Piecewise linear sheaves

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Abstract

On a finite-dimensional real vector space, we give a microlocal characterization of (derived) piecewise linear sheaves (PL sheaves) and prove that the triangulated category of such sheaves is generated by sheaves associated with convex polyhedra. We then give a similar theorem for PL γ-sheaves, that is, PL sheaves associated with the γ-topology, for a closed convex polyhedral proper cone γ. Our motivation is that convex polyhedra may be considered as building blocks for higher dimensional barcodes.

Contents

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Introduction

Persistent homology is an essential tool of Topological Data Analysis appearing in numerous papers. To our opinion it may be interpreted as follows (see [KS18]). One has some data on a manifold X which define a constructible sheaf F on X, one has a function $f\colon X \to \mathbb{R}$ (playing the role of a Morse function) and one can calculate the direct image by f of the data, that is, the derived direct image $\mathrm{R}f_*F$. Assuming that f is proper on the support of F, one gets a constructible (derived) sheaf on $\mathbb{R}$ and a variant of a theorem of Crawley-Boevey [CB14] (see also [Gui16] and, for the non compact case, [KS18, Th. 1.17 and Cor. 1.20]) asserts that such an object is nothing but a graded barcode. Moreover, in practice, the data on X are associated with an order and it follows that the barcodes are half-closed intervals (e.g., closed on the left and open on the right). In the langage of sheaves, this means that one gets a γ-sheaf, where γ is the cone $\{t \in \mathbb{R}; t \leq 0\}$.

However it is natural in many problems to replace the ordered set $(\mathbb{R}, \leq)$ with an ordered finite dimensional vector space $\mathbb{V}$ and the order may be deduced from the data of a closed convex proper cone γ with non–empty interior. Then we are lead to the study of (derived) constructible γ-sheaves on $\mathbb{V}$ and the category of such sheaves is no more equivalent to any natural category of barcodes. The aim of this paper, a kind of continuation of [KS18], is to find a substitute to this non existing equivalence.

First, we replace constructible sheaves with PL sheaves (PL for piecewise linear) which are much easier to manipulate. Recall that it has been proved in loc. cit. that constructible sheaves may be approximated, for a kind of derived bottleneck distance, by PL sheaves, and similarly for constructible γ-sheaves. We start this paper with a systematic study of PL-sheaves, showing in particular that a sheaf is PL if and only if its microsupport is a PL Lagrangian variety or, equivalently, is contained in such a Lagrangian variety. We also show that the six Grothendieck operations hold for PL sheaves.

Then we prove our main result, namely that the triangulated category of PL γ-sheaves is generated by the additive category of finite direct sums of constant sheaves on γ-locally closed convex polyhedra. What makes the interest of this result is that such direct sums are a natural higher dimensional analogue to barcodes.

Remark 0.1. Some notations and conventions in this paper differ from those of [KS18].

(a) A polyhedron was called a polytope in loc. cit.

(b) A PL set or a PL sheaf in loc. cit. is called here an LPL set or an LPL sheaf.

(c) The microsupport of a sheaf F is denoted here by $\mathrm{SS}(F)$ as in [KS90], instead of $\mu\mathrm{supp}(F)$ in [KS18].

1 PL geometry

1.1 PL sets and PL stratifications

Let $\mathbb{V}$ be a real finite-dimensional vector space.

Definition 1.1. (a) A *convex polyhedron* P in $\mathbb{V}$ is the intersection of a finite family of open or closed affine half-spaces.

(b) A PL set is a finite union of convex polyhedra.

(c) A locally PL set (an LPL set for short) is a locally finite union of convex polyhedra.

Note that an LPL set is subanalytic.
The next result is obvious.

Lemma 1.2. (i) *The family of* PL *sets in* $\mathbb{V}$ *is stable by finite unions and finite intersections.*

(ii) *If* Z *is* PL*, then its closure* $\overline{Z}$*, its interior* $\mathrm{Int}(Z)$ *and its complementary set* $\mathbb{V}\setminus Z$ *are* PL*.*

(iii) *Any connected component of a* PL *set is* PL*.*

(iv) *Let* $u\colon \mathbb{V}\to\mathbb{W}$ *be a linear map.*

(a) *If* $S\subset\mathbb{V}$ *is* PL*, then* $u(S)\subset\mathbb{W}$ *is* PL*.*
(b) *If* $Z\subset\mathbb{W}$ *is* PL*, then* $u^{-1}(Z)\subset\mathbb{V}$ *is* PL*.*

(v) *The preceding results still hold when when replacing* PL *with* LPL*, except* (iva) *in which case one has to assume that u is proper on* $\overline{S}$*.*

For a locally closed submanifold $Z\subset\mathbb{V}$, one sets for short

$$T^*_Z\mathbb{V} := T^*_ZU \text{ where } U \text{ is an open subset of } \mathbb{V} \text{ containing } Z \text{ as a closed subset.}$$

Definition 1.3. A PL-stratification of a set S of $\mathbb{V}$ is a finite family $Z=\{Z_a\}_{a\in A}$ of non-empty convex polyhedra such that

(i) $S=\bigcup_{a\in A} Z_a$,

(ii) each Z_a is a locally closed submanifold,

(iii) $Z_a\cap Z_b=\varnothing$ for $a\neq b$,

(iv) $Z_a\cap\overline{Z}_b\neq\varnothing$ implies $Z_a\subset\overline{Z}_b$.

Replacing "a finite family" with "a locally finite family" we get the notion of an LPL-stratification.

Recall the operation $\widehat{+}$ and the notion of a μ-stratification of [KS90, Def. 6.2.4, 8.3.19].

Proposition 1.4. *Let* $Z=\{Z_a\}_{a\in A}$ *be an* LPL *stratification. Then*

(i) $\{Z_a\}_{a\in A}$ *is a* μ*-stratification, that is,* $Z_a\subset\overline{Z}_b$ *implies* $(T^*_{Z_a}\mathbb{V}\,\widehat{+}\,T^*_{Z_b}\mathbb{V})\cap\pi^{-1}(Z_a)\subset T^*_{Z_a}\mathbb{V}$*.*

(ii) *Set* $\Lambda = \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$. *Then* $\Lambda \mathbin{\widehat{+}} \Lambda = \Lambda$.

Proof. We shall prove both statements together.

Assume that $Z_a \subset \overline{Z}_b \cap \overline{Z}_c$. We may assume (in a neighborhood of a point of Z_a) that $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}'$ for two linear spaces $\mathbb{W}$ and $\mathbb{W}'$ and $Z_a = \mathbb{W}$. Then $Z_b = \mathbb{W} \times S$ and $Z_c = \mathbb{W} \times L$ where S is open in some linear subspace $\mathbb{W}''$ of $\mathbb{W}'$ and similarly for L. Then one immediately checks that $(T^*_{Z_b}\mathbb{V} \mathbin{\widehat{+}} T^*_{Z_c}\mathbb{V}) \cap \pi^{-1}(Z_a) \subset T^*_{Z_a}\mathbb{V}$.

This proves (ii). Choosing $c = a$ we get (i). □

Proposition 1.5. *Consider a finite family* $\{P_b\}_{b\in B}$ *of convex polyhedra. Then there exists a* PL-*stratification* $\mathbb{V} = \bigsqcup_{a\in A} Z_a$ *such that each* P_b *is a union of strata.*

In the sequel, an interval of $\mathbb{R}$ means a convex subset of $\mathbb{R}$.

Proof. There exists a finite family $\{f_1, \dots, f_l\}$ of linear forms and a finite family $\{I_c\}_{c\in C}$ such that each I_c is either an open interval or a point, $\mathbb{R} = \bigsqcup_{c\in C} I_c$ and for all $b \in B$,

$$P_b = \bigcap_{1\le j\le l} f_j^{-1}(J_j), \text{ where } J_j \text{ is a union of some } I_c,\ c \in C.$$

For any family $d = \{c_1, \dots, c_l\} \in C^l$, set

$$Z_d = \bigcap_{j=1}^{l} f_j^{-1}(I_{c_l}).$$

Then the family $\{Z_d\}_{d\in C^l}$ is a PL-stratification of $\mathbb{V}$ finer that the family $\{P_b\}_{b\in B}$. □

1.2 PL Lagrangian subvarieties

Recall that the notions of co-isotropic, isotropic and Lagrangian subanalytic subvarieties are given in [KS90, Def. 6.5.1, 8.3.9].

Proposition 1.6. *Let* Λ *be a locally closed conic* LPL *isotropic subset of* $T^*\mathbb{V}$. *Then for any* $p \in \Lambda_{\mathrm{reg}}$ *there exists a linear affine subspace* $L \subset \mathbb{V}$ *with* $\Lambda \subset T^*_L\mathbb{V}$ *in a neighborhood of* p.

Proof. If λ is a linear affine isotropic subspace of $T^*\mathbb{V}$, then there exists a linear affine subspace L of $\mathbb{V}$ such that $\lambda \subset T^*_L\mathbb{V}$. □

Lemma 1.7. *Let* $\{L_a\}_{a\in A}$ *be a finite family of affine linear subspaces in a vector space* $\mathbb{W}$. *Set* $X = \bigcup_{a\in A} L_a$ *and let* S *be a closed subset of* X. *Assume that* $S \cap X_{\mathrm{reg}}$ *is open in* X_{reg} *and* S *is the closure of* $S \cap X_{\mathrm{reg}}$. *Then* S *is* PL.

Proof. Indeed $S \cap X_{\mathrm{reg}}$ is a locally finite union of connected components of X_{reg}. □

Theorem 1.8. (a) *Let* Λ *be a locally closed conic* PL *isotropic subset of* $T^*\mathbb{V}$. *Then there exists a* PL *stratification* $\{P_a\}_{a\in A}$ *of* $\mathbb{V}$ *such that* $\Lambda \subset \bigsqcup_{a\in A} T^*_{P_a}\mathbb{V}$.

(b) *Let Λ be a locally closed conic subanalytic Lagrangian subset of $T^*\mathbb{V}$ and assume that Λ is contained in a closed conic* PL *isotropic subset. Then Λ is* PL*.*

Proof. (a) Let $\{\Omega_i\}_{i\in I}$ be the family of connected components of Λ_{reg}. Note that the Ω_i's are PL. Then there exists an affine linear subspace L_i such that $\Omega_i =\subset T^*_{L_i}\mathbb{V}$ by Proposition 1.6. Choose a PL stratification $\{P_a\}_{a\in A}$ finer than the family $\{L_i\}_{i\in I}$. Then $\Lambda_{\mathrm{reg}} \subset \bigsqcup T^*_{P_a}\mathbb{V}$ and Proposition 1.4 (ii) implies that this last set is closed, hence contains Λ.

(b) follows from Lemma 1.7 with $\mathbb{W} = T^*\mathbb{V}$. □

Remark 1.9. In Lemma 1.7 and Theorem 1.8, all statements remain true when replacing everywhere "finite" with "locally finite" and "PL" with "LPL".

2 PL **sheaves**

2.1 Review on sheaves

Let us recall some definitions extracted from [KS90] and a few notations.

- Throughout this paper, $\mathbf{k}$ denotes a field. We denote by $\mathrm{Mod}(\mathbf{k})$ the abelian category of $\mathbf{k}$-vector spaces.

- For an abelian category $\mathscr{C}$, we denote by $\mathrm{D^b}(\mathscr{C})$ its bounded derived category. However, we write $\mathrm{D^b}(\mathbf{k})$ instead of $\mathrm{D^b}(\mathrm{Mod}(\mathbf{k}))$.

- If $\pi\colon E \to M$ is a vector bundle over M, we identify M with the zero-section of E and we set $\dot{E} := E \setminus M$. We denote by $\dot{\pi}\colon \dot{E} \to M$ the restriction of π to $\dot{E}$.

- For a vector bundle $E \to M$, we denote by $a\colon E \to E$ the antipodal map, $a(x, y) = (x, -y)$. For a subset $Z \subset E$, we simply denote by Z^a its image by the antipodal map. In particular, for a cone γ in E, we denote by $\gamma^a = -\gamma$ the opposite cone. For such a cone, we denote by γ° the polar cone (or dual cone) in the dual vector bundle E^*:

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v\rangle \geq 0 \text{ for all } v \in \gamma_x\}. \tag{2.1}$$

- Let M be a real manifold of dimension $\dim M$. We shall use freely the classical notions of microlocal sheaf theory, referring to [KS90]. We denote by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathrm{D^b}(\mathbf{k}_M)$ its bounded derived category. For short, an object of $\mathrm{D^b}(\mathbf{k}_M)$ is called a "sheaf" on M.

- For a locally closed subset $Z \subset M$, one denotes by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z extended by 0 on $M \setminus Z$. One defines similarly the sheaf L_Z for $L \in \mathrm{D^b}(\mathbf{k})$.

- We denote by or_M the orientation sheaf on M and by ω_M the dualizing complex on M. Recall that $\omega_M \simeq \mathrm{or}_M[\dim M]$. One shall use the duality functors

$$\mathrm{D}'_M(\,\bullet\,) = \mathrm{R}\mathscr{H}om(\,\bullet\,, \mathbf{k}_M), \quad \mathrm{D}_M(\,\bullet\,) = \mathrm{R}\mathscr{H}om(\,\bullet\,, \omega_M). \tag{2.2}$$

- For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ we denote by $\mathrm{SS}(F)$ its singular support, or microsupport, a closed conic co-isotropic subset of T^*M.

Constructible sheaves

We refer the reader to [KS90] for terminologies not explained here.

Definition 2.1. Let M be a real analytic manifold and let $F \in \mathrm{Mod}(\mathbf{k}_M)$. One says that F is weakly $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigsqcup_{a\in A} M_a$ such that for each stratum M_a, the restriction $F|_{M_a}$ is locally constant. If moreover, the stalk F_x is of finite rank for all $x \in M$, then one says that F is $\mathbb{R}$-constructible.

Notation 2.2. (i) One denotes by $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves, a thick abelian subcategory of $\mathrm{Mod}(\mathbf{k}_M)$.
(ii) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves with $\mathbb{R}$-constructible cohomology and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c,c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ consisting of sheaves with compact support.

Recall that the natural functor $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ is an equivalence of categories.

2.2 Microlocal characterization of PL sheaves

Recall Remark 0.1.

Definition 2.3. One says that $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ is PL if there exists a finite family $\{P_a\}_{a\in A}$ of convex polyhedra such that $\mathbb{V} = \bigcup_{a\in A} P_a$ and $F|_{P_a}$ is constant of finite rank for any $a \in A$.

Replacing the finite family $\{P_a\}_{a\in A}$ with a locally finite family, we get the notion of an LPL sheaf.

By this definition

(2.3) $\quad F$ is PL if and only if $H^j(F)$ is PL for all $j \in \mathbb{Z}$.

One sets

$$\begin{cases} \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}});\ F \text{ is PL}\}, \\ \mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}). \end{cases} \tag{2.4}$$

Of course, $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ and $\mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a subcategory of $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$.

Proposition 2.4. *The natural functor* $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ *is an equivalence.*

Proof. There exists a triangulation $\mathbb{S} = (S, \Delta)$ and a homeomorphism $f\colon |\mathbb{S}| \to \mathbb{V}$ such that its restriction to $|\sigma|$ is linear for any $\sigma \in \Delta$. Then the result follow from [KS90, Th. 8.1.10]. ☐

Theorem 2.5. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$. Then the conditions below are equivalent.*

(a) $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$,

(b) $\mathrm{SS}(F)$ *is a closed conic* PL *Lagrangian subset of* $T^*\mathbb{V}$,

(c) $\mathrm{SS}(F)$ *is contained in a closed conic* PL *isotropic subset of* $T^*\mathbb{V}$.

Proof. (a)⇒(c) Consider a covering $\{P_b\}_{b\in B}$ by convex polyhedra such that $F|_{P_b}$ is constant and choose a finer PL stratification $\mathbb{V} = \bigsqcup_{a\in A} Z_a$. This is a μ-stratification and this implies $\mathrm{SS}(F) \subset \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$ by [KS90, Prop. 8.4.1].

(b)⇒(a) By Theorem 1.8 (a), there exists a PL stratification $\mathbb{V} = \bigsqcup_{a\in A} Z_a$ such that $\mathrm{SS}(F) \subset \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$. Then $F|_{Z_a}$ is locally constant for each $a \in A$ by [KS90, Prop. 8.4.1].

(b)⇔(c) in view of Theorem 1.8 (b). □

The next result immediately follows from Definition 2.3. It can also easily be deduced from [KS90], Theorem 1.8 and Theorem 2.5.

Corollary 2.6. (i) *The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a full triangulated subcategory of the category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ and the category $\mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a full thick abelian subcategory of the category $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}})$.*

(ii) *If F_1 and F_2 are* PL, *then so are $F_1 \otimes F_2$ and $\mathrm{R}\mathscr{H}om(F_1, F_2)$.*

(iii) *Let $f\colon \mathbb{V} \to \mathbb{W}$ be a linear map.*

(a) *If G is a PL sheaf on $\mathbb{W}$, then $f^{-1}G$ and $f^!G$ are* PL *sheaves on $\mathbb{V}$.*

(b) *If F is a* PL *sheaf on $\mathbb{V}$ then $\mathrm{R}f_*F$ and $\mathrm{R}f_!F$ are* PL *sheaves on $\mathbb{W}$.*

Recall that the convolution functor $\star\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ is defined by the formula:

$$F \star G := \mathrm{R}s_!(F \boxtimes G),$$

where $s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is the map $(x, y) \mapsto x + y$.

Corollary 2.7. *The convolution functor induces a functor $\star\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$.*

Proposition 2.8. *Let Z be a locally closed subset of $\mathbb{V}$. Then Z is* PL *if and only if $\mathrm{SS}(\mathbf{k}_Z)$ is* PL.

Proof. (i) Assume that Z is PL. Then the sheaf $\mathbf{k}_Z$ is PL and its microsupport is PL by Theorem 2.5.

(ii) Conversely, assume that $\mathrm{SS}(\mathbf{k}_Z)$ is PL. Set $\partial Z = \overline{Z} \setminus Z$. Since Z is locally closed, ∂Z is closed.

(ii)–(a) First, notice that $\overline{Z} = \pi(\mathrm{SS}(\mathbf{k}_Z))$ is PL.

(ii)–(b) Now consider the exact sequence of sheaves $0 \to \mathbf{k}_Z \to \mathbf{k}_{\overline{Z}} \to \mathbf{k}_{\partial Z} \to 0$. Since $\mathbf{k}_Z$ and $\mathbf{k}_{\overline{Z}}$ are PL sheaves, the sheaf $\mathbf{k}_{\partial Z}$ is PL. Therefore, ∂Z is PL and it follows that Z is PL. □

2.3 Distance and approximation

The bottleneck distance for persistent modules is a classical subject that we shall not review here, referring to [CSEH07] and [CCSG+09, CdSGO16, Cur13, EH08, Ghr08, Les15]. Here, we use a convolution distance for sheaves similar to that of [KS18] and slightly different from the classical ones since it is defined in the derived setting. Note that this "derived" distance has recently been studied with great details in [BG18] in case of dimension one.

Assume that the vector space $\mathbb{V}$ is endowed with a norm $\|\cdot\|$ (see Remark 2.9 below). We define a family of sheaves $\{K_a\}_{a\in\mathbb{R}}$ as follows:

$$K_a \simeq \begin{cases} \mathbf{k}_{\{\|x\|\leq a\}} & \text{for } a \geq 0, \\ \mathbf{k}_{\{\|-x\|<-a\}}[n] & \text{for } a < 0. \end{cases} \tag{2.5}$$

There are natural morphisms and isomorphisms

$$\begin{aligned} &\chi_{a,b}\colon K_b \to K_a \text{ for } a \leq b \in \mathbb{R}, \\ &K_a \star K_b \simeq K_{a+b} \text{ for } a, b \in \mathbb{R} \end{aligned} \tag{2.6}$$

such that $\chi_{a,b} \circ \chi_{b,c} = \chi_{a,c}$ for $a \leq b \leq c$.

Remark 2.9. In [KS18] we have used the Euclidean norm, but the argument works for any norm, since (2.6) remains true. Here a norm $\|\cdot\|\colon \mathbb{V} \to \mathbb{R}_{\geq 0}$ satisfies (1) $\|x+y\| \leq \|x\| + \|y\|$, (2) $\|ax\| = a\|x\|$ for $a \geq 0$, and (3) $\|x\| = 0$ implies $x = 0$. Hence norms correspond bijectively with open relatively compact convex neighborhood of 0. Note that we do not ask $\|x\| = \|-x\|$ and that is why we define K_a for $a < 0$ as above, in order that the sheaf K_a is the dual of K_{-a}.

Definition 2.10. ([KS18, Def. 2.2]) Let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ and let $a \geq 0$. One says that F and G are *a-isomorphic* if there are morphisms $f\colon K_a \star F \to G$ and $g\colon K_a \star G \to F$ which satisfies the following compatibility conditions: the composition $K_{2a} \star F \xrightarrow{K_a \star f} K_a \star G \xrightarrow{g} F$ coincides with the natural morphism $\chi_{0,2a} \star F\colon K_{2a} \star F \to F$ and the composition $K_{2a} \star G \xrightarrow{K_a \star g} K_a \star F \xrightarrow{f} G$ coincides with the natural morphism $\chi_{0,2a} \star G\colon K_{2a} \star G \to G$.

One sets

$$\mathrm{dist}(F,G) = \inf\Big(\{+\infty\} \cup \{a \in \mathbb{R}_{\geq 0}\,;\, F \text{ and } G \text{ are } a\text{-isomorphic}\}\Big)$$

and calls $\mathrm{dist}(\,\bullet\,,\,\bullet\,)$ the *convolution distance*.

In loc. cit. we have proved that any object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ can be approximated with LPL-sheaves. A similar result holds for constructible sheaves.

Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ consisting of objects F such that there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{P}})$ with $F \simeq G|_{\mathbb{V}}$.

Theorem 2.11 (The approximation theorem)**.** *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$. For each $\varepsilon > 0$ there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ such that* $\mathrm{dist}(F,G) \leq \varepsilon$ *and* $\mathrm{supp}(G) \subset \mathrm{supp}(F) + B_\varepsilon$.

The proof is the same as in [KS18] after noticing that one can choose the simplicial complex $\mathbf{S} = (S, \Delta)$ (with the notations of loc. cit.) to be finite, tanks to the fact that $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$.

2.4 The stability theorem and an application

Recall first the stability theorem of [KS18, Th. 2.7], a derived version of a classical theorem (see [CSEH07]).

For a set X and a map $f\colon X \to \mathbb{V}$, one sets

$$\|f\| = \sup_{x\in X} \|f(x)\|.$$

Theorem 2.12 (The stability theorem)**.** *Let X be a locally compact space and let $f_1, f_2\colon X \to \mathbb{V}$ be two continuous maps. Then, for any $F \in \mathrm{D^b}(\mathbf{k}_X)$, we have*

$$\mathrm{dist}(\mathrm{R}f_{1*}F, \mathrm{R}f_{2*}F) \leq \|f_1 - f_2\| \quad \text{and} \quad \mathrm{dist}(\mathrm{R}f_{1!}F, \mathrm{R}f_{2!}F) \leq \|f_1 - f_2\|.$$

Here we give an application which did not appear in [KS18]. Of course, this application is well-known for the classical (non-derived) distance.

Let (M, d) be a subanalytic metric space and let K_1 and K_2 be two subanalytic compact subsets. Define for $i = 1, 2$

$$\begin{aligned}
&f_i(\cdot) = d(\cdot, K_i),\\
&\Gamma_i = \{(x,t) \in M \times \mathbb{R}; d(x, K_i) = t\}, \quad G_i = \mathbf{k}_{\Gamma_i},\\
&Z_i = \{(x,t) \in M \times \mathbb{R}; d(x, K_i) \leq t\}, \quad F_i = \mathbf{k}_{Z_i}.
\end{aligned}$$

Hence, Γ_i is the graph of f_i and Z_i is the epigraph of f_i.

Lemma 2.13. *One has $d(K_1, K_2) = \|f_1 - f_2\|$.*

Lemma 2.14. *One has $\mathrm{dist}(\mathrm{R}f_{1*}\mathbf{k}_M, \mathrm{R}f_{2*}\mathbf{k}_M) \leq \|f_1 - f_2\|$.*

Proof. This is a particular case of Theorem 2.12. □

Lemma 2.15. *Denote by γ the cone $\{t \leq 0\}$ in $\mathbb{R}$ and still denote by φ_γ the map $M \times \mathbb{R} \to M \times \mathbb{R}_\gamma$. Then $F_i \simeq \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_i$.*

Lemma 2.16. *One has $\mathrm{dist}(\mathrm{R}p_*\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_1, \mathrm{R}p_*\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_2) \leq \mathrm{dist}(\mathrm{R}p_*G_1, \mathrm{R}p_*G_2)$.*

Proof. The two functors $\mathrm{R}p_*$ and $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}$ commute (with obvious notations). Then the result follows from [KS18, Prop. 2.6]. □

Applying these lemmas, one gets a derived version of a result of [CSEH07].

Theorem 2.17. *One has $\mathrm{dist}(\mathrm{R}p_*F_1, \mathrm{R}p_*F_2) \leq d(K_1, K_2)$.*

Proof. We have

$$\mathrm{R}p_*G_i \simeq \mathrm{R}f_{i*}\mathbf{k}_M.$$

Applying Lemma 2.14, we get

$$\mathrm{dist}(\mathrm{R}p_*G_1, \mathrm{R}p_*G_2) \leq \|f_1 - f_2\|.$$

To conclude, apply Lemma 2.15, 2.16 and 2.13.

□

2.5 Generators for PL sheaves

Consider a triangulated category $\mathscr{D}$ and a family of objects $\mathscr{G}$. Consider the full subcategory $\mathscr{T}$ of $\mathscr{D}$ defined as follows. An object $F \in \mathscr{D}$ belongs to $\mathscr{T}$ if there exists a finite sequence $F_0, \dots, F_N$ in $\mathscr{D}$ with $F_0 = 0$, $F_N = F$ and distinguished triangles $F_k \to F_{k+1} \to G_k[m_k] \xrightarrow{+1}$, $0 \le k < N$ with $m_k \in \mathbb{Z}$ and $G_k \in \mathscr{G}$. Clearly, $\mathscr{T}$ is a triangulated subcategory of $\mathscr{D}$. In this paper, we shall say that $\mathscr{G}$ generates $\mathscr{D}$ if $\mathscr{T} = \mathscr{D}$.

Theorem 2.18. *The triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is generated by the family $\{\mathbf{k}_P\}$ where P ranges over the family of locally closed convex polyhedra.*

Proof. (i) We denote by $\mathscr{G}$ the family of sheaves isomorphic to some $\mathbf{k}_P$, P a locally closed convex polyhedron, and denote by $\mathscr{T}$ the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ generated by $\mathscr{G}$, that is, the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ which contains $\mathscr{G}$.

(ii) We argue by induction on $\dim \mathbb{V}$. The case where $\dim \mathbb{V}$ is 0 or 1 is clear.

(iii) Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$. By truncation, we may reduce to the case where F is concentrated in degree 0.

(iv) There exists a finite family $\{H_a\}_{a\in A}$ of affine hyperplanes such that, setting $U = \mathbb{V} \setminus \bigcup_a H_a$, the restriction of F to U is locally constant. Let $U = \bigsqcup_i U_i$ be the decomposition of U into connected component. Each U_i is an open convex polyhedron. Set $Z = \bigcup_a H_a$ and consider the exact sequence $0 \to F_U \to F \to F_Z \to 0$. The sheaf F_U is a finite direct sum of sheaves of the type $\mathbf{k}_{U_i}$. Hence $F_U \in \mathscr{T}$ and it remains to show that F_Z belongs to $\mathscr{T}$.

(v) We argue by induction on $\#A$. If $\#A = 1$, then the result follows from the induction hypothesis on the dimension of $\mathbb{V}$ since we may identify F_{H_a} with a sheaf on the affine space H_a. Let $a \in A$ and define G by the exact sequence $0 \to G \to F_Z \to F_{H_a} \to 0$. By the induction hypothesis G and F_{H_a} belong to $\mathscr{T}$ and the result follows. □

2.6 PL sheaves on the projective space

Let $\mathbb{V} \hookrightarrow \mathbb{P}$ be the projective compactification of $\mathbb{V}$. Hence, setting $\mathbb{W} = \mathbb{V} \times \mathbb{R}$,

$$\mathbb{P} \simeq (\mathbb{W} \setminus \{0\})/\mathbb{R}^{\times},$$

where $\mathbb{R}^{\times}$ is the multiplicative group $\mathbb{R}\setminus\{0\}$. Denote by $\pi\colon (\mathbb{W}\setminus\{0\}) \to \mathbb{P}$ the projection and by $\iota\colon \mathbb{W} \setminus \{0\} \hookrightarrow \mathbb{W}$ the embedding.

We shall say that a subset A of $\mathbb{P}$ is PL if $\iota(\pi^{-1}(A))$ is PL in $\mathbb{W}$.

Similarly, one defines the category of PL sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{P}})$ as the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{P}})$ consisting of objects F such that $\iota_!\pi^{-1}F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{W}})$.

Denote by $j\colon \mathbb{V} \hookrightarrow \mathbb{P}$ the open embedding and by $H_\infty = \mathbb{P} \setminus j(\mathbb{V})$ the hyperplane at infinity. The next result is easy and its proof is left to the reader.

Proposition 2.19. *The functor $j_!\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{P}})$ is well-defined, is fully faithful and its essential image consists of objects F such that $F|_{H_\infty} \simeq 0$.*

3 PL γ-sheaves

3.1 Review on γ-sheaves

In this subsection we shall review some definitions and results extracted from [KS90, KS18]. The so-called γ-topology has been studied with some details in [KS90, § 3.4].

Let $\mathbb{V}$ be a finite-dimensional real vector space. We denote by $a\colon \mathbb{V} \to \mathbb{V}$ the antipodal map $x \mapsto -x$.

Hence, for two subsets A, B of $\mathbb{V}$, one has $A + B = s(A \times B)$. A subset A of V is called a cone if $0 \in A$ and $\mathbb{R}_{>0}A \subset A$. A convex cone A is proper if $A \cap A^a = \{0\}$.

Throughout the paper, we consider a cone $\gamma \subset \mathbb{V}$ and we assume:

(3.1) *γ is closed proper convex with non-empty interior.*

In § 3.2 we shall make the extra assumption that γ is *polyhedral*, meaning that it is a finite intersection of closed half-spaces.

We say that a subset A of $\mathbb{V}$ is *γ-invariant* if $A + \gamma = A$. Note that a subset A is γ-invariant if and only if $\mathbb{V} \setminus A$ is γ^a-invariant.

The family of γ-invariant open subsets of $\mathbb{V}$ defines a topology, which is called the *γ-topology* on $\mathbb{V}$. One denotes by $\mathbb{V}_\gamma$ the space $\mathbb{V}$ endowed with the γ-topology and one denotes by

$$\varphi_\gamma\colon \mathbb{V} \to \mathbb{V}_\gamma \tag{3.2}$$

the continuous map associated with the identity. Note that the closed sets for this topology are the γ^a-invariant closed subsets of $\mathbb{V}$.

Definition 3.1. Let A be a subset of $\mathbb{V}$.

(a) One says that A is *γ-open* (resp. *γ-closed*) if A is open (resp. closed) for the γ-topology.

(b) One says that A is *γ-locally closed* if A is the intersection of a γ-open subset and a γ-closed subset.

(c) One says that A is *γ-flat* if $A = (A + \gamma) \cap (A + \gamma^a)$.

(d) One says that a closed set A is *γ-proper* if the addition map s is proper on $A \times \gamma^a$.

Remark that a closed subset A is γ-proper if and only if $A \cap (x + \gamma)$ is compact for any $x \in \mathbb{V}$.

Proposition 3.2 ([KS18, Prop. 3.4])**.** *The set of γ-flat open subsets Ω of $\mathbb{V}$ and the set of γ-locally closed subsets Z of $\mathbb{V}$ are isomorphic by the correspondence*

$$\Omega \longmapsto (\Omega + \gamma) \cap \overline{\Omega + \gamma^a}$$
$$\mathrm{Int}(Z) \longleftarrow Z.$$

In particular, γ-locally closed subsets are γ-flat.

We shall use the notations:

$$
\begin{cases}
\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}); \mathrm{SS}(F) \subset \mathbb{V} \times \gamma^{\circ a}\}, \\
\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\
\mathrm{Mod}_{\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\
\mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{Mod}_{\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}).
\end{cases} \tag{3.3}
$$

We call an object of $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ *a γ-sheaf.*

It follows from [KS90, Prop. 5.4.14] that for $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and $H \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ}}(\mathbf{k}_{\mathbb{V}})$, the sheaves $F \otimes G$ and $\mathrm{R}\mathscr{H}om(H, F)$ belong to $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$.

The next result is implicitly proved in [KS90] and explicitly in [KS18]. (In this statement, the hypothesis that $\mathrm{Int}(\gamma)$ is non empty is not necessary.)

Theorem 3.3. *Let γ be a closed convex proper cone in $\mathbb{V}$. The functor $\mathrm{R}\varphi_{\gamma_*}\colon \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of triangulated categories with quasi-inverse φ_γ^{-1}. Moreover, this equivalence preserves the natural t-structures of both categories. In particular, for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$, the condition $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ is equivalent to the condition: $\mathrm{SS}(H^j(F)) \subset \mathbb{V} \times \gamma^{\circ a}$ for any $j \in \mathbb{Z}$.*

Thanks to this theorem, the reader may ignore microlocal sheaf theory, at least in a first reading.

Corollary 3.4 ([KS18, Cor. 1.8])**.** *Let A be a γ-locally closed subset of $\mathbb{V}$. Then $\mathrm{SS}(\mathbf{k}_A) \subset \mathbb{V} \times \gamma^{\circ a}$.*

Proposition 3.5. *Assume (3.1). Let $U = U + \gamma$ be a γ-open set and let $x_0 \in \partial U$. Then there exist a linear coordinate system $(x_1, \dots, x_n)$ on $\mathbb{V}$, an open neighborhood V of x_0, an open subset W of $\mathbb{V}$ and a bi-Lipschitz isomorphism $\varphi\colon V \xrightarrow{\sim} W$ such that $\varphi(V \cap U) = W \cap \{x \in \mathbb{V}; x_n > 0\}$.*

Proof. The proofs of [GS16, Lem. 2.36, 2.37] (which were formulated for subanalytic open subsets) extend immediately to our situation. ☐

Recall the duality functor D'_M of (2.2).

Corollary 3.6. *Let Z be a γ-locally closed subset of $\mathbb{V}$. Then, $\mathrm{D}'_M(\mathbf{k}_Z)$ is concentrated in degree 0. Moreover, $\mathrm{D}'_M(\mathbf{k}_Z) \simeq \mathbf{k}_S$ with $\Omega = \mathrm{Int}(Z)$ and $S = \overline{\Omega + \gamma} \cap (\Omega + \gamma^a)$.*

Proof. It follows from Proposition 3.5 that $\mathrm{D}'_M(\mathbf{k}_{\Omega+\gamma}) \simeq \mathbf{k}_{\overline{\Omega+\gamma}}$ and $\mathrm{D}'_M(\mathbf{k}_{\overline{\Omega+\gamma^a}}) \simeq \mathbf{k}_{\Omega+\gamma^a}$.

Set $A = \Omega + \gamma$ and $B = \overline{\Omega + \gamma^a}$. Then $\mathbf{k}_A$ and $\mathbf{k}_B$ are cohomologically constructible. By Corollary 3.4, $\mathrm{SS}(\mathbf{k}_A) \cap \mathrm{SS}(\mathbf{k}_B) \subset T^*_{\mathbb{V}}\mathbb{V}$. Then $\mathrm{D}'_M(\mathbf{k}_A \otimes \mathbf{k}_B) \simeq \mathrm{D}'_M(\mathbf{k}_A) \otimes \mathrm{D}'_M(\mathbf{k}_B)$ by [KS90, Cor. 6.4.3]. ☐

3.2 Review on PL γ-sheaves

From now on, we shall assume that the cone γ satisfies:

(3.4) *γ is a closed proper convex polyhedral cone with non-empty interior.*

Definition 3.7. Assume (3.4).

(a) A PL γ-*barcode* (A, Z) in $\mathbb{V}$ is the data of a *finite* set of indices A and a family $Z = \{Z_a\}_{a\in A}$ of non-empty, γ-locally closed, convex polyhedra of $\mathbb{V}$.

(b) A PL γ-*partition* (A, Z) is a γ-barcode (A, Z) such that $Z_a \cap Z_b = \varnothing$ for $a \neq b$.

(c) The support of a γ-barcode (A, Z), denoted by $\mathrm{supp}(A, Z)$, is the set $\bigcup_{a\in A} \overline{Z_a}$.

Remark 3.8. In [KS18], we defined a PL γ-stratification of a closed set S as a barcode (A, Z) such that $\mathrm{supp}(A, Z) = S$ and $Z_a \cap Z_b = \varnothing$ for $a \neq b$. However, since a PL γ-stratification is not a PL stratification (see Definition 1.3), we prefer here to avoid this terminology and use the notion of a PL γ-partition.

We shall use the notations:

$$\begin{cases} \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}). \end{cases} \tag{3.5}$$

Note that, in view of (2.3) and Theorem 3.3:

$$F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \Leftrightarrow H^j(F) \in \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \text{ for all } j \in \mathbb{Z}. \tag{3.6}$$

Proposition 3.9. *Assume* (3.4). *If* $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ *then* $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$.

Proof. By Theorem 3.3, it remains to prove that $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F$ is PL. Denote by $Z(\gamma)$ the set $\{(x, y) \in \mathbb{V}\times\mathbb{V}; y - x \in \gamma\}$ and denote by q_1 and q_2 the first and second projections defined on $\mathbb{V}\times\mathbb{V}$. Then (see [KS90, Prop. 3.5.4]):

$$\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \simeq \mathrm{R}q_{1*}(\mathbf{k}_{Z(\gamma)} \otimes q_2^{-1}F)$$

and the result follows from Corollary 2.6. □

Corollary 3.10. *Let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}_\infty})$. *For each* $\varepsilon > 0$ *there exists* $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ *such that* $\mathrm{dist}(F, G) \leq \varepsilon$.

Proof. We shall apply Theorem 2.11. There exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ such that $\mathrm{dist}(F, G) \leq \varepsilon$. Then, $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ by Proposition 3.9. Moreover, by [KS18, Prop. 2.6] :

$$\mathrm{dist}\big(\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F, \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G\big) \leq \mathrm{dist}(F, G) \leq \varepsilon.$$

Since F is a γ-sheaf, one has $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \simeq F$. □

The three theorems below are the main results of [KS18]. We recall them for the reader's convenience.

Theorem 3.11 ([KS18, Th. 3.10])**.** *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. *Then for each* $x \in \mathbb{V}$, *there exists an open neighborhood* U *of* x *such that* $F|_{(x+\gamma^a)\cap U}$ *is constant.*

Theorem 3.12 ([KS18, Th. 3.14])**.** *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$*. Let* Ω *be a* γ*-flat open set and let* $Z = (\Omega + \gamma) \cap \overline{\Omega + \gamma^a}$ *, a* γ*-locally closed subset. Assume that* $F|_{\Omega}$ *is locally constant. Then* $F|_Z$ *is locally constant.*

Theorem 3.13 ([KS18, Lem. 3.16, Th. 3.17])**.** *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$*. Then there exists a* PL γ*-partition* (A, Z) *with* $\mathrm{supp}(A, Z) = \mathrm{supp}(F)$ *and such that* $F|_{Z_a}$ *is constant for each* $a \in A$*. Moreover,* $F_x \simeq 0$ *for* $x \notin \bigsqcup_{a \in A} Z_a$*.*

(In fact, Theorem 3.13 was proved for LPL sheaves but the proof can easily be adapted to PL sheaves.)

3.3 Generators for PL γ-sheaves

In [KS18] we have constructed a category $\mathbf{Bar}_\gamma$ whose objects are the γ-barcodes and a fully faithful functor

$$\Psi\colon \mathbf{Bar}_\gamma \to \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}), \quad Z = \{Z_a\}_{a \in A} \mapsto \bigoplus_{a \in A} \mathbf{k}_{Z_a}. \tag{3.7}$$

However, as shown in [KS18, Ex. 2.14, 2.15], the functor Ψ is not essentially surjective as soon as $\dim \mathbb{V} > 1$.

Definition 3.14. An object of $\mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is a barcode γ-sheaf if it is in the essential image of Ψ.

In [KS18] we made the following conjecture.

Conjecture 3.15. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ and assume that F has compact support. Then there exists a bounded complex $F^\bullet \in \mathrm{C}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}_\gamma}))$ whose image in $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is isomorphic to F and such that each component F^j of $F^\bullet$ is a barcode γ-sheaf with compact support.

As usual, for an additive category $\mathscr{C}$, $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ denotes the category of bounded complexes of objects of $\mathscr{C}$.

In this subsection, we shall prove a weaker form of this conjecture, namely:

Theorem 3.16. *The triangulated category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ *is generated by the family* $\{\mathbf{k}_P\}_P$ *where* P *ranges over the family of* γ*-locally closed convex polyhedra.*

In particular, the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is generated by the barcodes γ-sheaves.

Proof. Let $F \in \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$. There exists $\{\xi_k\}_{1 \le k \le l}$ in $\gamma^{0,a}$ and $\{c_j\}_{0 \le j \le N}$ in $\mathbb{R}$ with $-\infty = c_0 < c_1 < \cdots < c_{N-1} < c_N = +\infty$ such that, setting

$$H_{k,j} = \{x \in \mathbb{V}; \langle x, \xi_k \rangle = c_j\}, \quad U := \mathbb{V} \setminus \bigsqcup_{k,j} H_{k,j},$$

the sheaf $F|_U$ is locally constant.

For $n = (n_1, \dots, n_l)$ with $0 \le n_k < N$, define

$$Z_n = \bigcap_{k=1}^{l} \{x; c_{n_k} \le \langle x, \xi_k \rangle < c_{1+n_k}\}, \quad \Omega_n = \bigcap_{k=1}^{l} \{x; c_{n_k} < \langle x, \xi_k \rangle < c_{1+n_k}\}.$$

Then $Z_n = \overline{\Omega_n + \gamma^{0,a}} \cap (\Omega_n + \gamma)$ and Z_n is γ-locally closed.

Since F_{Ω_n} is constant, F_{Z_n} is constant by Theorem 3.12.

Now we have $\mathbb{V} = \bigsqcup_{n \in [0,N-1]^l} Z_n$. Set $A = \{n \in [0, N-1]^l\,; F|_{Z_n} \not\simeq 0\}$. Then $\operatorname{supp}(F) = \bigcup_{n \in A} \overline{Z_n}$.

Lemma 3.17. *There exists $n \in A$ such that Z_n is open in* $\operatorname{supp}(F)$*.*

Proof of the lemma. We order the set of n's by $n \le n'$ if $n_j \le n'_j$ for all $j \in [1, \dots, l]$. Let n be a minimal element of A. Then Z_n is open in $\operatorname{supp}(F)$. Indeed,

$$\begin{aligned} Z_n = \operatorname{supp}(F) \cap Z_n &= W \cap \bigcap_k \{x; c_{n_k} \le \langle x, \xi_k \rangle < c_{n_k+1}\} \\ &= \operatorname{supp}(F) \cap \bigcap_k \{x; \langle x, \xi_k \rangle < c_{n_k+1}\}. \end{aligned}$$

Note that the last equality is true since otherwise there exists $n' < n$ in A such that $\operatorname{supp}(F) \cap Z_{n'} \neq \varnothing$, and n would not be minimal. □

Now we can complete the proof of Theorem 3.16.

Let us take $n \in A$ such that $Z_n \subset \operatorname{supp}(F)$ is open in $\operatorname{supp}(F)$. Then we have an exact sequence

$$0 \to \mathbf{k}_{Z_n} \otimes F(Z_n) \to F \to F'' \to 0$$

and $\operatorname{supp} F'' \subset \operatorname{supp}(F) \setminus Z_n$. Then the proof goes by induction on $\#A$. □

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