Microlocal study of ind-sheaves I: micro-support and regularity

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Abstract

We define the notions of micro-support and regularity for ind-sheaves, and prove their invariance by contact transformations. We apply the results to the ind-sheaves of temperate holomorphic solutions of $\mathcal{D}$-modules. We prove that the micro-support of such an ind-sheaf is the characteristic variety of the corresponding $\mathcal{D}$-module and that the ind-sheaf is regular if the $\mathcal{D}$-module is regular holonomic.

Contents

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1 Introduction

Recall that a system of linear partial differential equations on a complex manifold X is the data of a coherent module $\mathcal{M}$ over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators. Let F be a complex of sheaves on X with $\mathbb{R}$-constructible cohomologies (one says an $\mathbb{R}$-constructible sheaf, for short). The complex of "generalized functions" associated with F is described by the complex $R\mathcal{H}om(F, \mathcal{O}_X)$, and the complex of solutions of $\mathcal{M}$ with values in this complex is described by the complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X)).$$

One may also microlocalize the problem by replacing $R\mathcal{H}om(F, \mathcal{O}_X)$ with $\mu hom(F, \mathcal{O}_X)$. In [4] one shows that most of the properties of this complex, especially those related to propagation or Cauchy problem, are encoded in two geometric objects, both living in the cotangent bundle T^*X, the characteristic variety of the system $\mathcal{M}$, denoted by $\mathrm{char}(\mathcal{M})$, and the micro-support of F, denoted by $SS(F)$.

The complex $R\mathcal{H}om(F, \mathcal{O}_X)$ allows us to treat various situations. For example if M is a real analytic manifold and X is a complexification of M, by taking as F the dual $D'(\mathbb{C}_M)$ of the constant sheaf on M, one obtains the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. If Z is a complex analytic hypersurface of X and $F = \mathbb{C}_Z[-1]$ is the (shifted) constant sheaf on Z, one obtains the sheaf of holomorphic functions with singularities on Z. However, the complex $R\mathcal{H}om(F, \mathcal{O}_X)$ does not allow us to treat sheaves associated with holomorphic functions with growth conditions. So far this difficulty was overcome in two cases, the temperate case including Schwartz's distributions and meromorphic functions with poles on Z and the dual case including C^∞-functions and the formal completion of $\mathcal{O}_X$ along Z. The method was two construct specific functors, the functor $T\mathcal{H}om$ of [2] and the functor $\overset{\mathrm{w}}{\otimes}$ of [5].

There is a more radical method, which consists in replacing the too narrow framework of sheaves by that of ind-sheaves, as explained in [6]. For example, the presheaf of holomorphic temperate functions on a complex manifold X (which, to a subanalytic open subset of X, associates the space of holomorphic functions with temperate growth at the boundary) is clearly not a sheaf. However it makes sense as an object (denoted by $\mathcal{O}_X^t$) of the derived category of ind-sheaves on X. Then it is natural to ask if the microlocal theory of sheaves, in particular the theory of micro-support, applies in this general setting.

In this paper we give the definition and the elementary properties of the micro-support of ind-sheaves as well as the notion of regularity.

We prove in particular that the micro-support $SS(\cdot)$ and the regular micro-support $SS_{reg}(\cdot)$ of ind-sheaves behave naturally with respect to distinguished triangles and that these micro-supports are invariant by "quantized contact transformations" (in the framework of sheaf theory, as explained in [4]).

When X is a complex manifold and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we study the ind-sheaf $Sol^t(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^t)$. We prove that

(i) $SS(Sol^t(\mathcal{M})) = \mathrm{char}(\mathcal{M})$,

(ii) if $\mathcal{M}$ is holonomic, $Sol^t(\mathcal{M})$ is regular if $\mathcal{M}$ is regular holonomic.

Finally, we treat an example: we calculate the ind-sheaf of the temperate holomorphic solutions of an irregular differential equation.

This paper is the first one of a series. In Part II, we shall introduce the microlocalization functor for ind-sheaves, and in Part III we shall study the functorial behavior of micro-supports.

2 Notations and review

We will mainly follow the notations in [4] and [6].

Geometry.
In this paper, all manifolds will be real analytic (sometimes, complex analytic). Let X be a manifold. One denotes by $\tau\colon TX \to X$ the tangent bundle to X and by $\pi\colon T^*X \to X$ the cotangent bundle. One denotes by $a\colon T^*X \to T^*X$ the antipodal map. If $S \subset T^*X$, one denotes by $\dot{S}$ the set $S \setminus T^*_XX$, and one denotes by S^a the image of S by the antipodal map. In particular, $\dot{T}^*X = T^*X \setminus X$, the set T^*X with the zero-section removed. One denotes by $\dot{\pi}\colon \dot{T}^*X \to X$ the projection.

For a smooth submanifold Y of X, T_YX denotes the normal bundle to Y and T^*_YX the conormal bundle. In particular, T^*_XX is identified with X, the zero-section.

For a submanifold Y of X and a subset S of X, we denote by $C_Y(S)$ the Whitney normal cone to S along Y, a conic subset of T_YX.

If S is a locally closed subset of T^*X, we say that S is $\mathbb{R}^+$-conic (or simply "conic", for short) if it is locally invariant under the action of $\mathbb{R}^+$. If S is smooth, this is equivalent to saying that the Euler vector field on T^*X is tangent to S.

Let $f\colon X\to Y$ be a morphism of real manifolds. One has two natural maps

$$T^*X \underset{f_d}{\longleftarrow} X\times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y \tag{2.1}$$

(In [4], f_d is denoted by ${}^tf'$.) We denote by q_1 and q_2 the first and second projections defined on $X\times Y$.

Sheaves.

Let k be a field. We denote by $\mathrm{Mod}(k_X)$ the abelian category of sheaves of k-vector spaces and by $D^b(k_X)$ its bounded derived category.

We denote by $\mathbb{R}\text{-}\mathrm{C}(k_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of k-vector spaces on X, and by $D^b_{\mathbb{R}-\mathrm{c}}(k_X)$ (resp. $D^b_{\mathrm{w}-\mathbb{R}-\mathrm{c}}(k_X)$) the full triangulated subcategory of $D^b(k_X)$ consisting of objects with $\mathbb{R}$-constructible (resp. weakly $\mathbb{R}$-constructible) cohomology. On a complex manifold, one defines similarly the categories $D^b_{\mathbb{C}-\mathrm{c}}(k_X)$ and $D^b_{\mathrm{w}-\mathbb{C}-\mathrm{c}}(k_X)$ of $\mathbb{C}$-constructible and weakly $\mathbb{C}$-constructible sheaves.

If Z is a locally closed subset of X and if F is a sheaf on X, recall that F_Z is a sheaf on X such that $F_Z|_Z\simeq F|_Z$ and $F_Z|_{X\setminus Z}\simeq 0$. One writes k_{XZ} instead of $(k_X)_Z$ and one sometimes writes k_Z instead of k_{XZ}.

If $f\colon X\to Y$ is a morphism of manifolds, one denotes by $\omega_{X/Y}$ the relative dualizing complex on X and if $Y=\{\mathrm{pt}\}$ one simply denotes it by ω_X. Recall that

$$\omega_X\simeq \mathrm{or}_X[\dim_{\mathbb{R}} X]$$

where or_X is the orientation sheaf and $\dim_R X$ is the dimension of X as a real manifold. We denote by D'_X and D_X the duality functors on $D^b(k_X)$, defined by

$$D'_X(F)=R\mathcal{H}om\,(F,k_X),\;\; D_X(F)=R\mathcal{H}om\,(F,\omega_X).$$

If F is an object of $D^b(k_X)$, $SS(F)$ denotes its micro-support, a closed conic involutive subset of T^*X. For an open subset U of T^*X, one denotes by $D^b(k_X;U)$ the localization of the category $D^b(k_X)$ with respect to the triangulated subcategory consisting of sheaves F such that $SS(F)\cap U=\emptyset$.

We shall also use the functor μhom as well as the operation $\widehat{+}$ and refer to loc. cit. for details.

$\mathcal{O}$ and $\mathcal{D}$

On a complex manifold X we consider the structural sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf $\mathcal{D}_X$ of linear holomorphic differential operators of finite order.

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the abelian category of coherent $\mathcal{D}_X$-modules. We denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules and by $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $D^b_{\mathrm{hol}}(\mathcal{D}_X)$), $D^b_{\mathrm{rh}}(\mathcal{D}_X)$) its full triangulated category consisting of objects with coherent cohomologies (resp. holonomic cohomologies, regular holonomic cohomologies).

Categories. In this paper, we shall work in a given universe $\mathcal{U}$, and a category means a $\mathcal{U}$-category. If $\mathcal{C}$ is a category, $\mathcal{C}^\wedge$ denotes the category of functors from $\mathcal{C}^{\mathrm{op}}$ to **Set**. The category $\mathcal{C}^\wedge$ admits inductive limits, however, in case $\mathcal{C}$ also admits inductive limits, the Yoneda functor $h^\wedge\colon \mathcal{C} \to \mathcal{C}^\wedge$ does not commute with such limits. Hence, one denotes by $\varinjlim$ the inductive limit in $\mathcal{C}$ and by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$.

One denotes by $\mathrm{Ind}(\mathcal{C})$ the category of ind-objects of $\mathcal{C}$, that is the full subcategory of $\mathcal{C}^\wedge$ consisting of objects F such that there exist a small filtrant category I and a functor $\alpha\colon I \to \mathcal{C}$, with

$$F \simeq \text{“}\varinjlim\text{”}\, \alpha, \text{ i.e., } F \simeq \text{“}\varinjlim_{i\in I}\text{”}\, F_i, \text{ with } F_i \in \mathcal{C}.$$

The category $\mathcal{C}$ is considered as a full subcategory of $\mathrm{Ind}(\mathcal{C})$.

If $\varphi\colon \mathcal{C} \to \mathcal{C}'$ is a functor, it defines a functor $I\varphi\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ which commutes with "$\varinjlim$".

If $\mathcal{C}$ is an additive category, we denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$ and by $K(\mathcal{C})$ the associated homotopy category. If $\mathcal{C}$ is abelian, one denotes by $D(\mathcal{C})$ its derived category. One defines as usual the full subcategories $C^*(\mathcal{C}), K^*(\mathcal{C}), D^*(\mathcal{C})$, with $* = +, -, b$. One denotes by Q the localization functor:

$$Q\colon K^*(\mathcal{C}) \to D^*(\mathcal{C}).$$

We keep the same notation Q to denote the composition $C^*(\mathcal{C}) \to K^*(\mathcal{C}) \to D^*(\mathcal{C})$.

One denotes by $C^{[a,b]}(\mathcal{C})$ the full subcategory of $C(\mathcal{C})$ consisting of objects $F^\bullet$ satisfying $F^i = 0$ for $i \notin [a,b]$. If $a, b \in \mathbb{Z}$ with $a \leq b$, there is a natural isomorphism

$$\mathrm{Ind}(C^{[a,b]}(\mathcal{C})) \xrightarrow{\sim} C^{[a,b]}(\mathrm{Ind}(\mathcal{C})).$$

Ind-sheaves. Here, X is a Hausdorff locally compact space with a countable base of open sets and k is a field. One denotes by $\mathrm{I}(k_X)$ the abelian category of ind-sheaves of k-vector spaces on X, that is, $\mathrm{I}(k_X) = \mathrm{Ind}(\mathrm{Mod}^c(k_X))$, the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves with compact support on X. We denote by $D^b(\mathrm{I}(k_X))$ the bounded derived category of $\mathrm{I}(k_X)$.

There is a natural fully faithful exact functor

$$\begin{aligned} \iota_X \colon \mathrm{Mod}(k_X) &\to \mathrm{I}(k_X), \\ F &\mapsto \text{“}\varinjlim_{U \subset\subset X}\text{”}\, F_U \ (U \text{ open}). \end{aligned}$$

Most of the time, we shall not write this functor and identify $\mathrm{Mod}(k_X)$ with a full abelian subcategory of $\mathrm{I}(k_X)$ and $D^b(k_X)$ with a full triangulated subcategory of $D^b(\mathrm{I}(k_X))$.

The category $\mathrm{I}(k_X)$ admits an internal hom denoted by $\mathcal{I}hom$ and this functor admits a left adjoint, denoted by $\otimes$. If $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$ and $G \simeq \text{“}\varinjlim_j\text{”}\, G_j$, then

$$\begin{aligned} \mathcal{I}hom\,(G, F) &\simeq \varprojlim_j \text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(G_j, F_i) \\ G \otimes F &\simeq \text{“}\varinjlim_i\text{”}\ \text{“}\varinjlim_j\text{”}(G_j \otimes F_i). \end{aligned}$$

The functor ι_X admits a left adjoint

$$\alpha_X \colon \mathrm{I}(k_X) \to \mathrm{Mod}(k_X),$$

To $F = \text{“}\varinjlim_{i\in I}\text{”}\, F_i$, this functor associates $\alpha_X(F) = \varinjlim_{i\in I} F_i$. This functor also admits a left adjoint

$$\beta_X \colon \mathrm{Mod}(k_X) \to \mathrm{I}(k_X),$$

and both functors α_X and β_X are exact. The functor β_X is not so easy to describe. For example, for an open subset U and a closed subset Z, one has;

$$\begin{aligned} \beta_X(k_{XU}) &\simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\, k_{XV} \ (V \text{ open}), \\ \beta_X(k_{XZ}) &\simeq \text{“}\varinjlim_{Z \subset V}\text{”}\, k_{X\overline{V}} \ (V \text{ open}). \end{aligned}$$

One sets

$$\mathcal{H}om\,(G,F) = \alpha_X \mathcal{I}hom\,(G,F) \in \mathrm{Mod}(k_X).$$

One has

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F) = \Gamma(X;\mathcal{H}om\,(G,F)).$$

The functors $\mathcal{I}hom$ and $\mathcal{H}om$ are left exact and admit right derived functors $R\mathcal{I}hom$ and $R\mathcal{H}om$.

Let $f\colon X \to Y$ be a morphism of topological spaces (Y satisfies the same assumptions as X). There are natural functors

$$\begin{aligned}
&f^{-1}\colon \mathrm{I}(k_Y)) \to \mathrm{I}(k_X)\\
&f_*\colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)\\
&f_{!!}\colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y).
\end{aligned}$$

The proper direct image functor is denoted by $f_{!!}$ instead of $f_!$ because it does not commute with ι, that is $\iota_Y f_! \neq f_{!!}\iota_X$ in general..

These functors induce derived functors, and moreover the functor $Rf_{!!}$ admits a right adjoint denoted by $f^!$:

$$\begin{aligned}
&f^{-1}\colon D^b\mathrm{I}(k_Y)) \to D^b(\mathrm{I}(k_X)),\\
&Rf_*\colon D^b(\mathrm{I}(k_X)) \to D^b(\mathrm{I}(k_Y)),\\
&Rf_{!!}\colon D^b(\mathrm{I}(k_X)) \to D^b(\mathrm{I}(k_Y)),\\
&f^!\colon D^b(\mathrm{I}(k_Y)) \to D^b(\mathrm{I}(k_X)).
\end{aligned}$$

Let $a_X\colon X \to \{\mathrm{pt}\}$ denote the canonical map. We also introduce a notation. We set

$$\begin{aligned}
\mathrm{I}\Gamma(X;\cdot) &= a_{X*}(\cdot),\\
R\mathrm{I}\Gamma(X;\cdot) &= Ra_{X*}(\cdot).
\end{aligned}$$

Ind-sheaves on real manifolds. Let X be a real analytic manifold. Among all ind-sheaves, there are those which are ind-objects of the category of $\mathbb{R}$-constructible sheaves, and we shall encounter them in our applications.

We denote by $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$ the full abelian subcategory of $\mathbb{R}\text{-}\mathrm{C}(k_X)$ consisting of $\mathbb{R}$-constructible sheaves with compact support. We set

$$\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(k_X))$$

and denote by $D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ the full subcategory of $D^b(\mathrm{I}(k_X))$ consisting of objects with cohomology in $\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)$. (Note that in [6], $\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)$ was denoted by $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$.)

Theorem 2.1. *The natural functor* $D^b(\mathrm{I}\mathbb{R}\text{–c}(k_X)) \to D^b_{\mathrm{I}\mathbb{R}\text{–c}}(\mathrm{I}(k_X))$ *is an equivalence.*

There is an alternative construction of $\mathrm{I}\mathbb{R}\text{–c}(k_X)$, using Grothendieck topologies. Denote by Op_X the category of open subsets of X (the morphisms $U \to V$ are the inclusions), and by by $\mathrm{Op}_{X_{sa}}$ its full subcategory consisting of open subanalytic subsets of X. One endows this category with a Grothendieck topology by deciding that a family $\{U_i\}_i$ in $\mathrm{Op}_{X_{sa}}$ is a covering of $U \in \mathrm{Op}_{X_{sa}}$ if for any compact subset K of X, there exists a finite subfamily which covers $U \cap K$. In other words, we consider families which are locally finite in X. One denotes by X_{sa} the site defined by this topology.

Sheaves on X_{sa} are easy to construct. Indeed, consider a presheaf F of k-vector spaces defined on the subcategory $\mathrm{Op}^c_{X_{sa}}$ of relatively compact open subanalytic subsets of X and assume that the sequence

$$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$$

is exact for any U and V in $\mathrm{Op}^c_{X_{sa}}$. Then there exists a unique sheaf $\tilde{F}$ on X_{sa} such that $\tilde{F}(U) \simeq F(U)$ for all $U \in \mathrm{Op}^c_{X_{sa}}$. Sheaves on X_{sa} define naturally ind-sheaves on X. Indeed:

Theorem 2.2. *There is a natural equivalence of abelian categories*

$$\mathrm{I}\mathbb{R}\text{–c}(k_X) \xrightarrow{\sim} \mathrm{Mod}(k_{X_{sa}}),$$

given by

$$\mathrm{I}\mathbb{R}\text{–c}(k_X) \ni F \mapsto \big(\mathrm{Op}^c_{X_{sa}} \ni U \mapsto \mathrm{Hom}_{\mathrm{I}\mathbb{R}\text{–c}(k_X)}(k_U, F)\big).$$

As usual, we denote by $\mathcal{C}^\infty_X$ the sheaf of complex-valued functions of class $\mathcal{C}^\infty$, by $\mathcal{D}b_X$ (resp. $\mathcal{B}_X$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathcal{D}_X$ the sheaf of analytic finite-order differential operators.

Let U be an open subset of X. One sets $\mathcal{C}^\infty_X(U) = \Gamma(U; \mathcal{C}^\infty_X)$.

Definition 2.3. Let $f \in \mathcal{C}^\infty_X(U)$. One says that f has *polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \tag{2.2}$$

It is obvious that f has polynomial growth at any point of U. We say that f is *tempered* at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point.

For an open subanalytic set U in X, denote by $\mathcal{C}^{\infty,t}_X(U)$ the subspace of $\mathcal{C}^\infty_X(U)$ consisting of tempered functions. Denote by $\mathcal{D}b^t_X(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{X\setminus U}(X;\mathcal{D}b_X) \to \Gamma(X;\mathcal{D}b_X) \to \mathcal{D}b^t_X(U) \to 0.$$

It follows from the results of Lojasiewicz [8] that $U \mapsto \mathcal{C}^\infty_X(U)$ and $U \mapsto \mathcal{D}b^t_X(U)$ are sheaves on the subanalytic site X_{sa}, hence define ind-sheaves.

Definition 2.4. We call $\mathcal{C}^{\infty,t}_X$ (resp. $\mathcal{D}b^t_X$) the ind-sheaf of tempered $\mathcal{C}^\infty$-functions (resp. tempered distributions).

One can also define the ind-sheaf of Whitney $\mathcal{C}^\infty$-functions, but we shall not recall here its construction. These ind-sheaves are well-defined in the category $\mathrm{Mod}(\beta_X\mathcal{D}_X)$. Roughly speaking, it means that if P is a differential operator defined on the closure $\bar{U}$ of an open subset U, then it acts on $\mathcal{C}^{\infty,t}_X(U)$ and $\mathcal{D}b^t_X(U)$.

Let now X be a complex manifold. We denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators, not to be confused with $\mathcal{D}_{X^{\mathbb{R}}}$. We set

$$\mathcal{O}^t_X \;\; := \;\; R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^{\mathbb{R}}})$$

One can prove that the natural morphism

$$R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X^{\mathbb{R}}}) \to R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^{\mathbb{R}}})$$

is an isomorphism. One calls $\mathcal{O}^t_X$ the ind-sheaf of tempered holomorphic functions. One shall be aware that in fact, $\mathcal{O}^t_X$ is not an ind-sheaf but an object of the derived category $D^b(\mathrm{I}(\mathbb{C}_X))$, or better, of $D^b(\beta_X\mathcal{D}_X)$. It is not concentrated in degree 0 as soon as $\dim X > 1$.

Let $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. It follows from the construction of $\mathcal{O}^t_X$ that:

$$R\mathcal{H}om\,(G, \mathcal{O}^t_X) \simeq T\mathcal{H}om\,(G, \mathcal{O}_X),$$

where $T\mathcal{H}om\,(\cdot, \mathcal{O}_X)$ denotes the functor of temperate cohomology of [2] (see also [5] for a detailed construction and [1] for its microlocalization).

3 Complements of homological algebra

The results of this section are extracted from [7]. Let $\mathcal{C}$ denote a small abelian category. We shall study some links between the derived category $D^b(\mathrm{Ind}(\mathcal{C}))$ and the category $\mathrm{Ind}(D^b(\mathcal{C}))$.

We define the functor $J\colon D^b(\mathrm{Ind}(\mathcal{C})) \to (D^b(\mathcal{C}))^\wedge$ by setting for $F \in D^b(\mathrm{Ind}(\mathcal{C}))$ and $G \in D^b(\mathcal{C})$

$$J(F)(G) = \mathrm{Hom}_{D^b(\mathrm{Ind}(\mathcal{C}))}(G, F). \tag{3.1}$$

Theorem 3.1. (i) *The functor J takes its values in $\mathrm{Ind}(D^b(\mathcal{C}))$.*

(ii) *Consider a small and filtrant category I, integers $a \leq b$ and a functor $I \to C^{[a,b]}(\mathcal{C})$, $i \mapsto F_i$. If $F \in D^b(\mathrm{Ind}(\mathcal{C}))$, $F \simeq Q(\text{“}\varinjlim_i\text{”} F_i)$ and $G \in D^b(\mathcal{C})$, then:*

(a) $J(F) \simeq \text{“}\varinjlim_i\text{”}\, Q(F_i)$,

(b) $\mathrm{Hom}_{D^b(\mathrm{Ind}(\mathcal{C}))}(G, F) \simeq \varinjlim_i \mathrm{Hom}_{D^b(\mathcal{C})}(G, F_i)$.

(iii) *For each $k \in \mathbb{Z}$, the diagram below commutes.*

$$\begin{array}{ccc} D^b(\mathrm{Ind}(\mathcal{C})) & \xrightarrow[J]{} & \mathrm{Ind}(D^b(\mathcal{C})) \\ & {\scriptstyle H^k}\searrow \quad \swarrow{\scriptstyle IH^k} & \\ & \mathrm{Ind}(\mathcal{C}) & \end{array}$$

Lemma 3.2. *Assume that $\mathcal{C}$ has finite homological dimension. Let $\varphi\colon X \to Y$ be a morphism in $\mathrm{Ind}(D^b(\mathcal{C}))$ and assume that φ induces an isomorphism $IH^k(\varphi)\colon IH^k(X) \xrightarrow{\sim} IH^k(Y)$ for every $k \in \mathbb{Z}$. Then φ is an isomorphism.*

Theorem 3.3. *Let $\psi\colon D^b(\mathrm{Ind}(\mathcal{C})) \to D^b(\mathrm{Ind}(\mathcal{C}'))$ be a triangulated functor which satisfies: if $F \in D^b(\mathrm{Ind}(\mathcal{C}))$, $F \simeq Q(\text{“}\varinjlim_i\text{”} F_i)$ with $F_i \in C^{[a,b]}(\mathcal{C})$, then $H^k\psi(F) \simeq \text{“}\varinjlim_i\text{”} H^k\psi(Q(F_i))$. Assume moreover that the homological dimension of $\mathcal{C}'$ is finite. Then there exists a unique functor $J\psi\colon \mathrm{Ind}(D^b(\mathcal{C})) \to \mathrm{Ind}(D^b(\mathcal{C}'))$ which commutes with “$\varinjlim$” and such that the diagram below commutes:*

$$\begin{array}{ccc} D^b(\mathrm{Ind}(\mathcal{C})) & \xrightarrow[\psi]{} & D^b(\mathrm{Ind}(\mathcal{C}')) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathrm{Ind}(D^b(\mathcal{C})) & \xrightarrow{J\psi} & \mathrm{Ind}(D^b(\mathcal{C}')). \end{array}$$

Remark 3.4. The functor $J\colon D^b(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(D^b(\mathcal{C}))$ is neither full nor faithful. Indeed, let $\mathcal{C} = \mathrm{Mod}^c(k_X)$ and let $F \in \mathrm{Mod}(k_X)$ considered as a full subcategory of $\mathrm{I}(k_X)$. Then

$$\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(k_X, F[n]) \simeq H^n(X; F).$$

On the other hand,

$$\mathrm{Hom}_{\mathrm{Ind}(D^b(\mathrm{Mod}^c(k_X)))}(J(k_X), J(F[n])) \simeq \varprojlim_{U \subset\subset X} H^n(U; F).$$

Let $\mathcal{T}$ be a full triangulated subcategory $D^b(\mathcal{C})$. One identifies $\mathrm{Ind}(\mathcal{T})$ with a full subcategory of $\mathrm{Ind}(D^b(\mathcal{C}))$.

Let $F \in D^b(\mathrm{Ind}(\mathcal{C}))$. Let us denote by $\mathcal{T}_F$ the category of arrows $G \to F$ in $D^b(\mathrm{Ind}(\mathcal{C}))$ with $G \in \mathcal{T}$. The category $\mathcal{T}_F$ is filtrant.

Lemma 3.5. *For $F \in D^b(\mathrm{Ind}(\mathcal{C}))$, the conditions below are equivalent.*

(i) $J(F) \in \mathrm{Ind}(\mathcal{T})$,

(ii) *for each $k \in \mathbb{Z}$, one has* $H^k(F) \simeq \text{“}\varinjlim_{G \to F \in \mathcal{T}_F}\text{”}\, H^k(G)$.

Definition 3.6. Let $\mathcal{T}$ be a full triangulated subcategory of $D^b(\mathcal{C})$. One denotes by $J^{-1}\mathrm{Ind}(\mathcal{T})$ the full subcategory of $D^b(\mathrm{Ind}(\mathcal{C}))$ consisting of objects $F \in D^b(\mathrm{Ind}(\mathcal{C}))$ such that $J(F) \in \mathrm{Ind}(\mathcal{T})$.

Proposition 3.7. *The category $J^{-1}\mathrm{Ind}(\mathcal{T})$ is a triangulated subcategory of $D^b(\mathrm{Ind}(\mathcal{C}))$.*

We will apply these results to the category $\mathrm{I}(k_X) = \mathrm{Ind}(\mathrm{Mod}^c(k_X))$. Hence J is the functor:

$$J\colon D^b(\mathrm{I}(k_X)) \longrightarrow \mathrm{Ind}(D^b(\mathrm{Mod}^c(k_X))).$$

By the definition one has

$$J(F) \simeq \text{“}\varinjlim_{U \subset\subset X}\text{”}\, J(F_U) \qquad \text{for any } F \in D^b(\mathrm{I}(k_X)).$$

As a corollary of Theorem 3.3, one gets:

Proposition 3.8. *For $G \in D^b(k_X)$ and $F \in D^b(\mathrm{I}(k_X))$, assume that $J(F) \simeq \text{“}\varinjlim_i\text{”}\, J(F_i)$ with $F_i \in D^b(k_X)$. Then there are natural isomorphisms:*

$$J(G \otimes F) \simeq \text{“}\varinjlim_i\text{”}\, J(G \otimes F_i), \tag{3.2}$$

$$J(R\mathcal{I}hom\,(G, F)) \simeq \text{“}\varinjlim_i\text{”}\, J(R\mathcal{I}hom\,(G, F_i)). \tag{3.3}$$

4 Micro-support and regularity

Let γ be a closed convex proper cone in an affine space X. One denotes by γ° its polar cone,

$$\gamma^\circ = \{\xi \in X^*; \langle x, \xi\rangle \geq 0 \text{ for all } x \in \gamma\}.$$

Let $W \subset X$ be an open subset. We introduce the functor $\Phi_{\gamma,W}\colon D^b(\mathrm{I}(k_X)) \to D^b(\mathrm{I}(k_X))$ as follows. Denote by $q_1, q_2\colon X \times X \to X$ the first and second projections and denote by $s\colon X \times X \to X$ the map $(x, y) \mapsto x - y$. One sets

$$\Phi_{\gamma,W}(F) = Rq_{1!!}(k_{s^{-1}\gamma\cap q_1^{-1}W\cap q_2^{-1}W} \otimes q_2^{-1}F).$$

One writes Φ_γ instead of $\Phi_{\gamma,X}$. Define the functor $\Phi^-_{\gamma,W}$ by replacing the kernel $k_{s^{-1}\gamma\cap q_1^{-1}W\cap q_2^{-1}W}$ with the complex $k_{s^{-1}\gamma\cap q_1^{-1}W\cap q_2^{-1}W} \to k_{s^{-1}(0)}$ in which $k_{s^{-1}(0)}$ is situated in degree 0. We have a distinguished triangle in $D^b(\mathrm{I}(k_X))$

$$\Phi_{\gamma,W}(F) \to F \to \Phi^-_{\gamma,W}(F) \xrightarrow{+1}.$$

Note that if $F \in D^b(k_X)$, then

$$\begin{cases} \mathrm{supp}(\Phi_{\gamma,W}(F)) \subset \overline{W}, \\ \Phi_\gamma(F) \to F \text{ is an isomorphism on } X \times \mathrm{Int}\gamma^\circ, \\ SS(\Phi_\gamma(F)) \subset X \times \gamma^\circ. \\ SS(\Phi^-_{\gamma,W}(F)) \bigcap W \times \mathrm{Int}\gamma^\circ = \emptyset \end{cases}$$

Lemma 4.1. *Let $F \in D^b(\mathrm{I}(k_X))$ and let $p \in T^*X$. The conditions* (1a)–(4b) *below are all equivalent. Moreover, if $F \in D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$, these conditions are equivalent to* (5a)*.*

(1a) *Assume that for a small and filtrant category I, integers $a \leq b$ and a functor $I \to C^{[a,b]}(\mathrm{Mod}(k_X))$, $i \mapsto F_i$ one has $F \simeq Q(\text{“}\varinjlim_{i\in I}\text{”}\, F_i)$. Then there exists a conic open neighborhood U of p in T^*X such that for any $i \in I$ there exists a morphism $i \to j$ in I which induces the zero-morphism $0 : F_i \to F_j$ in $D^b(k_X; U)$.*

(1b) *There exist a conic open neighborhood U of p in T^*X, a small and filtrant category I, integers $a \leq b$ and a functor $I \to C^{[a,b]}(\mathrm{Mod}(k_X))$, $i \mapsto F_i$, such that $SS(F_i) \cap U = \emptyset$ and $F \simeq Q(\text{“}\varinjlim_{i}\text{”}\, F_i)$ in a neighborhood of $\pi(p)$.*

(2a) *Assume that for a small and filtrant category I, integers $a \leq b$ and a functor $I \to D^{[a,b]}(k_X)$, $i \mapsto F_i$ one has $J(F) \simeq$ "$\varinjlim_{i\in I}$" $J(F_i)$. Then there exists a conic open neighborhood U of p in T^*X such that for any $i \in I$ there exists a morphism $i \to j$ in I which induces the zero-morphism $0 : F_i \to F_j$ in $D^b(k_X; U)$.*

(2b) *There exist a conic open neighborhood U of p in T^*X, a small and filtrant category I, integers $a \leq b$, a functor $I \to D^b(k_X)$, $i \mapsto F_i$ and F' isomorphic to F in neighborhood of $\pi(p)$ such that $SS(F_i) \cap U = \emptyset$ and $J(F') \simeq$ "$\varinjlim_{i}$" $J(F_i)$.*

(3a) *There exists a conic open neighborhood U of p in T^*X such that for any $G \in D^b(k_X)$ with $\mathrm{supp}(G) \subset\subset \pi(U)$, $SS(G) \subset U \cup T^*_X X$, one has $\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(G, F) = 0$.*

(3b) *There exists a conic open neighborhood U of p in T^*X such that for any $G \in D^b(k_X)$ with $\mathrm{supp}(G) \subset\subset \pi(U)$, $SS(G) \subset U^a \cup T^*_X X$, one has $R\mathrm{I}\Gamma(X; G \otimes F) = 0$.*

Assume now that X is an affine space and let $p = (x_0; \xi_0)$.

(4a) *There exist a relatively compact open neighborhood W of x_0 and a closed convex proper cone γ with $\xi_0 \in \mathrm{Int}\gamma^\circ$ such that $\Phi_{\gamma,W}(F) \simeq 0$.*

(4b) *There exist $F' \in D^b(\mathrm{I}(k_X))$ with $F' \simeq F$ in a neighborhood of x_0 and F' has compact support, and a closed convex proper cone γ as in* (4a) *such that $\Phi_\gamma(F') \simeq 0$ in a neighborhood of x_0.*

(5a) *Same condition as* (3a) *with $G \in D^b_{\mathbb{R}-c}(k_X)$.*

Proof. The plan of the proof is as follows:

$$\begin{array}{ccccc}
(2a) & \Longleftarrow & (3a) & \Longleftarrow & (2b) \\
\Downarrow & & \Downarrow & & \Uparrow \\
(1a) & & (5a) & & (1b) \\
\Downarrow & \swarrow\!\!\!\swarrow & & & \Uparrow \\
(3b) & \Longrightarrow & (4a) & \Longrightarrow & (4b)
\end{array}$$

(2a) $\Rightarrow$ (1a) follows from $F \simeq Q(\text{"}\varinjlim_{i}\text{"}\, F_i) \Rightarrow J(F) \simeq \text{"}\varinjlim_{i}\text{"}\, J(Q(F_i))$.

(1a) $\Rightarrow$ (3b). Let $F \simeq Q(\text{"}\varinjlim_{i}\text{"}\, F_i)$ and let $i \in I$. There exists $i \to j$ such that the morphism $F_i \to F_j$ in $D^b(k_X)$ is zero in $D^b(k_X; U)$. Hence, there exists a morphism $F_j \to F'_{ij}$ in $D^b(k_X)$ which is an isomorphism on U and such that the composition $F_i \to F_j \to F'_{ij}$ is the zero-morphism in $D^b(k_X)$. Consider the commutative diagram in which the row on the bottom is a distinguished triangle in $D^b(k_X)$ and $SS(F_{ij}) \cap U = \emptyset$:

$$\begin{array}{ccccccc} & & F_i & & & & \\ & \swarrow\cdots & \downarrow & \overset{0}{\searrow} & & & \\ F_{ij} & \longrightarrow & F_j & \longrightarrow & F'_{ij} & \xrightarrow{+1} & \end{array}$$

Since the arrow $F_i \to F'_{ij}$ is zero, the dotted arrow may be completed, making the diagram commutative. Hence, we may assume from the beginning that for any $i \in I$ there exists $i \to j$ such that the morphism $F_i \to F_j$ factorizes as $F_i \to F_{ij} \to F_j$ with $SS(F_{ij}) \cap U = \emptyset$.

We may assume X is affine and $U = W \times \lambda$ where W is open and relatively compact and λ is an open convex cone. Then $SS(G \otimes F_{ij}) \cap U = \emptyset$, and the sheaf $G \otimes F_{ij}$ has compact support. Hence, $R\Gamma(X; G \otimes F_{ij}) \simeq 0$ which implies $H^j R\text{I}\Gamma(X; G \otimes F) \simeq$ "$\varinjlim_i$" $H^j R\Gamma(X; G \otimes F_i) \simeq 0$ for all j. We conclude therefore $R\text{I}\Gamma(X; G \otimes F) \simeq 0$.

(3b) $\Rightarrow$ (4a). Let $F = Q($"$\varinjlim_i$" $F_i)$, with $F_i \in C^{[a,b]}(\mathrm{Mod}(k_X))$. Set

$$H_\varepsilon = \{x; \langle x - x_0; \xi_0 \rangle > -\varepsilon\}$$

and let $K \subset\subset \pi(U)$ be a compact neighborhood of x_0. Then there exist an open convex cone γ and an open neighborhood W of x_0 satisfying the following conditions:

$$\begin{cases} W \subset H_\varepsilon \cap K, \\ (x + \gamma) \cap H_\varepsilon \subset W \text{ for all } x \in W, \\ \overline{W} \times \gamma^\circ \subset U \cup T^*_X X. \end{cases}$$

Set

$$G_x = k_{(x+\gamma^a)\cap H_\varepsilon}, \quad G = \bigoplus_{x \in W} G_x.$$

Since $\mathrm{supp}(G) \subset\subset \pi(U)$ and $SS(G) \subset \overline{W} \times \gamma^{\circ a}$, we get by the hypothesis:

$$\text{“}\varinjlim_i\text{”}\, H^k R\Gamma(X; G \otimes F_i) \simeq 0.$$

Hence,

$$\text{“}\varinjlim_i\text{”}\,(\bigoplus_{x\in W} H^kR\Gamma(X;G_x\otimes F_i)) \;\simeq\; 0.$$

Hence one obtains:

$$\begin{cases}\text{for any } i\in I, \text{ there exists } i\to j \text{ such that } H^kR\Gamma(X;G_x\otimes F_i)\to\\ H^kR\Gamma(X;G_x\otimes F_j) \text{ is zero for any } x\in W \text{ and any } k\in\mathbb{Z}.\end{cases}$$

On the other-hand,

$$H^k(\Phi_{\gamma,W}(F_i))_x \simeq H^kR\Gamma(X;G_x\otimes F_i).$$

Therefore, for any $i\in I$ there exists $i\to j$ such that for any $k\in\mathbb{Z}$, the morphism $H^k(\Phi_{\gamma,W}(F_i))\to H^k(\Phi_{\gamma,W}(F_j))$ is the zero morphism, and this implies

$$H^k(\Phi_{\gamma,W}(F)) \simeq \text{“}\varinjlim_i\text{”}\, H^k\Phi_{\gamma,W}(F_i)\simeq 0.$$

This gives the desired result: $\Phi_{\gamma,W}(F)=0$.

(4a) $\Rightarrow$ (4b) is obvious by taking F_W as F'.

(4b) $\Rightarrow$ (1b). Let W be an open relatively compact neighborhood of x_0 such that $F|_W\simeq F'|_W$ and $\Phi_\gamma(F')|_W\simeq 0$.

Then one has a distinguished triangle:

$$Rq_{1!!}(k_{s^{-1}(\gamma\setminus\{0\})\cap q_1^{-1}W}\otimes q_2^{-1}F')\to\Phi_\gamma(F')_W\to F'_W\xrightarrow{+1},$$

and hence one obtains $Rq_{1!!}(k_{s^{-1}(\gamma\setminus\{0\})\cap q_1^{-1}W}[1]\otimes q_2^{-1}F')\simeq F'_W$. Let $F'=Q(\text{“}\varinjlim_i\text{”}\,F_i)$ with $F_i\in C^{[a,b]}(\mathrm{Mod}(k_X))$, and take a finite injective resolution I of $k_{s^{-1}(\gamma\setminus\{0\})\cap q_1^{-1}W}[1]$. Since $I\otimes F_i$ is a finite complex of soft sheaves, $Rq_{1!}(k_{s^{-1}(\gamma\setminus\{0\})\cap q_1^{-1}W}[1]\otimes q_2^{-1}F_i)$ is represented by $F'_i := q_{1!}(I\otimes q_2^{-1}F_i)$. Hence one has

$$Rq_{1!!}(k_{s^{-1}(\gamma\setminus\{0\})\cap q_1^{-1}W}\otimes q_2^{-1}F')\simeq Q(\text{“}\varinjlim_i\text{”}\,F'_i).$$

Since $SS(F'_i)\cap W\times\mathrm{Int}\gamma^\circ=\emptyset$, we obtain the desired result.

(1b) $\Rightarrow$ (2b) is obvious.

(2b) $\Rightarrow$ (3a). Let $J(F) \simeq$ "$\varinjlim_i$" $J(F_i)$. If $G \in D^b(k_X)$, we get the isomorphism:

$$\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(G, F) \simeq \varinjlim_i \mathrm{Hom}_{D^b(k_X)}(G, F_i).$$

We may assume that X is affine and $U = W \times \lambda$ where W is open and λ is an open convex cone. Then the micro-support of $R\mathcal{H}om\,(G, F_i)$ is contained in $SS(F_i) + \bar{\lambda}^a$ and this set does not intersect $X \times \lambda$. Since $R\mathcal{H}om\,(G, F_i)$ has compact support, $\mathrm{Hom}\,(G, F_i)$ is zero.

(3a) $\Rightarrow$ (2a). We may assume that X is affine, $p = (x_0; \xi_0)$ and $U = X' \times \mathrm{Int}\gamma^\circ$, with $\xi_0 \in \mathrm{Int}\gamma^\circ$ for a neighborhood X' of x_0. Let V be an open neighborhood of x_0 and let $W = \{x; \langle x - x_0; \xi_0 \rangle > -\varepsilon\}$. Then by taking V and ε small enough, the sheaf $\Phi_\gamma(H_W)_V$ satisfies the condition in (3a) for any $H \in D^b(k_X)$. Let $J(F) =$ "$\varinjlim_i$" $J(F_i)$. Then $\varinjlim_i \mathrm{Hom}_{D^b(k_X)}(G, F_i) \simeq 0$ for any $G = \Phi_\gamma(H_W)_V$. Let $i \in I$ and choose $H = F_i$. There exists $i \to j$ such that the composition $(\Phi_\gamma(F_{iW}))_V \to F_i \to F_j$ is zero. The morphism $(\Phi_\gamma((F_{iW}))_V \to F_i$ is an isomorphism on $U' := (V \cap W) \times \mathrm{Int}\gamma^\circ$. Therefore, $F_i \to F_j$ is zero in $D^b(k_X; U')$.

(3a) $\Rightarrow$ (5a) is obvious.

(5a) $\Rightarrow$ (3b). (Assuming $F \in D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$.) Let (2a-rc) denote the condition (2a) in which one asks moreover that $F_i \in D^{[a,b]}_{\mathbb{R}-c}(k_X)$. Define similarly (1a-rc). Then the same proof of (3a) $\Rightarrow$ (2a) $\Rightarrow$ (1a) $\Rightarrow$ (3b) can be applied to show (5a) $\Rightarrow$ (2a-rc) $\Rightarrow$ (1a-rc) $\Rightarrow$ (3b).

q.e.d.

Definition 4.2. Let $F \in D^b(\mathrm{I}(k_X))$. The micro-support of F, denoted by $SS(F)$, is the closed conic subset of T^*X whose complementary is the set of points $p \in T^*X$ such that one of the equivalent conditions in Lemma 4.1 is satisfied.

Proposition 4.3. (i) *For $F \in D^b(\mathrm{I}(k_X))$, one has $SS(F) \cap T^*_X X = \mathrm{supp}(F)$.*

(ii) *Let $F \in D^b(k_X)$. Then $SS(\iota_X F) = SS(F)$.*

(iii) *Let $F \in D^b(\mathrm{I}(k_X))$. Then $SS(\alpha_X F) \subset SS(F)$.*

(iv) *Let $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ be a distinguished triangle in $D^b(\mathrm{I}(k_X))$. Then $SS(F_i) \subset SS(F_j) \cup SS(F_k)$ if $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. (i) $\mathrm{supp}(F) \subset SS(F)$ follows for example from (1b) of Lemma 4.1. The other inclusion is obvious.

(ii) The inclusion $SS(F) \subset SS(\iota_X F)$ follows from (2a) since $J(F)$ is "$\varinjlim$" F. The converse inclusion follows from (1b).

(iii) is obvious, using condition (3b).

(iv) is obvious by (3b).

q.e.d.

Definition 4.4. Let $\Lambda_i, i \in I$ be a family of closed conic subsets of T^*X, indexed by the objects of a small and filtrant category I. One sets

$$\underset{i}{\mathit{lim}}\ \Lambda_i = \bigcap_{J \subset I} \overline{\bigcup_{j \in J} \Lambda_j}$$

where J ranges over the family of cofinal subcategories of I.

In other words, $p \in T^*X$ does not belong to $\underset{i}{\mathit{lim}}\ \Lambda_i$ if there exists an open neighborhood U of p and a cofinal subset J of I such that $\Lambda_j \cap U = \emptyset$ for every $j \in J$.

It follows immediately from the definition that if $J(F) \simeq$ "$\varinjlim_i$" $J(F_i)$, then

$$SS(F) \subset \underset{i}{\mathit{lim}}\ SS(F_i). \tag{4.1}$$

It follows from Proposition 3.8 that if $G \in D^b(k_X)$, one has the inclusions

$$\begin{aligned} SS(G \otimes F) &\subset \underset{i}{\mathit{lim}}\,(SS(G) \hat{+} SS(F_i)), \\ SS(R\mathcal{I}hom\,(G, F)) &\subset \underset{i}{\mathit{lim}}\,(SS(G)^a \hat{+} SS(F_i)). \end{aligned} \tag{4.2}$$

Example 4.5. Let $X = \mathbb{R}^2$ endowed with coordinates (x, y) and denote by $(x, y; \xi, \eta)$ the associated coordinates on T^*X. Let

$$\begin{aligned} Y &= \{(x, y); y = 0\}, \\ U &= \{(x, y); x^2 < y\}, \\ Z_\varepsilon &= \{(x, y); x^2 < y \le \varepsilon^2\}. \end{aligned}$$

Set $F_\varepsilon = k_{Z_\varepsilon}$ and $F = k_U \otimes \beta_X(k_{\{0\}}) \simeq$ "$\varinjlim_\varepsilon$" F_ε. Then

$$\begin{aligned}
SS(k_Y) &= T^*_Y X = \{(x,y;\xi,\eta)\,;\, y=\xi=0\}, \\
SS(F_\varepsilon) &= \{(x,y;0,0)\,;\, x^2 \le y \le \varepsilon^2\} \\
&\qquad \bigcup \{(x,y;\xi,\eta)\,;\, y = x^2, |x| \le \varepsilon,\ \xi = -2x\eta,\ \eta \le 0\} \\
&\qquad\quad \bigcup \{(x,y;\xi,\eta)\,;\, y = \varepsilon^2, |x| \le \varepsilon,\ \xi = 0,\ \eta \le 0\} \\
&\qquad\qquad \bigcup \{(\pm\varepsilon,\varepsilon^2;\xi,\eta)\,;\, 0 \le \pm\xi \le -2\varepsilon\eta,\ \eta \le 0\}, \\
SS(F) &= \{(x,y;\xi,\eta); x = y = \xi = 0, \eta \le 0\}.
\end{aligned}$$

On the other-hand, one has

$$\begin{aligned}
&SS(F) = \lim_\varepsilon SS(F_\varepsilon), \\
&R\mathcal{H}om\,(k_Y, F) \simeq k_{\{0\}}\,[-2]\,, \\
&\lim_\varepsilon\,(T^*_Y X \widehat{+} SS(F_\varepsilon)) = T^*_{\{0\}} X, \\
&T^*_Y X \widehat{+} SS(F) = \{(x,y;\xi,\eta); x = y = \xi = 0\} \\
&\qquad\qquad\qquad \subsetneqq SS(R\mathcal{H}om\,(k_Y, F)).
\end{aligned}$$

Note that $SS(F)$ is not involutive.

Recall that subanalytic isotropic subsets of T^*X are defined in [4]. Let us say for short that a conic locally closed subset Λ of T^*X is isotropic if Λ is contained in a conic locally closed subanalytic isotropic subset.

Definition 4.6. (i) We denote by $D^b_{\mathrm{w-\mathbb{R}-c}}(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ consisting of objects F such that $SS(F)$ is isotropic. We call an object of this category a weakly $\mathbb{R}$-constructible ind-sheaf.

(ii) Let us denote by $D^b_{\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b_{\mathrm{w-\mathbb{R}-c}}(\mathrm{I}(k_X))$ consisting of objects F such that one has $R\mathcal{H}om\,(G,F) \in D^b_{\mathbb{R}-\mathrm{c}}(k_X)$ for any $G \in D^b_{\mathbb{R}-\mathrm{c}}(k_X)$. We call an object of this category an $\mathbb{R}$-constructible ind-sheaf.

Note that the functor α_X induces functors

$$\begin{aligned}
&\alpha_X\colon D^b_{\mathrm{w-\mathbb{R}-c}}(\mathrm{I}(k_X)) \to D^b_{\mathrm{w-\mathbb{R}-c}}(k_X), \\
&\alpha_X\colon D^b_{\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)) \to D^b_{\mathbb{R}-\mathrm{c}}(k_X).
\end{aligned}$$

The last property follows from $\alpha_X(F) = R\mathcal{H}om\,(\mathbb{C}_X, F)$.

Conjecture 4.7. *Let* $F \in D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ *and let* $G \in D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(k_X)$*. Then* $R\mathcal{I}hom\,(G, F)$ *and* $G \otimes F$ *belong to* $D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$*.*

Example 4.5 shows that the knowledge of $SS(F)$ and $SS(G)$ does not allows us to estimate the micro-support of $R\mathcal{H}om\,(F, G)$ by the one for sheaves, and that is one reason for the definition below.

Definition 4.8. Let $F \in D^b(\mathrm{I}(k_X))$.

(i) Let $S \subset T^*X$ be a locally closed conic subset and let $p \in T^*X$. We say that F is regular along S at p if there exist F' isomorphic to F in a neighborhood of $\pi(p)$, an open neighborhood U of p with $S \cap U$ closed in U, a small and filtrant category I and a functor $I \to D^{[a,b]}(k_X), i \mapsto F_i$ such that $J(F') \simeq \text{“}\varinjlim_i\text{”}\, J(F_i)$ and $SS(F_i) \cap U \subset S$.

(ii) If U is an open subset of T^*X and F is regular along S at each $p \in U$, we say that F is regular along S on U.

(iii) Let $p \in T^*X$. We say that F is regular at p if F is regular along $SS(F)$ at p.

If F is regular at each $p \in SS(F)$, we say that F is regular.

(iv) We denote by $SS_{reg}(F)$ the conic open subset of $SS(F)$ consisting of points p such that F is regular at p, and we set

$$SS_{irr}(F) = SS(F) \setminus SS_{reg}(F).$$

Note that $SS_{irr}(F) = SS(F)$ for F in Example 4.5.

Proposition 4.9. (i) *Let* $F \in D^b(\mathrm{I}(k_X))$*. Then* F *is regular along any locally closed set* S *at each* $p \notin SS(F)$*.*

(ii) *Let* $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ *be a distinguished triangle in* $D^b(\mathrm{I}(k_X))$*. If* F_j *and* F_k *are regular along* S*, so is* F_i *for* $i, j, k \in \{1, 2, 3\}, j \neq k$*.*

(iii) *Let* $F \in D^b(k_X)$*. Then* $\iota_X F$ *is regular.*

Proof. (i) and (iii) are obvious and the proof of (ii) is similar to that of Proposition 4.3 (iv). q.e.d.

It is possible to localize the category $D^b(\mathrm{I}(k_X))$ with respect to the micro-support, exactly as for usual sheaves.

Let V be a subset of T^*X and let $\Omega = T^*X \setminus V$. We shall denote by $D^b_V(k_X)$ the full triangulated subcategory of $D^b(k_X)$ consisting of objects

F such that $SS(F) \subset V$, and by $D^b(k_X;\Omega)$ the localization of $D^b(k_X)$ by $D^b_V(k_X)$.

Similarly, we denote by $D^b_V(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b(\mathrm{I}(k_X))$ consisting of objects F such that $SS(F) \subset V$.

Definition 4.10. One sets

$$D^b(\mathrm{I}(k_X;\Omega)) = D^b(\mathrm{I}(k_X))/D^b_V(\mathrm{I}(k_X)),$$

the localization of $D^b(\mathrm{I}(k_X))$ by $D^b_V(\mathrm{I}(k_X))$.

Let F_1 and F_2 are two objects of $D^b(\mathrm{I}(k_X))$ whose images in $D^b(\mathrm{I}(k_X;\Omega))$ are isomorphic. There exist a third object $F_3 \in D^b(\mathrm{I}(k_X;\Omega))$ and distinguished triangles in $D^b(\mathrm{I}(k_X))$: $F_i \to F_3 \to G_i \xrightarrow{+1}$ $(i = 1,2)$ such that $SS(G_i) \cap \Omega = \emptyset$. It follows that $SS(F_1) \cap \Omega = SS(F_3) \cap \Omega = SS(F_2) \cap \Omega$.

Therefore if $F \in D^b(\mathrm{I}(k_X;\Omega))$, the subsets $SS(F)$ and $SS_{irr}(F)$ of Ω are well-defined.

5 Invariance by contact transformations

It is possible to define contact transformations on ind-sheaves. We shall follow the notations in [4] Chapter VII.

We denote by p_1 and p_2 the first and second projections defined on $T^*(X \times Y) \simeq T^*X \times T^*Y$, and we denote by p_2^a the composition of p_2 with the antipodal map on T^*Y.

We denote by $r\colon X \times Y \to Y \times X$ the canonical map and we keep the same notation to denote its inverse.

By a kernel K on $X \times Y$ we mean an object of $D^b(k_{X\times Y})$. To a kernel K one associates the kernel on $Y \times X$

$$K^* := r_* R\mathcal{H}om\,(K, \omega_{X\times Y/Y}).$$

One defines the functor

$$\begin{aligned}(5.1)\qquad \Phi_K\colon D^b(k_Y) &\to D^b(k_X)\\ G &\mapsto Rq_{1!}(K \otimes q_2^{-1}G).\end{aligned}$$

Consider another manifold Z and a kernel L on $Y \times Z$. One defines the projection q_{12} from $X \times Y \times Z$ to $X \times Y$, and similarly with q_{23}, q_{13}.

One sets

$$(5.2)\qquad K \circ L = Rq_{13!}(q_{12}^{-1}K \otimes q_{23}^{-1}L).$$

Choosing $Z = \{\mathrm{pt}\}$, one has $\Phi_K(G) = K \circ G$ for $G \in D^b(k_Y)$.

Let Ω_X and Ω_Y be two conic open subsets of T^*X and T^*Y, respectively. One denotes by $N(\Omega_X, \Omega_Y)$ the full subcategory of $D^b(k_{X\times Y}; \Omega_X \times T^*Y)$ of objects K satisfying;

$$(5.3) \qquad \begin{cases} SS(K) \cap (\Omega_X \times T^*Y) \subset \Omega_X \times \Omega_Y^a, \\ p_1\colon SS(K) \cap (\Omega_X \times T^*Y) \to \Omega_X \text{ is proper.} \end{cases}$$

Let us recall some results of loc. cit.

(i) Let $K \in N(\Omega_X, \Omega_Y)$. Then the functor Φ_K induces a well-defined functor: $\Phi_K^\mu\colon D^b(k_Y; \Omega_Y) \to D^b(k_X; \Omega_X)$.

(ii) Let $L \in N(\Omega_Y, \Omega_Z)$. Then $K \circ L \in N(\Omega_X, \Omega_Z)$. Moreover, the two functors $\Phi_{K\circ L}^\mu$ and $\Phi_K^\mu \circ \Phi_L^\mu$ from $D^b(k_Z; \Omega_Z)$ to $D^b(k_X; \Omega_X)$ are isomorphic.

We construct the functor analogous to the functor Φ_K for ind-sheaves by defining

$$(5.4) \qquad \begin{array}{rcl} \tilde{\Phi}_K\colon D^b(\mathrm{I}(k_Y)) & \to & D^b(\mathrm{I}(k_X)) \\ G & \mapsto & Rq_{1!!}(K \otimes q_2^{-1}G). \end{array}$$

Applying Theorem 3.3, we get:

Lemma 5.1. *Let $G \in D^b(\mathrm{I}(k_Y))$ and assume that $J(G) \simeq$ "$\varinjlim_i$" $J(G_i)$, with I small and filtrant and $G_i \in D^b(k_Y)$. Then $J(\tilde{\Phi}_K(G)) \simeq$ "$\varinjlim_i$" $J(\Phi_K(G_i))$.*

Now assume that $\dim X = \dim Y$ and that there exists a smooth conic Lagrangian submanifold $\Lambda \subset \Omega_X \times \Omega_Y^a$ such that $p_1\colon \Lambda \to \Omega_X$ and $p_2^a\colon \Lambda \to \Omega_Y$ are isomorphisms. In other words, Λ is the graph of a homogeneous symplectic isomorphism $\chi\colon \Omega_Y \xrightarrow{\sim} \Omega_X$.

Let K be a kernel satisfying the assumptions of Theorem 7.2.1 of loc. cit., that is:

$$(5.5) \qquad \begin{cases} K \text{ is cohomologically constructible,} \\ (p_1^{-1}(\Omega_X) \cup p_2^{a-1}(\Omega_Y)) \cap SS(K) \subset \Lambda, \\ k_\Lambda \xrightarrow{\sim} \mu hom(K, K) \quad \text{on } \Omega_X \times \Omega_Y^a. \end{cases}$$

Theorem 5.2. *Assume* (5.5).

(i) *The functor* $\tilde{\Phi}_K$ *induces a well-defined functor:* $\tilde{\Phi}^\mu_K\colon D^b(\mathrm{I}(k_Y;\Omega_Y)) \to D^b(\mathrm{I}(k_X;\Omega_X))$. *Similarly, the functor* $\tilde{\Phi}_{K^*}$ *induces a well-defined functor:* $\tilde{\Phi}^\mu_{K^*}\colon D^b(\mathrm{I}(k_X;\Omega_X)) \to D^b(\mathrm{I}(k_Y;\Omega_Y))$.

(ii) *The functor* $\tilde{\Phi}^\mu_K\colon D^b(\mathrm{I}(k_Y;\Omega_Y)) \to D^b(\mathrm{I}(k_X;\Omega_X))$ *and the functor* $\tilde{\Phi}^\mu_{K^*}\colon D^b(\mathrm{I}(k_X;\Omega_X)) \to D^b(\mathrm{I}(k_Y;\Omega_Y))$ *are equivalences of categories inverse one to each other.*

(iii) *If* $G \in D^b(\mathrm{I}(k_Y))$, *then* $SS(\tilde{\Phi}_K(G)) \cap \Omega_X = \chi(SS(G) \cap \Omega_Y)$.

(iv) *If* G *is regular at* $p \in \Omega_Y$, *then* $\tilde{\Phi}_K(G)$ *is regular at* $\chi(p) \in \Omega_X$. *In other words,* $SS_{irr}(\Phi_K(G)) \cap \Omega_X = \chi(SS_{irr}(G) \cap \Omega_Y)$.

Proof. (i) Let $G \in D^b(\mathrm{I}(k_Y))$ and assume that $SS(G) \cap \Omega_Y = \emptyset$. Let us prove that $SS(\tilde{\Phi}_K(G)) \cap \Omega_X = \emptyset$. Let $p_X \in \Omega_X$ and let $p_Y = \chi^{-1}(p_X)$. There exist an open neighborhood U_Y of p_Y in Ω_Y and an inductive system such that $J(G) \simeq$ "$\varinjlim_{i\in I}$" $J(G_i)$, and for any $i \in I$ there exists $i \to j$ such that the morphism $G_i \to G_j$ is zero in $D^b(k_Y;U_Y)$. Applying Lemma 5.1 we find that $J(\tilde{\Phi}_K(G)) \simeq$ "$\varinjlim_{i}$" $J(\Phi_K(G_i))$. Since the morphism $\Phi_K(G_i) \to \Phi_K(G_j)$ is zero in $D^b(k_X;U_X)$, the result follows.

(ii) One has the isomorphism $K \circ K^* \simeq k_{\Delta_X}$ in $N(\Omega_X,\Omega_X)$ and the isomorphism $K^* \circ K \simeq k_{\Delta_Y}$ in $N(\Omega_Y,\Omega_Y)$. Hence, it is enough to remark that

$$\tilde{\Phi}^\mu_K \circ \tilde{\Phi}^\mu_{K^*} \simeq \tilde{\Phi}^\mu_{K\circ K^*}, \tag{5.6}$$

which follows from the fact that the two functors $\tilde{\Phi}_K \circ \tilde{\Phi}_{K^*}$ and $\tilde{\Phi}_{K\circ K^*}$, from $D^b(\mathrm{I}(k_X))$ to $D^b(\mathrm{I}(k_X))$ are isomorphic.

(iii) For an open subset $U_Y \subset \Omega_Y$, set $U_X = \chi(U_Y)$. Then $K \in N(U_X,U_Y)$ and K satisfies (5.5) with Ω replaced with U. Let $G \in D^b(\mathrm{I}(k_Y))$ with $SS(G) = \emptyset$ in a neighborhood of $p_Y \in \Omega_Y$. By the proof of (i), $SS(\tilde{\Phi}_K(G)) = \emptyset$ in a neighborhood of $\chi(p_Y)$.

(iv) The proof is similar to that of (iii). q.e.d.

6 Ind-sheaves and $\mathcal{D}$-modules

Let now X be a complex manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We set for short

$$\begin{aligned} Sol(\mathcal{M}) &= R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X), \\ Sol^t(\mathcal{M}) &= R\mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M},\mathcal{O}^t_X). \end{aligned}$$

Theorem 6.1. *One has*

$$SS(Sol^t(\mathcal{M})) = \mathrm{char}(\mathcal{M}).$$

Proof. (i) The inclusion $\mathrm{char}(\mathcal{M}) \subset SS(Sol^t(\mathcal{M}))$ follows from

$$SS(Sol(\mathcal{M})) = \mathrm{char}(\mathcal{M}), \quad \alpha_X(Sol^t(\mathcal{M})) \simeq Sol(\mathcal{M}).$$

and Proposition 4.3 (ii).

(ii) Let us prove the converse inclusion using condition (5a) of Lemma 4.1. Assume that $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ satisfy $SS(G) \cap \mathrm{char}(\mathcal{M}) \subset T^*_X X$. One has the morphisms

$$\begin{aligned} R\mathcal{H}om\,(G, R\mathcal{I}hom_{\beta_X \mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}^t_X)) &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mathcal{H}om\,(G, \mathcal{O}_X)) \\ &\to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(G, \mathcal{O}_X)). \end{aligned}$$

It follows from [1, Corollary 4.2.5] that the second morphism is an isomorphism. Hence the result follows from $SS(Sol(\mathcal{M})) = \mathrm{char}(\mathcal{M})$ and Lemma 4.1 (5a). q.e.d.

The following conjecture is a consequence of Conjecture 4.7.

Conjecture 6.2. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $Sol^t(\mathcal{M})$ belongs to $D^b_{\mathbb{R}-\mathrm{c}}(\mathrm{I}(\mathbb{C}_X))$.*

Theorem 6.3. *If $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then $Sol^t(\mathcal{M}) \to Sol(\mathcal{M})$ is an isomorphism.*

Proof. This is a reformulation of a result of [2] which asserts that for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, the natural morphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mathcal{H}om\,(G, \mathcal{O}_X)) \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(G, \mathcal{O}_X))$$

is an isomorphism. q.e.d.

We conjecture the following statement in which "only if " part is a consequence of the theorem above.

Conjecture 6.4. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular holonomic if and only if $Sol^t(\mathcal{M})$ is regular.*

7 An example

In this section $X = \mathbb{C}$ endowed with the holomorphic coordinate z, and we shall study the ind-sheaf of temperate holomorphic solutions of the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X \exp(1/z) = \mathcal{D}_X/\mathcal{D}_X(z^2\partial_z + 1)$. We set for short

$$\begin{aligned} \mathcal{S}^t &:= H^0(Sol^t(\mathcal{M})) &&\simeq \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X), \\ \mathcal{S} &:= H^0(Sol(\mathcal{M})) &&\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Notice first that $\mathcal{O}^t_X$ is concentrated in degree 0 (since $\dim X = 1$), and it is a sub-ind-sheaf of $\mathcal{O}_X$. It follows that the morphism $\mathcal{S}^t \to \mathcal{S}$ is a monomorphism.

Moreover,

$$\mathcal{S} \simeq \mathbb{C}_{X,X\setminus\{0\}} \cdot \exp(1/z).$$

Lemma 7.1. *Let $V \subset X$ be a connected open subset. Then $\Gamma(V;\mathcal{S}^t) \neq 0$ if and only if $V \subset X \setminus \{0\}$ and $\exp(1/z)|_V$ is tempered.*

Proof. The space $\Gamma(V;\mathcal{S})$ has dimension one and is generated by the function $\exp(1/z)$. Hence, the subspace $\Gamma(V;\mathcal{S}^t) \simeq \Gamma(V;\mathcal{S}) \cap \Gamma(V;\mathcal{O}^t)$ is not zero if and only if $\exp(1/z) \in \Gamma(V;\mathcal{O}^t_X)$, that is, if and only if $\exp(1/z)|_V$ is tempered. q.e.d.

Let us set $z = x + iy$.

Lemma 7.2. *Let W be an open subanalytic subset of $\mathbb{P}^1(\mathbb{C})$ with $\infty \notin W$. Assume that there exist positive constants C and A such that*

$$\exp(x) \leq C(1 + x^2 + y^2)^N \text{ on } W. \tag{7.1}$$

Then there exists a constant B such that $x \leq B$ on W.

Proof. If x is not bounded on W, then there exists a real analytic curve $\gamma\colon [0,\varepsilon[\to \mathbb{P}^1(\mathbb{C})$ such that $\operatorname{Re}\gamma(0) = \infty$ and $\gamma(t) \in W$ for $t > 0$. Writing $\gamma(t) = (x(t), y(t))$, one has

$$y(t) = cx(t)^q + \mathrm{O}(x(t)^{q-\varepsilon}).$$

for some $q \in \mathbb{Q}$, $c \in \mathbb{R}$ and $\varepsilon > 0$. Then (7.1) implies that $\exp(x)$ has a polynomial growth when $x \to \infty$, which is a contradiction. q.e.d.

Let $\bar{B}_\varepsilon$ denote the closed ball with center $(\varepsilon, 0)$ and radius ε and set $U_\varepsilon = X \setminus \bar{B}_\varepsilon$.

Proposition 7.3. *One has the isomorphism*

$$\text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{XU_\varepsilon} \xrightarrow{\sim} \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X). \tag{7.2}$$

Proof. It follows from Lemma 7.2 that $\exp(1/z)$ is temperate (in a neighborhood of 0) on an open subanalytic subset $V \subset X \setminus \{0\}$ if and only if $\mathrm{Re}(1/z)$ is bounded on V, that is, if and only if $V \subset U_\varepsilon$ for some $\varepsilon > 0$.

Let V be a connected relatively compact subanalytic open subset of $X \setminus \{0\}$. Then a morphism $\mathbb{C}_V \to \mathbb{C}_{X\setminus\{0\}} \cdot \exp(1/z)$ factorizes through a morphism $\mathbb{C}_V \to \mathcal{S}^t$ if and only if it factorizes through $\mathbb{C}_{U_\varepsilon}$. Hence we get the isomorphism (7.2) by Theorem 2.2. q.e.d.

Remark 7.4. In fact one can show

$$H^1(Sol^t(\mathcal{M})) \xrightarrow{\sim} H^1(Sol(\mathcal{M})) \simeq \mathbb{C}_0.$$

The isomorphism $H^1(Sol(\mathcal{M})) = \mathcal{O}_X/(z^2\partial_z + 1)\mathcal{O}_X \xrightarrow{\sim} \mathbb{C}_0$ is given by

$$(\mathcal{O}_X)_0 \ni v(z) \mapsto \oint v(z)z^{-2}\exp(-1/z)\, dz.$$

Note that $\varphi(z){:=}z^{-2}\exp(-1/z)$ is a solution to the adjoint equation

$$(-\partial_z z^2 + 1)\varphi(z) = 0.$$

The distinguished triangle

$$\mathcal{S}^t \to Sol^t(\mathcal{M}) \to H^1(Sol^t(\mathcal{M}))[-1] \xrightarrow{+1}$$

gives a non-zero element of $Ext^2(\mathbb{C}_0, \mathcal{S}^t) \xrightarrow{\sim} Ext^2(\mathbb{C}_0, \mathbb{C}_X) \simeq \mathbb{C}$.

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