS90, Rem. 4.3.2], the isomorphisms in Definition 3.1 hold as soon as the objects $\underset{x\in U}{\text{“colim”}}\,\mathrm{R}\Gamma(U;F)$ and $\underset{x\in U}{\text{“lim”}}\,\mathrm{R}\Gamma_c(U;F)$ are representable.

Proposition 3.3. *The category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ *is triangulated.*

Proof. (i) Remark first that for $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ and $j \in \mathbb{Z}$, one has

$$\text{“}\mathrm{colim}_{x\in U}\text{”}\, H^j(U;F) \xrightarrow{\sim} H^j(F)_x.$$

(ii) Clearly, if $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$, then so does the shifted sheaf $F[j]$ for $j \in \mathbb{Z}$.

(iii) Consider a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume that $F', F'' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$. We get the morphism of long exact sequence in the abelian category $\mathrm{Mod}(\mathbf{k})$

$$\tag{3.1} \begin{array}{ccccccccc} \cdots \longrightarrow & H^j(U;F') & \longrightarrow & H^j(U;F) & \longrightarrow & H^j(U;F'') & \longrightarrow & H^{j+1}(U;F') & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^j(F')_x & \longrightarrow & H^j(F)_x & \longrightarrow & H^j(F'')_x & \longrightarrow & H^{j+1}(F')_x & \longrightarrow \cdots . \end{array}$$

Applying the functor “$\mathrm{colim}_{x\in U}$” and using [KS06, Th. 8.6.5], the first line gives rise to the long exact sequence in the abelian category $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}))$:

$$\tag{3.2} \cdots \to H^j(F')_x \to \text{“}\mathrm{colim}_{x\in U}\text{”}\, H^j(U;F) \to H^j(F'')_x \to H^{j+1}(F')_x \to \cdots$$

We shall apply [KS06, Lem. 15.4.6], following its notations, to the category $\mathscr{C} = \mathrm{Mod}(\mathbf{k})$. Consider the morphism

$$\varphi\colon \text{“}\mathrm{colim}_{x\in U}\text{”}\, \mathrm{R}\Gamma(U;F) \to F_x.$$

It follows from (3.1) that $IH^j(\varphi)$ is an isomorphism for all $j \in \mathbb{Z}$ and therefore φ is an isomorphism by loc. cit.

(iv) The proof for $\mathrm{R}\Gamma_x F$ is the same and we do not repeat it. □

Proposition 3.4. *Let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$*. Then* $\mathrm{D}_X F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$*. Moreover, one has the isomorphisms* $\mathrm{R}\Gamma_x \mathrm{D}_X F \simeq \mathrm{D}(F_x)$ *and* $(\mathrm{D}_X F)_x \simeq \mathrm{D}(\mathrm{R}\Gamma_x F)$*.*

We shall adapt the proof of [KS90, Prop. 4.3.4 (iii)].

Proof. (i) The first isomorphism holds without any hypothesis, see [KS90, Exe. viii 3].

(ii) Recall first the isomorphism for U open and $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$:

$$\mathrm{R}\Gamma(U;\mathrm{D}_X F) \simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;F),\mathbf{k}).$$

Now assume that $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$. Applying the functor “$\mathrm{colim}_{x\in U}$”, we get the isomorphisms

$$\begin{aligned} \text{“}\mathrm{colim}_{x\in U}\text{”}\, \mathrm{R}\Gamma(U;\mathrm{D}_X F) &\simeq \mathrm{RHom}\,(\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;F),\mathbf{k}) \\ &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_x F,\mathbf{k}) = \mathrm{D}(\mathrm{R}\Gamma_x F). \end{aligned}$$

This proves that $\text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U;\mathrm{D}_X F)$ is representable, hence isomorphic to $(\mathrm{D}_X F)_x$. This also proves the second isomorphism.

(iii) Let K be a compact neighborhood of x. One has

$$\begin{aligned}\mathrm{R}\Gamma_K(X,\mathrm{D}_X F) &\simeq \mathrm{RHom}\,(\mathbf{k}_{XK},\mathrm{D}_X F)\\ &\simeq \mathrm{RHom}\,(F_K,\omega_X) \simeq \mathrm{RHom}\,(\mathrm{R}\Gamma(X;F_K),\mathbf{k}).\end{aligned}$$

Now, denote by $\dot{K}$ the interior of K. We have

$$\begin{aligned}\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;\mathrm{D}_X F) &\simeq \text{“}\lim_{x\in \dot{K}}\text{”}\, \mathrm{R}\Gamma_K(X;\mathrm{D}_X F)\\ &\simeq \mathrm{RHom}\,(\text{“colim”}_{x\in\dot{K}}\, \mathrm{R}\Gamma(X;F_K),\mathbf{k}) \simeq \mathrm{RHom}\,(\text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U;F),\mathbf{k})\\ &\simeq \mathrm{D}(F_x).\end{aligned}$$

This proves that $\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;\mathrm{D}_X F)$ is representable, hence isomorphic to $\mathrm{R}\Gamma_x \mathrm{D}_X F$. ☐

Proposition 3.5. *Let $F\in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$, let $L\in\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ and let $x\in X$. Then $F\overset{\mathrm{L}}{\otimes}L$ and $\mathrm{R}\mathscr{H}om\,(L_X,F)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$. Moreover one has the isomorphisms*

$$\mathrm{R}\mathscr{H}om\,(L_X,F)_x \xrightarrow{\sim} \mathrm{RHom}\,(L,F_x),\quad (\mathrm{R}\Gamma_x F)\overset{\mathrm{L}}{\otimes}L \xrightarrow{\sim} \mathrm{R}\Gamma_x(F\otimes L_X),$$

Proof. (i) One has

$$\begin{aligned}\mathrm{RHom}\,(L,F_x) &\simeq \mathrm{RHom}\,(L,\text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U;F)) \simeq \text{“colim”}_{x\in U}\, \mathrm{RHom}\,(L,\mathrm{R}\Gamma(U;F))\\ &\simeq \text{“colim”}_{x\in U}\, \mathrm{RHom}\,(L_{XU},F) \simeq \text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U;\mathrm{R}\mathscr{H}om\,(L_X,F)).\end{aligned}$$

This proves that $\text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U;\mathrm{R}\mathscr{H}om\,(L_X,F))$ is representable as well as the first isomorphism.

(ii) One has

$$\begin{aligned}(\mathrm{R}\Gamma_x F)\overset{\mathrm{L}}{\otimes}L &\simeq (\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;F))\overset{\mathrm{L}}{\otimes}L \simeq \text{“}\lim_{x\in U}\text{”}(\mathrm{R}\Gamma_c(U;F)\overset{\mathrm{L}}{\otimes}L)\\ &\simeq \text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;F\overset{\mathrm{L}}{\otimes}L_X).\end{aligned}$$

This proves that $\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;F\otimes L_X)$ is representable as well as the second isomorphism. ☐

All along this paper, we shall consider the hypothesis

(3.3) For any $M\in\mathrm{D}^{\mathrm{b}}(\mathbf{k})$, if $\mathrm{D}M\simeq 0$, then $M\simeq 0$.

Since $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ is triangulated, this is equivalent to saying that the functor $\mathrm{D}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k})^{\mathrm{op}}\to \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ is conservative.

Example 3.6. (i) Hypothesis (3.3) is obviously satisfied if $\mathbf{k}$ is a field. Indeed, in this case, $M\simeq\bigoplus_j H^j(M)\,[-j]$ and we are reduced to the case where M is a vector space. The result then follows since the map $M\to \mathrm{DD}M$ is injective.

(ii) This property is satisfied when $\mathbf{k}=\mathbb{Z}$. See [KS90, Exe. I.31].

Proposition 3.7. *Assume* (3.3)*. Then the functor* $\prod_{x\in X} \mathrm{R}\Gamma_x(\bullet)\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ *is conservative.*

Proof. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$ and assume that $\mathrm{R}\Gamma_x F \simeq 0$ for all $x \in X$. By Proposition 3.4, we get that $\mathrm{D}_X F \simeq 0$. Hence $\mathrm{R}\Gamma_x \mathrm{D}_X F \simeq 0$ and by the same proposition, we get that $\mathrm{D}(F_x) \simeq 0$. Using the hyptohesis (3.3), we get $F \simeq 0$. □

4 Weakly $\mathbb{R}$-constructible sheaves

The property of being weakly cohomologically constructible is not stable by the six operations. That is why we shall consider instead weakly $\mathbb{R}$-constructible sheaves. Hence, from now on, all manifolds and morphisms of manifolds will be real analytic.

Recall (see [KS90, Exe. I.30]) that $M \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ is *perfect* if it is isomorphic to a bounded complex of finitely generated projective $\mathbf{k}$-modules. If M is perfect, then so is $\mathrm{D}(M)$ and the morphism $M \to \mathrm{DD}(M)$ is an isomorphism. We shall denote by $\mathrm{D}^{\mathrm{b}}_{f}(\mathbf{k})$ the full triangulated category of $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ consisting of perfect objects.

Let X be a real analytic manifold. As already mentioned, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$) the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of weakly $\mathbb{R}$-constructible (resp. $\mathbb{R}$-constructible) sheaves on X.

Recall that $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ if and only if its micro-support $\mathrm{SS}(F)$ is contained in a conic subanalytic isotropic subset of T^*X and this is equivalent to the fact that $\mathrm{SS}(F)$ is a conic subanalytic Lagrangian subset.

If $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, then $F_1 \overset{\mathrm{L}}{\otimes} F_2$ and $\mathrm{R}\mathscr{H}om\,(F_1, F_2$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. Moreover, if $f\colon X \to Y$ is a morphism of real analytic manifolds and $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_Y)$, then $f^{-1}G$ and $f^!G$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. If $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ and f is proper on $\mathrm{supp}(F)$, then $\mathrm{R}f_!F \xrightarrow{\sim} \mathrm{R}f_*F$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_Y)$. This follows from [KS90, Prop. 8.4.6].

Finally recall ([KS90, § 8.4] that $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ is $\mathbb{R}$-constructible if for any $x \in X$, F_x is perfect.

Lemma 4.1. *Weakly* $\mathbb{R}$*-constructible sheaves are weakly cohomologically constructible. In other words, the category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ *is a full triangulated subcategory of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\mathbf{k}_X)$*.*

This result is implicitly proved in [KS90, Pro. 8.4.9]. For the reader's convenience, we repeat the proof (this is basically a slightly more detailed version of the proof of [KS90, Lemma 8.4.7]).

Proof. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. We want to prove that F is weakly cohomologically constructible, and this is a local problem, so we can assume $X = \mathbb{R}^n$.

Let $x \in X$ and consider the real analytic function $\varphi\colon X \to \mathbb{R}, y \mapsto |y-x|$. Since F is weakly $\mathbb{R}$-constructible, its micro-support $SS(F)$ is a closed conic subanalytic isotropic subset of T^*X (see [KS90, Th. 8.4.2]). We can therefore apply the microlocal Bertini-Sard theorem (see [KS90, Prop. 8.3.12]), which shows that there exists $b \in \mathbb{R}$ such that for all $y \in X$ with $0 < \varphi(y) < b$ we have $d\varphi(y) \notin SS(F)$.

Now, it follows from the microlocal Morse lemma (see [KS90, Cor. 5.4.19]) that for any $a \in \mathbb{R}$ with $0 < a < b$, the natural morphisms

$$\mathrm{R}\Gamma(B_b(x); F) \to \mathrm{R}\Gamma(B_a(x); F) \to F_x$$

are isomorphisms. (The second one is not directly part of the lemma, but is easily deduced, cf. e.g [KS90, Remark 2.6.9]).

Similarly (using $-\varphi$ instead), we get that for a suitable $b \in \mathbb{R}$ and $0 < a < b$ the natural morphisms

$$\mathrm{R}\Gamma_x F \to \mathrm{R}\Gamma_{\overline{B_a(x)}}(X;F) \to \mathrm{R}\Gamma_{\overline{B_b(x)}}(X;F)$$

and

$$\mathrm{R}\Gamma_c(B_a(x);F) \to \mathrm{R}\Gamma_c(B_b(x);F)$$

are isomorphisms.

Since the balls $B_a(x)$ make up a fundamental system of open neighborhoods of x, we obtain

$$\text{“}\mathrm{colim}_{x\in U}\text{”}\, \mathrm{R}\Gamma(U;F) \simeq \text{“}\mathrm{colim}_{a\longrightarrow 0}\text{”}\, \mathrm{R}\Gamma(B_a(x);F) \simeq F_x,$$

$$\text{“}\lim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;F) \simeq \text{“}\lim_{a\longrightarrow 0}\text{”}\, \mathrm{R}\Gamma_c(B_a(x);F) \simeq \text{“}\lim_{a\longrightarrow 0}\text{”}\, \mathrm{R}\Gamma_{\overline{B_a(x)}}(X;F) \simeq \mathrm{R}\Gamma_x F.$$

This completes the proof. □

Inverse images

Theorem 4.2. *Let $f\colon X \to Y$ be a morphism of real analytic manifolds. Let $L_Y, G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_Y)$ and assume that L_Y is locally constant. Then*

(a) $f^{-1}\mathrm{R}\mathscr{H}om(L_Y, G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(f^{-1}L_Y, f^{-1}G)$.

(b) *Assume* (3.3). *Then* $f^!G\overset{\mathrm{L}}{\otimes}f^{-1}L_Y \xrightarrow{\sim} f^!(G\overset{\mathrm{L}}{\otimes}L_Y)$.

Proof. Since the problem is local on Y, we may assume that L_Y is the constant sheaf associated with for some $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$. Hence $f^{-1}L_Y \simeq L_X$.

(a) Let $x \in X$, and set $y = f(x)$. Applying Proposition 3.5, one gets

$$\begin{aligned}\mathrm{R}\mathscr{H}om(L_X, f^{-1}G)_x &\simeq \mathrm{R}\mathscr{H}om(L, (f^{-1}G)_x)\\ &\simeq \mathrm{R}\mathscr{H}om(L, G_y) \simeq \mathrm{R}\mathscr{H}om(L_Y, G)_y \simeq (f^{-1}\mathrm{R}\mathscr{H}om(L_Y, G))_x.\end{aligned}$$

(b) Remark first that for any sheaf H on Y, one has $\mathrm{R}\Gamma_x(f^!H) \simeq \mathrm{R}\Gamma_y H$. Then using Proposition 3.5, one has

$$\begin{aligned}\mathrm{R}\Gamma_x(f^!G\overset{\mathrm{L}}{\otimes}L_X) &\simeq (\mathrm{R}\Gamma_x f^!G)\overset{\mathrm{L}}{\otimes}L \simeq (\mathrm{R}\Gamma_y G)\overset{\mathrm{L}}{\otimes}L\\ \mathrm{R}\Gamma_x f^!(G\overset{\mathrm{L}}{\otimes}L_Y) &\simeq \mathrm{R}\Gamma_y(G\overset{\mathrm{L}}{\otimes}L_Y) \simeq (\mathrm{R}\Gamma_y G)\overset{\mathrm{L}}{\otimes}L.\end{aligned}$$

Set $A = f^!G\overset{\mathrm{L}}{\otimes}L_X$ and $B = f^!(G\overset{\mathrm{L}}{\otimes}L_Y)$. We have proved that the morphism $A \to B$ induces for all $x \in X$ an isomorphism $\mathrm{R}\Gamma_x A \simeq \mathrm{R}\Gamma_x B$. Then $A \simeq B$ by Proposition 3.7. □

Tensor product and hom

We shall make use of the following result, well-known among specialists. However, we shall give a proof for the reader's convenience.

Lemma 4.3. *Let $L, M \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ and let $N \in \mathrm{D}^{\mathrm{b}}_{f}(\mathbf{k})$. Then*

$$\mathrm{RHom}\,(L, M)\overset{\mathrm{L}}{\otimes}N \xrightarrow{\sim} \mathrm{RHom}\,(L, M\overset{\mathrm{L}}{\otimes}N). \tag{4.1}$$

Proof. By Hypothesis, we may represent N with a bounded complex of projective modules of finite rank.
(i) Assume first that $N = P$ is concentrated in a single degree. If P is of finite rank, there exists an integer n and an epimorphism $\mathbf{k}^n \twoheadrightarrow P$. If moreover P is projective, then this epimorphism has a retract and we get $\mathbf{k}^n \simeq P \oplus Q$. This proves the result in this case.
(ii) Now assume that N is represented by the complex $0 \to P^0 \to \cdots \to P^m \to 0$ with all P^j's projective of finite rank. Here we assume for simplicity in the notations that P^0 is in degree 0. Assume that the result is proved Let us use the so-called "stupid truncation". Denote by N_0 the complex $0 \to P^0 \to \cdots \to P^{m-1} \to 0$ and by $u\colon N \to N_0$ the natural morphism. We have an exact sequence of complexes $0 \to P^m[-m] \to N \xrightarrow{u} N_0 \to 0$ and it follows that the triangle $P^m[-m] \to N \xrightarrow{u} N_0 \xrightarrow{+1}$ is distinguished. Arguing by induction on m the proof is complete. □

Let X and Y be real analytic manifolds. As usual, one denotes by q_1 and q_2 the projections from $X \times Y$ to X and Y, respectively. One denotes $\delta\colon X \to X \times X$ the diagonal morphism. One denotes by $\overset{\mathrm{L}}{\boxtimes}$ the external product

$$F \overset{\mathrm{L}}{\boxtimes} G := q_1^{-1}F\overset{\mathrm{L}}{\otimes}q_2^{-1}G.$$

Recall [KS90, Prop. 3.4.4] that for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ and $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$, one has the isomorphism

$$\mathrm{D}_X F \overset{\mathrm{L}}{\boxtimes} G \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(q_1^{-1}F, q_2^{!}G). \tag{4.2}$$

Also note the isomorphism, for $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$:

$$(F_1 \overset{\mathrm{L}}{\boxtimes} F_2)\overset{\mathrm{L}}{\otimes}(G_1 \overset{\mathrm{L}}{\boxtimes} G_2) \simeq (F_1\overset{\mathrm{L}}{\otimes}G_1) \overset{\mathrm{L}}{\boxtimes} (F_2\overset{\mathrm{L}}{\otimes}G_2). \tag{4.3}$$

Theorem 4.4. *Let $L_X, F_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, with L_X locally constant and let $F_2 \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. Then*

(a) $\mathrm{R}\mathscr{H}om\,(L_X, F_1)\overset{\mathrm{L}}{\otimes}F_2 \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(L_X, F_1\overset{\mathrm{L}}{\otimes}F_2)$.

(b) *Assume* (3.3). *Then* $\mathrm{R}\mathscr{H}om\,(F_2, F_1)\overset{\mathrm{L}}{\otimes}L_X \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(F_2, F_1\overset{\mathrm{L}}{\otimes}L_X)$.

Proof. We may assume that L_X is the constant sheaf associated with $L \in \mathrm{D^b}(\mathbf{k})$. The fact that $F_1 \overset{\mathrm{L}}{\otimes} F_2$ and $\mathrm{R}\mathscr{H}om(L_X, F_1 \overset{\mathrm{L}}{\otimes} F_2)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ follows from [KS90, Prop. 8.4.6].

(a) Let $x \in X$. One has

$$(\mathrm{R}\mathscr{H}om(L_X, F_1) \overset{\mathrm{L}}{\otimes} F_2)_x \simeq \mathrm{R}\mathscr{H}om(L_X, F_1)_x \overset{\mathrm{L}}{\otimes} F_{2x} \simeq \mathrm{RHom}(L, F_{1x}) \overset{\mathrm{L}}{\otimes} F_{2x} \\ \simeq \mathrm{RHom}(L, F_{1x} \overset{\mathrm{L}}{\otimes} F_{2x}) \simeq (\mathrm{R}\mathscr{H}om(L_X, F_1 \overset{\mathrm{L}}{\otimes} F_2))_x.$$

The second and fourth isomorphisms follow from Proposition 3.5 and the third one from Lemma 4.3.

(b) One has

$$\mathrm{R}\mathscr{H}om(F_2, F_1) \overset{\mathrm{L}}{\otimes} L_X \simeq \delta^!(\mathrm{D}_X F_2 \overset{\mathrm{L}}{\boxtimes} F_1) \overset{\mathrm{L}}{\otimes} \delta^{-1}(\mathbf{k}_X \overset{\mathrm{L}}{\boxtimes} L_X) \simeq \delta^!((\mathrm{D}_X F_2 \overset{\mathrm{L}}{\boxtimes} F_1) \overset{\mathrm{L}}{\otimes} (\mathbf{k}_X \overset{\mathrm{L}}{\boxtimes} L_X)) \\ \simeq \delta^!((\mathrm{D}_X F_2 \overset{\mathrm{L}}{\otimes} \mathbf{k}_X) \overset{\mathrm{L}}{\boxtimes} (F_1 \overset{\mathrm{L}}{\otimes} L_X)) \simeq \mathrm{R}\mathscr{H}om(F_2, F_1 \overset{\mathrm{L}}{\otimes} L_X).$$

Here, the second isomorphism follows from Theorem 4.2 (b). The other ones follow from (4.2) and (4.3). ☐

Direct images

Let $f: X \to Y$ be a morphism of real analytic manifolds. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ and $L_Y \in \mathrm{D^b}(\mathbf{k}_Y)$ being locally constant. One can ask if the morphism

$$\mathrm{R}f_* F \overset{\mathrm{L}}{\otimes} L_Y \to \mathrm{R}f_*(F \overset{\mathrm{L}}{\otimes} f^{-1} L_Y)$$

is an isomorphism. The answer is negative in general, even if we require F to be constructible, thanks to an example of [Hoh23, Rem. 4.6].

However, there is a positive answer when considering sheaves *constructible up to infinity*. Before proving the result for general direct images, let us establish it in the particular case of open embeddings.

Lemma 4.5. *Let $j: U \hookrightarrow X$ be the open embedding of a subanalytic relatively compact open subset U of X. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_U)$ and assume that there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ with $j^{-1}G \simeq F$. Let $L_X \in \mathrm{D^b}(\mathbf{k}_X)$ be locally constant. Then*

(a) $\mathrm{R}j_! \mathrm{R}\mathscr{H}om(j^{-1}L_X, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(L_X, \mathrm{R}j_! F)$.

(b) *Assume (3.3). Then* $\mathrm{R}j_* F \overset{\mathrm{L}}{\otimes} L_X \xrightarrow{\sim} \mathrm{R}j_*(F \overset{\mathrm{L}}{\otimes} j^{-1} L_X)$.

Proof. As above, we may assume that L_X is the constant sheaf associated with $L \in \mathrm{D^b}(\mathbf{k})$. Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ be such that $j^{-1}G \simeq F$. Then $\mathrm{R}j_* F \simeq \mathrm{R}\Gamma_U G \simeq \mathrm{R}\mathscr{H}om(\mathbf{k}_{XU}, G)$ and $\mathrm{R}j_! F \simeq G_U \simeq \mathbf{k}_{XU} \otimes G$.

(a) Note that $j^{-1}\mathrm{R}\mathscr{H}om(L_X, G) \simeq \mathrm{R}\mathscr{H}om(j^{-1}L_X, F)$. Using Theorem 4.4 (a), we get

$$\begin{aligned}\mathrm{R}j_!\mathrm{R}\mathscr{H}om(j^{-1}L_X, F) &\simeq \mathbf{k}_{XU} \otimes \mathrm{R}\mathscr{H}om(L_X, G) \simeq \mathrm{R}\mathscr{H}om(L_X, G \otimes \mathbf{k}_{XU})\\ &\simeq \mathrm{R}\mathscr{H}om(L_X, \mathrm{R}j_!F).\end{aligned}$$

(b) Using Theorem 4.4 (b), we have

$$\begin{aligned}\mathrm{R}j_*F\overset{\mathrm{L}}{\otimes}L_X \simeq \mathrm{R}\mathscr{H}om(\mathbf{k}_{XU}, G)\overset{\mathrm{L}}{\otimes}L_X &\simeq \mathrm{R}\mathscr{H}om(\mathbf{k}_{XU}, G\overset{\mathrm{L}}{\otimes}L_X)\\ &\simeq \mathrm{R}j_*j^{-1}(G \otimes L_X) \simeq \mathrm{R}j_*(F\overset{\mathrm{L}}{\otimes}j^{-1}L_X).\end{aligned}$$

□

Recall the following notions extracted from [Sch23].

Definition 4.6. A b-analytic manifold X_∞ is a pair $(X, \widehat{X})$ with $X \subset \widehat{X}$ an open embedding of real analytic manifolds such that X is relatively compact and subanalytic in $\widehat{X}$.

A morphism $f\colon X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ of b-analytic manifolds is a morphism of real analytic manifolds $f\colon X \to Y$ such that the graph Γ_f of f in $X \times Y$ is subanalytic in $\widehat{X} \times \widehat{Y}$.

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. One says that F is "weakly constructible up to infinity" or simply weakly b-constructible, if $j_{X!}F$ (or, equivalently, $\mathrm{R}j_*F$) belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ consisting of weakly b-constructible sheaves.

Theorem 4.7. *Let $f\colon X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and let $L_Y \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ be locally constant. Then $\mathrm{R}f_!F$ and $\mathrm{R}f_*F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and*

(a) $\mathrm{R}f_!\mathrm{R}\mathscr{H}om(f^{-1}L_Y, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(L_Y, \mathrm{R}f_!F)$.

(b) *Assume (3.3). Then* $(\mathrm{R}f_*F)\overset{\mathrm{L}}{\otimes}L_Y \xrightarrow{\sim} \mathrm{R}f_*(F\overset{\mathrm{L}}{\otimes}f^{-1}L_Y)$.

Proof. Here again we may assume that L_Y is the constant sheaf associated with some $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$. Set for short $Z = \widehat{X} \times \widehat{Y}$ and denote by q_1 and q_2 the first and second projection from Z to $\widehat{X}$ and $\widehat{Y}$. Denote by $\Gamma_f \subset Z$ the graph of f. Note that, Γ_f is subanalytic in Z by definition, and relatively compact in Z since it is contained in the relatively compact subset $X \times Y$.

One has

$$\begin{aligned}\mathrm{R}f_!F &\simeq j_Y^{-1}\mathrm{R}q_{2!}(q_1^{-1}j_{X!}F\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f}),\\ \mathrm{R}f_*F &\simeq j_Y^!\mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om(\mathbf{k}_{\Gamma_f}, q_1^!\mathrm{R}j_{X*}F).\end{aligned}$$

Note that the supports of $q_1^{-1}j_{X!}F \otimes \mathbf{k}_{\Gamma_f}$ and $\mathrm{R}\mathscr{H}om(\mathbf{k}_{\Gamma_f}, q_1^!\mathrm{R}j_{X*}F)$ are contained in Γ_f and hence compact in Z.

(a) Let us apply the functor $\mathrm{R}\mathscr{H}om(L_Y, \bullet)$ to the first isomorphism. We get

$$\begin{aligned}
\mathrm{R}\mathscr{H}om(L_Y, \mathrm{R}f_!F) &\simeq \mathrm{R}\mathscr{H}om(j_Y^{-1}L_{\widehat{Y}}, j_Y^{-1}\mathrm{R}q_{2*}(q_1^{-1}j_{X!}F\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f}))\\
&\simeq j_Y^{-1}\mathrm{R}\mathscr{H}om(L_{\widehat{Y}}, \mathrm{R}q_{2*}(q_1^{-1}j_{X!}F\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f}))\\
&\simeq j_Y^{-1}\mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om(L_{\widehat{X}\times\widehat{Y}}, q_1^{-1}j_{X!}F\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f})\\
&\simeq j_Y^{-1}\mathrm{R}q_{2*}(\mathrm{R}\mathscr{H}om(q_1^{-1}L_{\widehat{X}}, q_1^{-1}j_{X!}F)\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f})\\
&\simeq j_Y^{-1}\mathrm{R}q_{2*}(q_1^{-1}\mathrm{R}\mathscr{H}om(L_{\widehat{X}}, j_{X!}F)\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f})\\
&\simeq j_Y^{-1}\mathrm{R}q_{2*}(q_1^{-1}j_{X!}\mathrm{R}\mathscr{H}om(L_X, F)\overset{\mathrm{L}}{\otimes}\mathbf{k}_{\Gamma_f})\\
&\simeq \mathrm{R}f_!\mathrm{R}\mathscr{H}om(f^{-1}L_Y, F).
\end{aligned}$$

The second and fifth isomorphism use Theorem 4.2 (a), the fourht uses Theorem 4.4 (a), and the sixth isomorphism uses Lemma 4.5 (a). On the other hand, the third isomorphims is classical (see [KS90, (2.6.15)]).
(b) The proof is completely analogous, using parts (b) of the statements mentioned above instead. ☐

Recall the isomorphisms which hold for any $F \in \mathrm{D^b}(\mathbf{k}_X)$ and $L \in \mathrm{D^b}(\mathbf{k})$. One has

$$\begin{aligned}
&\mathrm{R}\Gamma_c(X; F\overset{\mathrm{L}}{\otimes}L_X)) \xrightarrow{\sim} \mathrm{R}\Gamma_c(X;F)\overset{\mathrm{L}}{\otimes}L,\\
&\mathrm{R}\Gamma(X;\mathrm{R}\mathscr{H}om(L_X,F)) \simeq \mathrm{RHom}(L, \mathrm{R}\Gamma(X;F)).
\end{aligned} \tag{4.4}$$

Corollary 4.8. *Let $F \in \mathrm{D^b_{w\mathbb{R}c}}(\mathbf{k}_X)$ and let $L \in \mathrm{D^b}(\mathbf{k})$. Let U be an open relatively compact subanalytic subset of X. Then*

(a) $\mathrm{R}\Gamma_c(U;\mathrm{R}\mathscr{H}om(L_X,F)) \xrightarrow{\sim} \mathrm{RHom}(L, \mathrm{R}\Gamma_c(U;F))$.

(b) *Assume* (3.3). *Then* $\mathrm{R}\Gamma(U, F\overset{\mathrm{L}}{\otimes}L_X) \simeq \mathrm{R}\Gamma(U;F)\overset{\mathrm{L}}{\otimes}L$.

Proof. Apply Theorem 4.7 to the sheaf $F|_U$ with $X_\infty = (U, X)$, $Y = \mathrm{pt}$ and $f = a_U$. ☐

Note that this corollary applies in particular when $X_\infty = (X, \widehat{X})$ is b-analytic and $F \in \mathrm{D^b_{w\mathbb{R}c}}(\mathbf{k}_{X_\infty})$. In this case, one gets

$$\begin{aligned}
&\mathrm{R}\Gamma_c(X;\mathrm{R}\mathscr{H}om(L_X,F)) \xrightarrow{\sim} \mathrm{RHom}(L, \mathrm{R}\Gamma_c(X;F)),\\
&\mathrm{R}\Gamma(X, F\overset{\mathrm{L}}{\otimes}L_X) \simeq \mathrm{R}\Gamma(X;F)\overset{\mathrm{L}}{\otimes}L.
\end{aligned}$$

Duality

From our above result, we obtain slight generalisations of [KS90, Exe. VIII.3].

Recall that for any $G \in \mathrm{D^b}(\mathbf{k}_Y)$, one has $\mathrm{D}_X f^{-1}G \simeq f^!\mathrm{D}_Y G$.

Corollary 4.9. *Let $f\colon X\to Y$ be a morphism of real analytic manifolds. Let $G\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_Y)$ and assume that $L_Y\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ is locally constant. Then*

$$\mathrm{D}_X f^!(G\overset{\mathrm{L}}{\otimes}L_Y)\simeq f^{-1}\mathrm{D}_Y(G\overset{\mathrm{L}}{\otimes}L_Y).$$

Proof. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{D}_X f^!(G\overset{\mathrm{L}}{\otimes}L_Y)&\simeq \mathrm{R}\mathscr{H}om\,(f^!G\overset{\mathrm{L}}{\otimes}f^{-1}L_Y,\omega_X)\\&\simeq \mathrm{R}\mathscr{H}om\,(f^{-1}L_Y,\mathrm{D}_X(f^!G))\simeq \mathrm{R}\mathscr{H}om\,(f^{-1}L_Y,f^{-1}\mathrm{D}_YG)\\&\simeq f^{-1}\mathrm{R}\mathscr{H}om\,(L_Y,\mathrm{D}_YG)\simeq f^{-1}\mathrm{D}_Y(G\overset{\mathrm{L}}{\otimes}L_Y).\end{aligned}$$

□

Recall that for any $F\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one has $\mathrm{D}_Y\mathrm{R}f_!F\simeq\mathrm{R}f_*\mathrm{D}_XF$.

Corollary 4.10. *Let $f\colon X_\infty\to Y_\infty$ be a morphism of b-analytic manifolds. Let $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and let $L_Y\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ be locally constant. Then*

$$\mathrm{R}f_!\mathrm{D}_X(F\overset{\mathrm{L}}{\otimes}f^{-1}L_Y)\simeq\mathrm{D}_Y(\mathrm{R}f_*F\overset{\mathrm{L}}{\otimes}L_Y).$$

Proof. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{R}f_!\mathrm{D}_X(F\overset{\mathrm{L}}{\otimes}f^{-1}L_Y)&\simeq\mathrm{R}f_!\mathrm{R}\mathscr{H}om\,(f^{-1}L_Y,\mathrm{D}_XF)\\&\simeq\mathrm{R}\mathscr{H}om\,(L_Y,\mathrm{R}f_!\mathrm{D}_XF)\simeq\mathrm{R}\mathscr{H}om\,(L_Y,\mathrm{D}_Y\mathrm{R}f_*F)\\&\simeq\mathrm{D}_Y(\mathrm{R}f_*F\overset{\mathrm{L}}{\otimes}L_Y).\end{aligned}$$

□

Micro-support

Remark 4.11. Consider a field extension $\mathbf{l}$ of the field $\mathbf{k}$. Then of course all preceding results apply with $L=\mathbf{l}_X$ (the case of interest in [Hoh23]). Moreover, note that [KS90, Rem. 5.1.5] asserts that if *for* denotes the forgetful functor

$$\mathit{for}\colon\mathrm{D}^{\mathrm{b}}(\mathbf{l}_X)\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X),$$

and if $F\in\mathrm{D}^{\mathrm{b}}(\mathbf{l}_X)$, then the micro-support of F and that of $\mathit{for}(F)$ are the same.

To conclude, let us consider the action of the functors $L\otimes\bullet$ and $\mathrm{R}\mathscr{H}om\,(L,\bullet)$ on the micro-support. Remark first that for $F,L_X\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ with L_X locally constant, one has

$$\mathrm{SS}(L_X\overset{\mathrm{L}}{\otimes}F)\subset\mathrm{SS}(F),\quad\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(L_X,F))\subset\mathrm{SS}(F).\tag{4.5}$$

Indeed, one has by [KS90, Prop. 5.4.14]

$$\mathrm{SS}(L_X\overset{\mathrm{L}}{\otimes}F)\subset T^*_XX+\mathrm{SS}(F)=\mathrm{SS}(F),$$

and similarly with $\mathrm{R}\mathscr{H}om\,(L_X,F)$.

Proposition 4.12. *Assume that* $\mathbf{k}$ *is a field. Let* $F, L_X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ *with* L_X *locally constant,* $L_X \neq 0$*. Then*

$$\mathrm{SS}(L_X \otimes F) = \mathrm{SS}(F), \quad \mathrm{SS}(\mathrm{R}\mathscr{H}om\,(L_X, F)) = \mathrm{SS}(F).$$

Proof. The problem is local and we may assume that L_X is the constant sheaf associated with $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$. Then $L = \oplus_j H^j(L)\,[-j]$ and we may assume that $L \in \mathrm{Mod}(\mathbf{k})$. In this case, there exists $K \in \mathrm{Mod}(\mathbf{k})$ such that $L \simeq \mathbf{k} \oplus K$, hence $L_X \simeq \mathbf{k}_X \oplus K_X$ and the result follows (see e.g. [KS90, Prop. 5.1.3]). □

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Andreas Hohl
Université Paris Cité et Sorbonne Université,
CNRS, IMJ-PRG, F-75013 Paris, France
e-mail: andreas.hohl@imj-prg.fr
https://www.andreashohl.eu

Pierre Schapira
Sorbonne Université, CNRS IMJ-PRG
e-mail: pierre.schapira@imj-prg.fr
http://webusers.imj-prg.fr/~pierre.schapira/