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Thickening of the diagonal and interleaving distance

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Abstract

Given a topological space X, a thickening kernel is a monoidal presheaf on $(\mathbb{R}_{\geq 0}, +)$ with values in the monoidal category of derived kernels on X. A bi-thickening kernel is defined on $(\mathbb{R}, +)$. To such a thickening kernel, one naturally associates an interleaving distance on the derived category of sheaves on X. We prove that a thickening kernel exists and is unique as soon as it is defined on an interval containing 0, allowing us to construct (bi-)thickenings in two different situations. First, when X is a "good" metric space, starting with small usual thickenings of the diagonal. The associated interleaving distance satisfies the stability property and Lipschitz kernels give rise to Lipschitz maps. Second, by using (Guillermou et al. in Duke Math J 161:201–245, 2012), when X is a manifold and one is given a non-positive Hamiltonian isotopy on the cotangent bundle. In case X is a complete Riemannian manifold having a strictly positive convexity radius, we prove that it is a good metric space and that the two bi-thickening kernels of the diagonal, one associated with the distance, the other with the geodesic flow, coincide.

Keywords Sheaves · Interleaving distance · Persistent homology · Riemannian manifolds · Hamiltonian isotopies

Mathematics Subject Classification 55N99 · 18A99 · 35A27

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Introduction

The aim of this paper is to construct (and then to study) a kernel associated with a small thickening of the diagonal of a space X and, as a byproduct, an interleaving distance on the derived category of sheaves on X. Such a kernel is constructed in essentially two rather different situations: first when X is a metric space by using the distance, second when X is a manifold and one is given a non-positive Hamiltonian isotopy of the cotangent bundle. When X is a Riemannian manifold and the isotopy is associated with the geodesic flow, we prove that the two kernels coincide.

The interleaving distance introduced by Chazal et al. [9] has become a central element of TDA and has been actively studied since then [3–6]. It was generalised to multi-persistence modules by M. Lesnick in [23, 24]. Categorical frameworks for the interleaving distance have then been proposed in [7, 14]. In his thesis [12], J. Curry proposed an approach of persistence homology via sheaf theory. In [22], the author developed derived sheaf-technics for persistent homology and defined a new interleaving distance for the category of *derived* sheaves on a real normed vector space by considering thickenings associated with the convolution by closed balls of radius $a \geq 0$. This distance is sometimes called the convolution distance for sheaves and has recently been applied to question of symplectic topology (see for instance [1]). For a survey of the links between the (1-dimensional) interleaving distance, sheaf theory and symplectic topology, see the book by J. Zhang [32].

Let X be a "good" topological space and denote as usual by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on X, for a commutative unital ring of finite global dimension $\mathbf{k}$. We define a thickening kernel on X as a monoidal presheaf $\mathfrak{K}$ defined on the monoidal category $(\mathbb{R}_{\geq 0}, +)$ with values in the monoidal category $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X}), \circ)$ of kernels on X (see Definition 1.2.2). When this presheaf extends as a monoidal presheaf on $(\mathbb{R}, +)$, we call it a bi-thickening kernel of the diagonal.

To a thickening kernel, one naturally associates an interleaving distance dist_X on $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$.

Our first result (Theorem 1.2.3) asserts that a thickening kernel exists and is unique (up to isomorphism) as soon as it is constructed on some interval $[0, \alpha_X]$ (with $\alpha_X > 0$).

This theorem allows us to construct a (bi-)thickening kernel in two different situations. First in Sect. 2, when X is what we call here a *good metric space* (see Definition 2.1.1). Second in Sect. 3, when X is a real manifold and one is given a non-positive C^∞-function $h\colon \dot{T}^*X \to \mathbb{R}$, where $\dot{T}^*X$ is the cotangent bundle with the zero-section removed.

(1) Assume that (X, d_X) is a good metric space and denote by Δ_a the closed thickening of radius $a \geq 0$ of the diagonal. The hypothesis that (X, d_X) is good implies in particular that $\mathbf{k}_{\Delta_a} \circ \mathbf{k}_{\Delta_b} \simeq \mathbf{k}_{\Delta_{a+b}}$ for a, b sufficiently small (see (1.2) for the definition of $\circ$). Applying our first theorem, we get a thickening kernel $\mathfrak{K}$ on $(\mathbb{R}_{\geq 0}, +)$ or, under mild extra-hypotheses, a bi-thickening. In this case, for $a < 0$ small, $\mathfrak{K}_a$ is, up to a shift and an orientation, the kernel associated with an open thickening of the diagonal.

We obtain several results on the associated interleaving distance, some of them generalizing those of [22]. We prove in particular a stability theorem (Theorem 2.4.1) which asserts that given two kernels K_1 and K_2 on $Y \times X$ and a sheaf F on X, then $\mathrm{dist}_Y(K_1 \circ F, K_2 \circ F) \leq \mathrm{dist}_{Y\times X/X}(K_1, K_2)$ where $\mathrm{dist}_{Y\times X/X}$ is a relative distance. We also introduce the notion of a *δ-Lipschitz kernel* on $Y \times X$ and show that such a kernel induces a Lipschitz map for the interleaving distances (Theorem 2.5.4). In both cases (stability and Lipschitz) we also obtain similar results for non proper composition, but then we need to assume that our spaces are manifolds and the differential of the distance does not vanish. Indeed, in this situation, our proofs are based on Theorem 1.1.6 which asserts that under some microlocal hypotheses, non proper composition becomes associative.

(2) Assume now that X is a real manifold and one is given a C^∞-function $h\colon \dot{T}^*X \to \mathbb{R}$, homogeneous of degree 1 in the fiber such that the flow Φ of the Hamiltonian vector field of h is an Hamiltonian isotopy defined on $\dot{T}^*X \times I$ for some open interval I containing 0. This flow gives rise to a Lagrangian manifold $\Lambda \subset \dot{T}^*X \times \dot{T}^*X \times T^*I$. Thanks to the main theorem of [17], there exists a unique kernel $K^h \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{X\times X\times I})$ micro-supported by Λ and whose restriction to $t = 0$ is $\mathbf{k}_\Delta$. Moreover, since h is not time depending, this kernel satisfies $K^h_a \circ K^h_b \simeq K^h_{a+b}$ for a, b small. Assuming h is non-positive, there are natural morphisms $K^h_b \to K^h_a$ for $a \leq b$ and using our first theorem we get a bi-thickening kernel $\mathfrak{K}^h$.

When X is a complete Riemannian manifold having a strictly positive convexity radius, we prove (Theorem 3.2.3) that it is a good metric space and the associated thickening kernel is a bi-thickening, denoted here $\mathfrak{K}^{\mathrm{dist}}$. We have thus two bi-thickening kernels in this case, $\mathfrak{K}^{\mathrm{dist}}$ and $\mathfrak{K}^h$, the last one being associated with the geodesic flow (corresponding to $h(x,\xi) = -||\xi||_x$). We prove in Theorem 3.3.7 that these two kernels coincide.

In the course of the paper, we treat some easy examples and in particular we prove that the Fourier-Sato transform, an equivalence of categories for sheaves on a sphere and the dual sphere, is an isometry when endowing these spheres with their natural Riemannian metric. Indeed, the Fourier-Sato transform is nothing but the value at $\pi/2$ of the thickening kernel of the Riemannian sphere.

1 Sheaves and the interleaving distance

1.1 Sheaves

In the sequel, we denote by pt the topological space with a single element. For a topological space X, we denote by $a_X\colon X \to \mathrm{pt}$ the unique map from X to pt. We denote by Δ_X, or simply Δ, the diagonal of $X \times X$ and by δ_X or simply δ the diagonal embedding. If X is a C^∞-manifold, we denote by $\pi_X\colon T^*X \to X$ its cotangent bundle and by $\dot{T}^*X$ the cotangent bundle with the zero-section removed. Recall that a topological space X is good if it is Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

We consider a commutative unital ring of finite global dimension $\mathbf{k}$ and a good topological space X. We denote by $\mathrm{D}(\mathbf{k}_X)$ the derived category of sheaves of $\mathbf{k}$-modules on X and simply call an object of this category "a sheaf". We shall almost always work in the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ but we shall also need to consider the full subcategory $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_X)$ of $\mathrm{D}(\mathbf{k}_X)$ consisting of locally bounded objects, that is, objects whose restriction to any relatively compact open subset U of X belong to $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$ (see [17, Def. 1.12]).

We shall freely make use of the six Grothendieck operations on sheaves and refer to [20]. In particular, we denote by ω_X the dualizing complex and we use the duality functors

$$\mathrm{D}'_X = \mathrm{R}\mathscr{H}om(\bullet, \mathbf{k}_X), \quad \mathrm{D}_X = \mathrm{R}\mathscr{H}om(\bullet, \omega_X).$$

For a locally closed subset $A \subset X$, we denote by $\mathbf{k}_A$ the sheaf on X which is the constant sheaf with stalk $\mathbf{k}$ on A and 0 elsewhere. If F is a sheaf on X, one sets $F_A := F \otimes \mathbf{k}_A$. We also often simply denote by $F \otimes L$ the derived tensor product when L is of the type $\mathbf{k}_A$ up to a shift or an orientation. As usual, we denote by $\mathrm{R}\Gamma(X;\,\bullet)$ and $\mathrm{R}\Gamma_c(X;\,\bullet)$ the derived functors of global sections and global sections with compact supports.

When X is a C^∞-manifold, we shall make use of the microlocal theory of sheaves, following [20, Ch. V–VI]. Recall that the *micro-support* $\mathrm{SS}(F)$ of a sheaf F is a closed $\mathbb{R}^+$-conic subset of T^*X, co-isotropic for the homogeneous symplectic structure of

T^*X (we shall not use here this property). We shall also use the notation $\dot{\mathrm{SS}}(F) := \mathrm{SS}(F) \cap \dot{T}^*X$. We shall also encounter *cohomologically constructible* sheaves for which we refer to loc. cit. § 3.4. Recall that, on a real analytic manifold, $\mathbb{R}$-constructible sheaves (see loc. cit. Ch. VIII) are cohomologically constructible.

Kernels

Given topological spaces X_i ($i = 1, 2, 3$) we set $X_{ij} = X_i \times X_j$, $X_{123} = X_1 \times X_2 \times X_3$. We denote by $q_i\colon X_{ij} \to X_i$ and $q_{ij}\colon X_{123} \to X_{ij}$ the projections.

We shall often write for short D_i instead of D_{X_i}, as well as for similar notations such as for example D'_i or D_{ij}.

For $A \subset X_{12}$ and $B \subset X_{23}$ one sets $A \circ B = q_{13}(q_{12}^{-1}A \cap q_{23}^{-1}B)$:

$$\begin{tikzcd} & X_{123} \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ X_{12} & X_{13} & X_{23}. \end{tikzcd} \tag{1.1}$$

When the spaces X_i's are real manifolds, one denotes by $p_{ij}\colon T^*X_{123} \to T^*X_{ij}$ the projection and we also define

$$p_{i^a j}\colon T^*X_{123} \to T^*X_{ij}, \quad (x_1, x_2, x_3; \xi_1, \xi_2, \xi_3) \mapsto (x_i, x_j; -\xi_i, \xi_j)$$

the composition of p_{ij} with the antipodal map of T^*X_i.

For $A \subset T^*X_{12}$ and $B \subset T^*X_{23}$ one sets

$$A \overset{a}{\circ} B = p_{13}(p_{12}^{-1}A \cap p_{2^a 3}^{-1}B)$$

For good topological spaces X_i's as above, one often calls an object $K_{ij} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$ *a kernel*. One defines as usual the composition of kernels

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}(q_{12}^{-1}K_{12} \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_{23}). \tag{1.2}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

It is sometimes natural to permute the roles of X_i and X_j. We introduce the notation

$$\begin{aligned} &v\colon X_{12} \to X_{21}, \quad (x_1, x_2) \mapsto (x_2, x_1), \\ &\nu\colon X_{123} \to X_{321}, (x_1, x_2, x_3) \mapsto (x_3, x_2, x_1). \end{aligned} \tag{1.3}$$

Since v and ν are involutions, one has

$$v_* \simeq v_!,\ v^{-1} \simeq v^!, \quad \nu_* \simeq \nu_!,\ \nu^{-1} \simeq \nu^!. \tag{1.4}$$

Using (1.4), one immediately obtains:

Proposition 1.1.1 *Let $K_{ij} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$, $i = 1, 2$, $j = i+1$ and set $K_{ji} := v_* K_{ij}$. Then*

$$v_*(K_{12} \underset{2}{\circ} K_{23}) \simeq K_{32} \underset{2}{\circ} K_{21}.$$

In the sequel, we shall need to control the micro-support of the composition. Let X_i and K_{ij} be as above $i = 1, 2$, $j = i+1$. Let $A_{ij} = \mathrm{SS}(K_{ij}) \subset T^*X_{ij}$ and assume that

$$\begin{cases} \text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\,\mathrm{supp}(K_{12}) \cap q_{23}^{-1}\,\mathrm{supp}(K_{23}), \\ \text{(ii) } p_{12}^{-1}A_{12} \cap p_{2^a3}^{-1}A_{23} \cap (T^*_{X_1}X_1 \times T^*X_2 \times T^*_{X_3}X_3) \subset T^*_{X_{123}}X_{123}. \end{cases} \tag{1.5}$$

Proposition 1.1.2 *Assume (1.5). Then*

$$\mathrm{SS}(K_{12} \underset{2}{\circ} K_{23}) \subset A_{12} \overset{a}{\circ} A_{23}. \tag{1.6}$$

Proof This follows from the classical bounds to the micro-supports of proper direct images and non-characteristic inverse images of [20, § 5.4]. □

The next lemma will be useful.

Lemma 1.1.3 *Let $A \subset X_{12}$ and $B \subset X_{23}$ be two closed subsets.*

(a) *Assume that q_{13} is proper on $A \times_{X_2} B := q_{12}^{-1}A \cap q_{23}^{-1}B$. Then there is a natural morphism $\mathbf{k}_{A \circ B} \to \mathbf{k}_A \circ \mathbf{k}_B$.*
(b) *Assume moreover that the fibers of the map $q_{13}\colon A \times_{X_2} B \to A \circ B$ are contractible. Then $\mathbf{k}_{A \circ B} \xrightarrow{\sim} \mathbf{k}_A \circ \mathbf{k}_B$.*

Proof

(a) Set $C = q_{12}^{-1}A \cap q_{23}^{-1}B$. Then $q_{13}(C) = A \circ B$ and $\mathbf{k}_C \simeq q_{12}^{-1}\mathbf{k}_A \otimes q_{23}^{-1}\mathbf{k}_B$. By the hypothesis, the set $q_{13}^{-1}q_{13}(C)$ is closed and contains C. Therefore, the morphism $q_{13}^{-1}\mathbf{k}_{q_{13}(C)} \to \mathbf{k}_C$ defines by adjunction the morphism $\mathbf{k}_{A \circ B} \to \mathrm{R}q_{13*}(q_{12}^{-1}\mathbf{k}_A \otimes q_{23}^{-1}\mathbf{k}_B) \xleftarrow{\sim} \mathbf{k}_A \circ \mathbf{k}_B$ (recall that q_{13} is proper on C).
(b) is clear. □

It is easily checked, and well-known, that the composition of kernels is associative, namely given three kernels $K_{ij} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$, $i = 1, 2, 3$, $j = i+1$ one has an isomorphism

$$(K_{12} \underset{2}{\circ} K_{23}) \underset{3}{\circ} K_{34} \simeq K_{12} \underset{2}{\circ} (K_{23} \underset{3}{\circ} K_{34}), \tag{1.7}$$

this isomorphism satisfying natural compatibility conditions that we shall not make here explicit.

Of course, this construction applies in the particular cases where $X_i = \mathrm{pt}$ for some i. For example, if $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{12}})$ and $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_2})$, one usually sets $\Phi_K(F) = K \circ F$. Hence

$$\Phi_K(F) = K \circ F = \mathrm{R}q_{1!}(K \overset{\mathrm{L}}{\otimes} q_2^{-1} F). \tag{1.8}$$

It is natural to consider the right adjoint functor Ψ_K of the functor Φ_K (see [20, Prop. 3.6.2]) given by

$$\Psi_K(G) = \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(K, q_1^! G). \tag{1.9}$$

Given three spaces X_i $(i = 1, 2, 3)$ and kernels K_1 on X_{12} and K_2 on X_{23}, one has (by (1.7) or [20, Prop. 3.6.4])

$$\Phi_{K_2} \circ \Phi_{K_1} \simeq \Phi_{K_2 \circ K_1}, \quad \Psi_{K_1} \circ \Psi_{K_2} \simeq \Psi_{K_2 \circ K_1}. \tag{1.10}$$

Proposition 1.1.4 *Let* $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X})$ *and* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. *Then* $\mathrm{D}_X(\Phi_K(F)) \simeq \Psi_{v_* K}(\mathrm{D}_X F)$.

Proof One has the sequence of isomorphisms

$$\begin{aligned}
\mathrm{D}_X(\Phi_K(F)) &\simeq \mathrm{R}\mathscr{H}om\,(\mathrm{R}q_{1!}(K \overset{\mathrm{L}}{\otimes} q_2^{-1} F), \omega_X) \\
&\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(K \overset{\mathrm{L}}{\otimes} q_2^{-1} F, \omega_{X \times X}) \\
&\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(K, \mathrm{R}\mathscr{H}om\,(q_2^{-1} F, q_2^! \omega_X)) \\
&\simeq \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(K, q_2^! \mathrm{D}_X F).
\end{aligned}$$

□

Also note that when $X_2 = \mathrm{pt}$, that is, $F, K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, then

$$F \circ K \simeq \mathrm{R}\Gamma_c(X; F \overset{\mathrm{L}}{\otimes} K). \tag{1.11}$$

Non proper composition

In many situations, the non proper composition is useful. For $K_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{12}})$ and $K_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{23}})$, one sets

$$K_1 \overset{\mathrm{np}}{\circ} K_2 = \mathrm{R}q_{13*}(q_{12}^{-1} K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1} K_2). \tag{1.12}$$

One shall be aware that in general, this composition is not associative. However, under suitable hypotheses, it becomes associative.

Consider the diagram of good topological spaces

$$\begin{tikzcd}
 & & X_{123} \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & & \\
 & X_{12} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & X_{13} \arrow[dll, "p_1"'] \arrow[drr, "p_2"] & X_{23} \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X_1 & & X_2 & & X_3
\end{tikzcd} \tag{1.13}$$

Note that the squares $(X_{12}, X_2, X_{23}, X_{123})$, $(X_{12}, X_1, X_{13}, X_{123})$ and $(X_{13}, X_3, X_{23}, X_{123})$ are Cartesian.

Lemma 1.1.5 *Let X_i $(i = 1, 2, 3)$ be three C^∞-manifolds. Let $K_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{12}})$ and $K_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{23}})$. Assume that K_1 is cohomologically constructible and $\mathrm{SS}(K_1) \cap (T^*_{X_1}X_1 \times T^*X_2) \subset T^*_{X_{12}}X_{12}$. Then*

$$\mathrm{R}q_{12*}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2) \simeq K_1 \overset{\mathrm{L}}{\otimes} \mathrm{R}q_{12*}q_{23}^{-1}K_2.$$

Proof Applying [20, Prop. 5.4.1], we have

$$\begin{aligned}
&\mathrm{SS}(q_{23}^{-1}K_2) \subset T^*_{X_1}X_1 \times T^*X_{23},\\
&\mathrm{SS}(q_2^{!}\mathrm{R}r_{1*}K_2) \subset T^*_{X_1}X_1 \times T^*X_2.
\end{aligned}$$

Since $\mathrm{R}q_{12*}q_{23}^{!}\, K_2 \simeq q_2^{!}\,\mathrm{R}r_{1*}K_2$ and $\mathrm{SS}(\mathrm{R}q_{12*}q_{23}^{-1}K_2) = \mathrm{SS}(\mathrm{R}q_{12*}q_{23}^{!}\, K_2)$, we get:

$$\mathrm{SS}(\mathrm{R}q_{12*}q_{23}^{-1}K_2) \subset T^*_{X_1}X_1 \times T^*X_2. \tag{1.14}$$

Applying [20, Cor. 6.4.3] we get by the hypothesis and (1.14)

$$K_1 \overset{\mathrm{L}}{\otimes} \mathrm{R}q_{12*}q_{23}^{-1}K_2 \simeq \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_{12}K_1, \mathrm{R}q_{12*}q_{23}^{-1}K_2). \tag{1.15}$$

Moreover, the hypothesis implies $\mathrm{SS}(\mathrm{D}'_{12}K_1) \cap (T^*_{X_1}X_1 \times T^*X_2) \subset T^*_{X_{12}}X_{12}$, hence

$$\mathrm{SS}(q_{12}^{-1}\mathrm{D}'_{12}K_1) \cap T^*_{X_1}X_1 \times T^*X_{23} \subset T^*_{X_{123}}X_{123}.$$

The sheaf K_1 being cohomologically constructible on X_{12}, the sheaf $q_{12}^{-1}K_1 \simeq K_1 \boxtimes \mathbf{k}_{X_3}$ is cohomologically constructible on X_{123}. Applying again [20, Cor. 6.4.3], we get

$$\begin{aligned}
\mathrm{R}\mathscr{H}om\,(q_{12}^{-1}\mathrm{D}'_{12}K_1, q_{23}^{-1}K_2) &\simeq \mathrm{D}'_{123}q_{12}^{-1}\mathrm{D}'_{12}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2\\
&\simeq q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2.
\end{aligned}$$

To conclude, note that

$$\begin{aligned}\mathrm{R}\mathscr{H}om\,(\mathrm{D}'_{12}K_1, \mathrm{R}q_{12*}q_{23}^{-1}K_2) &\simeq \mathrm{R}q_{12*}\mathrm{R}\mathscr{H}om\,(q_{12}^{-1}\mathrm{D}'_{12}K_1, q_{23}^{-1}K_2)\\ &\simeq \mathrm{R}q_{12*}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2).\end{aligned}$$

Using (1.15), the proof is complete. □

Theorem 1.1.6 *Let* X_i $(i = 1, 2, 3, 4)$ *be four* C^∞*-manifolds and let* $K_i \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{i,i+1}})$ $(i = 1, 2, 3)$. *Assume that* K_1 *is cohomologically constructible,* q_1 *is proper on* $\operatorname{supp}(K_1)$ *and* $\mathrm{SS}(K_1) \cap (T^*_{X_1}X_1 \times T^*X_2) \subset T^*_{X_{12}}X_{12}$. *Then*

$$K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} (K_2 \overset{\mathrm{np}}{\underset{3}{\circ}} K_3) \simeq (K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} K_2) \overset{\mathrm{np}}{\underset{3}{\circ}} K_3.$$

Proof We shall assume for simplicity that $X_4 = \mathrm{pt}$. Consider Diagram 1.13. Then:

$$\begin{aligned}K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} (K_2 \overset{\mathrm{np}}{\underset{3}{\circ}} K_3) &= \mathrm{R}q_{1*}\big(K_1 \overset{\mathrm{L}}{\otimes} q_2^{-1}(K_2 \overset{\mathrm{np}}{\circ} K_3)\big)\\ &= \mathrm{R}q_{1*}\big(K_1 \overset{\mathrm{L}}{\otimes} q_2^{-1}\mathrm{R}r_{1*}(K_2 \overset{\mathrm{L}}{\otimes} r_2^{-1}K_3)\big)\\ &\simeq \mathrm{R}q_{1*}\big(K_1 \overset{\mathrm{L}}{\otimes} \mathrm{R}q_{12*}q_{23}^{-1}(K_2 \overset{\mathrm{L}}{\otimes} r_2^{-1}K_3)\big)\\ &\simeq \mathrm{R}q_{1*}\mathrm{R}q_{12*}\big(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}r_2^{-1}K_3\big)\\ &\simeq \mathrm{R}p_{1*}\mathrm{R}q_{13*}\big(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2 \overset{\mathrm{L}}{\otimes} q_{13}^{-1}p_2^{-1}K_3\big)\\ &\simeq \mathrm{R}p_{1*}(\mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2) \overset{\mathrm{L}}{\otimes} p_2^{-1}K_3\big)\\ &\simeq \mathrm{R}p_{1*}\big((K_1 \underset{2}{\circ} K_2) \overset{\mathrm{L}}{\otimes} p_2^{-1}K_3\big) \simeq \mathrm{R}p_{1*}\big((K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} K_2) \overset{\mathrm{L}}{\otimes} p_2^{-1}K_3\big).\end{aligned}$$

In the first isomorphism, we have used $q_2^{-1}\mathrm{R}r_{1*} \simeq \mathrm{R}q_{12*}q_{23}^{-1}$, which follows from the isomorphism $q_2^{!}\mathrm{R}r_{1*} \simeq \mathrm{R}q_{12*}q_{23}^{!}$. In the second isomorphism, we have used Lemma 1.1.5. In the fourth isomorphism, we have used the fact that q_{13} is proper on $\operatorname{supp}(q_{12}^{-1}K_1)$. Finally, in the sixth isomorphism we have again used the fact that q_{13} is proper on $\operatorname{supp} q_{12}^{-1}(K_1)$.

Note that the same proof holds without assuming $X_4 = \mathrm{pt}$. In this case replace X_i, X_{ij} and X_{123} with X_{i4}, X_{ij4} and X_{1234}, respectively. □

1.2 Monoidal presheaves

We shall use the theory of monoidal categories and refer to [19] and [21, Ch. IV]. Note that

- monoidal categories are called tensor categories in [21],

- to a monoidal category $(\mathscr{C}, \otimes)$ is naturally attached an isomorphism of functors ([21, Def. 4.2.1]) $\mathbf{a}(X, Y, Z)\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ satisfying the usual compatibility conditions,
- to a monoidal category with unit $(\mathscr{C}, \otimes, \mathbf{1})$ are naturally attached two functorial isomorphisms $\mathbf{r}\colon X \otimes \mathbf{1} \to X$ and $\mathbf{l}\colon \mathbf{1} \otimes X \to X$, denoted respectively α and β in [21, Lem. 4.2.6].

Example 1.2.1

(i) We regard the ordered set $(\mathbb{R}, \leq)$ as a category that we simply denote by $\mathbb{R}$ and we regard $\mathbb{R}_{\geq 0}$ as a full subcategory. The categories $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ endowed with the addition map $+$ are monoidal categories with unit, denoted $(\mathbb{R}, +)$ and $(\mathbb{R}_{\geq 0}, +)$, respectively.

(ii) Let X be a good topological space. The category $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X}), \circ)$ is a monoidal category with unit the sheaf $\mathbf{k}_\Delta$.

(iii) If $\mathscr{A}$ is a category, then the category $(\mathrm{Fct}(\mathscr{A}, \mathscr{A}), \circ)$ is a monoidal category with unit the object $\mathrm{id}_{\mathscr{A}}$.

Let I be a closed interval of $\mathbb{R}$. We assume

$$\text{either } I = [0, \alpha] \text{ or } I = [-\alpha, \alpha] \text{ for some } \alpha > 0. \tag{1.16}$$

We consider I as an ordered set and we denote by $I_{\leq}$ or simply I the associated category, a full subcategory of $(\mathbb{R}, \leq)$. Hence, $\mathrm{Ob}(I_{\leq}) = I$ and $\mathrm{Hom}_{I_{\leq}}(a, b) = \mathrm{pt}$ or $= \varnothing$ according whether $a \leq b$ or not. Although it has not been precisely defined, we shall look at $I_{\leq}$ as a "partially monoidal subcategory of $(\mathbb{R}, +)$".

Let $(\mathscr{C}, \otimes)$ be a monoidal category and consider a presheaf K on $I_{\leq}$ with values in $\mathscr{C}$. For $a \in I$, we write K_a instead of $K(a)$. Hence, we have "restriction" morphisms $\rho_{a,b}\colon K_b \to K_a$ for $a, b \in I$, $a \leq b$ satisfying the usual compatibility relations $\rho_{a,b} \circ \rho_{b,c} = \rho_{a,c}$ for $a \leq b \leq c$ and $\rho_{a,a} = \mathrm{id}$.

Definition 1.2.2 Let $(\mathscr{C}, \otimes, \mathbf{1})$ be a monoidal category with unit.

(a) A monoidal presheaf (K, ϕ_0, ϕ_2) on $I_{\leq}$ with values in $\mathscr{C}$ is the data of:

(1) a presheaf K on $I_{\leq}$ with value in $\mathscr{C}$,
(2) an isomorphism $\phi_0\colon \mathbf{1} \xrightarrow{\sim} K_0$,
(3) an isomorphism $\phi_2(a, b)\colon K_a \otimes K_b \xrightarrow{\sim} K_{a+b}$, for a, b such that $a, b, a+b \in I$, these data satisfy the following conditions:
(i) the diagram below commutes for all $a, b, a', b' \in I$ such that $a \leq a'$, $b \leq b'$, $a, b, a', b', a+b, a'+b' \in I$:

$$\begin{tikzcd}
K_{a'} \otimes K_{b'} \arrow[r, "\phi_2(a',b')", "\sim"'] \arrow[d, "\rho_{a,a'} \otimes \rho_{b,b'}"'] & K_{a'+b'} \arrow[d, "\rho_{a+b,a'+b'}"] \\
K_a \otimes K_b \arrow[r, "\phi_2(a,b)", "\sim"'] & K_{a+b}.
\end{tikzcd}$$

Here, the vertical arrows are induced by the restriction morphisms.

(ii) For all $a, b, c \in I$ such that $a+b, b+c, a+b+c \in I$, the diagram below commutes

$$\begin{tikzcd}
(K_a \otimes K_b) \otimes K_c \arrow[rr, "\mathbf{a}(K_a, K_b, K_c)"] \arrow[d, "\phi_2(a,b) \otimes \mathrm{id}"'] & & K_a \otimes (K_b \otimes K_c) \arrow[d, "\mathrm{id} \otimes \phi_2(b,c)"] \\
K_{a+b} \otimes K_c \arrow[d, "\phi_2(a+b,c)"'] & & K_a \otimes K_{b+c} \arrow[d, "\phi_2(a,b+c)"] \\
K_{a+b+c} \arrow[rr, equal] & & K_{a+b+c} .
\end{tikzcd}$$

The notation $\mathbf{a}(K_a, K_b, K_c)$ is defined in the second item above Example 1.2.1.

(iii) For all $a \in I$, the diagrams below commute

$$\begin{tikzcd}
\mathbf{1} \otimes K_a \arrow[r, "\mathbf{l}_{K_a}"] \arrow[d, "\phi_0 \otimes \mathrm{id}_{K_a}"'] & K_a \arrow[d, equal] \\
K_0 \otimes K_a \arrow[r, "\phi_2(0,a)"] & K_a ,
\end{tikzcd}
\qquad
\begin{tikzcd}
K_a \otimes \mathbf{1} \arrow[r, "\mathbf{r}_{K_a}"] \arrow[d, "\mathrm{id}_{K_a} \otimes \phi_0"'] & K_a \arrow[d, equal] \\
K_a \otimes K_0 \arrow[r, "\phi_2(a,0)"] & K_a .
\end{tikzcd}$$

(b) Let K and K' be two monoidal presheaves on $I_{\leq}$. A morphism of monoidal presheaves $\eta\colon K \to K'$ is a morphism such that for every $a, b \in I$ such that $a+b \in I$ the following diagram commutes

$$\begin{tikzcd}
K_a \otimes K_b \arrow[r, "\eta_a \otimes \eta_b"] \arrow[d, "\phi_2(a,b)"'] & K'_a \otimes K'_b \arrow[d, "\phi'_2(a,b)"] \\
K_{a+b} \arrow[r, "\eta_{a+b}"] & K'_{a+b} .
\end{tikzcd}$$

(c) We denote by $\mathrm{Fun}^{\otimes}(I^{\mathrm{op}}, \mathscr{C})$ the category whose objects are the monoidal presheaves on $I_{\leq}$ with values in $\mathscr{C}$ and the morphisms are the morphisms of monoidal presheaves.

Assuming that $I = [0, \alpha]$, the inclusion functor $i_\alpha\colon I_{\leq} \hookrightarrow \mathbb{R}_{\geq 0}$ induces a functor

$$i_\alpha^*\colon \mathrm{Fun}^{\otimes}(\mathbb{R}_{\geq 0}^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}^{\otimes}(I^{\mathrm{op}}, \mathscr{C}), \;\; F \mapsto F \circ i_\alpha. \tag{1.17}$$

Similarly, if $I = [-\alpha, \alpha]$, the inclusion functor $j_\alpha\colon I_{\leq} \hookrightarrow \mathbb{R}_{\geq 0}$ induces a functor

$$j_\alpha^*\colon \mathrm{Fun}^{\otimes}(\mathbb{R}^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Fun}^{\otimes}(I^{\mathrm{op}}, \mathscr{C}), \;\; F \mapsto F \circ j_\alpha. \tag{1.18}$$

Theorem 1.2.3 *Assuming that $I = [0, \alpha]$, the functor i_α^* in (1.17) is an equivalence of categories. Similarly, assuming that $I = [-\alpha, \alpha]$, the functor j_α^* in (1.18) is an equivalence of categories.*

Proof (A) Let us first treat the case $I = [0, \alpha]$.
It follows from [19, Ch XI.5] that we can assume that $\mathscr{C}$ is a strict monoidal category. We set $\lambda = \frac{\alpha}{2}$.
(i) We start by showing that i_α^* is essentially surjective. For that purpose, given a monoidal presheaf K on I, we will construct a monoidal presheaf $\mathfrak{K} \colon \mathbb{R}_{\geq 0} \to \mathscr{C}$ such that $i_\alpha^* \mathfrak{K} \simeq K$.
(i)–(a) For $a \geq 0$ we write $a = n\lambda + r_a$ with $0 \leq r_a < \lambda$. Then, one sets

$$\mathfrak{K}_a := \underbrace{K_\lambda \otimes \cdots \otimes K_\lambda}_{n} \otimes K_{r_a}. \tag{1.19}$$

(i)–(b) We now construct the restriction morphisms $\rho_{a,b}$. For $a \leq b \leq \lambda$, $\rho_{a,b}$ is given by the definition of the presheaf K. Let us write $a = m \cdot \lambda + r_a$ and $b = n \cdot \lambda + r_b$ with $0 \leq r_a, r_b < \lambda$. Since $0 \leq a \leq b$, $m \leq n$. If $m = n$, then $r_a \leq r_b$ and we set $\rho_{a,b} := (\mathrm{id}_{K_\lambda})^{\circ m} \circ \rho_{r_a, r_b}$.

Now assume $m > n$. Notice that

$$\begin{aligned}
\mathfrak{K}_b &\simeq (K_\lambda)^{\circ m} \circ K_\lambda \circ (K_\lambda)^{\circ(n-m-1)} \circ K_{r_b} \\
\mathfrak{K}_a &\simeq (K_\lambda)^{\circ m} \circ K_{r_a} \circ (K_0)^{\circ(n-m-1)} \circ K_0.
\end{aligned}$$

Hence, we set $\rho_{a,b} := (\mathrm{id}_{K_\lambda})^{\circ m} \circ \rho_{r_a,\lambda} \circ (\rho_{0,\lambda})^{\circ n-m-1} \circ \rho_{0,r_b}$.
(i)–(c) Let us construct the isomorphisms $\phi_2(a_1, a_2) \colon \mathfrak{K}_{a_1} \otimes \mathfrak{K}_{a_2} \to \mathfrak{K}_{a_1+a_2}$, for $a_1, a_2 \in \mathbb{R}_{\geq 0}$. Write

$$a_i = n_i \cdot \alpha + r_i, \quad 0 \leq r_i < \lambda, \quad i = 1, 2.$$

Since $r_i + \lambda \leq \alpha$, $K_{r_i} \otimes K_\lambda \overset{\phi_2(r_i,\lambda)}{\simeq} K_{r_i+\lambda} \overset{\phi_2^{-1}(\lambda,r_i)}{\simeq} K_\lambda \otimes K_{r_i}$. We set

$$s_i := \phi_2^{-1}(\lambda, r_i) \circ \phi_2(r_i, \lambda)$$

Let $n \in \mathbb{N}$ and consider the map

$$\psi_{i,n} := (\mathrm{id}_{K_\lambda}^{\otimes n-1} \otimes s_i) \circ \ldots \circ (\mathrm{id}_{K_\lambda}^{\otimes p} \otimes s_i \otimes \mathrm{id}_{K_\lambda}^{\otimes n-1-p}) \circ \ldots \circ (s_i \otimes \mathrm{id}_{K_\lambda}^{\otimes n-1}).$$

We now define the map $\phi_2(a_1, a_2) \colon \mathfrak{K}_{a_1} \otimes \mathfrak{K}_{a_2} \to \mathfrak{K}_{a_1+a_2}$ by setting

$$\phi_2(a_1, a_2) := (\mathrm{id}_{K_\lambda^{\otimes(n_1+n_2)}} \otimes \phi_2(r_1, r_2)) \circ (\mathrm{id}_{K_\lambda}^{\otimes n_1} \otimes \psi_{1,n_2} \otimes \mathrm{id}_{K_{r_2}}).$$

By construction, $\phi_2(a_1, a_2)$ is an isomorphism.

It is straightforward to check that $\mathfrak{K}$ is a monoidal presheaf on $\mathbb{R}_{\geq 0}$ and that $i_\alpha^* \mathfrak{K} \simeq K$.
(ii)-(a) Let us prove that i_α^* is faithful. Let $f, g \colon \mathfrak{K} \to \mathfrak{K}'$ be two monoidal morphisms between monoidal presheaves on $\mathbb{R}_{\geq 0}$. Assume that $i_\alpha^*(f) = i_\alpha^*(g)$. Hence, for every $0 \leq a \leq \alpha$, $f_a = g_a$ and it follows from the definition of a monoidal morphism that for every $b \in \mathbb{R}_{\geq 0}$, $f_b = g_b$.

(ii)-(b) Let us show that i_α^* is full. Let $\mathfrak{K}, \mathfrak{K}' \in \mathrm{Fun}^{\otimes}(\mathbb{R}_{\geq 0}^{\mathrm{op}}, \mathscr{C})$ and let $f : i_\alpha^* \mathfrak{K} \to i_\alpha^* \mathfrak{K}'$ be a monoidal morphism. For $a \in \mathbb{R}_{\geq 0}$, we write $a = n\lambda + r_a$ with $0 \leq r_a < \lambda$. We define the morphism $\mathfrak{f}_a$ as the composition

$$\mathfrak{K}_a \simeq \mathfrak{K}_\lambda^{\otimes n} \otimes \mathfrak{K}_{r_a} \xrightarrow{f_\lambda^{\otimes n} \otimes f_{r_a}} \mathfrak{K}_\lambda'^{\otimes n} \otimes \mathfrak{K}'_{r_a} \simeq \mathfrak{K}'_a.$$

The family of morphisms $(\mathfrak{f}_a)_{a \in \mathbb{R}_{\geq 0}}$ defines a monoidal morphism $\mathfrak{f} : \mathfrak{K} \to \mathfrak{K}'$ such that $i_\alpha(\mathfrak{f}) = f$.
(B) Assume now that $I = [-\alpha, \alpha]$. Part (A) of the proof applies when replacing the interval $[0, \alpha]$ and $\mathbb{R}_{\geq 0}$ with the interval $[-\alpha, 0]$ and $\mathbb{R}_{\leq 0}$. Then combine these two cases. □

1.3 Thickening kernels and interleaving distance

Let us first recall that a categorical axiomatic for interleaving distances was developed in [7, 14]. Here, we do not work in an abstract categorical setting but restrict ourselves to the study of kernels for sheaves, a natural framework for applications.

Definition 1.3.1 Let X be a good topological space.

(a) A thickening kernel is a monoidal presheaf $\mathfrak{K}$ on $(\mathbb{R}_{\geq 0}, +)$ with values in the monoidal category $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X}), \circ)$.
(b) The thickening kernel $\mathfrak{K}$ is a bi-thickening kernel if it extends as a monoidal presheaf on $(\mathbb{R}, +)$.

In the sequel, for a thickening (resp. a bi-thickening) kernel $\mathfrak{K}$, one sets $\mathfrak{K}_a = \mathfrak{K}(a)$ for $a \geq 0$ (resp. for $a \in \mathbb{R}$).

In other words, a thickening kernel is a family of kernels $\mathfrak{K}_a \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X})$ satisfying

$$\mathfrak{K}_a \circ \mathfrak{K}_b \simeq \mathfrak{K}_{a+b}, \quad \mathfrak{K}_0 \simeq \mathbf{k}_\Delta \text{ for } a \in \mathbb{R}_{\geq 0}$$

and the compatibility conditions of Definition 1.2.2.

We shall often simply write "a thickening" instead of "a thickening kernel ".

Remark 1.3.2 Let $I = [0, \alpha]$ with $\alpha > 0$. Note that if the thickening (or the bi-thickening) $\mathfrak{K}$ exists, then it is uniquely defined by its restriction to $[0, \alpha]$, up to isomorphism. More precisely, given two thickenings $\mathfrak{K}_1$ and $\mathfrak{K}_2$ and an isomorphism of monoidal presheaves

$$\theta : \mathfrak{K}_1|_I \xrightarrow{\sim} \mathfrak{K}_2|_I,$$

then there exists a unique isomorphism of monoidal presheaves $\lambda : \mathfrak{K}_1 \xrightarrow{\sim} \mathfrak{K}_2$ such that $\lambda|_I = \theta$.

Example 1.3.3

(i) The constant presheaf $a \mapsto \mathbf{k}_\Delta$ is a thickening kernel called the constant thickening on X and simply denoted $\mathbf{k}_\Delta$ (or $\mathbf{k}_{\Delta_X}$ if necessary).

(ii) Let X_i ($i = 1, 2$) be two good topological spaces and let $\mathfrak{K}_i$ be a thickening kernel on X_i. Then $\mathfrak{K}_1 \boxtimes \mathfrak{K}_2$ is a thickening kernel on $X_1 \times X_2$. This applies in particular when $\mathfrak{K}_i$ is the constant thickening on X_1 or X_2.

(iii) Let (X, d_X) be a metric space. We shall prove in Theorem 2.1.6 below that, under suitable hypotheses, there exists a thickening kernel $\mathfrak{K}$ with $\mathfrak{K}_a = \mathbf{k}_{\Delta_a}$ for $0 \le a \le \alpha_X$. For S a good topological space, we sometimes denote by $\mathfrak{K}_{S\times X/S}$ the thickening kernel $\mathbf{k}_{\Delta_S} \boxtimes \mathfrak{K}$.

(iv) Another example of a thickening kernel will be given in Sect. 3.1 in which we use the kernel of [17] associated with a Hamiltonian isotopy.

The next definition is mimicking [22, Def. 2.2].

Definition 1.3.4 Let $\mathfrak{K}$ be a thickening kernel on X, let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and let $a \ge 0$.

(a) One says that F and G are *a-isomorphic* if there are morphisms $f : \mathfrak{K}_a \circ F \to G$ and $g : \mathfrak{K}_a \circ G \to F$ which satisfy the following compatibility conditions: the composition

$$\mathfrak{K}_{2a} \circ F \xrightarrow{\mathfrak{K}_a \circ f} \mathfrak{K}_a \circ G \xrightarrow{g} F$$

and the composition

$$\mathfrak{K}_{2a} \circ G \xrightarrow{\mathfrak{K}_a \circ g} \mathfrak{K}_a \circ F \xrightarrow{f} G$$

coincide with the morphisms induced by the canonical morphism $\rho_{0,2a} : \mathfrak{K}_{2a} \to \mathfrak{K}_0$.

(b) One sets

$$\mathrm{dist}_{\mathfrak{K}}(F, G) = \inf\Big(\{+\infty\} \cup \{a \in \mathbb{R}_{\ge 0}\,;\ F \text{ and } G \text{ are } a\text{-isomorphic}\}\Big)$$

and calls $\mathrm{dist}_{\mathfrak{K}}(\bullet, \bullet)$ the interleaving distance (associated with $\mathfrak{K}$).

Note that if F and G are a-isomorphic, then they are b-isomorphic for any $b \ge a$.

The next result show that the interleaving distance $\mathrm{dist}_{\mathfrak{K}}$ is a pseudo-distance on $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$.

Proposition 1.3.5 *Let $\mathfrak{K}$ be a thickening kernel on X and let $F, G, H \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then*

(i) *F and G are 0-isomorphic if and only if $F \simeq G$,*

(ii) $\mathrm{dist}_{\mathfrak{K}}(F, G) = \mathrm{dist}_{\mathfrak{K}}(G, F)$,

(iii) $\mathrm{dist}_{\mathfrak{K}}(F, G) \le \mathrm{dist}_{\mathfrak{K}}(F, H) + \mathrm{dist}_{\mathfrak{K}}(H, G)$.

The proof is straightforward.

Remark 1.3.6 It is proved in [27] that if X_∞ is a b-analytic manifold (see [30]) endowed with a good distance, then, under suitable hypotheses, the pseudo-distance $\mathrm{dist}_{\mathfrak{K}}$ becomes a distance when restricted to the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{X_\infty})$ of sheaves constructible up to infinity. In particular, on any real analytic manifold X, $\mathrm{dist}_{\mathfrak{K}}$ becomes a distance when restricted to constructible sheaves with compact support. Let us also mention the paper [11] in which the completeness of the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^n})$ is discussed in the case of the convolution distance.

2 The interleaving distance on metric spaces

From now on and until the end of this section, unless otherwise stated, we assume that X is a good topological space and that $\mathbf{k}$ is either a field or $\mathbf{k} = \mathbb{Z}$.

2.1 Thickening of the diagonal

Let (X, d_X) be a metric space. For $a \geq 0$, $x_0 \in X$ and some α_X to be defined in Definition 2.1.1, set

$$\begin{cases} B_a(x_0) = \{x \in X; d_X(x_0, x) \leq a\}, \\ B_a^\circ(x_0) = \{x \in X; d_X(x_0, x) < a\}, \text{ (here, } a > 0), \\ \Delta_a = \{(x_1, x_2) \in X \times X; d_X(x_1, x_2) \leq a\}, \\ \Delta_a^\circ = \{(x_1, x_2) \in X \times X; d_X(x_1, x_2) < a\}, \text{ (here, } a > 0), \\ Z = \{(x_1, x_2, t) \in X \times X \times \mathbb{R}_{\geq 0}; d_X(x_1, x_2) \leq t,\ t < \alpha_X\}, \\ \Omega^+ = \{(x_1, x_2, t) \in X \times X \times \mathbb{R}_{>0}; d_X(x_1, x_2) < t,\ t < \alpha_X\}. \end{cases} \tag{2.1}$$

Definition 2.1.1 A metric space (X, d_X) is good if the underlying topological space is good and moreover there exists some $\alpha_X > 0$ such that for all $0 \leq a, b$ with $a + b \leq \alpha_X$, one has

$$\begin{cases} \text{(i) for any } x_1, x_2 \in X,\ B_a(x_1) \cap B_b(x_2) \text{ is contractible or empty (in par-} \\ \text{ticular, for any } x \in X,\ B_a(x) \text{ is contractible),} \\ \text{(ii) the two projections } q_1 \text{ and } q_2 \text{ are proper on } \Delta_a, \\ \text{(iii) } \Delta_a \circ \Delta_b = \Delta_{a+b}. \end{cases} \tag{2.2}$$

Clearly, in this definition, α_X is not unique. In the sequel, if we want to mention which α_X we choose, we denote the good metric space by (X, d_X, α_X).

Let U be an open subset of a real C^0-manifold M. Recall (see [20, Exe. III.4]) that U is locally cohomologically trivial (l.c.t. for short) in M if for each $x \in \overline{U} \backslash U$, $(\mathrm{R}\Gamma_{\overline{U}}(\mathbf{k}_M))_x \simeq 0$ and $(\mathrm{R}\Gamma_U(\mathbf{k}_M))_x \simeq \mathbf{k}$.

We shall say that U is locally topologically convex (l.t.c. for short) in M if each $x \in M$ admits an open neighborhood W such that there exists a topological isomorphism $\phi \colon W \xrightarrow{\sim} V$, with V open in a real vector space, such that $\phi(W \cap U)$ is convex. Clearly, if U is l.t.c. then it is l.c.t.

The natural morphism $\mathbf{k}_U \to \mathbf{k}_M$ defines a section of $\mathcal{H}om\,(\mathbf{k}_U, \mathbf{k}_M) \simeq \mathcal{H}om\,(\mathbf{k}_U \otimes \mathbf{k}_{\overline{U}}, \mathbf{k}_M)$, hence defines the morphisms:

$$\mathbf{k}_U \to \mathrm{D}'_M \mathbf{k}_{\overline{U}}, \quad \mathbf{k}_{\overline{U}} \to \mathrm{D}'_M \mathbf{k}_U.$$

When U is l.c.t., then these morphisms are isomorphisms. If moreover, U is l.t.c., then these sheaves are cohomologically constructible.

We shall also encounter the hypotheses:

$$\begin{cases}\text{The good metric space } X \text{ is a } C^0\text{-manifold and} \\ \text{a. for } x \in X \text{ and } 0 < a \leq \alpha_X, \text{ the set } B_a^\circ(x) \text{ is l.t.c. in } X, \\ \text{b. for } 0 < a \leq \alpha_X, \text{ the set } \Delta_a^\circ \text{ is l.t.c. in } X \times X, \\ \text{c. the set } \Omega^+ \text{ is l.t.c. in } X \times X \times]-\infty, \alpha_X[. \\ \text{d. For } x, y \in X, \text{ setting } Z_a(x,y) = B_a(x) \cap B_a^\circ(y), \text{ one has} \\ \quad \mathrm{R}\Gamma(X; \mathbf{k}_{Z_a(x,y)}) \simeq 0 \text{ for } x \neq y \text{ and } 0 < a \leq \alpha_X. \end{cases} \tag{2.3}$$

Lemma 2.1.2 *Let (X, d_X) be a good metric space satisfying (2.3) and let $0 < a \leq \alpha_X$.*

(a) *For $x \in X$, the sheaves $\mathbf{k}_{B_a(x)}$ and $\mathbf{k}_{B_a^\circ(x)}$ are cohomologically constructible and dual one to each other for the duality functor D'_X.*
(b) *The sheaves $\mathbf{k}_{\Delta_a}$ and $\mathbf{k}_{\Delta_a^\circ}$ are cohomologically constructible and dual one to each other for the duality functor $\mathrm{D}'_{X\times X}$.*
(c) *The sheaves $\mathbf{k}_Z$ and $\mathbf{k}_{\Omega^+}$ are cohomologically constructible and dual one to each other for the duality functor $\mathrm{D}'_{X\times X\times\mathbb{R}}$.*
(d) *For $x \in X$ one has the isomorphism $\mathbf{k} \xrightarrow{\sim} \mathrm{R}\Gamma_c(B_a^\circ(x); \omega_X)$*

Proof (a)–(b)–(c) follow immediately from the hypothses.
(d) denote by $(\cdot)^*$ the duality functor $\mathrm{RHom}\,(\cdot, \mathbf{k})$. By Poincaré duality (see e.g., [20, (3.1.8)]) one has

$$\begin{aligned}\mathrm{R}\Gamma_c(B_a^\circ(x); \omega_X)^* &\simeq \mathrm{RHom}\,(\mathbf{k}_{B_a^\circ(x)}, \mathbf{k}_X) \\ &\simeq \mathrm{R}\Gamma(X; \mathbf{k}_{B_a(x)}) \simeq \mathbf{k}.\end{aligned}$$

The last isomorphism follows from the fact that $B_a(x)$ is contractible. This completes the proof when $\mathbf{k}$ is a field. Otherwise, when $\mathbf{k} = \mathbb{Z}$, use [20, Exe. I 31]. □

The next hypothesis will be used in order to apply Theorem 1.1.6 and we shall give in Lemma 2.1.3 below a natural criterion in order that it is satisfied.

$$\begin{cases}\text{The good metric space } X \text{ is a } C^\infty\text{-manifold and, for } 0 < a \leq \alpha_X, \\ \mathrm{SS}(\mathbf{k}_{\Delta_a}) \cap (T_X^*X \times T^*X) \subset T_{X\times X}^*X \times X.\end{cases} \tag{2.4}$$

Lemma 2.1.3 *Let (X, d_X) be a good metric space. Assume that X is a C^∞-manifold, the distance function $f := d_X : X \times X \to \mathbb{R}$ is of class C^1 on $W := \Delta_a^\circ \setminus \Delta$ for $a \leq \alpha_X$ and the partial differentials $d_x f$ and $d_y f$ do not vanish on W. Then (2.4) is satisfied.*

Proof Apply [20, Prop. 5.3.3]. □

We shall obtain in Theorems 2.6.1 and 3.2.3 large classes of examples in which hypotheses (2.2), (2.3) and (2.4) are satisfied.

Lemma 2.1.4 *Let (X, d_X) be a good metric space.*

(a) *For every $a, b \geq 0$, $\mathbf{k}_{\Delta_a} \circ \mathbf{k}_{\Delta_b} \simeq \mathbf{k}_{\Delta_b} \circ \mathbf{k}_{\Delta_a}$.*

(b) *For any* $0 \le a, b$ *with* $a + b \le \alpha_X$,

$$\mathbf{k}_{\Delta_a} \circ \mathbf{k}_{\Delta_b} \simeq \mathbf{k}_{\Delta_{a+b}}, \tag{2.5}$$

and the correspondence $a \mapsto \mathbf{k}_{\Delta_a}$ *defines a monoidal presheaf on* $[0, \alpha_X]$ *with values in the monoidal category* $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X}), \circ)$.

Proof

(a) Recall notations (1.3). Since $v^{-1}\mathbf{k}_{\Delta_a} \simeq \mathbf{k}_{v^{-1}(\Delta_a)} \simeq \mathbf{k}_{\Delta_a}$, the result follows.
(b) We shall follow the notations of (1.1) (with $X_i = X$ for all i). Setting $\Delta_a \times_2 \Delta_b = q_{12}^{-1}\Delta_a \cap q_{23}^{-1}\Delta_b$, we have

$$q_{12}^{-1}\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_{23}^{-1}\mathbf{k}_{\Delta_b} \simeq \mathbf{k}_{\Delta_a \times_2 \Delta_b}.$$

The map $q_{13} \colon \Delta_a \times_2 \Delta_b \to \Delta_{a+b}$ is proper, surjective and has contractible fibers by Hypothesis (2.2). Therefore, $\mathrm{R}q_{13!}\mathbf{k}_{\Delta_a \times_2 \Delta_b} \simeq \mathbf{k}_{\Delta_{a+b}}$ by Lemma 1.1.3. The other conditions in Definition 1.2.2 are easily checked. □

We shall refine Definition 1.3.1.

Definition 2.1.5 Let (X, d_X, α_X) be a good metric space.

(a) A *metric thickening kernel of the diagonal* is a thickening kernel whose restriction to $[0, \alpha_X]$ is isomorphic to the monoidal presheaf $a \mapsto \mathbf{k}_{\Delta_a}$ on $[0, \alpha_X]$.
(b) A metric bi-thickening kernel is a bi-thickening kernel whose restriction to $\mathbb{R}_{\ge 0}$ is a metric thickening kernel.

When there is no risk of confusion, (that is, almost always) we shall simply call a metric thickening kernel, "a thickening".

Note that if the metric thickening (or bi-thickening) exists, then it is unique up to isomorphism. This last isomorphism is unique in the sense of Remark 1.3.2.

Theorem 2.1.6 *Let* (X, d_X, α_X) *be a good metric space. There exists a metric thickening* $\mathfrak{K}$ *of the diagonal. Moreover, for each* $a \ge 0$*, the two projections* $q_1, q_2 \colon X \times X \to X$ *are proper on* $\operatorname{supp} \mathfrak{K}_a$.

Proof The first part of the statement follows from Lemma 2.1.4 and Theorem 1.2.3. The properness of q_1 and q_2 on $\operatorname{supp} \mathfrak{K}_a$ for $0 \le a \le \alpha$ follows from Hypothesis (2.2). The general case follows from the construction of the kernel. □

Corollary 2.1.7 *In the preceding situation, let* Y *be a good topological space and let* $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times Y})$. *Then*

$$\mathfrak{K}_a \circ L \xrightarrow{\sim} \mathfrak{K}_a \overset{\mathrm{np}}{\circ} L \text{ for } a \ge 0.$$

(*See* (1.12) *for the notation* $\overset{\mathrm{np}}{\circ}$.)

Non proper composition for the distance kernels

Proposition 2.1.8 *Let (X, d_X, α_X) be a good metric space satisfying (2.3) and (2.4). Then for $a \geq 0$, and for smooth real manifolds X_i $(i = 2, 3)$ setting $X = X_1$, we have for any $L_i \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$ with $i = 1, 2,\ j = i + 1$,*

$$\mathfrak{K}_a \circ (L_1 \overset{\mathrm{np}}{\circ} L_2) \simeq (\mathfrak{K}_a \circ L_1) \overset{\mathrm{np}}{\circ} L_2.$$

Proof

(i) Assume first that $0 \leq a < \alpha_X$. In this case, $\mathfrak{K}_a = \mathbf{k}_{\Delta_a}$ is cohomologically constructible and q_1 is proper on its support. Using hypothesis (2.4), we may apply Theorem 1.1.6.

(ii) Assume that the result has been proved for $\mathfrak{K}_b$ (for any kernels L_1 and L_2) for some $b \geq 0$ and let us prove that it is true for $\mathfrak{K}_{b+a}$ as soon as $0 \leq a < \alpha_X$. We have

$$\begin{aligned}\mathfrak{K}_{b+a} \circ (L_1 \overset{\mathrm{np}}{\circ} L_2) &\simeq \mathfrak{K}_b \circ (\mathfrak{K}_a \circ (L_1 \overset{\mathrm{np}}{\circ} L_2)) \simeq \mathfrak{K}_b \circ ((\mathfrak{K}_a \circ L_1) \overset{\mathrm{np}}{\circ} L_2) \\ &\simeq (\mathfrak{K}_b \circ (\mathfrak{K}_a \circ L_1)) \overset{\mathrm{np}}{\circ} L_2 \simeq (\mathfrak{K}_{a+b} \circ L_1) \overset{\mathrm{np}}{\circ} L_2\end{aligned}$$

$\square$

Thickening and convolution

In [22], the space X is the Euclidian space $\mathbb{R}^n$ and the composition $\mathbf{k}_{\Delta_a} \circ$ is replaced by the convolution $\mathbf{k}_{B_a} \star$ where B_a is the closed ball of center 0. One can proceed similarly if the good metric space (X, d_X) is a topological group.

Definition 2.1.9 A good metric group (X, d_X, m, e), or simply (X, d_X) for short, is a good metric space (X, d_X) which is a topological group for the topology induced by the distance, with multiplication m and neutral element e, and such that the distance is bi-invariant. In other words,

$$d_X(x_1, x_2) = d_X(x_1 x_3, x_2 x_3) = d_X(x_3 x_1, x_3 x_2) \text{ for } x_1, x_2, x_3 \in X.$$

One defines the convolution of $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ by

$$F \star G := \mathrm{R}m_!(F \boxtimes G).$$

Proposition 2.1.10 *Assume that X is a good metric group. Let B_a be the closed ball of radius a centered at the unit e. There is a canonical isomorphism of functor*

$$\mathbf{k}_{\Delta_a} \circ \simeq \mathbf{k}_{B_a} \star.$$

Proof Consider the map $v\colon X \times X \to X \times X, (x_1, x_2) \mapsto (x_1 x_2^{-1}, x_2)$. One has $\Delta_a = v^{-1} q_1^{-1}(B_a)$, $v^{-1} \circ q_2^{-1} \simeq q_2^{-1}$ and $m \circ v = q_1$. Therefore, for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$,

$$\begin{aligned}\mathbf{k}_{B_a} \star F &= \mathrm{R}m_!(\mathbf{k}_{B_a} \boxtimes F)\\ &\simeq \mathrm{R}m_! \mathrm{R}v_!(v^{-1} q_1^{-1} \mathbf{k}_{B_a} \otimes q_2^{-1} F) \simeq \mathbf{k}_{\Delta_a} \circ F.\end{aligned}$$

We have used $\mathrm{R}v_!(v^{-1} q_1^{-1} \mathbf{k}_{B_a} \otimes q_2^{-1} F) \simeq q_1^{-1} \mathbf{k}_{B_a} \otimes \mathrm{R}v_! q_2^{-1} F \simeq \mathbf{k}_{B_a} \boxtimes q_2^{-1} F$ which follows from $v_! \circ v^{-1} \simeq \mathrm{id}$. □

2.2 Bi-thickening of the diagonal

In this subsection, (X, d_X, α_X) is a good metric space satisfying (2.3). When necessary, we denote by X_i $(i = 1, 2, \dots)$ various copies of X.

For $a \geq 0$, we define the functors $\mathfrak{L}_a$ and $\mathfrak{R}_a$ by

$$\mathfrak{L}_a = \Phi_{\mathfrak{K}_a} = \mathfrak{K}_a \circ = \mathrm{R}q_{1!}\big(\mathfrak{K}_a \overset{\mathrm{L}}{\otimes} q_2^{-1}(\bullet)\big), \quad \mathfrak{R}_a = \Psi_{\mathfrak{K}_a} = \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,\big(\mathfrak{K}_a, q_1^!(\bullet)\big). \tag{2.6}$$

Recall that the functor $\mathfrak{R}_a$ is right adjoint to the functor $\mathfrak{L}_a$ (see [20, Proposition 3.6.2]).

Lemma 2.2.1 *Let (X, d_X, α_X) be a good metric space satisfying (2.3). For $0 < a \leq \alpha_X$, $\mathbf{k}_{\Delta_a} \circ (\mathbf{k}_{\Delta_a^\circ} \otimes q_2^{-1} \omega_X) \simeq \mathbf{k}_\Delta$.*

Proof Set $S_a = q_{12}^{-1} \Delta_a \cap q_{23}^{-1} \Delta_a^\circ$. We have

$$\mathbf{k}_{\Delta_a} \circ \mathbf{k}_{\Delta_a^\circ} = \mathrm{R}q_{13!}\big(q_{12}^{-1} \mathbf{k}_{\Delta_a} \otimes q_{23}^{-1} \mathbf{k}_{\Delta_a^\circ}\big) \simeq \mathrm{R}q_{13!} \mathbf{k}_{S_a}.$$

Let $(x_1, x_3) \in X_1 \times X_3$ and set $Z_a = q_{13}^{-1}(x_1, x_3) \cap S_a$. Then $Z_a = B_a(x_1) \cap B_a^\circ(x_3)$ and it follows from the hypothesis that $(\mathrm{R}q_{13!} \mathbf{k}_{S_a})_{(x_1, x_3)} \simeq \mathrm{R}\Gamma(X_2; \mathbf{k}_{Z_a}) \simeq 0$ for $x_1 \neq x_3$. Therefore, $\mathrm{R}q_{13!} \mathbf{k}_{S_a}$ is supported by $\Delta \subset X_{13}$ and we get

$$\begin{aligned}\mathrm{R}q_{13!}(\mathbf{k}_{S_a} \otimes q_2^{-1} \omega_X) &\simeq \mathrm{R}q_{13!}((\mathbf{k}_{S_a} \otimes q_{13}^{-1} \mathbf{k}_\Delta) \otimes q_2^{-1} \omega_X)\\ &\simeq \mathrm{R}q_{13!}(\mathbf{k}_{S_a \cap q_{13}^{-1} \Delta} \otimes q_2^{-1} \omega_X) \xrightarrow{\sim} \mathbf{k}_\Delta.\end{aligned}$$

The last morphism is associated with the morphism $\mathbf{k}_{S_a \cap q_{13}^{-1} \Delta} \otimes q_2^{-1} \omega_X \to q_{13}^! \mathbf{k}_\Delta$ which is deduced from the morphism $\mathbf{k}_{S_a \cap q_{13}^{-1} \Delta} \to q_{13}^{-1} \mathbf{k}_\Delta$. (Recall that $S_a \cap q_{13}^{-1} \Delta$ is open in $q_{13}^{-1} \Delta$.) It is an isomorphism by Lemma 2.1.2 (d). □

For $0 \leq a \leq \alpha_X$, set $\mathfrak{K}_a = \mathbf{k}_{\Delta_a}$ and for $0 < a \leq \alpha_X$, set $\mathfrak{K}_{-a} = \mathbf{k}_{\Delta_a^\circ} \otimes q_2^{-1} \omega_X$.

Lemma 2.2.2 *Let (X, d_X, α_X) be a good metric space satisfying (2.3). The map $a \mapsto \mathfrak{K}_a$ defines a monoidal presheaf on $[-\alpha_X, \alpha_X]$ with values in the monoidal category $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times X}), \circ)$.*

Proof

(i) For $0 < b \leq a$, $\mathbf{k}_{\Delta_a} \circ (\mathbf{k}_{\Delta_b^\circ} \otimes q_2^{-1}\omega_X) \simeq \mathbf{k}_{\Delta_{a-b}}$. This follows from Lemmas 2.1.4 and 2.2.1 and $\mathbf{k}_{\Delta_a} \circ \mathbf{k}_{\Delta_b^\circ} \simeq \mathbf{k}_{\Delta_{a-b}} \circ \mathbf{k}_{\Delta_b} \circ \mathbf{k}_{\Delta_b^\circ}$.

(ii) For $0 < a, b, a+b < \alpha_X$, $\mathbf{k}_{\Delta_b^\circ} \circ (\mathbf{k}_{\Delta_a^\circ} \otimes q_2^{-1}\omega_X) \simeq \mathbf{k}_{\Delta_{a+b}^\circ}$. This follows from (i), Lemma 2.1.4 and $(\mathbf{k}_{\Delta_b^\circ} \otimes q_2^{-1}\omega_X) \circ (\mathbf{k}_{\Delta_a^\circ} \otimes q_2^{-1}\omega_X) \overset{\mathrm{L}}{\otimes} \mathbf{k}_{\Delta_{a+b}} \simeq \mathbf{k}_\Delta$.

(iii) For $0 < b \leq a \leq \alpha_X$, $\mathbf{k}_{\Delta_a^\circ} \circ \mathbf{k}_{\Delta_b} \simeq \mathbf{k}_{\Delta_{a-b}^\circ}$. Indeed, apply $\mathbf{k}_{\Delta_{a-b}^\circ} \overset{\mathrm{L}}{\otimes} q_2^{-1}\omega_X \circ$ to both sides of (ii). □

Applying Theorem 1.2.3, we get:

Proposition 2.2.3 *Let (X, d_X, α_X) be a good metric space satisfying (2.3). Then $\mathfrak{K}$ extends as a metric bi-thickening kernel and, for $0 < a \leq \alpha_X$, one has $\mathfrak{K}_{-a} \simeq \mathbf{k}_{\Delta_a^\circ} \otimes q_2^{-1}\omega_X$. Moreover, $\mathfrak{R}_a \simeq \mathfrak{K}_{-a} \circ$ for $a \geq 0$.*

There is indeed a better result. Set

$$I = (-\alpha_X, \alpha_X). \tag{2.7}$$

Theorem 2.2.4 *Let (X, d_X, α_X) be a good metric space satisfying (2.3). There exists an object $K^d \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X\times I})$ and a distinguished triangle*

$$\mathbf{k}_{\{d_X(x,y)<-t\}} \otimes q_2^{-1}\omega_X \to K^d \to \mathbf{k}_{\{d_X(x,y)\leq t\}} \xrightarrow[\psi]{+1} . \tag{2.8}$$

In particular, $K^d|_{\{t=a\}} \simeq \mathfrak{K}_a$ for $a \in I$.

Proof We shall mimick the construction in [17, Exa. 3.10]. We have the isomorphism

$$\mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\Delta\times\{t=0\}}, \mathbf{k}_{X\times X\times\mathbb{R}}) \simeq \mathbf{k}_{\Delta\times\{t=0\}} \otimes q_2^{-1}\omega_X^{\otimes -1}\,[-1]. \tag{2.9}$$

Indeed, $\mathbf{k}_{\Delta\times\{t=0\}} \simeq \mathbf{k}_\Delta \boxtimes \mathbf{k}_{\{t=0\}}$ and it follows from [20, Prop. 3.4.4] that $\mathrm{D}'_{X\times X\times\mathbb{R}}(\mathbf{k}_\Delta \boxtimes \mathbf{k}_{\{t=0\}}) \simeq \mathrm{D}'_{X\times X}\mathbf{k}_\Delta \boxtimes \mathrm{D}'_{\mathbb{R}}\mathbf{k}_{\{t=0\}}$. Moreover, $\mathrm{D}'_{X\times X}\mathbf{k}_\Delta \simeq \delta_{X!}\delta_X^!\,\mathbf{k}_{X\times X} \simeq \mathbf{k}_\Delta \otimes q_2^{-1}\omega_X$ and $\mathrm{D}'_{\mathbb{R}}\mathbf{k}_{\{t=0\}} \simeq \mathbf{k}_{\{t=0\}}\,[-1]$.

By Lemma 2.1.2, we also have the isomorphism

$$\mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{d_X(x,y)\leq -t\}}, \mathbf{k}_{X\times X\times\mathbb{R}}) \simeq \mathbf{k}_{\{d_X(x,y)<-t\}} \quad t \in (-a, 0). \tag{2.10}$$

These isomorphisms together with the morphism $\mathbf{k}_{\{d_X(x,y)\leq -t\}} \to \mathbf{k}_{\Delta\times\{t=0\}}$ induce the morphism $\mathbf{k}_{\Delta\times\{t=0\}} \otimes q_2^{-1}\omega_X^{\otimes -1}[-1] \to \mathbf{k}_{\{d_X(x,y)<-t\}}$. Hence, we obtain

$$\mathbf{k}_{\{d_X(x,y)\leq t\}} \to \mathbf{k}_{\Delta\times\{t=0\}} \to \mathbf{k}_{\{d_X(x,y)<-t\}} \otimes q_2^{-1}\omega_X[+1]$$

Denoting by ψ the composition, we get the distinguished triangle (2.8). □

Remark 2.2.5 It would be possible to extend K^d to a sheaf $\mathfrak{K}^{\mathrm{dist}} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{X\times X\times\mathbb{R}})$ by using Theorem 1.2.3 and using the monoidal category $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X\times\mathbb{R}}), \overset{+}{\circ})$, where $\overset{+}{\circ}$ is an operation adapted from [31], composition with respect to X and convolution with respect to $\mathbb{R}$.

2.3 Properties of the interleaving distance

We shall extend to metric spaces a few results of [22, § 2.2]. In this section, (X, d_X) is a good metric space and $\mathfrak{K}$ is the metric thickening of the diagonal. Recall the interleaving distance $\mathrm{dist}_{\mathfrak{K}}$ of Definition 1.3.4. We set

$$\mathrm{dist}_X = \mathrm{dist}_{\mathfrak{K}}. \tag{2.11}$$

Lemma 2.3.1 *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and let $a \geq 0$. Then*

$$\mathrm{R}\Gamma(X; \mathfrak{K}_a \circ F) \xrightarrow{\sim} \mathrm{R}\Gamma(X; F) \quad \textit{and} \quad \mathrm{R}\Gamma_c(X; \mathfrak{K}_a \circ F) \xrightarrow{\sim} \mathrm{R}\Gamma_c(X; F).$$

Proof It follows from the definition of the functor $\mathfrak{K}_a$ that is it enough to check these isomorphisms for $0 \leq a \leq \alpha_X$, thus replacing $\mathfrak{K}_a$ with $\mathbf{k}_{\Delta_a}$. Consider the Cartesian diagram

$$\begin{tikzcd} & X \times X \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X \arrow[dr, "q_2'"'] & & X \arrow[dl, "q_1'"] \\ & \mathrm{pt} & \end{tikzcd}$$

Using the fact that q_1 and q_2 are proper on Δ_a we get the isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma(X; \mathbf{k}_{\Delta_a} \circ F) &\simeq \mathrm{R}q'_{2*}\mathrm{R}q_{1!}(\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \simeq \mathrm{R}q'_{2*}\mathrm{R}q_{1*}(\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \\ &\simeq \mathrm{R}q'_{1*}\mathrm{R}q_{2*}(\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \simeq \mathrm{R}q'_{1*}\mathrm{R}q_{2!}(\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \\ &\simeq \mathrm{R}q'_{1*}(\mathrm{R}q_{2!}\mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} F) \\ &\simeq \mathrm{R}q'_{1*}F \simeq \mathrm{R}\Gamma(X; F). \end{aligned}$$

Here we use the isomorphism $\mathrm{R}q_{2!}\mathbf{k}_{\Delta_a} \simeq \mathbf{k}_X$ which follows from the fact that the fibers of $q_2 \colon \Delta_a \to X$ are compact and contractible.

A similar proof holds for $\mathrm{R}\Gamma_c(X; F)$. □

Proposition 2.3.2 *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. If $\mathrm{dist}_X(F, G) < +\infty$, then $\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}\Gamma(X; G)$ and $\mathrm{R}\Gamma_c(X; F) \simeq \mathrm{R}\Gamma_c(X; G)$.*

Proof This follows immediately from the definition of the distance and Lemma 2.3.1. □

Proposition 2.3.3 *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume that* $\mathrm{supp}(F) \subset B(x_0, a)$ *with* $a \leq \alpha_X$. *Set $M = \mathrm{R}\Gamma(X; F)$ and denote by M_{x_0} the sky-scraper sheaf at $\{x_0\}$ with stalk M. Then* $\mathrm{dist}_X(F, M_{x_0}) \leq a$.

We shall mimick the proof of [22, Exa. 2.4].

Proof We have

$$\mathbf{k}_{\Delta_a} \circ M_{x_0} \simeq M_{B(x_0,a)},$$

the constant sheaf on $B(x_0, a)$ with stalk M extended by 0 outside of $B(x_0, a)$.

Denote by $a_X : X \to \mathrm{pt}$ the unique map from X to pt. The morphism $a_X^{-1}\mathrm{R}a_{X*}F \to F$ defines the map $M_X \to F$ and F being supported in $B(x_0, a)$, we get the morphism $g\colon \mathbf{k}_{\Delta_a} \circ M_{x_0} \simeq M_{B(x_0,a)} \to F$.

On the other hand, we have

$$\begin{aligned}
(\mathbf{k}_{\Delta_a} \circ F)_{x_0} &\simeq \mathrm{R}\Gamma(q_1^{-1}(x_0); \mathbf{k}_{\Delta_a} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \\
&\simeq \mathrm{R}\Gamma(\{x_0\} \times X; \{x_0\} \times \mathbf{k}_{B(x_0,a)} \overset{\mathrm{L}}{\otimes} q_2^{-1}F) \\
&\simeq \mathrm{R}\Gamma(B(x_0, a); F) \simeq M
\end{aligned} \tag{2.12}$$

which defines $f\colon \mathbf{k}_{\Delta_a} \circ F \to M_{x_0}$. One easily checks that f and g satisfy the compatibility conditions in Definition 1.3.4. Therefore $\mathrm{dist}_X(F, M_{x_0}) \leq a$. □

In particular, a non-zero object can be a-isomorphic (see Definition 1.3.4) to the zero object.

Corollary 2.3.4 *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume that there exists a ball $B_{x_0}(a)$ with $a \leq \alpha_X$ which contains the supports of F and G. Then* $\mathrm{dist}_X(F, G) < \infty$ *if and only if* $\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}\Gamma(X; G)$.

Proof

(i) Assume $M := \mathrm{R}\Gamma(X; F) \simeq \mathrm{R}\Gamma(X; G)$. Then

$$\mathrm{dist}_X(F, G) \leq \mathrm{dist}_X(F, M_{x_0}) + \mathrm{dist}_X(G, M_{x_0})$$

and it remains to apply Proposition 2.3.3.

(ii) The converse assertion is nothing but Proposition 2.3.2. □

Corollary 2.3.5 *Consider two distinguished triangles $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ and $G_1 \to G_2 \to G_3 \xrightarrow{+1}$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that there exists a ball $B_{x_0}(a)$ with $a \leq \alpha_X$ which contains the supports of all sheaves F_i, G_i $(i = 1, 2, 3)$ and also assume that* $\mathrm{dist}_X(F_i, G_i) < \infty$ *for $i = 1, 2$. Then* $\mathrm{dist}_X(F_3, G_3) < \infty$.

Proof It follows from Corollary 2.3.4 that $\mathrm{R}\Gamma(X; F_i) \simeq \mathrm{R}\Gamma(X; G_i)$ for $i = 1, 2$. Since the functor $\mathrm{R}\Gamma(X; \bullet)$ is triangulated, this isomorphism still holds for $i = 3$. Then the result follows again from Corollary 2.3.4. □

Locally constant sheaves

Recall that an object $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ is locally constant (resp. constant) if, for all $j \in \mathbb{Z}$, $H^j(L)$ is a locally constant (resp. constant) sheaf.

Lemma 2.3.6 *Let $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume that L is locally constant. Let $a \geq 0$. Then $\mathfrak{K}_a \circ L \xrightarrow{\sim} L$.*

Proof We may choose a such that $a < \alpha_X$ and replace $\mathfrak{K}_a$ with $\mathbf{k}_{\Delta_a}$. It is then enough to prove that, for $x \in X$, the natural morphism $(\mathbf{k}_{\Delta_a} \circ L)_x \to L_x$ is an isomorphism. We may also assume that L is a constant sheaf in a neighborhood of $B_a(x)$. Then by (2.12), we get

$$(\mathbf{k}_{\Delta_a} \circ L)_x \simeq \mathrm{R}\Gamma(B_a(x); L) \simeq L_x.$$

$\square$

Proposition 2.3.7 *Let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that F is locally constant and that $\mathrm{dist}_X(F, G)$ is finite. Then F is a direct summand of G. In particular, if both F and G are locally constant, then $F \simeq G$.*

Proof By the hypothesis and Lemma 2.3.6 we have morphisms $F \to G \to F$ such that the composition is an isomorphism. $\square$

It follows that the interleaving distance is not really interesting when considering locally constant sheaves.

2.4 The stability theorem

Let X be a good topological space and let (Y, d_Y) be a good metric space. We denote by $\mathfrak{K}_a^Y$ the kernel on $Y \times Y$. It defines an endofunctor of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times Y})$, $K \mapsto K \circ \mathfrak{K}_a^Y$. We then get a pseudo-distance on $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times Y})$ that we call a relative distance and denote by $\mathrm{dist}_{X \times Y/X}$ (see Example 1.3.3).

Theorem 2.4.1 (The stability theorem) *Let X be a good topological space and let (Y, d_Y) be a good metric space. Let $K_1, K_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y \times X})$ and let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then*

(a) $\mathrm{dist}_Y(K_1 \circ F, K_2 \circ F) \leq \mathrm{dist}_{Y \times X/X}(K_1, K_2)$.
(b) *Assume moreover that X and Y are C^∞-manifolds and that (Y, d_Y) satisfies* (2.3) *and* (2.4). *Then* $\mathrm{dist}_Y(K_1 \overset{\mathrm{np}}{\circ} F, K_2 \overset{\mathrm{np}}{\circ} F) \leq \mathrm{dist}_{Y \times X/X}(K_1, K_2)$.

Proof

(a) We have

$$\mathfrak{K}_a^Y \circ (K_i \circ F) \simeq (\mathfrak{K}_a^Y \circ K_i) \circ F, \quad i = 1, 2.$$

Then the result follows immediately from Definition 1.3.4.

(b) The proof is the same as in (a) after replacing $\circ$ with $\overset{\mathrm{np}}{\circ}$ and using Proposition 2.1.8. ◻

Let X and Y be as above and let $f_1, f_2 \colon X \to Y$ be two continuous maps. As usual, one sets

$$\operatorname{dist}(f_1, f_2) = \sup_{x\in X} d_Y(f_1(x), f_2(x)).$$

Corollary 2.4.2 ((The metric stability theorem, see [22, Th. 2.7])) *Let X be a good topological space and let Y be a (real, finite dimensional) normed vector space, d_Y the associated distance. Then $\operatorname{dist}_Y(\mathrm{R}f_{1!}F, \mathrm{R}f_{2!}F) \le \operatorname{dist}(f_1, f_2)$. If X is a C^∞-manifold and Y is an Euclidian vector space, the same result holds with $\mathrm{R}f_!$ replaced with $\mathrm{R}f_*$.*

Proof Let $a = \operatorname{dist}(f_1, f_2)$. Of course, we may assume that $a < \infty$. Denote by Γ_i the graph of f_i in $Y \times X$. Then

$$\Gamma_{f_i} \subset \Delta_a^Y \circ \Gamma_{f_j},\ i, j \in \{1, 2\}. \tag{2.13}$$

Moreover, for $f = f_1$ or $f = f_2$, one has

$$\mathbf{k}_{\Delta_a^Y} \circ \mathbf{k}_{\Gamma_f} \simeq \mathbf{k}_{\Delta_a^Y \circ \Gamma_f}. \tag{2.14}$$

Set $K_i = \mathbf{k}_{\Gamma_{f_i}}$ $(i = 1, 2)$. By (2.13) and (2.14), we get morphisms $\mathbf{k}_{\Delta_a^Y} \circ K_{f_1} \to K_{f_2}$ and $\mathbf{k}_{\Delta_a^Y} \circ K_{f_2} \to K_{f_1}$ satisfying the conditions of Definition 1.3.4. Therefore,

$$\operatorname{dist}_{Y\times X/X}(K_{f_1}, K_{f_2}) \le a = \operatorname{dist}(f_1, f_2). \tag{2.15}$$

Since $\mathrm{R}f_{i!}F \simeq K_i \circ F$ and $\mathrm{R}f_{i*}F \simeq K_i \overset{\mathrm{np}}{\circ} F$, the result follows from Theorem 2.4.1 since hypotheses (2.3) and (2.4) are satisfied if Y is an Euclidian vector space. ◻

Remark 2.4.3 In [22, Th. 2.7] the proof for $\mathrm{R}f_*$ and $\mathrm{R}f_!$ is almost the same and X is only assumed to be a good topological space. The reason why the non proper case is easier in the situation of [22] is that these authors use the convolution functor $\mathbf{k}_{B_a}\star$ instead of $\mathbf{k}_{\Delta_a}\circ$.

More precisely, consider the diagram in which Y is a real finite dimensional normed vector space, Y_1 and Y_2 are two copies of Y and s is the map $(y_1, y_2) \mapsto y_1 + y_2$, s_{13} is the map $(y_1, x, y_2) \mapsto (y_1 + y_2, x)$:

$$\begin{tikzcd}
& Y_1 \times X \times Y_2 \arrow[dl, "p_{12}"'] \arrow[d, "s_{13}"] \arrow[dr, "p_{23}"] & & \\
Y_{12} \arrow[dr, "s"'] & Y \times X \arrow[d, "p_1"'] \arrow[dr, "p_2"'] & X \times Y_2 \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\
& Y & X & Y_2.
\end{tikzcd}$$

Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y_2 \times X})$ and denote by B_a the closed ball of Y_1 with center 0 and radius $a \geq 0$. Set for short $\mathbf{k}_B := \mathbf{k}_{B_a}$. Then

$$\begin{aligned}
\mathbf{k}_B \star (K \overset{\mathrm{np}}{\circ} F) &\simeq \mathrm{R}s_*(\mathbf{k}_B \boxtimes \mathrm{R}q_{2*}(K \overset{\mathrm{L}}{\otimes} q_1^{-1} F)) \\
&\simeq \mathrm{R}s_* \mathrm{R}p_{12*}(\mathbf{k}_B \boxtimes (K \overset{\mathrm{L}}{\otimes} q_1^{-1} F)) \\
&\simeq \mathrm{R}p_{1*} \mathrm{R}s_{13*}(\mathbf{k}_B \boxtimes (K \overset{\mathrm{L}}{\otimes} q_1^{-1} F)) \\
&\simeq \mathrm{R}p_{1*} \mathrm{R}s_{13*}((\mathbf{k}_B \boxtimes K) \overset{\mathrm{L}}{\otimes} s_{13}^{-1} p_2^{-1} F) \\
&\simeq \mathrm{R}p_{1*}(\mathrm{R}s_{13*}(\mathbf{k}_B \boxtimes K) \overset{\mathrm{L}}{\otimes} p_2^{-1} F) \simeq (\mathbf{k}_B \star K) \overset{\mathrm{np}}{\circ} F.
\end{aligned}$$

Here, the 2nd isomorphism follows from the fact that $\mathbf{k}_B$ being cohomologically constructible, the functor $\mathbf{k}_B \boxtimes \bullet$ commutes with (non proper) direct images thanks to [20, Prop. 3.4.4]. The 5th isomorphism follows from the fact that s is proper on $\mathrm{supp}(\mathbf{k}_B \boxtimes K)$.

2.5 Lipschitz kernels

A general setting

We consider two good metric spaces (X, d_X) and (Y, d_Y). To avoid confusion, we denote by α_X and α_Y the constants appearing in (2.2), by Δ_a^X and Δ_a^Y the thickenings of the diagonals, by $\mathfrak{K}_a^{\mathrm{X}}$ and $\mathfrak{K}_a^{\mathrm{Y}}$ the associated thickening kernels and by $\rho_{a,b}^X$ and $\rho_{a,b}^Y$ the restriction functors. Recall the notation for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$

$$\Phi_K(F) = K \circ F.$$

Definition 2.5.1 Let $\delta > 0$ and let $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y \times X})$. We say that K is a δ-Lipschitz kernel from X to Y if there exists $\rho > 0$ such that $\rho \leq \alpha_X$ and $\delta\rho \leq \alpha_Y$ and there are morphisms of sheaves $\sigma_a : \mathfrak{K}_{\delta a}^{\mathrm{Y}} \circ K \to K \circ \mathfrak{K}_a^{\mathrm{X}}$ for $0 \leq a \leq \rho$ satisfying the following compatibility relations:

(i) for $0 \leq a \leq b \leq \rho$, the diagram of sheaves commutes:

$$\begin{tikzcd}
\mathfrak{K}_{\delta b}^{\mathrm{Y}} \circ K \arrow[r, "\sigma_b"] \arrow[d, "\rho_{\delta a, \delta b}^{Y}"'] & K \circ \mathfrak{K}_b^{\mathrm{X}} \arrow[d, "\rho_{a,b}^{X}"] \\
\mathfrak{K}_{\delta a}^{\mathrm{Y}} \circ K \arrow[r, "\sigma_a"] & K \circ \mathfrak{K}_a^{\mathrm{X}},
\end{tikzcd} \tag{2.16}$$

(ii) for $0 \le a,\ b$ and $a + b \le \rho$, the diagram of sheaves commutes:

$$\begin{tikzcd} \mathfrak{K}^{\mathrm{Y}}_{\delta(a+b)} \circ K \arrow[r, "\mathfrak{K}^{\mathrm{Y}}_{\delta b} \circ \sigma_a"] \arrow[rr, bend right, "\sigma_{a+b}"'] & \mathfrak{K}^{\mathrm{Y}}_{\delta b} \circ K \circ \mathfrak{K}^{\mathrm{X}}_a \arrow[r, "\sigma_b \circ \mathfrak{K}^{\mathrm{X}}_a"] & K \circ \mathfrak{K}^{\mathrm{X}}_{a+b}. \end{tikzcd} \tag{2.17}$$

A Lipschitz kernel is a δ-Lipschitz kernel for some $\delta > 0$.

Note that thanks to the hypothesis that $a \le \alpha_X$, we could have written $\mathbf{k}_{\Delta^X_a}$ instead of $\mathfrak{K}^{\mathrm{X}}_a$ and similarly with Y instead of X. We have chosen to use the notation $\mathfrak{K}$ thanks to the next lemma.

Remark 2.5.2 Of course, a Lipschitz kernel from X to Y is not necessarily a Lipschitz kernel from Y to X. However, when there is no risk of confusion, we shall simply call K "a Lipschitz kernel".

Lemma 2.5.3 *If K is a Lipschitz kernel, then for all $a \ge 0$ there are morphisms of sheaves $\sigma_a : \mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \to K \circ \mathfrak{K}^{\mathrm{X}}_a$ and moreover (2.16) and (2.17) are satisfied for all $a, b \ge 0$.*

Sketch of proof Assume we have constructed the morphisms σ_a for $a \le A$ and let $0 \le b \le \rho$. One defines the morphism

$$\begin{aligned} \sigma_{a+b} : \mathfrak{K}^{\mathrm{Y}}_{\delta(a+b)} \circ K &\simeq \mathbf{k}_{\Delta^Y_{\delta b}} \circ \mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \\ &\to \mathbf{k}_{\Delta^Y_{\delta b}} \circ K \circ \mathfrak{K}^{\mathrm{X}}_a \\ &\to K \circ \mathbf{k}_{\Delta^X_b} \circ \mathfrak{K}^{\mathrm{X}}_a \simeq K \circ \mathfrak{K}^{\mathrm{X}}_{a+b}. \end{aligned}$$

The fact that σ_a is well-defined and the verification of the compatibility relations (2.16) and (2.17) are left to the reader. □

The next result is essentially a reformulation in the language of kernels of [14, Th. 4.3].

Theorem 2.5.4 (The functorial Lipschitz theorem) *Let (X, d_X) and (Y, d_Y) be good metric spaces and let $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y \times X})$ be a δ-Lipschitz kernel from X to Y. Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$.*

(a) *One has* $\mathrm{dist}_Y(K \circ F_1, K \circ F_2) \le \delta \cdot \mathrm{dist}_X(F_1, F_2)$.
(b) *Assume moreover that X and Y are C^∞-manifolds satisfying* (2.3) *and* (2.4). *Then* $\mathrm{dist}_Y(K \overset{\mathrm{np}}{\circ} F_1, K \overset{\mathrm{np}}{\circ} F_2) \le \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

Proof

(a) Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ and assume that F_1 and F_2 are a-isomorphic. Hence, there are morphisms

$$f : \mathfrak{K}^{\mathrm{X}}_a \circ F_1 \to F_2, \quad g : \mathfrak{K}^{\mathrm{X}}_a \circ F_2 \to F_1$$

satisfying the conditions of Definition 1.3.4. Applying the functor $K \circ$ we get the morphisms given by the dotted arrows

$$\begin{tikzcd}
K \circ \mathfrak{K}^{\mathrm{X}}_a \circ F_1 \arrow[r, "\Phi_K(f)"] & K \circ F_2 \\
\mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \circ F_1 \arrow[u, "\sigma_a"] \arrow[ur, dotted] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K \circ \mathfrak{K}^{\mathrm{X}}_a \circ F_2 \arrow[r, "\Phi_K(g)"] & K \circ F_1 \\
\mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \circ F_2 \arrow[u, "\sigma_a"] \arrow[ur, dotted] &
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
K \circ \mathfrak{K}^{\mathrm{X}}_{2a} \circ F_1 \arrow[r, "\Phi_K(\mathfrak{L}_a(f))"] & K \circ \mathfrak{K}^{\mathrm{X}}_a \circ F_2 \arrow[r, "\Phi_K(g)"] & K \circ F_1. \\
\mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \circ \mathfrak{K}^{\mathrm{X}}_a \circ F_1 \arrow[u, "\mathfrak{L}^X_a(\sigma_a)"] \arrow[r, "\mathfrak{L}^Y_{\delta a}(\Phi_K(f))"] & \mathfrak{K}^{\mathrm{Y}}_{\delta a} \circ K \circ F_2 \arrow[u, "\sigma_a"] \arrow[ur, dotted] & \\
\mathfrak{K}^{\mathrm{Y}}_{2\delta a} \circ K \circ F_1 \arrow[u, "\mathfrak{L}^Y_{\delta a}(\sigma_a)"] \arrow[ur, dotted] & &
\end{tikzcd}$$

The two diagrams with dotted arrows commute by the definition of the dotted arrows and the square diagram commutes by Definition 2.5.1 (i). The composition of the two vertical arrows is given by σ_{2a} by Definition 2.5.1 (ii). The composition of the two horizontal arrows is given by $\rho^X_{0,2a}$. Therefore, the composition of the two dotted arrows is given by $\rho^X_{0,2a}\sigma_{2a} = \rho^Y_{0,2\delta a}$. The same result holds when interchanging the roles of F_1 and F_2.

(b) The proof is the same as in (a) after replacing $\circ$ with $\overset{\mathrm{np}}{\circ}$ and using Proposition 2.1.8.

□

In particular, we get:

Corollary 2.5.5 *Assume that $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y\times X})$ is a δ-Lipschitz kernel from X to Y and that there exists a δ^{-1}-Lipschitz kernel $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times Y})$ from Y to X such that $\Phi_{L\circ K} \simeq \mathrm{id}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)}$. Then for $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, one has $\mathrm{dist}_Y(K \circ F_1, K \circ F_2) = \delta \cdot \mathrm{dist}_X(F_1, F_2)$.*

If X and Y are C^∞-manifolds satisfying (2.3) and (2.4), then the same result holds for $K \circ F$ replaced with $K \overset{\mathrm{np}}{\circ} F$.

Lipschitz correspondences

As above, we denote by X_i and Y_i $(i = 1, 2)$ two copies of X or Y. We keep the assumptions and notations of the beginning of this section.

We assume to be given a subset S of $Y \times X$ and consider the diagram

$$\begin{tikzcd}
& Y_{12}\times X_1 \arrow[dl, "p_{12}"'] \arrow[dd, "p_{13}"] \arrow[dr, "p_{23}"] & & Y_2\times X_{12} \arrow[dl, "q_{12}"'] \arrow[dd, "q_{13}"] \arrow[dr, "q_{23}"] & \\
\Delta_b^Y \subset Y_{12} & & S\subset Y_2\times X_1 & & \Delta_a^X\subset X_{12} \\
& Y_1\times X_1 \arrow[rr, equal] & & Y_2\times X_2 &
\end{tikzcd} \tag{2.18}$$

We set

$$\Delta_b^Y \times_Y S = p_{12}^{-1}(\Delta_b^Y) \cap p_{23}^{-1}(S) \subset Y_{12} \times X_1,$$
$$S \times_X \Delta_a^X = q_{12}^{-1}(S) \cap q_{23}^{-1}(\Delta_a^X) \subset Y_2 \times X_{12}.$$

Note that $\Delta_b^Y \circ S = p_{13}(\Delta_b^Y \times_Y S)$ and $S \circ \Delta_a^X = q_{13}(S \times_X \Delta_a^X)$ are contained in $Y_1 \times X_1 = Y_2 \times X_2 = Y \times X$. We shall consider one of the hypotheses (2.19) or (2.20) below for some constants $\rho, \delta > 0$ such that $\rho \le \alpha_X$ and $\delta\rho \le \alpha_Y$.

$$\begin{cases} \text{(a) } S \text{ is a closed subset of } Y \times X, \\ \text{(b) the fibers of the projection } p_{13}\colon \Delta_b^Y \times_Y S \to \Delta_b^Y \circ S \text{ are contractible or empty} \\ \text{for } 0 \le b \le \alpha_Y, \\ \text{(c) } S \circ \Delta_a^X \subset \Delta_{\delta a}^Y \circ S \text{ for } a \le \rho. \end{cases} \tag{2.19}$$

$$\begin{cases} \text{(a) } S \text{ is a closed subset of } Y \times X, \\ \text{(b) there a closed embedding } \iota\colon Y_2 \times X_{12} \hookrightarrow Y_{12} \times X_1 \text{ such that } p_{13} \circ \iota = q_{13}, \\ \text{(c) } \iota(S \times_X \Delta_a^X) \subset \Delta_{\delta a}^Y \times_Y S \text{ for } a \le \rho. \end{cases} \tag{2.20}$$

Theorem 2.5.6 *Let $S \subset Y \times X$ and consider constants $\rho, \delta > 0$ such that $\rho \le \alpha_X$ and $\delta\rho \le \alpha_Y$. One makes either hypothesis (2.19) or hypothesis (2.20). Then $\mathbf{k}_S \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y\times X})$ is a δ-Lipschitz kernel from X to Y.*

Proof (i) It is enough to construct a natural morphism of sheaves

$$\mathbf{k}_{\Delta_{\delta a}^Y} \circ \mathbf{k}_S \to \mathbf{k}_S \circ \mathbf{k}_{\Delta_a^X} \text{ for } a \le \rho \text{ (which implies } \delta a \le \alpha_Y). \tag{2.21}$$

(ii)–(a) Assume (2.19). Since the closed set $\Delta_{\delta a}^Y \circ S$ contains the closed set $S \circ \Delta_a^X$, we have a morphism of sheaves

$$\mathbf{k}_{\Delta_{\delta a}^Y \circ S} \to \mathbf{k}_{S \circ \Delta_a^X}. \tag{2.22}$$

By Lemma 1.1.3 and the hypothesis, there is an isomorphism and a morphism

$$\mathbf{k}_{\Delta_{\delta a}^Y \circ S} \simeq \mathbf{k}_{\Delta_{\delta a}^Y} \circ \mathbf{k}_S, \quad \mathbf{k}_{S \circ \Delta_a^X} \to \mathbf{k}_S \circ \mathbf{k}_{\Delta_a^X}.$$

Together with (2.22), this defines (2.21).

(ii) –(b) Assume (2.20). By this hypothesis, there is a natural morphism

$$\mathbf{k}_{\Delta^Y_{\delta a}\times_Y S} \to \iota_*\mathbf{k}_{S\times_X\Delta^X_a}. \tag{2.23}$$

Now remark that

$$\mathbf{k}_{S\times_X\Delta^X_a} \simeq q_{12}^{-1}\mathbf{k}_S \overset{\mathrm{L}}{\otimes} q_{23}^{-1}\mathbf{k}_{\Delta^X_a}, \quad \mathbf{k}_{\Delta^Y_{\delta a}\times_Y S} \simeq p_{12}^{-1}\mathbf{k}_{\Delta^X_a} \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathbf{k}_S.$$

By (2.23), we get the morphisms

$$\begin{aligned}\mathbf{k}_{\Delta^Y_{\delta a}} \circ \mathbf{k}_S &\simeq \mathrm{R}p_{13!}(p_{12}^{-1}\mathbf{k}_{\Delta^X_a} \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathbf{k}_S) \simeq \mathrm{R}p_{13!}\mathbf{k}_{\Delta^Y_{\delta a}\times_Y S}\\ &\to \mathrm{R}p_{13!}\iota_*\mathbf{k}_{S\times_X\Delta^X_a} \simeq \mathrm{R}p_{13!}\iota_*(q_{12}^{-1}\mathbf{k}_S \overset{\mathrm{L}}{\otimes} q_{23}^{-1}\mathbf{k}_{\Delta^X_a})\\ &\simeq \mathrm{R}q_{13!}(q_{12}^{-1}\mathbf{k}_S \overset{\mathrm{L}}{\otimes} q_{23}^{-1}\mathbf{k}_{\Delta^X_a}) \simeq \mathbf{k}_S \circ \mathbf{k}_{\Delta^X_a}.\end{aligned}$$

We have thus constructed the morphism (2.21). □

Let $f : X \to Y$ be a continuous map. We set $\Gamma_f = \{(f(x), x) \in Y \times X\}$.

Corollary 2.5.7 *Let $f : (X, d_X) \to (Y, d_Y)$ be a δ-Lipschitz map. Then $\mathbf{k}_{\Gamma_f}$ is a δ-Lipschitz kernel from X to Y.*

Proof

(i) We shall check (2.19) with $S = \Gamma_f$. Of course, this set is closed in $Y \times X$.
(ii) Let us check (2.19) (b). One has

$$\Delta^Y_b \times_Y S = \{(y_1, y_2, x) \in Y \times Y \times X; d_Y(y_1, y_2) \le b, y_2 = f(x)\}.$$

For $(y_1, x) \in \Delta^Y_b \circ S$, $q_{13}^{-1}(y_1, x) \cap \Delta^Y_b \times_Y S$ is the set $(y_1, y_2 = f(x), x)$ if $d_Y(y_1, y_2) \le b$ and is empty otherwise.
(iii) Let us check (2.19) (c). One has

$$\begin{aligned}\Delta^Y_{\delta a} \circ S &= \{(y, x) \in Y \times X; \exists y' \in Y,\ d_Y(y, y') \le \delta a,\ y' = f(x)\},\\ S \circ \Delta^X_a. &= \{(y, x) \in Y \times X; \exists x' \in X,\ d_X(x, x') \le a,\ y = f(x')\}.\end{aligned}$$

Let $(y, x) \in S \circ \Delta^X_a$ and let $x' \in X$ be such that $d_X(x, x') \le a$, $y = f(x')$. Set $y' = f(x)$. Then $d_Y(y, y') \le \delta a$ since f is δ-Lipschitz and therefore $(y, x) \in \Delta^Y_{\delta a} \circ S$. □

Example 2.5.8 Let $X = \mathbb{S}^1$, $Y = \mathbb{R}^2$ and denote by S the graph of the embedding $j : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$. Then $\mathbf{k}_S \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y\times X})$ is a δ-Lipschitz kernel from X to Y with $\delta = \frac{\pi}{\sqrt{2}}$ and defines a fully faithful functor.

Corollary 2.5.9 *Let (X, d_X) and (Y, d_Y) be good metric spaces and let $f : X \to Y$ be a δ-Lipschitz map. Let $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$.*

(a) *One has* $\mathrm{dist}_Y(\mathrm{R}f_!F_1, \mathrm{R}f_!F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.
(b) *If moreover, X and Y are C^∞-manifolds satisfying hypotheses* (2.3) *and* (2.4)*, then*
$\mathrm{dist}_Y(\mathrm{R}f_*F_1, \mathrm{R}f_*F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

Proof First remark that for every $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, $\mathrm{R}f_!F \simeq \mathbf{k}_{\Gamma_f} \circ F$ and $\mathrm{R}f_*F \simeq \mathbf{k}_{\Gamma_f} \overset{\mathrm{np}}{\circ} F$. Then apply Corollary 2.5.7 and Theorem 2.5.4. □

2.6 Some elementary examples

Vector spaces

The interleaving distance for sheaves on a (finite dimensional) real normed vector space has been studied with great details in [22] and in fact this paper is a special case and a guide for the present one. In loc. cit. the composition $\mathbf{k}_{\Delta_a} \circ$ was replaced by the convolution $\mathbf{k}_{B_a} \star$ which is equivalent (see Proposition 2.1.10). When the norm is not Euclidian, we get an example where the whole theory developed here applies although the metric space is not associated with a Riemannian manifold.

The next result is obvious.

Proposition 2.6.1 *Let $X = \mathbb{V}$ be a real n-dimensional Euclidian vector space and let d_X be the associated distance. Then (X, d_X) satisfies hypotheses* (2.2)*,* (2.3) *and* (2.4)*.*

In the situation of Proposition 2.6.1, the bi-thickening kernel is given by

$$\mathfrak{K}_a \simeq \begin{cases} \mathbf{k}_{\Delta_a} \text{ if } a \geq 0, \\ \mathbf{k}_{\Delta^\circ_{-a}}[n] \text{ if } a < 0. \end{cases}$$

More precisely, in this situation, the sheaf $\mathfrak{K}^{\mathrm{dist}}$ is described, up to isomorphism, in [17, Exa. 3.11] by the distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}})$:

$$\mathbf{k}_{\{|x-y|<-t\}}[n] \to \mathfrak{K}^{\mathrm{dist}} \to \mathbf{k}_{\{|x-y|\leq t\}} \xrightarrow{+1}$$

The real line

Let $X = \mathbb{R}$ be the real line. Recall that, $\mathbf{k}$ being a field, one has an isomorphism

$$F \simeq \bigoplus_j H^j(F)[-j] \text{ for } F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X). \tag{2.24}$$

Hence, the study of objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ is reduced to that of objects of $\mathrm{Mod}(\mathbf{k}_X)$. But, as it is well-known, there exist non zero morphisms between objects concentrated in different degrees.

Constructible sheaves with compact support on $\mathbb{R}$ (over a field) are classified via the famous theorem of Crawley-Boevey [10]. See also [16] for a formulation in the language of constructible sheaves and see [22, Th. 1.17] for the case of not necessarily compactly supported sheaves. Distances on such sheaves are studied with great details in [3]. Recall that in this setting the thickening of the identity is provided by the following family of endofunctors of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$, $\mathbf{k}_{B_a}\star$, $a \geq 0$, where $B_a = [-a, a]$.

2.7 Example: the Fourier–Sato transform

Consider first the topological n-sphere ($n > 0$) defined as follows. Let $\mathbb{V}$ be a real vector space of dimension $n+1$, set $\dot{\mathbb{V}} = \mathbb{V}\backslash\{0\}$ and $\mathbf{S} := \dot{\mathbb{V}}/\mathbb{R}^+$ where $\mathbb{R}^+$ is the multiplicative group $\mathbb{R}_{>0}$. Define similarly the dual sphere $\mathbf{S}^*$, starting with $\mathbb{V}^*$. The sets

$$P = \{(y, x) \in \mathbf{S}^* \times \mathbf{S};\ \langle y, x\rangle \geq 0\}, \quad I = \{(y, x) \in \mathbf{S}^* \times \mathbf{S};\ \langle y, x\rangle > 0\}, \tag{2.25}$$

are well-defined. We define the kernel

$$K_I = \mathbf{k}_I \overset{\mathrm{L}}{\otimes} (\omega_{\mathbf{S}^*} \boxtimes \mathbf{k}_{\mathbf{S}}). \tag{2.26}$$

Note that $K_I \simeq \mathrm{R}\mathscr{H}om\,(\mathbf{k}_P, \omega_{\mathbf{S}^*} \boxtimes \mathbf{k}_{\mathbf{S}})$, which is in accordance with [17, eq (1.21)]. Moreover, $K_I \simeq \mathbf{k}_I\,[n]$ up to the choice of an orientation on $\mathbb{S}^*$.

The Fourier-Sato transform $\mathfrak{F}^\wedge$ and its inverse $\mathfrak{F}^\vee$ are the functors

$$\mathfrak{F}^\wedge := \mathbf{k}_P \circ \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbf{S}}) \rightleftarrows \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbf{S}^*}) \colon \circ K_I := \mathfrak{F}^\vee \tag{2.27}$$

Theorem 2.7.1 (see [29]) *The functor $\mathfrak{F}^\wedge$ and the functor $\mathfrak{F}^\vee$ are equivalences of categories quasi-inverse to each other.*

We shall give a proof of this result at the same time as we shall prove Theorem 2.7.4 below.

Now, we consider the n-sphere $\mathbb{S}^n$ of radius 1 embedded in the Euclidian space $\mathbb{R}^{n+1}$ and endowed with its canonical Riemannian metric. Denoting by $||\cdot||$ the Euclidian norm on $\mathbb{R}^{n+1}$, the map

$$\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n, \quad x \mapsto x/||x||$$

identifies the topological sphere $\mathbf{S}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$ and the Euclidian sphere $\mathbb{S}^n$.

The isomorphism $\mathbb{R}^n \simeq \mathbb{R}^{n*}$ induces the isomorphism $\mathbb{S}^n \simeq \mathbb{S}^{n*}$ and we shall identify these two spaces. When there is no risk of confusion, we write for short $\mathbb{S} := \mathbb{S}^n$. Recall that (using the notations defined in (3.8)):

$$r_{\mathrm{inj}}(\mathbb{S}^n) = \pi, \quad r_{\mathrm{conv}}(\mathbb{S}^n) = \pi/2.$$

The next result is obvious and is also a corollary of Theorem 3.2.3.

Proposition 2.7.2 *The metric space* $\mathbb{S}$ *satisfies* (2.2), (2.3) *and* (2.4) *when choosing* $\alpha_{\mathbb{S}} < \pi/2$.

In particular, $\mathbb{S}^n$ admits a bi-thickening $\{\mathfrak{L}_b\}_{b\in\mathbb{R}}$.

Lemma 2.7.3 *For* $0 < a \leq b \leq \pi/2$, *one has* $\mathbf{k}_{\Delta_a^\circ} \circ \mathbf{k}_{\Delta_b}[n] \simeq \mathbf{k}_{\Delta_{b-a}}$.

Proof Consider the diagram

$$\begin{tikzcd} & \mathbb{S}\times\mathbb{S}\times\mathbb{S} \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ \Delta_a^\circ \subset \mathbb{S}\times\mathbb{S} & \mathbb{S}\times\mathbb{S} & \mathbb{S}\times\mathbb{S}\supset\Delta_b \end{tikzcd}$$

For $x_1, x_3 \in \mathbb{S}$, set for short

$$P_{x_3}^b = \Delta_b \cap (\mathbb{S}\times\{x_3\}), \quad I_{x_1}^a = \Delta_a^\circ \cap (\{x_1\}\times\mathbb{S}).$$

Denote by $\widetilde{q}_{13}$ the restriction of q_{13} to $\Delta_a^\circ \times_{\mathbb{S}} \Delta_b$. Then

$$\widetilde{q}_{13}^{-1}(x_1, x_3) = \{x_2 \in \mathbb{S};\, d_{\mathbb{S}}(x_1, x_2) < a,\, d_{\mathbb{S}}(x_2, x_3) \leq b\}.$$

In other words, $\widetilde{q}_{13}^{-1}(x_1, x_3)$ is the intersection of an open ball of radius a and a closed ball of radius b with $a \leq b$. It follows that

$$\mathrm{R}\Gamma_c(I_{x_1}^a \times_{\mathbb{S}} P_{x_3}^b; \mathbf{k}_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}}) = \begin{cases} \mathbf{k}[-n] & \text{if } d_{\mathbb{S}}(x_1, x_3) \leq b-a, \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 2.7.4 *The equivalence* $\mathfrak{F}^\wedge$ *given by* Theorem 2.7.1 *induces an isometry*

$$(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{S}}), \mathrm{dist}_{\mathbb{S}}) \xrightarrow{\sim} (\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{S}^*}), \mathrm{dist}_{\mathbb{S}^*}).$$

Proof of both Theorems 2.7.1 and 2.7.4. Let us identify $\mathbb{S}^n$ and the dual sphere $\mathbb{S}^{n*}$. Then the sets P and I of (2.25) may be also defined as:

$$P = \{(x, y) \in \mathbb{S}\times\mathbb{S};\, d_{\mathbb{S}}(x, y) \leq \pi/2\}, \quad I = \{(x, y) \in \mathbb{S}\times\mathbb{S};\, d_{\mathbb{S}}(x, y) < \pi/2\}. \tag{2.28}$$

Since $\mathbf{k}_{\Delta_{\pi/2}} \simeq \mathbf{k}_{\Delta_{\pi/4}} \circ \mathbf{k}_{\Delta_{\pi/4}}$ we have $\mathbf{k}_P \simeq \mathfrak{K}_{\pi/2}$. (It was not possible to deduce directly this result form (2.28) since $\alpha_{\mathbb{S}} < \pi/2$.) Therefore $\mathbf{k}_P \circ$ is an isometry and the inverse of $\mathbf{k}_P$ is given by $\mathfrak{K}_{-\pi/2}$ which is isomorphic to K_I. □

Remark 2.7.5 A similar result holds for the Radon transform on real projective spaces.

3 The interleaving distance associated with a Hamiltonian isotopy

3.1 General case

Let us briefly recall the main result of [17] § 3. Consider a real C^∞-manifold X, its cotangent bundle $\pi_X\colon T^*X \to X$ endowed with the Liouville form α_X and an open interval I of $\mathbb{R}$ containing 0. Set as above $\dot{T}^*X = T^*X \setminus T^*_X X$, where $T^*_X X$ is the zero-section, and still denote by $\pi_X\colon \dot{T}^*X \to X$ the projection. When there is no risk of confusion, we may write π instead of π_X.

Assume to be given a real C^∞-function $h\colon \dot{T}^*X \times I \to \mathbb{R}$ homogeneous of degree 1 with respect to the fiber variable. Let Φ_h denote the flow associated with the Hamiltonian vector field H_h. We assume that Φ_h is well-defined on the open interval $I \subset \mathbb{R}$. Hence,

$$\Phi_h\colon \dot{T}^*X \times I \to \dot{T}^*X \tag{3.1}$$

and [17, hypothesis (3.1)] is satisfied, that is, setting $\varphi_{h,t} = \Phi_h(\cdot, t)$, $\varphi_{h,t}$ is a homogeneous symplectic isomorphism of $\dot{T}^*X$ for each $t \in I$ and $\varphi_{h,0} = \mathrm{id}_{\dot{T}^*X}$. To Φ_h, one associates

$$v_{\Phi_h} = \frac{\partial \Phi_h}{\partial t}\colon \dot{T}^*X \times I \to T\dot{T}^*X.$$

One recovers h by $h = \langle \alpha_X, v_{\Phi_h} \rangle$.

Denote by $\Lambda_h \subset \dot{T}^*X \times \dot{T}^*X \times T^*I$ the smooth conic Lagrangian manifold associated with Φ_h (see [17, Lem. A.2]):

$$\Lambda_h = \{(\Phi_h(x,\xi,t), (x,-\xi), (-h(\Phi_h(x,\xi,t),t))); (x,\xi) \in \dot{T}^*X, t \in I\}. \tag{3.2}$$

The main result of loc. cit. (see [17, Th. 3.7]) is the existence of an object $K^h \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{X\times X\times I})$ (denoted K therein) characterized by the two properties:

$$\mathrm{SS}(K^h) \subset \Lambda_h \cup T^*_{X\times X\times I}(X \times X \times I) \text{ and } K^h|_{\{t=0\}} \simeq \mathbf{k}_\Delta. \tag{3.3}$$

Now we assume that

$$\begin{cases} h \text{ is not time-depending, homogeneous of degree 1 with respect to the fiber} \\ \text{variable and the hamiltonian flow } \Phi \text{ is well-defined on } \dot{T}^*X \times \mathbb{R}. \end{cases} \tag{3.4}$$

Note that since h is not time-depending, the hamiltonian flow Φ is well-defined on $\dot{T}^*X \times \mathbb{R}$ as soon as it is well-defined on $\dot{T}^*X \times I$ for some open interval I containing 0.

One has

$$\phi_{h,a} \circ \phi_{h,b} = \phi_{h,a+b}. \tag{3.5}$$

Therefore the object K^h belongs to $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{X\times X\times\mathbb{R}})$.

For $a\in\mathbb{R}$, we set $K^h_a = K^h|_{t=a}$.

Lemma 3.1.1 *Assuming (3.4), we have the isomorphisms*

$$K^h_a \circ K^h_b \simeq K^h_{a+b} \textit{ for } a, b \in \mathbb{R}. \tag{3.6}$$

Proof By (3.5), the two isotopies $\{\Phi_{h,a}\circ\Phi_{h,t}\}_{t\in\mathbb{R}}$ and $\{\Phi_{h,a+t}\}_{t\in I}$ coincide. Their associated kernels are respectively $K^h_a\circ K^h$ and $T_{a*}(K^h)$, where T_a is the translation $(x,x',t)\mapsto(x,x',t+a)$. These two kernels are micro-supported by Λ and their restriction at $t=-a$ are isomorphic to $\mathbf{k}_\Delta$. They are thus isomorphic by the unicity of kernels satisfying (3.3) and restricting to $t=b$, we get (3.6). □

Now we assume:

$$\text{the function } h \text{ is non-positive.} \tag{3.7}$$

In the sequel, we denote by $(t;\tau)$ the coordinates on $T^*\mathbb{R}$. Therefore, $\Lambda_h\subset\dot{T}^*X\times\dot{T}^*X\times T^*_{\tau\geq0}\mathbb{R}$ and it follows from [17, Prop. 4.8] that for $a\leq b\in\mathbb{R}$ there are natural morphisms

$$\rho_{a,b}\colon K^h_b\to K^h_a,$$

satisfying the compatibility conditions of Theorem 1.2.2. Therefore we have:

Theorem 3.1.2 *Assume to be given a real non-positive C^∞-function $h\colon\dot{T}^*X\to\mathbb{R}$ homogeneous of degree 1 in the fiber variable such that the associated flow Φ_h is defined on $\dot{T}^*X\times I$ for an open interval I containing 0. Then the family $\{K^h_a\}_{a\in\mathbb{R}}$ defines a monoidal presheaf $\mathfrak{K}^h$ on $(\mathbb{R},+)$ with values in $(\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X}),\circ)$.*

(Recall that for a monoidal presheaf $\mathfrak{K}$ on $(\mathbb{R},+)$ one sets $\mathfrak{K}_a:=\mathfrak{K}(a)$.)

Remark 3.1.3 One shall not confuse the monoidal presheaf $\mathfrak{K}^h$, a presheaf on the monoidal ordered set $(\mathbb{R}_\geq,+)$ with values in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X})$ and K^h, an object of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{X\times X\times\mathbb{R}})$. The object K^h is explicitly calculated in [17, Exa. 3.10, 3.11] for the cases of the Euclidean space and the Euclidian sphere.

Definition 3.1.4 Let $h\colon\dot{T}^*X\to\mathbb{R}$ be a real non-positive C^∞-function homogeneous of degree 1 in the fiber variable such that the associated flow Φ_h is defined on $\dot{T}^*X\times I$ for an open interval I containing 0. We denote by dist_h the pseudo-distance on $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ associated with the monoidal presheaf $\mathfrak{K}^h$ (see Definition 1.3.4).

Remark 3.1.5 The notion of non-positive isotopy is due to [15]. Let us also mention that several distances naturally appear in symplectic topology (see for example the recent paper [28]). As far as we know, the pseudo-distance dist_h on sheaves on X is new.

3.2 The case of Riemannian manifolds

In this Section, we shall use some classical results of Riemannian geometry, referring to [8, 13].

Consider a Riemannian manifold (X, g) of class C^∞ and denote by d_X its associated distance. We assume

$$(X, g) \text{ is complete and has a strictly positive convexity radius } r_{\mathrm{conv}}, \text{ hence strictly positive injectivity radius } r_{\mathrm{inj}}. \tag{3.8}$$

Recall that $r_{\mathrm{conv}} \le \frac{r_{\mathrm{inj}}}{2}$ (see [2]).

$$\text{For } (X, g) \text{ satisfying (3.8), we choose } 0 < \alpha_X < r_{\mathrm{conv}}. \tag{3.9}$$

Note that a compact Riemannian manifold satisfies hypothesis (3.8).

Consider the cotangent bundle T^*X and its zero-section $T^*_X X$. The isomorphism $TX \xrightarrow{\sim} T^*X$ endows T^*X with a metric and we denote by $||\xi||_x$ the norm of the vector $\xi \in T^*_x X$.

For the reader's convenience, we recall some of the notations (2.1) and introduce some new ones:

$$\begin{cases} B_a(x_0) = \{x \in X; d_X(x_0, x) \le a\}, \\ B^\circ_a(x_0) = \{x \in X; d_X(x_0, x) < a\}, \\ \Delta_a = \{(x_1, x_2) \in X \times X; d_X(x_1, x_2) \le a\}, \\ \Delta^\circ_a = \{(x_1, x_2) \in X \times X; d_X(x_1, x_2) < a\}, \\ S_a(x_0) = \{x \in X; d_X(x_0, x) = a\}, \\ B^*_X(r) = \{(x; \xi) \in T^*X; ||\xi||_x < r\}, \\ S^*_X(r) = \{(x; \xi) \in T^*X; ||\xi||_x = r\}. \end{cases} \tag{3.10}$$

We also introduce the sets:

$$\begin{cases} I =]-r_{\mathrm{inj}}, r_{\mathrm{inj}}[, \quad I^+ =]0, r_{\mathrm{inj}}[, \quad I^- =]-r_{\mathrm{inj}}, 0[, \\ J = X \times X \times I, \quad J^\pm = X \times X \times I^\pm, \\ Z = \{(x, y, t) \in J; d_X(x, y) \le t < r_{\mathrm{inj}}\}, \\ \Omega^+ = \{(x, y, t) \in J; d_X(x, y) < t\}, \\ \Omega^- = \{(x, y, t) \in J; d_X(x, y) < -t\}, \\ A = \{((x; \xi), t) \in T^*X \times I; ||\xi||_x \le t < r_{\mathrm{inj}}\} \end{cases} \tag{3.11}$$

Let us recall the construction of the exponential map. Consider the function

$$f : T^*X \to \mathbb{R}, \quad f(x, \xi) = -\frac{1}{2}||\xi||^2_x. \tag{3.12}$$

Denote by X_f the Hamiltonian vector fields of f and by Φ_f the flows associated to this vector fields. In the literature (see e.g., [25, Exa. 1.1.23], [26, p. 15]), the flow Φ_f is known (via the isomorphism $TX \simeq T^*X$) as *the geodesic flow* of the Riemannian manifold (X, g).

The exponential map e_f, given by

$$e_f(x, \xi, t) = \pi_X \circ \Phi_f(x, \xi, t),$$

is well-defined for $t \in \mathbb{R}$. The well-known theorem (see loc. cit.) which asserts that the geodesic flow coincides with the Hamiltonian flow of the function f may be translated as follows.

Lemma 3.2.1 *The map*

$$E_f : T^*X \times I \to J = X \times X \times I, \quad E_f(x, \xi, t) = (e_f(x, \xi, 1), x, t) \tag{3.13}$$

is well-defined and induces C^∞-isomorphisms

$$B_X^*(r) \times \{t\} \simeq \Delta_r^\circ \times \{t\} \textit{ for } r < r_{\mathrm{inj}} \textit{ and all } t.$$

The proof of the next lemma is due to Stéphane Guillermou. It is much simpler than an earlier proof of ours.

Lemma 3.2.2 *Let (X, g) be a Riemannian manifold satisfying (3.8) and let α_X be as in (3.9). Let x and y in X with $x \neq y$ and set $Z_a(x, y) = B_a^\circ(x) \cap B_a(y)$. Then* $\mathrm{R}\Gamma(X; \mathbf{k}_{Z_a(x,y)}) \simeq 0$. *In other words, (2.3)(d) is satisfied.*

Proof

(i) We may assume

$$\begin{cases} \text{for any } x_1, x_2 \text{ in } W \text{ with } x_1 \neq x_2, \text{ there exists a unique geodesic } l(x_1, x_2) \subset W \\ \text{with } x_1, x_2 \in l(x_1, x_2), \\ \text{for } x_1, x_2, x_3 \text{ in } W, \text{ if } d(x_1, x_3) = d(x_1, x_2) + d(x_2, x_3) \text{ then } x_2 \in l(x_1, x_3). \end{cases} \tag{3.14}$$

Let us introduce some notations:

$$\begin{aligned} &Z_a = Z_a(x, y), \\ &M = \{z; d(x, z) = d(y, z)\}, \\ &M_x = \{z; d(x, z) < d(y, z)\}, \quad M_y = \{z; d(x, z) > d(y, z)\}, \\ &Z' = M_x \cap B_a(y), \quad Z'' = B_a^\circ(x) \cap \overline{M}_y. \end{aligned}$$

Note that $Z_a = Z' \sqcup Z''$, Z' is open in Z_a and Z'' is closed in Z_a.

(ii) It follows from (3.14) that

$$\begin{cases} \text{for any geodesic } l(x, z), l(x, z) \cap M \text{ has at most one point, and similarly with} \\ l(y, z). \end{cases} \tag{3.15}$$

Indeed, let $z_1, z_2 \in l(x, z) \cap M$. Then $d(x, z_1) = d(x, z_2) + d(z_2, z_1)$ or $d(x, z_2) = d(x, z_1) + d(z_1, z_2)$ or $d(z_1, z_2) = d(z_1, x) + d(x, z_2)$. Assume for example the first equality. Since $z_1, z_2 \in M$, we get $d(y, z_1) = d(y, z_2) + d(z_2, z_1)$ which implies that the geodesic (y, z_1) contains z_2. Since there is at most one geodesic containing both z_1 and z_2, we find that $y \in l(x, z)$ which implies $z_1 = z_2$.

(iii) Let us prove that $\mathrm{R}\Gamma(X; \mathbf{k}_{Z'}) \simeq 0$. Let $p\colon B_a(y) \setminus \{y\} \to S_a(y)$ be the map which sends $z \in B_a(y) \setminus \{y\}$ to $p(z) \in l(y, z) \cap S_a(y)$. It follows from (3.15) that the fibers of p intersect Z' along a unique interval and this interval is half-open. Since $y \notin \overline{Z}'$, we have $\mathrm{R}\Gamma(X; \mathbf{k}_{Z'}) \simeq \mathrm{R}\Gamma(B_a(y); \mathbf{k}_{Z'}) \simeq \mathrm{R}\Gamma(B_a(y)\setminus\{y\}; \mathbf{k}_{Z'})$. Moreover, $\mathrm{R}\Gamma(B_a(y)\setminus\{y\}; \mathbf{k}_{Z'}) \simeq \mathrm{R}\Gamma(S_a(y); \mathrm{R}p_!\mathbf{k}_{Z'}) \simeq 0$.

(iv) Let us prove that $\mathrm{R}\Gamma(X; \mathbf{k}_{Z''}) \simeq 0$. Let $q\colon B_a(x) \setminus \{x\} \to S_a(x)$ be the map which sends $z \in B_a(x) \setminus \{y\}$ to $p(z) \in l(x, z) \cap S_a(x)$. It follows from (3.15) that the fibers of q intersect Z'' along a unique interval and this interval is half-open. Since $x \notin \overline{Z}''$, we have $\mathrm{R}\Gamma(X; \mathbf{k}_{Z''}) \simeq \mathrm{R}\Gamma(B_a(x); \mathbf{k}_{Z''}) \simeq \mathrm{R}\Gamma(B_a(x) \setminus \{x\}; \mathbf{k}_{Z''})$. Moreover, $\mathrm{R}\Gamma(B_a(x) \setminus \{x\}; \mathbf{k}_{Z''}) \simeq \mathrm{R}\Gamma(S_a(x); \mathrm{R}q_!\mathbf{k}_{Z''}) \simeq 0$.

(v) The result then follows from the distinguished triangle $\mathbf{k}_{Z'} \to \mathbf{k}_{Z_a} \to \mathbf{k}_{Z''} \xrightarrow{+1}$.

□

Theorem 3.2.3 *Let (X, g) be a real Riemannian manifold satisfying (3.8) and let α_X be as in (3.9). Then hypotheses (2.2), (2.3) and (2.4) are satisfied.*

Proof (A) Let us prove (2.2).
(a)–(i) Let x_1 and x_2 in X. Since $a, b \leq \alpha_X < r_{\mathrm{conv}}$, the ball $B_a(x_1)$ and $B_a(x_2)$ are geodesically convex. Hence, their intersection is either empty or also geodesically convex and geodesically convex sets are contractible.
(a)–(ii) The closed and bounded subsets are compact by the Hopf–Rinow Theorem. Therefore, condition (ii) is satisfied.
(a)–(iii) Let us prove that for $(x_1, x_3) \in \Delta_{a+b}$, there exists $x_2 \in X$ such that $d_X(x_1, x_2) \leq a$ and $d_X(x_2, x_3) \leq b$. Without loss of generality we can assume that $d_X(x_1, x_3) = a + b$. Since X is complete, it follows from the Hopf–Rinow Theorem that x_1 and x_3 can be joined by a minimal geodesic $\gamma\colon [0, 1] \to X$. Then $d(x_1, \gamma(t))$ will take all values between 0 and $a + b$. Let $t_2 \in [0, 1]$ such that $d(x_1, \gamma(t_2)) = a$. Since γ is also minimal on every subinterval of $[0, 1]$ it is minimal on $[t_2, 1]$. Then, $d_X(x_2, x_3) = b$.
(B) Let us prove (2.3)(c). The set Ω^+ is, in a neighborhood of $\Delta \times \{0\}$ and locally in $X \times X \times \mathbb{R}$, C^∞-isomorphic to the open set $\{(x, \xi, t); ||\xi||_x < t\}$. By the Morse lemma with parameters (see [18, Lem. C.6.1 and its proof]) this last set is locally topologically convex since, in a local chart, it is isomorphic to a constant cone $\{((x; \xi), t); ||\xi|| < t\}$ associated with the standart Euclidian metric.
(C) Let us prove (2.3)(a) and (b). By Lemma 3.2.1, we are reduced to prove the result after replacing Δ_a with $B_X^*(a)$ in which case the proof is similar to (B).
(D) The hypothesis (2.3)(d) is satisfied thanks to Lemma 3.2.2.
(E) The hypothesis (2.4) follows from Lemma 2.1.3. Indeed, the distance function $f := d_X\colon X \times X \to \mathbb{R}$ is of class C^∞ on $W := \Delta_a^\circ \setminus \Delta$ for $a \leq \alpha_X$ and we are reduced to check that for any given $y \in X$, the differential of the function $x \mapsto g(x) = d_X(y, x)$ does not vanish for $0 < d_X(x, y) < \alpha_X$. By composing with the exponential map, we are reduced to prove the same result on $T_y^* X$ in which case it is clear. □

Notation 3.2.4 We shall denote by $a \mapsto \mathfrak{K}_a^{\mathrm{dist}}$, $a \in \mathbb{R}$ the bithickening of the diagonal given by Theorems 3.2.3 and Proposition 2.2.3.

3.3 Comparison of the two kernels on Riemannian manifolds

In this subsection, (X, g) denotes a Riemannian manifold with associated distance d_X. We shall always assume (3.8).

Recall the function f and the flow Φ_f defined in (3.12), and consider the function

$$h\colon \dot{T}^*X \to \mathbb{R}, \quad h(x,\xi) = -||\xi||_x. \tag{3.16}$$

Denote by X_h the Hamiltonian vector fields of h and by Φ_h the flow associated to this vector fields. Since h is homogeneous of degree 1 in ξ and f is homogeneous of degree 2 in ξ, we have for $\lambda > 0$

$$\begin{cases} \Phi_h(x,t;\lambda\xi) = \lambda \cdot \Phi_h(x,t;\xi), \\ \Phi_f(x,t;\lambda\xi) = \lambda \cdot \Phi_f(x,\lambda t;\xi). \end{cases} \tag{3.17}$$

(Of course, in the formula above, λ acts on the fiber variables.)

Since $f = -\frac{1}{2}h^2$, the Hamiltonian vector fields of f and h are related by $X_f = -hX_h = ||\xi||X_h$. In particular, we see that X_f and X_h are tangent to the unit co-sphere $S_X^*(r)$ and their restrictions to $S_X^*(1)$ coincide. It follows that $\Phi_h(x,t;\xi) = \Phi_f(x,t;\xi)$ if $||\xi|| = 1$ and, by homogeneity, using (3.17)

$$\Phi_h(x,t;\xi) = ||\xi||_x \cdot \Phi_f\left(x,t;\frac{\xi}{||\xi||_x}\right) = ||\xi||_x \cdot \Phi_f\left(x,1;\frac{t}{||\xi||_x}\xi\right) \text{ for } \xi \neq 0. \tag{3.18}$$

By the hypothesis (3.8), we get

Lemma 3.3.1 *Hypothesis* (3.4) *is satisfied for* h.

Denote as above by Λ_h the Lagrangian manifold given by (3.2). One has

$$\Lambda_h = \{(\Phi_h(x,\xi,t), (x,-\xi), (t, ||\xi||_x)); (x,\xi) \in \dot{T}^*X, t \in \mathbb{R}\}. \tag{3.19}$$

Denote by K^h the quantization of Λ_h and by $\mathfrak{K}^h$ the monoidal presheaf on $(\mathbb{R}, +)$ with values in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X\times X}, \circ)$ associated with K^h constructed in Theorem 3.1.2 and denote by $\mathfrak{K}^{\mathrm{dist}}$ the monoidal presheaf associated with the good metric space (X, d_X) (see Theorem 3.2.3 and Notation 3.2.4).

With Notations (3.11), the distinguished triangle (2.8) reads as

$$\mathbf{k}_{\Omega^-} \otimes q_2^{-1}\omega_X \to K^d \to \mathbf{k}_Z \xrightarrow{+1}. \tag{3.20}$$

Lemma 3.3.2 *Assume* (3.8). *One has* $\Lambda_h \cap T^*J^+ = \dot{\mathrm{SS}}(\mathbf{k}_Z) \cap T^*J^+$.

Proof

(i) Recall that

$$\Lambda_h = \{(\Phi_h(x,\xi,t),(x,-\xi),(t,-h(\Phi_h(x,\xi,t));(x,\xi)) \in \dot{T}^*X, t \in I\}. \tag{3.21}$$

In particular,

$$\pi_{J^+}(\Lambda_h \cap T^*J^+) = E_f(\{||\xi||_x \le t\}) = \partial\Omega^+.$$

(ii) The set $\partial\Omega^+$ is a smooth hypersurface of J^+ and it follows from [20, Prop. 8.3.10] that $\Lambda_h \cap T^*J^+$ is one half of $\dot{T}^*_{\partial\Omega^+}J^+$. Since $\Lambda_h \subset \{\tau \ge 0\}$, Λ_h is the interior conormal to $\partial\Omega^+$. □

Denote by $j\colon J^+ \hookrightarrow J$ the open embedding.

Lemma 3.3.3 *One has* $\mathbf{k}_Z \simeq \mathrm{R}j_*j^{-1}\mathbf{k}_Z$.

Proof One has $\mathbf{k}_{\Omega^+} \simeq j_!j^{-1}\mathbf{k}_{\Omega^+}$. Applying the duality functor $\mathrm{D}'_{X\times X\times\mathbb{R}}$ we get the result by Lemma 2.1.2. (Recall that, setting $M = X \times X \times \mathbb{R}$, $\mathrm{D}'_M \circ j_! \simeq \mathrm{R}j_* \circ \mathrm{D}'_M$.) □

In the proof of the next lemma, we shall use the operation $\widehat{+}$ defined in [20, § 6.2].

Lemma 3.3.4 *One has*

(a) $\mathrm{SS}(\mathbf{k}_Z) \cap \pi^{-1}(X \times X \times \{0\}) \subset \{(x,x,0;\xi,-\xi,\tau); \tau \ge ||\xi||_x\}$,
(b) *One has* $\mathrm{SS}(\mathbf{k}_{\Omega^-}) \cap \pi^{-1}(X \times X \times \{0\}) \subset \{(x,x,0;\xi,-\xi,\tau); \tau \ge ||\xi||_x\}$.

Proof

(a) Recall (3.19). We have in a neighborhood of $t = 0$

$$\Lambda_h = \Big\{(x - \frac{t}{||\xi||_x}\xi + t^2\epsilon(x,t,\xi), x, t; \xi + t\eta(x,t,\xi), -\xi, ||\xi||_x); (x,\xi) \in \dot{T}^*X, t \in \mathbb{R}\Big\}.$$

This implies

$$(\Lambda_h \cap T^*J^+)\widehat{+}\{(x,y,0;0,0,\tau \ge 0)\} \subset \{(x,x,0;\xi,-\xi,\tau); \tau \ge ||\xi||_x\}.$$

To conclude, apply [20, Th. 6.3.1] together with Lemmas 3.3.2 and 3.3.3.

(b) follows from (a) by applying the duality functor (using Lemma 2.1.2) together with v_* where v is the map $(x,y,t) \mapsto (x,y,-t)$. □

Lemma 3.3.5 *Let* $p = (x,x,0;\xi,-\xi,\tau)$ *with* $\tau > ||\xi||_x$. *Then*

(a) *the natural morphism* $\mathbf{k}_Z \to \mathbf{k}_{\Delta\times\{0\}}$ *is an isomorphism in* $\mathrm{D^b}(\mathbf{k}_J;p)$.

(b) *the natural morphism* $\mathbf{k}_{\Delta\times\{t=0\}} \otimes q_2^{-1}\omega_X^{\otimes -1}[-1] \to \mathbf{k}_{\{d_X(x,y)<-t\}}$ *is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_J; p)$.

Proof

(a) Similarly as in part (C) of the proof of Theorem 3.2.3, the set Z is, in a neighborhood of $\Delta\times\{0\}$ and locally on $X\times X\times\mathbb{R}$, C^∞-isomorphic to the set A of (3.11). We are thus reduced to prove a similar result with Z and $\Delta\times\{0\}$ replaced with A and $T^*_XX\times\{0\}$. In this case, the result follows from Lemma 3.3.6 below.
(b) follows from (a) by applying the duality functor, using Lemma 2.1.2. □

Lemma 3.3.6 *Let E be a vector bundle over X and let $\gamma\subset E$ be a closed convex proper cone containing the zero-section X. Let $p\in T^*E\times_E X$ with $p\in\mathrm{Int}(\gamma^\circ)$. Then the natural morphism $\mathbf{k}_\gamma\to\mathbf{k}_X$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E; p)$.*

Proof We may assume that $E = X\times\mathbb{V}$ for a real vector space $\mathbb{V}$. Let us choose local coordinates on X and identify $T^*\mathbb{V}$ with $\mathbb{V}\times\mathbb{V}^*$. Then $p = ((x;\xi),(0,\eta))\in T^*X\times\mathbb{V}\times\mathbb{V}^*$. By [20, Lem. 3.7.10], the Fourier-Sato transform interchanges the two objects $\mathbf{k}_\gamma$ and $\mathbf{k}_{X\times\{0\}}$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ with the two objects $\mathbf{k}_{\mathrm{Int}\gamma^\circ}$ and $\mathbf{k}_{E^*}$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E^*})$. Hence, applying Theorem 5.5.5 and formula (5.5.6) of loc. cit., we are reduced to prove that the natural morphism $\mathbf{k}_{\mathrm{Int}(\gamma^\circ)}\to\mathbf{k}_E$ is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{E^*}; q)$ with $q = ((x;\xi),(\eta,0))\in T^*X\times\mathbb{V}^*\times\mathbb{V}$, which is obvious since the two sheaves are isomorphic in a neighborhood of any point $(x,\eta)\in X\times\mathrm{Int}(\gamma^\circ)$. □

Recall the sheaf K^d constructed in Theorem 2.2.4 and the monoidal presheaf $\mathfrak{K}^{\mathrm{dist}}$.

Theorem 3.3.7 *Let (X,g) be a complete Riemannian manifold satisfying* (3.8)*. Then*

(a) *One has the isomorphism $K^h|_J \simeq K^d|_J$.*
(b) *the two monoidal presheaves $\mathfrak{K}^h$ and $\mathfrak{K}^{\mathrm{dist}}$ are isomorphic.*

Proof

(i) Of course, (b) follows from (a). By the unicity result in [17, Prop. 3.2 (iii)], it remains to prove that

$$\dot{\mathrm{S}}\mathrm{S}(K^d)\subset\Lambda_h. \tag{3.22}$$

(ii) It follows from the distinguished triangle (3.20) that $K^d|_{J^+}\simeq\mathbf{k}_Z|_{J^+}$ and it then follows from Lemma 3.3.2 that (3.22) is true on J^+. Moreover, $\dot{\mathrm{S}}\mathrm{S}(K^d|_{J^-}) = v(\dot{\mathrm{S}}\mathrm{S}(K^d|_{J^+}))$ where v is the map $(x,y,t;\xi,\eta,\tau)\mapsto(y,x,-t;\eta,\xi,\tau)$. Since $v(\Lambda_h)=\Lambda_h$, we get that (3.22) is true on J^-.
(iii) One has $\mathrm{SS}(K^d)\cap\pi^{-1}(X\times X\times\{0\})\subset\{(x,x,0;\xi,-\xi,\tau);\tau\geq||\xi||_x\}$ thanks to Lemma 3.3.5. The natural morphism $\psi\colon\mathbf{k}_Z\to\mathbf{k}_{\Omega^-}\otimes q_2^{-1}\omega_X[+1]$ is an isomorphism by Lemma 3.3.5. This implies (3.22). □

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