Persistent homology and microlocal sheaf theory

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Abstract

We interpret some results of persistent homology and barcodes (in any dimension) with the language of microlocal sheaf theory. For that purpose we study the derived category of sheaves on a real finite-dimensional vector space $\mathbb{V}$. By using the operation of convolution, we introduce a pseudo-distance on this category and prove in particular a stability result for direct images. Then we assume that $\mathbb{V}$ is endowed with a closed convex proper cone γ with non empty interior and study γ-sheaves, that is, constructible sheaves with microsupport contained in the antipodal to the polar cone (equivalently, constructible sheaves for the γ-topology). We prove that such sheaves may be approximated (for the pseudo-distance) by "piecewise linear" γ-sheaves. Finally we show that these last sheaves are constant on stratifications by γ-locally closed sets, an analogue of barcodes in higher dimension.

Contents

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Introduction

Persistent homology and barcodes are recent concrete applications of algebraic topology. The aim of this paper is to show that many results of this theory are easily interpreted in the language of microlocal sheaf theory and that, in this formulation, one may extend the theory to higher dimension.

Although the theory is quite new, there is already a vast literature on persistent homology. See in particular the survey papers [EH08, Ghr08, Oud15].

We understand persistent homology as follows. One has a set S in a normed space X and one wants to understand its structure. For that purpose, one replaces each point $x \in S$ with a closed ball of center x and radius t and makes t going to infinity. The union of these balls is a closed set $Z \subset X \times \mathbb{R}$ and to understand how the homology of the union of the balls varies with t is equivalent to project to $\mathbb{R}$ the constant sheaf $\mathbf{k}_Z$ associated with Z. From this point of view, we are not far from Morse theory for sheaves (see [GM88, KS90]). Moreover, the sheaf one obtains has particular properties. It is constructible and is associated with a topology whose open sets are the intervals $]-\infty, a[$ with $a \in [-\infty, +\infty]$. As we shall see, the category of such sheaves is equivalent to a category (that we shall define) of barcodes.

As described above, persistent homology takes its values on $\mathbb{R}$ and barcodes are defined on the ordered space $(\mathbb{R}, \leq)$. However, the necessity of treating more than one parameter t naturally appears (see for *e.g.* [Les15, LW15]).

A higher dimensional generalization of the ordered space $(\mathbb{R}, \leq)$ is the data of a finite-dimensional real vector space $\mathbb{V}$ and a closed convex proper cone $\gamma \subset \mathbb{V}$ with non-empty interior. We call here a γ-sheaf an object of the derived category of sheaves on $\mathbb{V}$ whose microsupport is contained in $\gamma^{\circ a}$, the antipodal to the polar cone to γ. As we shall see, this category is equivalent to the derived category of sheaves on $\mathbb{V}_\gamma$, the space $\mathbb{V}$ endowed with the so-called γ-topology.

The main goal of this paper is to describe constructible γ-sheaves on $\mathbb{V}$. Constructible sheaves on real analytic manifolds are now well-understood (the story began on complex manifolds with [Kas75]) but, as we shall see, γ-sheaves have a very specific behavior and are not so easy to treat. We shall mainly restrict ourselves to piecewise linear sheaves (PL-sheaves for short), those which are locally constant on a stratification associated with a locally finite family of hyperplanes.

A final remark: most of the authors define (higher dimensional) persistent homology modules as functors from the order set $(\mathbb{R}^n, \leq)$ to the category of (finite dimensional) vector spaces over a given field $\mathbf{k}$. Thanks to a result of Justin Curry [Cur13, Th. 4.2.10], this is equivalent to considering sheaves or cosheaves on a topological space associated with the order.

The main results of this paper may be described as follows.

(a) In § 1.2 we recall and complete some results of [KS90, § 3.5] on the γ-topology, showing that the category of γ-sheaves is equivalent to the derived category of sheaves on $\mathbb{V}_\gamma$.

Then in § 1.3 and 1.4 we recall, in our language, some basic results of persistent homology (that is, essentially Morse theory for sheaves) and construct the category of barcodes on $\mathbb{R}$. This last category is proved, thanks to a theorem of Guillermou [Gui16] (see also Crawley-Boevey [CB14] for a very similar result), to be equivalent to that of constructible γ-sheaves, for $\gamma = \mathbb{R}_{\leq 0}$.

(b) In § 2, using the convolution of sheaves (in the derived setting), we introduce a kind of pseudo-distance, denoted by dist, and show a stability result for direct images, namely, given two maps $f_1, f_2 \colon X \to \mathbb{V}$ and a sheaf F on X, if $\|f_1 - f_2\| \leq \varepsilon$, then $\mathrm{dist}(\mathrm{R}f_{1*}F, \mathrm{R}f_{2*}F) \leq \varepsilon$.

We also introduce the notion of piecewise linear sheaves. Then, we show that any constructible sheaf may be approximated by a PL-sheaf and we also prove a similar result for γ-sheaves.

(c) In § 3, we prove that given a PL-γ-sheaf, there exists a stratification of $\mathbb{V}$ by γ-locally closed polytopes on which the sheaf is constant. This stratification plays the role of higher dimensional barcodes, but there is no more an equivalence of categories, as in dimension one.

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1 Persistent homology

1.1 Sheaves

The aim of this subsection is simply to fix a few notations.

- Throughout the paper, $\mathbf{k}$ denotes a field. We denote by $\mathrm{Mod}(\mathbf{k})$ the abelian category of $\mathbf{k}$-vector spaces.
- For an abelian category $\mathscr{C}$, we denote by $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ its bounded derived category. However, we write $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ instead of $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbf{k}))$.
- If $\pi \colon E \to M$ is a vector bundle over M, we identify M with the zero-section of E and we set $\dot{E} := E \setminus M$. We denote by $\dot{\pi} \colon \dot{E} \to M$ the restriction of π to $\dot{E}$.

- For a vector bundle $E \to M$, we denote by $a\colon E \to E$ the antipodal map, $a(x,y) = (x,-y)$. For a subset $Z \subset E$, we simply denote by Z^a its image by the antipodal map. In particular, for a cone γ in E, we denote by $\gamma^a = -\gamma$ the opposite cone. For such a cone, we denote by γ° the polar cone (or dual cone) in the dual vector bundle E^*:

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v\rangle \geq 0 \text{ for all } v \in \gamma_x\}. \tag{1.1}$$

- Let M be a real manifold M of dimension $\dim M$. We shall use freely the classical notions of microlocal sheaf theory, referring to [KS90]. We denote by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathrm{D^b}(\mathbf{k}_M)$ its bounded derived category. For short, an object of $\mathrm{D^b}(\mathbf{k}_M)$ is called a "sheaf" on M.

- For a locally closed subset $Z \subset M$, one denotes by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z extended by 0 on $M \setminus Z$. One defines similarly the sheaf L_Z for $L \in \mathrm{D^b}(\mathbf{k})$.

- We denote by or_M the orientation sheaf on M and by ω_M the dualizing complex on M. Recall that $\omega_M \simeq \mathrm{or}_M\,[\dim M]$. One shall use the duality functors

$$\mathrm{D}'_M(\,\bullet\,) = \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbf{k}_M), \quad \mathrm{D}_M(\,\bullet\,) = \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \omega_M). \tag{1.2}$$

- For $F \in \mathrm{D^b}(\mathbf{k}_M)$ we denote by $\mu\mathrm{supp}(F)$[1] its microsupport, a closed conic co-isotropic subset of T^*M.

- For $F \in \mathrm{D^b}(\mathbf{k}_M)$, one denotes by $\mathrm{Sing}(F)$ the singular locus of F, that is, the complement of the largest open subset on which F is locally constant.

Constructible sheaves

We refer the reader to [KS90] for terminologies not explained here.

Definition 1.1. Let M be a real analytic manifold and let $F \in \mathrm{Mod}(\mathbf{k}_M)$. One says that F is weakly $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigsqcup_\alpha M_\alpha$ such that for each stratum M_α, the restriction $F|_{M_\alpha}$ is locally constant. If moreover, the stalk F_x is of finite rank for all $x \in M$, then one says that F is $\mathbb{R}$-constructible.

Notation 1.2. (i) One denotes by $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves and by $\mathrm{Mod}_{\mathbb{R}\mathrm{c,c}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ consisting of sheaves with compact support. Both are thick abelian subcategories of $\mathrm{Mod}(\mathbf{k}_M)$.
(ii) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D^b}(\mathbf{k}_M)$ consisting of sheaves with $\mathbb{R}$-constructible cohomology and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c,c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ consisting of sheaves with compact support.

A theorem of [Kas84] (see also [KS90, Th. 8.4.5]) asserts that the natural functor $\mathrm{D^b}(\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ is an equivalence of categories.

When $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$, $\mu\mathrm{supp}(F)$ and $\mathrm{Sing}(F)$ are subanalytic. The first result is proved in loc. cit. and the second one follows from $\mathrm{Sing}(F) = \dot{\pi}(\mu\mathrm{supp}(F) \cap \dot{T}^*M)$. (Recall that $\dot{\pi}$ is the projection $\dot{T}^*M := T^*M \setminus M \to M$.)

[1] $\mu\mathrm{supp}(F)$ was denoted by $\mathrm{SS}(F)$ in [KS90].

1.2 γ-topology

The so-called γ-topology has been studied with some details in [KS90, § 3.4].

Let $\mathbb{V}$ be a finite-dimensional real vector space. We denote by s the addition map.

$$s\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}, \quad (x,y)\mapsto x+y,$$

and by $a\colon \mathbb{V}\to\mathbb{V}$ the antipodal map $x\mapsto -x$.

Hence, for two subsets A, B of $\mathbb{V}$, one has $A+B=s(A\times B)$. A subset A of V is called a cone if $0\in A$ and $\mathbb{R}_{>0}A\subset A$. A convex cone A is proper if $A\cap A^a=\{0\}$.

Throughout the paper, we consider a cone $\gamma\subset\mathbb{V}$ and we assume:

(1.3) *γ is closed proper convex with non-empty interior.*

Sometimes we shall make the extra assumption that γ is *polyhedral*, meaning that it is a finite intersection of closed half-spaces.

We say that a subset A of $\mathbb{V}$ is *γ-invariant* if $A+\gamma=A$. Note that a subset A is γ-invariant if and only if $\mathbb{V}\setminus A$ is γ^a-invariant.

The family of γ-invariant open subsets of $\mathbb{V}$ defines a topology, which is called the *γ-topology* on $\mathbb{V}$. One denotes by $\mathbb{V}_\gamma$ the space $\mathbb{V}$ endowed with the γ-topology and one denotes by

$$\varphi_\gamma\colon \mathbb{V}\to\mathbb{V}_\gamma \tag{1.4}$$

the continuous map associated with the identity. Note that the closed sets for this topology are the γ^a-invariant closed subsets of $\mathbb{V}$.

Definition 1.3. Let A be a subset of $\mathbb{V}$.

(a) One says that A is *γ-open* (resp. *γ-closed*), if A is open (resp. closed) for the γ-topology.

(b) One says that A is *γ-locally closed* if A is the intersection of a γ-open subset and a γ-closed subset.

(c) One says that A is *γ-flat* if $A=(A+\gamma)\cap(A+\gamma^a)$.

(d) One says that a closed set A is *γ-proper* if the map s is proper on $A\times\gamma^a$.

Remark 1.4. (i) A closed subset A is γ-proper if and only if $A\cap(x+\gamma)$ is compact for any $x\in\mathbb{V}$.

(ii) Let A be a subset of $\mathbb{V}$ and assume that $\overline{A}$ is γ-proper. Then $\overline{A+\gamma^a}=\overline{A}+\gamma^a$.

(iii) If A is closed and if there exist a closed convex proper cone γ_1 with $\gamma\subset\operatorname{Int}(\gamma_1)\cup\{0\}$ and $x\in\mathbb{V}$ such that $A\cap(x+\gamma_1)=\varnothing$, then A is γ-proper.

(iv) One has $\operatorname{Int}(\gamma)=\operatorname{Int}(\gamma)+\gamma$ and $\overline{\operatorname{Int}(\gamma)}=\gamma$.

We shall use the notations:

$$
(1.5)\qquad \begin{cases} \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}); \mu\mathrm{supp}(F) \subset \mathbb{V} \times \gamma^{\circ a}\}, \\ \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}). \end{cases}
$$

We call an object of $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ *a γ-sheaf.*

It follows from [KS90, Prop. 5.4.14] that for $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and $H \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ}}(\mathbf{k}_{\mathbb{V}})$, the sheaves $F \otimes G$ and $\mathrm{R}\mathscr{H}om(H, F)$ belong to $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$.

The next result is implicitly proved in [KS90], without assuming that $\mathrm{Int}(\gamma)$ is non empty.

Theorem 1.5. *Let γ be a closed convex proper cone in $\mathbb{V}$. The functor $\mathrm{R}\varphi_{\gamma_*}\colon \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of triangulated categories with quasi-inverse φ_γ^{-1}.*

Proof. (i) The proof of [KS90, Prop. 5.2.3] shows that $\varphi_\gamma^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma}) \to \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ is well defined. The same statement asserts that for $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$, one has $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \xrightarrow{\sim} F$.
(ii) By [KS90, Prop. 3.5.3 (iii)], for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$, one has $F \xrightarrow{\sim} \mathrm{R}\varphi_{\gamma_*}\varphi_\gamma^{-1}F$. □

Corollary 1.6. *The functor $\varphi_{\gamma_*}\colon \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of abelian categories with quasi-inverse φ_γ^{-1}.*

Proof. (i) By Theorem 1.5, the functor $\varphi_\gamma^{-1}\colon \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma}) \to \mathrm{Mod}(\mathbf{k}_{\mathbb{V}})$ takes its values in $\mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. Moreover, the same statement asserts that for $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$, $F \xrightarrow{\sim} \mathrm{R}\varphi_{\gamma_*}\varphi_\gamma^{-1}F$. Therefore, $F \xrightarrow{\sim} \varphi_{\gamma_*}\varphi_\gamma^{-1}F$.
(ii) By taking the 0-th cohomology of the isomorphism $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \xrightarrow{\sim} F$ and using the fact that φ_γ^{-1} commutes with H^0, we get the isomorphism $\varphi_\gamma^{-1}\varphi_{\gamma_*}F \xrightarrow{\sim} F$. □

Corollary 1.7. *The equivalence of categories in* Theorem 1.5 *preserves the natural t-structures of both categories. In particular, for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$, the condition $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ is equivalent to the condition: $\mu\mathrm{supp}(H^j(F)) \subset \gamma^{\circ a}$ for any $j \in \mathbb{Z}$.*

Proof. This follows from Corollary 1.6. □

Corollary 1.8. *Let A be a γ-locally closed subset of $\mathbb{V}$. Then $\mu\mathrm{supp}(\mathbf{k}_A) \subset \mathbb{V} \times \gamma^{\circ a}$.*

Proof. The subset A is locally closed in V_γ. Let us denote by $\mathbf{k}_{A,\gamma} \in \mathrm{Mod}(\mathbf{k}_{V_\gamma})$ the constant sheaf supported on A. Then we have $\mathbf{k}_A \simeq \varphi_\gamma^{-1}(\mathbf{k}_{A,\gamma})$. □

1.3 Persistent homology

Let $\mathbb{V}$ be a real vector space and let γ be a cone satisfying hypothesis (1.3). We also assume that γ is subanalytic.

Let M be a real analytic manifold and let $f\colon M \to \mathbb{V}$ be a continuous subanalytic map. We denote by $\Gamma_f^+ \subset M \times \mathbb{V}$ the γ-epigraph of f.

$$
\begin{aligned} \Gamma_f^+ &= \{(x, y) \in M \times \mathbb{V}; f(x) - y \in \gamma\} \\ &= \Gamma_f + \gamma^a. \end{aligned}
$$

We denote by $p\colon M \times \mathbb{V} \to \mathbb{V}$ the projection.

Lemma 1.9. *One has* $\mu\mathrm{supp}(\mathbf{k}_{\Gamma_f^+}) \subset T^*M \times (\mathbb{V} \times \gamma^{\circ a})$.

Proof. The set Γ_f^+ being γ-closed, the result follows from Corollary 1.8. ☐

Theorem 1.10. *Let M be a real analytic manifold and let $f\colon M \to \mathbb{V}$ be a continuous subanalytic map. Assume that*

(1.6) *for each $K \subset \mathbb{V}$ compact, the set $\{x \in M; f(x) \in K + \gamma\}$ is compact.*

Then $\mathrm{R}p_*\mathbf{k}_{\Gamma_f^+}$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ\mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$.

Proof. Let $K \subset \mathbb{V}$ be a compact subset. Then

$$\begin{aligned}\{(x,y) \in \Gamma_f^+; y \in K\} &= \{(x,y) \in M \times \mathbb{V}; f(x) \in y + \gamma, y \in K\}\\ &\subset \{x \in M; f(x) \in K + \gamma\} \times K.\end{aligned}$$

Hence, the map p is proper on Γ_f^+.

Applying [KS90, Prop. 5.4.4], we get that $\mathrm{R}p_*\mathbf{k}_{\Gamma_f^+}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. Moreover, this object is $\mathbb{R}$-constructible by loc. cit. Prop. 8.4.8. ☐

Example 1.11. Let M and f be as above with $\mathbb{V} = \mathbb{R}$ and $\gamma = \{t \leq 0\}$. In this case, $\Gamma_f^+ = \{(x,t) \in M \times \mathbb{R}; f(x) \leq t\}$ is the epigraph of f. Hypothesis (1.6) is translated as:

(1.7) for each $t \in \mathbb{R}$, the set $\{x \in M; f(x) \leq t\}$ is compact.

Set $K = \{x \in M; f(x) \leq 0\}$. By the hypothesis, K is compact. Let $a = \inf_{x \in K} f(x)$. Then $f(x) \geq a$ for all $x \in M$.

Example 1.12. Let $\mathbb{V} = \mathbb{R}^n$, $\gamma = (\mathbb{R}_{\leq 0})^n$ and let $f = (f_1, \cdots, f_n)$, each f_i being continuous and subanalytic and satisfying hypothesis (1.7). Then f satisfies the hypothesis of Theorem 1.10. Moreover, in this case, $\mathrm{supp}(\mathrm{R}p_*\mathbf{k}_{\Gamma_f^+}) \subset y + \gamma^a$ for some $y \in \mathbb{V}$.

The aim of this paper is to describe the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ\mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$. We first treat the case of the dimension one, where things are particularly simple.

1.4 The case of dimension one

In this subsection, we shall study the category of γ-sheaves and construct a category of barcodes in dimension one, proving the equivalence of these categories in Theorem 1.22. Note that various constructions of categories of barcodes already exist in the literature. See in particular [BS14, BdSS15, BL16].

We denote by t a coordinate on $\mathbb{R}$ and by $(t;\tau)$ the associated homogeneous coordinates on $T^*\mathbb{R}$. Therefore, $F \in \mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{R}})$ if there exists a discrete set $Z \subset \mathbb{R}$ such that F is locally constant on $\mathbb{R} \setminus Z$ and moreover, the stalk of F at each point of $\mathbb{R}$ is finite-dimensional.

The next result is an important theorem of Stéphane Guillermou who deduces it from a theorem of Gabriel on the representations of quivers.

In the sequel, an interval means a non-empty convex subset of $\mathbb{R}$.

Theorem 1.13 ([Gui16, Cor. 7.3])**.** *Let $F \in \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$ and assume that F has compact support. Then, there exist a finite set A and a family of intervals $\{I_\alpha\}_{\alpha\in A}$ such that $F \simeq \bigoplus_{\alpha\in A} \mathbf{k}_{I_\alpha}$. Moreover such a decomposition is unique.*

We shall extend Guillermou's theorem to the non compact case. Note that Theorem 1.14 below is very similar to a theorem of Crawley-Boevey [CB14, Theorem 1.1].

Theorem 1.14. *Let $F \in \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then, there exists a locally finite family of intervals $\{I_\alpha\}_{\alpha\in A}$ such that $F \simeq \bigoplus_{\alpha\in A} \mathbf{k}_{I_\alpha}$. Moreover such a decomposition is unique.*

Proof. For $n \in \mathbb{Z}_{>0}$, set $U_n =]-n, n[$ and $F_n = F \otimes \mathbf{k}_{U_n}$. Then by the theorem above, there exists a finite family $\{I_\alpha^{(n)}\}_{\alpha\in A_n}$ of intervals in U_n such that $F_n \simeq \oplus_{\alpha\in A_n}\mathbf{k}_{I_\alpha^{(n)}}$. Then there exists an injective map $A_n \rightarrowtail A_{n+1}$ (hereafter we identify A_n as a subset of A_{n+1} by this injective map) such that $I_\alpha^{(n)} = I_\alpha^{(n+1)} \cap U_n$ for $\alpha \in A_n$. Set $A = \bigcup_{n\in\mathbb{Z}_{>0}} A_n$, Then, for any $\alpha \in A$, there exists a unique interval I_α such that $I_\alpha \cap U_n = I_\alpha^{(n)}$ for any n such that $\alpha \in A_n$. Then $\{I_\alpha\}_{\alpha\in A}$ is a locally finite family of intervals. Note that $A_n = \{\alpha \in A\,;\, I_\alpha \cap U_n \neq \varnothing\}$ and $I_\alpha^{(n)} = I_\alpha \cap U_n$ for $\alpha \in A_n$.

Let us show that F is isomorphic to $\bigoplus_{\alpha\in A} \mathbf{k}_{I_\alpha}$. For any n, there exists an isomorphism

$$\varphi_n\colon F|_{U_n} \xrightarrow{\sim} \bigoplus_{\alpha\in A} \mathbf{k}_{I_\alpha}|_{U_n} \simeq \bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_n}.$$

The restriction map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{U_m})}\big(\mathbf{k}_{I_\alpha}|_{U_m}, \mathbf{k}_{I_\beta}|_{U_m}\big) \to \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{U_n})}\big(\mathbf{k}_{I_\alpha}|_{U_n}, \mathbf{k}_{I_\beta}|_{U_n}\big)$$

is injective for $m \geq n$ and $\alpha, \beta \in A_n$. Indeed, the injectivity follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{U_m})}\big(\mathbf{k}_{I_\alpha}|_{U_m}, \mathbf{k}_{I_\beta}|_{U_m}\big) & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{U_n})}\big(\mathbf{k}_{I_\alpha}|_{U_n}, \mathbf{k}_{I_\beta}|_{U_n}\big) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{I_\alpha\cap I_\beta\cap U_m})}\big(\mathbf{k}_{I_\alpha\cap I_\beta\cap U_m}, \mathbf{k}_{I_\alpha\cap I_\beta\cap U_m}\big) & \rightarrowtail & \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_{I_\alpha\cap I_\beta\cap U_n})}\big(\mathbf{k}_{I_\alpha\cap I_\beta\cap U_n}, \mathbf{k}_{I_\alpha\cap I_\beta\cap U_n}\big)
\end{array}$$

Here the injectivity of the bottom arrows follows from the fact that $I_\alpha \cap I_\beta \cap U_m$ is empty if $I_\alpha \cap I_\beta \cap U_n$ is empty.

Hence the restriction map

$$\mathrm{End}_{\mathrm{Mod}(\mathbf{k}_{U_m})}\big(\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_m}\big) \longrightarrow \mathrm{End}_{\mathrm{Mod}(\mathbf{k}_{U_n})}\big(\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_n}\big) \tag{1.8}$$

is injective for $m \geq n$. Therefore, if an automorphism f of $\bigoplus_{\alpha\in A} \mathbf{k}_{I_\alpha}|_{U_n} \simeq \bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_n}$, as well as f^{-1}, lifts to an endomorphism of $\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_m}$, then it lifts to an automorphism of $\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_m}$.

By the injectivity of (1.8) and the fact that $\dim\big(\mathrm{End}_{\mathrm{Mod}(\mathbf{k}_{U_n})}\big(\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_n}\big)\big) < \infty$, there exists $m \geq n$ such that

$$\mathrm{End}_{\mathrm{Mod}(\mathbf{k}_{U_k})}\big(\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_k}\big) \longrightarrow \mathrm{End}_{\mathrm{Mod}(\mathbf{k}_{U_m})}\big(\bigoplus_{\alpha\in A_n} \mathbf{k}_{I_\alpha}|_{U_m}\big)$$

is an isomorphism for any $k \geq m$.

Thus we conclude that the image $K_{k,n}$ of the restriction map

$$\mathrm{Aut}\,\big(\bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}|_{U_k}\big) \longrightarrow \mathrm{Aut}\,\big(\bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}|_{U_n}\big)$$

is equal to $K_{m,n}$ for any $k \geq m$. Let P_n be the set of isomorphisms $F|_{U_n} \xrightarrow{\sim} \bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}|_{U_n}$. Then $\{P_n\}_{n\in\mathbb{Z}_{>0}}$ is a projective system of non-empty sets. Moreover, for any n, there exists $m \geq n$ such that $\mathrm{Im}(P_k \to P_n) = \mathrm{Im}(P_m \to P_n)$ for any $k \geq m$. Set $\tilde{P}_n = \mathrm{Im}(P_m \to P_n) \subset P_n$. Then $\{\tilde{P}_n\}_{n\in\mathbb{Z}_{>0}}$ is a projective system of non-empty sets such that the map $\tilde{P}_m \to \tilde{P}_n$ is surjective for any $m \geq n$.

Hence, by replacing φ_n, we can choose $\varphi_n \in \tilde{P}_n$ inductively so that we have $\varphi_{n+1}|_{U_n} = \varphi_n$ for every n. Thus we conclude that $F \simeq \bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}$. ☐

Corollary 1.15. *Let $F, G \in \mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then $\mathrm{Ext}^j(F,G) = 0$ for $j > 1$.*

Proof. By Theorem 1.14, we may assume that $F = \bigoplus_{a\in A} F_a$ and $G = \prod_{b\in B} G_b$ with $F_a = \mathbf{k}_{I_a}$ and $G = \mathbf{k}_{J_b}$ where I_a and J_b are intervals. Since

$$\mathrm{Ext}^j\big(\bigoplus_{a\in A} F_a, \prod_{b\in B} G_b\big) \simeq \prod_{(a,b)\in A\times B} \mathrm{Ext}^j(F_a, G_b),$$

we are reduced to prove the result with F and G replaced with F_a and G_b and in this case, the result is obvious. ☐

Example 1.16. In Corollary 1.15, one cannot replace $j > 1$ with $j \geq 1$. Indeed, one has $\mathrm{Ext}^1(\mathbf{k}_{[0,+\infty[}, \mathbf{k}_{]-\infty,0[}) \simeq \mathbf{k}$. (See Lemma 3.13 for a generalization in higher dimension.)

Corollary 1.15 classically implies

Corollary 1.17. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}})$. Then $F \simeq \bigoplus_j H^j(F)\,[-j]$.*

Barcodes

We denote by $\mathbb{R}_\gamma$ the set $\mathbb{R}$ endowed with the γ-topology where $\gamma =]-\infty, 0]$. Hence, the γ-open sets are the open subsets $\mathbb{R}_{<t} :=]-\infty, t[$ and the γ-closed sets are the closed subsets $\mathbb{R}_{\geq t} := [t, +\infty[$, with $t \in \overline{\mathbb{R}}$. The non-empty γ-locally closed sets are the intervals $[a, b[$ with $-\infty \leq a < b \leq +\infty$.

Definition 1.18. A γ-*barcode* (A, I), or simply a barcode, is the data of a set of indices A and a family $I = \{I_\alpha\}_{\alpha\in A}$ of intervals $I_\alpha = [a_\alpha, b_\alpha[\subset \mathbb{R}$ with $-\infty \leq a_\alpha < b_\alpha \leq +\infty$, these data satisfying

(1.9) the family $\{I_\alpha\}_{\alpha\in A}$ is locally finite on $\mathbb{R}$, that is, for any compact subset K of $\mathbb{R}$, $\{\alpha \in A\,;\, I_\alpha \cap K \neq \varnothing\}$ is finite.

In particular, A is countable. The support of the barcode (A, I), denoted by $\mathrm{supp}(A, I)$, is the closed set $\bigcup_{\alpha\in A}\overline{I_\alpha}$.

Note that we do not ask $I_\alpha \neq I_\beta$ for $\alpha \neq \beta$. Otherwise, we should have to endow each I_α with a multiplicity $m_\alpha \in \mathbb{N}$.

We shall now construct a category of γ-barcodes.

Let us say that a barcode (A, I) is *elementary* if $A \simeq \mathrm{pt}$. We shall identify the elementary barcode (pt, I) with the interval I.

Given two elementary barcodes $[a, b[$ and $[c, d[$, we set

$$\mathrm{Hom}_{\mathbf{Bar}_\gamma}([a, b[, [c, d[) = \begin{cases} \mathbf{k} & \text{if } a \leq c < b \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Note that

$$\mathrm{Hom}_{\mathbf{Bar}_\gamma}([a, b[, [c, d[) \simeq \mathrm{Hom}\,(\mathbf{k}_{[a,b[}, \mathbf{k}_{[c,d[}). \tag{1.11}$$

Given two barcodes (A, I) and (B, J) with $I_\alpha = [a_\alpha, b_\alpha[$ and $J_\beta = [c_\beta, d_\beta[$, we set

$$\mathrm{Hom}_{\mathbf{Bar}_\gamma}((A, I), (B, J)) = \prod_{(\alpha,\beta) \in A \times B} \mathrm{Hom}_{\mathbf{Bar}_\gamma}([a_\alpha, b_\alpha[, [c_\beta, d_\beta[). \tag{1.12}$$

For $u \in \mathrm{Hom}_{\mathbf{Bar}_\gamma}((A, I), (B, J))$ and $v \in \mathrm{Hom}_{\mathbf{Bar}_\gamma}((B, J), (C, K))$, the composition $v \circ u$ is defined as follows. Let $u = \{c_{\alpha,\beta}\}_{(\alpha,\beta) \in A \times B}$ and $v = \{c_{\beta,\gamma}\}_{(\beta,\gamma) \in B \times C}$ with $c_{\alpha,\beta}, c_{\beta,\gamma} \in \mathbf{k}$. One sets $v \circ u = \{c_{\alpha,\gamma}\}_{(\alpha,\gamma) \in A \times C}$ with

$$c_{\alpha,\gamma} = \sum_{\beta \in B} c_{\alpha,\beta} \cdot c_{\beta,\gamma} \quad \text{if } \mathrm{Hom}_{\mathbf{Bar}_\gamma}(I_\alpha, K_\gamma) = \mathbf{k}. \tag{1.13}$$

For a given (α, γ), the sum in (1.13) is finite. Indeed, given $[a, b[$ and $[e, f[$, consider the intervals $[c_\beta, d_\beta[$ with $-\infty \leq a \leq c_\beta < b \leq d_\beta \leq +\infty$ and $-\infty \leq c_\beta \leq e < d_\beta \leq f \leq +\infty$. If $b < +\infty$ or $-\infty < e$ then either b or e belongs to $[c_\beta, d_\beta[$ and the set of such β is finite thanks to (1.9). Now assume that $b = +\infty$ and $-\infty = e$. Then $c_\beta = -\infty$ and $d_\beta = +\infty$ and again the family of such β must be finite.

Notation 1.19. We denote by $\mathbf{Bar}_\gamma$ the category constructed above and call it the category of γ-barcodes.

Remark 1.20. The category $\mathbf{Bar}_\gamma$ contains much more morphisms than the category constructed in [BL16].

Lemma 1.21. *The category* $\mathbf{Bar}_\gamma$ *is additive.*

Proof. (i) The 0-barcode is $0 = (A, I)$ with $A = \varnothing$.
(ii) Given two barcodes (A, I) and (B, J), we set $(A, I) \oplus (B, J) = (A \sqcup B, I \sqcup J)$. $\square$

Then we define a functor $\Psi\colon \mathbf{Bar}_\gamma \to \mathrm{Mod}_{\mathbb{R}_{c,\gamma^{\circ a}}}(\mathbf{k}_\mathbb{R})$ as follows. We set

$$\Psi(A, I) = \bigoplus_{\alpha \in A} \mathbf{k}_{I_\alpha}. \tag{1.14}$$

By (1.11), both $\mathrm{Hom}_{\mathbf{Bar}_\gamma}([a,b[,[c,d[)$ and $\mathrm{Hom}\,(\mathbf{k}_{[a,b[},\mathbf{k}_{[c,d[})$ are simultaneously $\mathbf{k}$ or 0. Hence, we define $\Psi\colon \mathrm{Hom}_{\mathbf{Bar}_\gamma}([a,b[,[c,d[) \to \mathrm{Hom}\,(\mathbf{k}_{[a,b[},\mathbf{k}_{[c,d[})$ as the identity of $\mathbf{k}$.

Note that Ψ commutes with the composition of morphisms.

Then we extend Ψ by linearity:

$$\begin{aligned}\Psi\;:\;\mathrm{Hom}_{\mathbf{Bar}_\gamma}((A,I),(B,J)) &= \prod_{\alpha\in A,\beta\in B}\mathrm{Hom}_{\mathbf{Bar}_\gamma}(I_\alpha,J_\beta)\\ &\xrightarrow{\sim}\prod_{\alpha\in A,\beta\in B}\mathrm{Hom}\left(\mathbf{k}_{I_\alpha}\mathbf{k}_{J_\beta}\right)\\ &\simeq \mathrm{Hom}\,(\bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha},\bigoplus_{\beta\in B}\mathbf{k}_{J_\beta})\,.\end{aligned}$$

Here, we have used the fact that the sum is locally finite.

This construction shows that Ψ is additive and fully faithful.

Theorem 1.22. *The functor $\Psi\colon \mathbf{Bar}_\gamma \to \mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_\mathbb{R})$ is an equivalence of additive categories.*

Proof. By Theorem 1.14, there exist a locally finite family of intervals $\{I_\alpha\}_{\alpha\in A}$ such that $F\simeq\bigoplus_{\alpha\in A}\mathbf{k}_{I_\alpha}$. Since $\mu\mathrm{supp}(\mathbf{k}_{I_\alpha})\subset\mathbb{R}\times\gamma^{\circ a}$, the intervals I_α are of the type $[a,b[$ with $-\infty\le a<b\le+\infty$. ☐

Note that

$$\mathrm{supp}(\Psi(A,I)) = \mathrm{supp}(A,I) \quad \text{for a barcode } (A,I). \tag{1.15}$$

Let us denote by Ψ^{-1} a quasi-inverse of the functor Ψ.

Definition 1.23. The barcode functor $\mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_\mathbb{R}) \xrightarrow{\sim} \mathbf{Bar}_\gamma$ is the functor Ψ^{-1}.

Definition 1.23 extends to the derived category. Denote by $H^\bullet$ the functor $F\mapsto \bigoplus_j H^j(F)$ from $\mathrm{D^b}(\mathbf{k}_\mathbb{R})$ to $(\mathrm{Mod}(\mathbf{k}_\mathbb{R}))^{(\mathbb{Z})}$. One still calls

$$\Psi^{-1}\circ H^\bullet : \mathrm{D}^{\mathrm{b}}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_\mathbb{R}) \to \mathbf{Bar}_\gamma^{(\mathbb{Z})}$$

the barcode functor .

Example 1.24. Let M be a real analytic manifold endowed with a subanalytic distance d, let $K\subset M$ be a compact subanalytic subset and let $f(x)=d(x,K)$. Then

$$\Gamma_f^+ = \{(x,t); d(x,K)\le t\}$$

and f satisfies hypothesis 1.6. The barcodes of K are those of $\mathrm{R}f_*\mathbf{k}_{\Gamma_f^+}$, that is, the image of $\mathrm{R}f_*\mathbf{k}_{\Gamma_f^+}$ by the functor $\Psi^{-1}\circ H^\bullet$ of Definition 1.23.

A natural question is to extend the definition of barcodes to $\mathbb{R}^n$ when $n>1$. For that purpose, it is natural to replace the cone $\{t\le 0\}$ of $\mathbb{R}$ with a closed convex proper cone γ of $\mathbb{V}$ as in subsection 1.2.

2 Sheaves on a vector space

In this section, we denote by $\mathbb{V}$ a real vector space of finite dimension n.

We endow $\mathbb{V}$ with an Euclidean structure and denote by $\|\cdot\|$ the norm on $\mathbb{V}$. Let us denote the closed ball with the origin as the center and radius $a \geq 0$ by

$$B_a := \{x \in \mathbb{V}\,;\, \|x\| \leq a\}\,.$$

2.1 Convolution

References for this subsection are made to [Tam08, GS14].

Consider the maps

$$\begin{aligned}
&s\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}, \quad s(x,y)=x+y,\\
&q_i\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}\ (i=1,2), \quad q_1(x,y)=x,\ q_2(x,y)=y.
\end{aligned}$$

Recall that $a\colon \mathbb{V}\to\mathbb{V}$ denotes the antipodal map, $x\mapsto -x$. For a sheaf F, we set for short $F^a = a^{-1}F$.

One defines the functor of convolution $\star\colon \mathrm{D^b}(\mathbf{k}_\mathbb{V})\times\mathrm{D^b}(\mathbf{k}_\mathbb{V})\to\mathrm{D^b}(\mathbf{k}_\mathbb{V})$ and its adjoint $\mathscr{H}om^\star\colon (\mathrm{D^b}(\mathbf{k}_\mathbb{V}))^{\mathrm{op}}\times\mathrm{D^b}(\mathbf{k}_\mathbb{V})\to\mathrm{D^b}(\mathbf{k}_\mathbb{V})$ by the formulas:

$$\begin{aligned}
F\star G &:= \mathrm{R}s_!(F\boxtimes G),\\
\mathscr{H}om^\star(F,G) &:= \mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,(q_2^{-1}G, s^!F)\\
&\simeq \mathrm{R}s_*\mathrm{R}\mathscr{H}om\,(q_2^{-1}(G^a), q_1^!F).
\end{aligned}$$

For $F_1, F_2, F_3 \in \mathrm{D^b}(\mathbf{k}_\mathbb{V})$, we have

$$\mathrm{RHom}\,(F_1\star F_2, F_3)\simeq \mathrm{RHom}\,(F_1, \mathscr{H}om^\star(F_2,F_3)). \tag{2.1}$$

With $\star$ as a tensor product, $\mathrm{D^b}(\mathbf{k}_\mathbb{V})$ is a commutative monoidal category with a unit object $\mathbf{k}_{\{0\}}$: $\mathbf{k}_{\{0\}}\star F\simeq F$ functorially in $F\in\mathrm{D^b}(\mathbf{k}_\mathbb{V})$.

The functors $\star$ and $\mathscr{H}om^\star$ induce functors (see [GS14, Cor. 3.1.4])

$$\begin{aligned}
\bullet\star\bullet &:\ \mathrm{D^b}(\mathbf{k}_\mathbb{V})\times\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V})\to\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V}),\\
\mathscr{H}om^\star(\,\bullet\,,\,\bullet\,) &:\ (\mathrm{D^b}(\mathbf{k}_\mathbb{V}))^{\mathrm{op}}\times\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V})\to\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V}),\\
&:\ (\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V}))^{\mathrm{op}}\times\mathrm{D^b}(\mathbf{k}_\mathbb{V})\to\mathrm{D^b}_{\gamma^\circ a}(\mathbf{k}_\mathbb{V}).
\end{aligned}$$

We set $X=\mathbb{R}\times\mathbb{V}$ and denote by t the coordinate on $\mathbb{R}$. Following [GKS12, Exa 3.10] we recall that there exists $K\in\mathrm{D^b_{\mathbb{R}c}}(\mathbf{k}_X)$ unique up to isomorphism such that

$$\begin{aligned}
&\mu\mathrm{supp}(K)\subset\{(t,x;\tau,\xi);\tau=\|\xi\|\neq 0,\ x=-t\tau^{-1}\xi\}\cup\{\tau=\xi=0\},\\
&K|_{t=0}\simeq\mathbf{k}_{\{0\}}.
\end{aligned}$$

Moreover, there is a distinguished triangle

$$\mathbf{k}_{\{t<-\|x\|\}}\,[n]\to K\to\mathbf{k}_{\{\|x\|\leq t\}}\xrightarrow{+1}.$$

Set $K_a = K|_{t=a}$. Hence

$$K_a \simeq \begin{cases} \mathbf{k}_{\{\|x\| \leq a\}} & \text{for } a \geq 0, \\ \mathbf{k}_{\{\|x\| < -a\}}[n] & \text{for } a < 0. \end{cases}$$

We can easily check the isomorphism

$$K_a \star K_b \simeq K_{a+b}. \tag{2.2}$$

Hence, $K_a\star$ is an auto-equivalence of the category $\mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ as well as of the category $\mathrm{D^b_{\mathbb{R}c,\gamma^{\circ a}}}(\mathbf{k}_{\mathbb{V}})$. A quasi-inverse is given by $K_{-a}\star$.

Note that

$$\mathrm{Hom}\,(K_a \star F, G) \simeq \mathrm{Hom}\,(F, K_{-a} \star G). \tag{2.3}$$

and thus

$$K_a \star F \simeq \mathscr{H}om^{\star}(K_{-a}, F).$$

For $c \geq 0$ we have a canonical morphism $K_c \to \mathbf{k}_{\{0\}}$, which induces canonical morphisms:

$$\chi_{b,a} : \; K_a \longrightarrow K_b,$$
$$\chi_{b,a} \star F : \; K_a \star F \to K_b \star F$$

for $a \geq b$. In particular, one has

$$K_a \star F \xrightarrow{\chi_{0,a} \star F} F \xrightarrow{\chi_{b,0} \star F} K_b \star F \quad \text{for } a \geq 0 \geq b. \tag{2.4}$$

Moreover, one has (recall that $\mathrm{D}_{\mathbb{V}}$ denotes the duality functor, see (1.2)) :

$$\mathrm{D}_{\mathbb{V}}(K_a) \simeq K_{-a}. \tag{2.5}$$

Lemma 2.1. *For $F \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$, one has*

$$\mathrm{D}_{\mathbb{V}}(K_a \star F) \simeq K_{-a} \star \mathrm{D}_{\mathbb{V}}(F). \tag{2.6}$$

Proof. One has

$$\begin{aligned} \mathrm{D}_{\mathbb{V}}(K_a \star F) &\simeq \mathrm{R}\mathscr{H}om\,(\mathrm{R}s_!(K_a \boxtimes F), \omega_{\mathbb{V}}) \\ &\simeq \mathrm{R}s_*\mathrm{R}\mathscr{H}om\,(K_a \boxtimes F, s^!\omega_{\mathbb{V}}) \\ &\simeq \mathrm{R}s_*\mathrm{R}\mathscr{H}om\,(K_a \boxtimes F, \omega_{\mathbb{V}\times\mathbb{V}}) \\ &\simeq \mathrm{R}s_*(\mathrm{D}_{\mathbb{V}}(K_a) \boxtimes \mathrm{D}_{\mathbb{V}}F) \simeq K_{-a} \star \mathrm{D}_{\mathbb{V}}F. \end{aligned}$$

Note that the last isomorphism follows from the fact that K_a has a compact support. □

2.2 Distance and the stability theorem

There is a classical distance on barcodes introduced first in [CCSG⁺09] (see also [CdSGO16]), recently generalized to higher dimension in [Les15, LW15].

We shall interpret it in the language of sheaves, replacing $(\mathbb{R}, \leq)$ with $(\mathbb{V}, \gamma)$. Note that our construction makes an essential use of the language of derived categories.

Definition 2.2. Let $F, G \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ and let $a \geq 0$. One says that F and G are *a-isomorphic* if there are morphisms $f\colon K_a \star F \to G$ and $g\colon K_a \star G \to F$ which satisfies the following compatibility conditions: the composition $K_{2a} \star F \xrightarrow{K_a \star f} K_a \star G \xrightarrow{g} F$ coincides with the natural morphism $\chi_{0,2a} \star F\colon K_{2a} \star F \to F$ and the composition $K_{2a} \star G \xrightarrow{K_a \star g} K_a \star F \xrightarrow{f} G$ coincides with the natural morphism $\chi_{0,2a} \star G\colon K_{2a} \star G \to G$.

Note that if F and G are a-isomorphic, then they are b-isomorphic for any $b \geq a$.
One sets

$$\mathrm{dist}(F, G) = \inf\Big(\{+\infty\} \cup \{a \in \mathbb{R}_{\geq 0}\,;\, F \text{ and } G \text{ are } a\text{-isomorphic}\}\Big)$$

and calls $\mathrm{dist}(\,\bullet\,,\,\bullet\,)$ the *convolution distance*.

Note that for $F, G, H \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$,

- F and G are 0-isomorphic if and only if $F \simeq G$,
- $\mathrm{dist}(F, G) = \mathrm{dist}(G, F)$,
- $\mathrm{dist}(F, G) \leq \mathrm{dist}(F, H) + \mathrm{dist}(H, G)$.

Remark 2.3. We don't know if F and G are a-isomorphic when $\mathrm{dist}(F, G) \leq a$.

Example 2.4. Let $F \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ and assume that $\mathrm{supp}(F) \subset B_a$. Set $L := \mathrm{R}\Gamma(\mathbb{V}; F)$. Recall that, for a closed subset S of $\mathbb{V}$, one denotes by $L_S \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ the constant sheaf with stalk L on S extended by 0 on $\mathbb{V} \setminus S$. Let $G := L_{\{0\}}$. We have

$$K_a \star G \simeq L_{B_a}.$$

Denote by $q\colon \mathbb{V} \to \mathrm{pt}$ the unique map from $\mathbb{V}$ to pt. The morphism $q^{-1}\mathrm{R}q_* F \to F$ defines the map $L_{\mathbb{V}} \to F$ and F being supported by B_a, we get the morphism $g\colon K_a \star G \simeq L_{B_a} \to F$. On the other hand,we have $(K_a \star F)_0 \simeq L$ which defines $f\colon K_a \star F \to G$. One easily checks that f and g satisfy the compatibility conditions in Definition 2.2. Therefore

$$\mathrm{dist}(F, L_{\{0\}}) \leq a.$$

In particular, a non-zero object can be a-isomorphic to the zero object.

Remark 2.5. (i) If $\mathrm{dist}(F,G) < +\infty$, then we have

$$\mathrm{R}\Gamma(\mathbb{V};F) \simeq \mathrm{R}\Gamma(\mathbb{V};G) \quad \text{and} \quad \mathrm{R}\Gamma_{\mathrm{c}}(\mathbb{V};F) \simeq \mathrm{R}\Gamma_{\mathrm{c}}(\mathbb{V};G).$$

This follows from $\mathrm{R}\Gamma(\mathbb{V};K_a\star F) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbb{V};F)$ and $\mathrm{R}\Gamma_{\mathrm{c}}(\mathbb{V};K_a\star F) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{c}}(\mathbb{V};F)$ for $a \geq 0$.

Together with Example 2.4, for $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ with $\mathrm{supp}(F)$, $\mathrm{supp}(G) \subset B_a$, the condition $\mathrm{dist}(F,G) \leq 2a$ holds if and only if $\mathrm{R}\Gamma(\mathbb{V};F) \simeq \mathrm{R}\Gamma(\mathbb{V};G)$.

(ii) Let A and B be closed convex subsets of $\mathbb{V}$. Then $\mathrm{dist}(\mathbf{k}_A, \mathbf{k}_B) \leq a$ if and only if $A \subset B + B_a$ and $B \subset A + B_a$.

Indeed, we may assume that A and B are non-empty. If $\mathbf{k}_A$ and $\mathbf{k}_B$ are a-isomorphic, then there exists a non-zero morphism $\mathbf{k}_{A+B_a} \simeq K_a \star \mathbf{k}_A \to \mathbf{k}_B$. Hence, we have $A + B_a \supset B$. Similarly we have $B + B_a \supset A$. The converse implication can be proved similarly.

(iii) Let γ and γ' be closed convex cones. If $\mathrm{dist}(\mathbf{k}_\gamma, \mathbf{k}_{\gamma'}) < +\infty$, then one has $\gamma = \gamma'$. This follows from (ii).

(iv) Let U and V be open convex subsets of $\mathbb{V}$. Then $\mathrm{dist}(\mathbf{k}_U, \mathbf{k}_V) \leq a$ if and only if $U \subset V + B_a$ and $V \subset U + B_a$.

This follows from (ii), $\mathrm{D}_{\mathbb{V}}(\mathbf{k}_U[n]) \simeq \mathbf{k}_{\overline{U}}$, $\mathrm{D}_{\mathbb{V}}(\mathbf{k}_{\overline{U}}) \simeq \mathbf{k}_U[n]$ and Proposition 2.6 (i) below.

Proposition 2.6. (i) *If* $\mathrm{dist}(F,G) \leq a$, *then* $\mathrm{dist}\big(\mathrm{D}_{\mathbb{V}}(F), \mathrm{D}_{\mathbb{V}}(G)\big) \leq a$.

(ii) *Assume that* $\mathrm{dist}(F_i, G_i) \leq a_i$ $(i = 1,2)$. *Then one has*

$$\mathrm{dist}(F_1\star F_2, G_1\star G_2) \leq a_1+a_2 \quad \textit{and} \quad \mathrm{dist}\big(\mathscr{H}om^\star(F_1,F_2), \mathscr{H}om^\star(G_1,G_2)\big) \leq a_1+a_2.$$

(iii) $\mathrm{dist}(\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F, \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}G) \leq \mathrm{dist}(F,G)$.

Proof. (i) follows from Lemma 2.1.
(ii) Let $f_i\colon K_{a_i} \star F_i \to G_i$ $(i = 1,2)$. We get a map

$$f_1 \star f_2\colon K_{a_1} \star F_1 \star K_{a_2} \star F_2 \to G_1 \star G_2$$

and $K_{a_1} \star F_1 \star K_{a_2} \star F_2 \simeq K_{a_1+a_2} \star (F_1 \star F_2)$. The end of the proof is straightforward and the case of $\mathscr{H}om^\star$ is similar.
(iii) follows from $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F \simeq \mathrm{R}s_*(\mathbf{k}_{\gamma^a} \boxtimes F) \simeq \mathscr{H}om^\star(\mathbf{k}_{\mathrm{Int}(\gamma)}[n], F)$ and (ii). ☐

For a set X and a map $f\colon X \to \mathbb{V}$, one sets

$$\|f\| = \sup_{x\in X} \|f(x)\|.$$

The next result should be compare to [Les15, Th 5.3]. Note however that the two results are different since our distance is defined using derived operations on sheaves.

Theorem 2.7 (The stability theorem)**.** *Let X be a locally compact space and let $f_1, f_2\colon X \to \mathbb{V}$ be two continuous maps. Then, for any $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we have*

$$\mathrm{dist}(\mathrm{R}f_{1*}F, \mathrm{R}f_{2*}F) \leq \|f_1 - f_2\| \quad \textit{and} \quad \mathrm{dist}(\mathrm{R}f_{1!}F, \mathrm{R}f_{2!}F) \leq \|f_1 - f_2\|.$$

Proof. (i) Since the proofs for $\mathrm{R}f_*$ and $\mathrm{R}f_!$ are similar, we shall only treat the case of $\mathrm{R}f_*$.
(ii) Set $a = \|f_1 - f_2\|$. Let us show that $\mathrm{R}f_{1*}F$ and $\mathrm{R}f_{2*}F$ are a-isomorphic. Let us introduce some notations. Let $p_1\colon \mathbb{V} \times X \to \mathbb{V}$ and $p_2\colon \mathbb{V} \times X \to X$ be the projections. We set

$$\begin{aligned} G_i &= \{(f_i(x), x) \in \mathbb{V} \times X \,;\, x \in X\}\,, \\ S_i &= \{(y, x) \in \mathbb{V} \times X \,;\, \|y - f_i(x)\| \leq a\}\,, \\ S_i' &= \{(y, x) \in \mathbb{V} \times X \,;\, \|y - f_i(x)\| \leq 2a\}\,. \end{aligned}$$

for $i = 1, 2$. Then we have

$$\begin{aligned} \mathrm{R}f_{i*}F &\simeq \mathrm{R}p_{1*}\big(\mathbf{k}_{G_i} \otimes p_2^{-1}F\big), \\ K_a \star \mathrm{R}f_{i*}F &\simeq \mathrm{R}p_{1*}\big(\mathbf{k}_{S_i} \otimes p_2^{-1}F\big), \\ K_{2a} \star \mathrm{R}f_{i*}F &\simeq \mathrm{R}p_{1*}\big(\mathbf{k}_{S_i'} \otimes p_2^{-1}F\big). \end{aligned}$$

Then $G_2 \subset S_1$ induces a morphism $\mathbf{k}_{S_1} \to \mathbf{k}_{G_2}$, which induces $K_a \star \mathrm{R}f_{1*}F \to \mathrm{R}f_{2*}F$. Similarly, we construct $K_a \star \mathrm{R}f_{2*}F \to \mathrm{R}f_{1*}F$. Then, the composition

$$K_{2a} \star \mathrm{R}f_{1*}F \to K_a \star \mathrm{R}f_{2*}F \to \mathrm{R}f_{1*}F$$

is the canonical one, since these morphisms are induced by

$$\mathbf{k}_{S_1'} \to \mathbf{k}_{S_2} \to \mathbf{k}_{G_1}.$$

The same argument holds when intertwining f_1 and f_2. □

2.3 Piecewise linear sheaves

A convex polytope P is, by definition, the intersection of a finite family of open or closed affine half-spaces.

Definition 2.8. One says that $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ is PL (piecewise linear) if there exists a locally finite family $\{P_a\}_{a \in A}$ of convex polytopes such that $\mathbb{V} = \bigcup_{a \in A} P_a$ and $F|_{P_a}$ is constant for any $a \in A$.

We shall use the notations:

$$\text{(2.7)} \qquad \begin{cases} \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}});\ F \text{ is PL}\}, \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}). \end{cases}$$

Since the following result is easy to prove, we omit the proof.

Theorem 2.9. *The category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ *is triangulated. Moreover:*

(i) *If F and F' are* PL*, then so are $F \otimes F'$ and* $\mathrm{R}\mathscr{H}om\,(F, F')$.

(ii) *Let $f \colon \mathbb{V} \to \mathbb{V}'$ be a linear map.*

(a) *If F' is a PL sheaf on $\mathbb{V}'$, then $f^{-1}F'$ is a* PL *sheaf on* $\mathbb{V}$.

(b) *If F is a* PL *sheaf on* $\mathbb{V}$ *and* $\mathrm{supp}(F)$ *is proper over* $\mathbb{V}'$*, then* $\mathrm{R}f_*F$ *is a* PL *sheaf on* $\mathbb{V}'$.

(iii) *Let γ be a cone as in* (1.3) *and assume that γ is polyhedral. If F is* PL *and* $\mathrm{supp}(F)$ *is γ-proper (see* Definition 1.3*), then* $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F$ *is* PL.

2.4 The approximation theorem

Theorem 2.10 (The approximation theorem)**.** *Let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$*. For each $\varepsilon > 0$ there exists* $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ *such that* $\mathrm{dist}(F, G) \le \varepsilon$ *and* $\mathrm{supp}(G) \subset \mathrm{supp}(F) + B_\varepsilon$.

Proof. We shall follow the notations of [KS90, § 8.1]. Recall that a simplicial complex $\mathbf{S} = (S, \Delta)$ is the data consisting of a set S and a set Δ of subsets of S, satisfying suitable conditions (see loc. cit. Def. 8.1.1). For $\sigma \in \Delta$, one sets

$$|\sigma| = \Big\{(x(p))_{p\in S} \in \mathbb{R}^S_{\ge 0}\,;\, \sum_{p\in S} x(p) = 1,\ x(p) = 0 \text{ for } p \notin \sigma \text{ and } x(p) > 0 \text{ for } p \in \sigma\Big\}.$$

Note that the $|\sigma|$'s are disjoint to each other. One also sets

$$|\mathbf{S}| = \bigcup_{\sigma\in\Delta} |\sigma|.$$

We endow $|\mathbf{S}|$ with the induced topology from $\mathbb{R}^S_{\ge 0}$.

There exist a simplicial complex $\mathbf{S} = (S, \Delta)$ and a homeomorphism $f \colon |\mathbf{S}| \xrightarrow{\sim} \mathbb{V}$ such that

$$(f^{-1}F)|_{|\sigma|} \text{ is constant for any } \sigma \in \Delta.$$

Replacing $\mathbf{S}$ with its successive barycentral subdivisions, we may assume further that $\|f(x) - f(y)\| \le \varepsilon$ for any $\sigma \in \Delta$ and $x, y \in |\sigma|$. Then we set

$$g(x) = \sum_{p\in S} x(p) f(p).$$

Here S is identified with a subset of $|\mathbf{S}|$ by $S \ni q \mapsto x(p) = \delta_{p,q} \in \mathbb{R}^S$.

The map $g \colon |\mathbf{S}| \to \mathbb{V}$ is piecewise linear and continuous and satisfies:

$$\|f(x) - g(x)\| \le \varepsilon.$$

Hence g is a proper map. Now we set $G = \mathrm{R}g_* f^{-1}F$. Then $\mathrm{supp}(G) \subset \mathrm{supp}(F) + B_\varepsilon$ and $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$, and since $F \simeq \mathrm{R}f_* f^{-1}F$, we have by Theorem 2.7

$$\mathrm{dist}(F, G) \le \varepsilon.$$

□

One can approximate any γ-sheaf with a PL-γ-sheaf. Indeed:

Corollary 2.11. *Assume that γ is polyhedral. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ such that* $\mathrm{supp}(F)$ *is γ-proper. For each $\varepsilon > 0$ there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ such that* $\mathrm{dist}(F, G) \leq \varepsilon$.

Proof. We shall apply Theorem 2.10. There exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ such that $\mathrm{dist}(F, G) \leq \varepsilon$ and $\mathrm{supp}(G) \subset \mathrm{Supp}(F) + B_\varepsilon$. Hence $\mathrm{supp}(G)$ is γ-proper. Then, $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ by Theorem 2.9. On the other hand, Proposition 2.6 (iii) implies that

$$\mathrm{dist}\big(\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F, \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G\big) \leq \mathrm{dist}(F, G) \leq \varepsilon.$$

Since F is a γ-sheaf, one has $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F \simeq F$. □

2.5 Barcodes (several-dimensional case)

Recall that a family of subsets $Z = \{Z_\alpha\}_{\alpha\in A}$ is locally finite if for any compact subset K of $\mathbb{V}$, the set $\{\alpha \in A\,;\, Z_\alpha \cap K \neq \varnothing\}$ is finite.

Definition 2.12. We assume that γ is polyhedral. A γ-*barcode* (A, Z) in $\mathbb{V}$, or simply, a barcode, is the data of a set of indices A and a family $Z = \{Z_\alpha\}_{\alpha\in A}$ of subsets of $\mathbb{V}$, these data satisfying

$$(2.8)\qquad \begin{cases} \text{(i) the family } Z = \{Z_\alpha\}_{\alpha\in A} \text{ is locally finite in } \mathbb{V}, \\ \text{(ii) the } Z_\alpha\text{'s are non-empty, } \gamma\text{-locally closed, convex polytopes.} \end{cases}$$

The support of the γ-barcode (A, Z), denoted by $\mathrm{supp}(A, Z)$, is the set $\bigcup_{\alpha\in A} \overline{Z_\alpha}$.

Let us say that a barcode (A, Z) is *elementary* if $A \simeq \mathrm{pt}$. We shall identify the elementary barcode (pt, Z) with the γ-locally closed convex polytope Z.

The barcodes are the objects of the additive category $\mathbf{Bar}_\gamma$ that we shall describe now.

- the zero-barcode is $0 = (A, Z)$ with $A = \varnothing$,
- for two barcodes (A, S) and (B, Z), we set $(A, S) \oplus (B, Z) = (A \sqcup B, S \sqcup Z)$. In other words, the sum of $\{S_\alpha\}_{\alpha\in A}$ and $\{Z_\beta\}_{\beta\in B}$ is the barcode $\{W_\gamma\}_{\gamma\in C}$ with $C = A \sqcup B$ and $W_\gamma = S_\alpha$ or $W_\gamma = Z_\beta$ according as $\gamma = \alpha \in A$ or $\gamma = \beta \in B$.
- For two elementary barcodes S and T one sets

  $$(2.9)\quad \mathrm{Hom}_{\mathbf{Bar}_\gamma}(S, T) = \begin{cases} \mathbf{k} & \text{if } S \cap T \text{ is non-empty, closed in } S \text{ and open in } T, \\ 0 & \text{otherwise.} \end{cases}$$

  Note that

  $$(2.10)\qquad \mathrm{Hom}_{\mathbf{Bar}_\gamma}(S, T) \simeq \mathrm{Hom}\,(\mathbf{k}_S, \mathbf{k}_T).$$

- One extends this construction to barcodes by linearity. For two barcodes $(A,S)=\{S_\alpha\}_{\alpha\in A}$ and $(B,T)=\{T_\beta\}_{\beta\in B}$, one sets

$$\begin{aligned}\operatorname{Hom}_{\mathbf{Bar}_\gamma}((A,S),(B,T)) &= \prod_{\alpha\in A,\beta\in B}\operatorname{Hom}_{\mathbf{Bar}_\gamma}(S_\alpha,T_\beta)\\ &\simeq \operatorname{Hom}\big(\bigoplus_{\alpha\in A}\mathbf{k}_{S_\alpha},\prod_{\beta\in B}\mathbf{k}_{T_\beta}\big)\\ &\simeq \operatorname{Hom}\big(\bigoplus_{\alpha\in A}\mathbf{k}_{S_\alpha},\bigoplus_{\beta\in B}\mathbf{k}_{T_\beta}\big). \end{aligned}\tag{2.11}$$

(The last isomorphism follows from the fact that the sum is locally finite, similarly to the one-dimension case.) The composition of $u\in\operatorname{Hom}_{\mathbf{Bar}_\gamma}((A,S),(B,T))$ and $v\in\operatorname{Hom}_{\mathbf{Bar}_\gamma}((B,T),(C,V))$ is defined similarly to the one-dimension case (see (1.13)), or by using (2.11).

There is a natural functor

$$\Psi\colon \mathbf{Bar}_\gamma\to \operatorname{Mod}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}),\quad Z=\{Z_\alpha\}_{\alpha\in A}\mapsto \bigoplus_{\alpha\in A}\mathbf{k}_{Z_\alpha}\tag{2.12}$$

and the functor Ψ is fully faithful in view of (2.11). We have seen that when $\dim\mathbb{V}=1$ the functor Ψ is an equivalence of categories. However, the functor Ψ is no more essentially surjective when $\dim\mathbb{V}>1$ as seen in the following examples.

Example 2.13. Denote by (x,y) the coordinates on $\mathbb{R}^2$ and consider the sets

$$\begin{aligned}&\gamma=\{(x,y)\,;\,x\le 0, y\le 0\},\\ &A=(1,0)+\gamma^a,\quad B=(0,1)+\gamma^a,\quad C=(2,2)+\gamma^a.\end{aligned}$$

Define the γ-sheaf F by the exact sequence $0\to F\to\mathbf{k}_A\oplus\mathbf{k}_B\to\mathbf{k}_C\to 0$. Then $F\simeq\mathbf{k}_A\oplus\mathbf{k}_B$ on $\mathbb{R}^2\setminus C$ and F has rank one on $\operatorname{Int}(C)$.

Let us show that $\operatorname{End}_{\operatorname{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{V}})}(F)\simeq\mathbf{k}$.
Let u and v be the generators of $\Gamma(\mathbb{V};\mathbf{k}_A)$ and $\Gamma(\mathbb{V};\mathbf{k}_B)$, respectively. Then one has $F(\mathbb{V})=\mathbf{k}(u-v)\subset\Gamma(\mathbb{V};\mathbf{k}_A\oplus\mathbf{k}_B)$. We have an exact sequence

$$0\longrightarrow\mathbf{k}_{A\setminus B}\to\mathbf{k}_{A\cup B}\oplus\mathbf{k}_{A\setminus C}\longrightarrow F\to 0.$$

Here, the composition $\mathbf{k}_{A\setminus C}\to F\to\mathbf{k}_B$ vanishes and the morphism $\mathbf{k}_{A\cup B}\to F$ is given by $u-v$. Let $f\in\operatorname{End}(F)$. We shall show $f\in\mathbf{k}\,\mathrm{id}_F$. We may assume that $f(u-v)=0$ from the beginning, i.e., the composition $g\colon\mathbf{k}_{A\cup B}\to F\xrightarrow{f}F$ vanishes. Then the composition $\mathbf{k}_{A\setminus C}\to F\xrightarrow{f}F\to\mathbf{k}_A$ vanishes, since it coincides with g on $A\setminus B$. The composition $\mathbf{k}_{A\setminus C}\to F\xrightarrow{f}F\to\mathbf{k}_B$ vanishes since $\operatorname{Hom}(\mathbf{k}_{A\setminus C},\mathbf{k}_B)\simeq 0$. Hence $f=0$.

Therefore F is indecomposable. Hence, F is not in the essential image of Ψ.

Example 2.14. Denote by (x,y,z) the coordinates on $\mathbb{R}^3$ and consider the sets

$$\begin{aligned}&\gamma=\{(x,y,z)\,;\,x\le-(|y|+|z|)\},\quad S=\{(x,y,z);x=0,|y|+|z|=1\},\\ &Z=(S+\gamma^a)\cap\{x<1\}.\end{aligned}$$

Then Z is a γ-barcode. Since $Z \cap \{(x,y,z)\,;\, y=z=0\} = \varnothing$, we consider the map $r\colon Z \to \mathbb{R}^3 \setminus \mathbb{R} \times \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{S}^1$. Then the composition $S \to Z \xrightarrow{r} \mathbb{S}^1$ is an isomorphism. Let L be a locally constant but non constant sheaf of rank 1 on $\mathbb{S}^1$. Then, the sheaf $F = r^{-1}L$ is locally isomorphic to $\mathbf{k}_Z$ but F is not isomorphic to $\mathbf{k}_Z$ since $r_*(F \otimes \mathbf{k}_S) \simeq L \not\simeq \mathbf{k}_{\mathbb{S}^1}$. Hence F is not in the essential image of Ψ.

3 γ-sheaves

3.1 Complements on the γ-topology

We still denote by γ a closed proper convex cone with non-empty interior, as in (1.3), and recall Definition 1.3.

Lemma 3.1.

(i) *for any subset A of $\mathbb{V}$, one has $A + \mathrm{Int}(\gamma) = \overline{A} + \mathrm{Int}(\gamma)$.*

(ii) *If U is an open subset of $\mathbb{V}$, then we have $U + \mathrm{Int}(\gamma) = U + \gamma$. In particular $U \subset U + \mathrm{Int}(\gamma)$.*

Proof. (i) For $x \in \overline{A}$ and $v \in \mathrm{Int}(\gamma)$, let us show $x + v \in A + \mathrm{Int}(\gamma)$. Since $x + v - \mathrm{Int}(\gamma)$ is a neighborhood of x, there exists $y \in A \cap \big(x + v - \mathrm{Int}(\gamma)\big)$. Then, one has $x + v \in y + \mathrm{Int}(\gamma) \subset A + \mathrm{Int}(\gamma)$.
(ii) Let us show $U \subset U + \mathrm{Int}(\gamma)$. For $x \in U$, there exists $y \in \mathrm{Int}(\gamma) \cap (x - U)$. Then $x \in y + U \subset U + \mathrm{Int}(\gamma)$.

To complete it is enough to remark that $U + \gamma \subset \big(U + \mathrm{Int}(\gamma)\big) + \gamma = U + \mathrm{Int}(\gamma)$, ☐

Lemma 3.2.

(a) *The intersection of a γ-invariant set and a γ^a-invariant set is γ-flat, i.e., $(B+\gamma) \cap (C + \gamma^a)$ is γ-flat for any $B, C \subset \mathbb{V}$.*

(b) *If A is γ-flat, then $\mathrm{Int}(A) = \big(A + \mathrm{Int}(\gamma)\big) \cap \big(A + \mathrm{Int}(\gamma^a)\big)$ and $\mathrm{Int}(A)$ is γ-flat.*

(c) *If U is γ-open, then one has $\mathrm{Int}(\overline{U}) = U$.*

(d) *if U is a γ-flat open subset of $\mathbb{V}$, then $Z := (U + \gamma) \cap \overline{U + \gamma^a}$ is γ-locally closed and $\mathrm{Int}(Z) = U$.*

Proof. (a) Set $D = (B + \gamma) \cap (C + \gamma^a)$. Then we have $D + \gamma \subset (B + \gamma) + \gamma = B + \gamma$. Similarly we have $D + \gamma^a \subset (C + \gamma^a) + \gamma^a = C + \gamma^a$. Hence, we obtain $(D+\gamma) \cap (D + \gamma^a) \subset (B + \gamma) \cap (C + \gamma^a) = D$.
(b) Set $U = \big(A + \mathrm{Int}(\gamma)\big) \cap \big(A + \mathrm{Int}(\gamma^a)\big)$. Then U is open and contained in A, hence, in $\mathrm{Int}(A)$. The other inclusion follows from Lemma 3.1 (ii).

Note that $\mathrm{Int}(A)$ is γ-flat by (a).
(c) Let $x \in \mathrm{Int}(\overline{U})$. Then there exists $v \in \mathrm{Int}(\gamma^a)$ such that $x + v \in \overline{U}$. Since $x + \mathrm{Int}(\gamma^a)$ is a neighborhood of $x + v$, there exists $y \in U \cap (x + \mathrm{Int}(\gamma^a))$. Hence we have $x \in y + \mathrm{Int}(\gamma) \subset U$.

(d) It is obvious that Z is γ-locally closed and $U \subset \mathrm{Int}(Z)$. Conversely, (c) implies that $\mathrm{Int}(Z) \subset (U+\gamma) \cap \mathrm{Int}(\overline{U+\gamma^a}) = (U+\gamma) \cap (U+\gamma^a) = U$. □

Lemma 3.3. *Let Z be a γ-locally closed subset of $\mathbb{V}$ and $\Omega = \mathrm{Int}(Z)$. Then*

(a) *for any $x \in Z$, there exists a neighborhood W of x such that $(x+\gamma^a) \cap W \subset Z$,*

(b) *if $x \in \mathbb{V}$ satisfies $x \in \overline{(x+\gamma^a) \cap Z}$, then one has $x \in Z$,*

(c) *Z is γ-flat,*

(d) $\Omega + \mathrm{Int}(\gamma) = Z + \mathrm{Int}(\gamma) = Z + \gamma$,

(e) $Z = (\Omega+\gamma) \cap \overline{\Omega+\gamma^a} = (\Omega+\gamma) \cap \overline{\Omega}$,

(f) $\Omega = (Z+\gamma) \cap \big(Z + \mathrm{Int}(\gamma^a)\big) = Z \cap \big(Z + \mathrm{Int}(\gamma^a)\big)$
$= \left\{ x \in \mathbb{V}\,;\ \begin{array}{l} \text{there exists an open neighborhood } W \text{ of } x \\ \text{such that } \big(x+\gamma\big) \cap W \subset Z \end{array} \right\}$,

(g) *Ω is γ-flat.*

Proof. Write $Z = A \cap B$ with a γ-open A and γ-closed B. Hence $A + \gamma = A$ and $B + \gamma^a = B$.
(a) Setting $W = A$, one has $(x+\gamma^a) \cap W \subset B \cap A = Z$.
(b) Assume that $x \in \overline{(x+\gamma^a) \cap Z}$. Then one has $x \in \overline{Z} \subset B$. Hence it remains to show $x \in A$. Since $(x+\gamma^a) \cap Z \neq \varnothing$, there exists $y \in (x+\gamma^a) \cap Z$. Then one has $x \in y + \gamma \subset A + \gamma = A$.
(c) follows from Lemma 3.2 (a).
(d) It is enough to show that $Z \subset \Omega + \mathrm{Int}(\gamma)$. Let $x \in Z$. Then $A \cap \big(x + \mathrm{Int}(\gamma^a)\big) \subset A \cap B = Z$, which implies that

$$A \cap \big(x + \mathrm{Int}(\gamma^a)\big) \subset \Omega \text{ for any } x \in Z. \tag{3.1}$$

Since $A \cap \big(x + \mathrm{Int}(\gamma)^a\big) \neq \varnothing$, there exists $y \in A \cap \big(x + \mathrm{Int}(\gamma)^a\big)$. Then one has $x \in y + \mathrm{Int}(\gamma) \subset \Omega + \mathrm{Int}(\gamma)$.
(e) One has $(\Omega+\gamma) \cap \overline{\Omega+\gamma^a} \subset A \cap B = Z$. Let us show $Z \subset (\Omega+\gamma) \cap \overline{\Omega}$. The inclusion $Z \subset \Omega + \gamma$ follows from (d). Finally, let us show that $x \in \overline{\Omega}$ for any $x \in Z$. By (3.1), one has $x \in \overline{A \cap \big(x + \mathrm{Int}(\gamma^a)\big)} \subset \overline{\Omega}$.
(f) We shall show $(Z+\gamma) \cap \big(Z + \mathrm{Int}(\gamma^a)\big) \subset \Omega \subset Z \cap \big(Z + \mathrm{Int}(\gamma^a)\big)$. We have

$$(Z+\gamma) \cap \big(Z + \mathrm{Int}(\gamma^a)\big) \subset (\Omega+\gamma) \cap \big(Z + \mathrm{Int}(\gamma^a)\big) = A \cap B = Z.$$

Here the first inclusion follows from (d). Hence we obtain $(Z+\gamma) \cap \big(Z + \mathrm{Int}(\gamma^a)\big) \subset \Omega$.

The inclusion $\Omega \subset Z \cap \big(Z + \mathrm{Int}(\gamma^a)\big)$ follows from $\Omega \subset \Omega + \mathrm{Int}(\gamma^a)$, which is a consequence of Lemma 3.1 (ii).

Let us prove the last equality. Let $x \in \mathbb{V}$ such that $\big(x+\gamma\big) \cap W \subset Z$ for an open neighborhood W of x. Take $y \in \big(x + \mathrm{Int}(\gamma)\big) \cap W \subset Z$. Then we have $x \in y + \mathrm{Int}(\gamma^a) \subset Z + \mathrm{Int}(\gamma^a)$.

(g) One has

$$(\Omega+\gamma)\cap(\Omega+\gamma^a) = (\Omega+\gamma)\cap\big(\Omega+\mathrm{Int}(\gamma^a)\big) \subset (\Omega+\gamma)\cap\big(Z+\mathrm{Int}(\gamma^a)\big) \subset \Omega.$$

Here the first equality is by Lemma 3.1 (ii) and the last inclusion follows from (f). □

The following proposition now follows from Lemma 3.2 and Lemma 3.3.

Proposition 3.4. *The set of γ-flat open subsets Ω of $\mathbb{V}$ and the set of γ-locally closed subsets Z of $\mathbb{V}$ are isomorphic by the correspondence*

$$\Omega \longmapsto (\Omega+\gamma)\cap\overline{\Omega+\gamma^a}$$
$$\mathrm{Int}(Z) \longleftarrow\!\shortmid Z.$$

Lemma 3.5. *Let U_i $(i=1,2)$ be γ-flat open subsets such that $U_1\cap U_2=\varnothing$. Set $Z_i=(U_i+\gamma)\cap\overline{U_i+\gamma^a}$. Then one has $Z_1\cap Z_2=\varnothing$.*

Proof. Since Z_i is γ-locally closed, $Z_1\cap Z_2$ is also γ-locally closed. Then, by Lemma 3.3, $Z_1\cap Z_2$ is contained in the closure of $\mathrm{Int}(Z_1\cap Z_2)\subset\mathrm{Int}(Z_1)\cap\mathrm{Int}(Z_2)=U_1\cap U_2=\varnothing$. □

3.2 Study of γ-sheaves

Lemma 3.6. *We assume that γ is polyhedral. Let $x\in\mathbb{V}$, let I be an open interval of $\mathbb{R}$ with $0\in I$ and let $c\colon I\to\mathbb{V}$ be a real analytic map. Assume that $c(t)\in(x+\gamma)\setminus\{x\}$ for $t\in I, t>0$ and $c(0)=x$. Then $c'(t)\in\gamma$ for all $t\geq 0$ in a neighborhood of 0.*

Proof. Since γ is polyhedral, we may assume that $\gamma=\{x\in\mathbb{V}\,;\,f(x)\geq 0\}$ for a linear function $f(x)$ on $\mathbb{V}$. Set $\varphi(t)=f(c(t))$. it is enough to prove that $\varphi'(t)\geq 0$ for $t\geq 0$ in a neighborhood of 0. If $\varphi=0$, the result is clear. Otherwise, there exists $m\in\mathbb{N}$, $m>0$ such that $\varphi(t)=t^m v+O(t^{m+1})$ with $v\neq 0$. Then $v>0$ and it follows that $\varphi'(t)>0$ for $t>0$ in a neighborhood of $t=0$. □

Remark 3.7. Lemma 3.6 is no more true without the assumption that the cone is polyhedral. Consider the cone $\gamma=\{(x,y,z)\in\mathbb{R}^3; x^2+y^2\leq z^2, z\geq 0\}$ and the curve $c(t)=(t\sqrt{1-t^2},t^2,t)$. One easily checks that $c'(t)\notin\gamma$ for $0<t<1$.

Proposition 3.8. *We assume that γ is polyhedral. Let $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_\mathbb{V})$. Then for each $x\in\mathbb{V}$, there exists an open neighborhood U of x such that $F|_{(x+\gamma^a)\cap U}$ is constant.*

Proof. (i) By Corollary 1.7, we may assume that F is concentrated in degree 0. Moreover, the sheaf $F\otimes\mathbf{k}_{x+\gamma^a}$ belongs to $\mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_\mathbb{V})$. Hence we may assume from the beginning that $\mathrm{supp}(F)\subset x+\gamma^a$.

(ii) By Theorem 1.5, for any $y\in\mathbb{V}$, $F(y+\gamma)\xrightarrow{\sim}F_y$. Since F is $\mathbb{R}$-constructible, there exists an open neighborhood V of x such that $\Gamma(V;F)\xrightarrow{\sim}F_x$. Let $y\in V\cap(x+\gamma^a)$. Then $x\in y+\gamma$. Consider the diagram

$$\begin{array}{ccc} F(V) & \rightarrowtail & F_y \xleftarrow{\sim} F(y+\gamma) \\ \sim\downarrow & & \downarrow \\ F_x & \twoheadleftarrow & F((y+\gamma)\cap V) \end{array}$$

Let $E := F_x$, a finite-dimensional $\mathbf{k}$-vector space. We have an injective map $E \to F_y$ for all $y \in (x+\gamma^a) \cap V$, hence a monomorphism $E_{(x+\gamma^a)\cap V} \hookrightarrow F$. Define G as the cokernel of this map. Then $G_x \simeq 0$, $\mathrm{supp}(G) \subset x+\gamma^a$ and $G \in \mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. It remains to show that $G \simeq 0$ in a neighborhood of x.

(iii) Assume that $G \neq 0$ in any neighborhood of x. Then $\{y \in \mathbb{V}\,;\, G_y \neq 0\}$ is a subanalytic set whose closure contains x. By the curve selection lemma, we find an analytic curve $c\colon I \to \gamma^a$ such that $c(0)=x$ and $G_{c(t)} \neq 0$ for any $t \in I$ such that $t>0$. By Lemma 3.6, $c'(t) \in \gamma^a$ for all $t \geq 0$ in a neighborhood of 0. Setting $\varphi(t) = c(t^2)$ for $t \geq 0$ and $\varphi(t) = x$ for $t \leq 0$, we find a curve of class C^1 and $\mathrm{supp}(\varphi^{-1}G) \subset \{t \geq 0\}$. Denote by $(t;\tau)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}$. Applying [KS90, Cor. 6.4.4], we get

$$\begin{cases}(0;\tau) \in \mu\mathrm{supp}(\varphi^{-1}G) \text{ implies that there exists a sequence } \{(x_n;\xi_n)\}_n \subset \\ \mathrm{SS}(G) \text{ with } x_n \xrightarrow{n} x,\ t_n \xrightarrow{n} 0,\ \langle \varphi'(t_n), \xi_n\rangle \xrightarrow{n} \tau.\end{cases}$$

Since $\varphi'(t_n) \in \gamma^a$ and $\xi_n \in \gamma^{\circ a}$, we get $\tau \geq 0$. Hence, $\varphi^{-1}G \simeq 0$ in a neighborhood of 0. This is a contradiction. □

For any sheaf $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$, there exists a largest open subset U of $\mathbb{V}$ such that $F|_U$ is locally constant, namely the union of all open subsets on which F is locally constant. Moreover, U is subanalytic in $\mathbb{V}$ since $U = \mathbb{V} \setminus \mathrm{Sing}(F)$. Note that $U \cap \mathrm{supp}(F)$ is again open in $\mathbb{V}$ and subanalytic. It is the largest open subset of $\mathbb{V}$ on which F is locally constant with strictly positive rank. Hence it is a union of connected components of U.

Corollary 3.9. *Let $F \in \mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and let $U = \mathrm{supp}(F) \setminus \mathrm{Sing}(F)$. Then U is an open subset of $\mathbb{V}$ and $F|_U$ is locally constant. Moreover, U is dense in $\mathrm{supp}(F)$. In particular, one has $\mathrm{supp}(F) = \overline{\mathrm{Int}(\mathrm{supp}(F))}$.*

Proof. We know already that U is an open subset of $\mathbb{V}$ and $F|_U$ is locally constant. It remains to prove that U is dense in $\{x \in \mathbb{V}\,;\, F_x \neq 0\}$. We may assume that γ is polyhedral. Let $x \in X$ such that $F_x \not\simeq 0$. Applying Proposition 3.8, we find an open neighborhood W of x such that F is constant on $(x+\gamma^a) \cap W$, hence on the open set $V := (x + \mathrm{Int}(\gamma^a)) \cap W$. Then x belongs to the closure of V and $V \subset U$. □

Proposition 3.10. *Let $F \in \mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. Then $S := \mathrm{Sing}(F)$ has pure codimension 1. Moreover, for any $x \in S_{\mathrm{reg}}$, one has $T_xS \cap \mathrm{Int}(\gamma) = \varnothing$, or equivalently $(T^*_S\mathbb{V})_x \subset \gamma^\circ \cup \gamma^{\circ a}$.*

Proof. Assume that there is a point where S has codimension ≥ 2. Take an open subset U such that $S \cap U$ is a non-empty submanifold of codimension ≥ 2. Note that $\Lambda := \mu\mathrm{supp}(F)$ is involutive ([KS90, Theorem .6.5.4]), and $U \cap \pi(\Lambda \cap \dot{T}^*M) = S \cap U$. Hence Lemma 3.11 below implies that $\pi^{-1}U \cap T^*_SM \subset \Lambda \subset \mathbb{V} \times \gamma^{\circ a}$. It contradicts the fact that $\gamma^{\circ a}$ is a proper closed convex cone.

The last assertion is a consequence of the fact that $\mu\mathrm{supp}(F) \cap \pi^{-1}U \subset T^*_{S_{\mathrm{reg}}}\mathbb{V}$ for an open dense subanalytic subset U of S_{reg} and hence, S being an hypersurface, $(T^*_{S_{\mathrm{reg}}}\mathbb{V} \cap \pi^{-1}U) \subset \mu\mathrm{supp}(F) \cup \mu\mathrm{supp}(F)^a$. □

Lemma 3.11. *Let M be a manifold and N a submanifold of M of codimension ≥ 2. Let Λ be a closed conic involutive subset of T^*M. If $\pi(\Lambda \cap \dot{T}^*M) = N$, then $T^*_NM \subset \Lambda$.*

Proof. We choose a local coordinate system $(x) = (t, y)$ such that $N = \{t = 0\}$, $(t) = (t_1, \dots, t_m)$. Let $(x; \xi) = (t, y; \tau, \eta)$ denote the associated coordinates on T^*M. Let $(0, y_0) \in N$. There exists $(\tau_0, \eta_0) \neq 0$ such that $(0, y_0; \tau_0, \eta_0) \in \Lambda$. Since $\Lambda \cap \dot{T}^*M$ is contained in $t = 0$, $\Lambda \cap \dot{T}^*M$ is invariant by $\frac{\partial}{\partial \tau_k}$ for $1 \leq k \leq m$ by [KS90, Proposition 6.5.2]. Since $m > 1$, this implies that $\Lambda \cap \dot{T}^*M$ contains $(0, y_0; \tau, \eta_0)$ for any τ. Moreover, since Λ is conic, this implies $(0, y_0; \tau, 0) \in \Lambda \cap \dot{T}^*M$ for any τ. ∎

Corollary 3.12. *Let $F \in \mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and $G \in \mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ}}(\mathbf{k}_{\mathbb{V}})$. Then $\mathscr{E}xt^j(G, F) \simeq 0$ for $j \neq 0$.*

Proof. Let U and V be the largest open subsets of $\mathbb{V}$ on which F and G are respectively locally constant. Then $W = U \cap V$ is open, dense and $\mathrm{R}\mathscr{H}om\,(G, F)$ is concentrated in degree 0 on W. Therefore, $\mathrm{supp}(\mathscr{E}xt^j(G, F))$ has empty interior for $j > 0$. Since this sheaf belongs to $\mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ by [KS90, Prop. 5.4.14] and Corollary 1.7, it must be 0 by Corollary 3.9. ∎

The result of Corollary 3.12 does not hold if both F and G belong to $\mathrm{Mod}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$.

Lemma 3.13. *One has $\mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\gamma^a}, \mathbf{k}_{\mathrm{Int}(\gamma)}\,[n]) \simeq \mathbf{k}_{\{0\}}$.*

Proof. Applying [KS90, Prop. 3.4.6]) we get

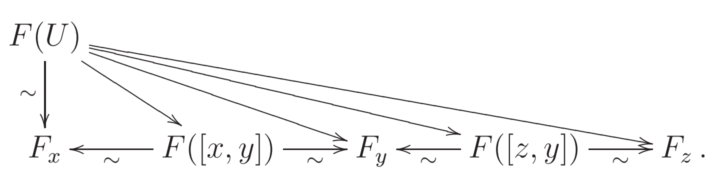
$$\begin{aligned}\mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{0\}}, \mathbf{k}_{\mathrm{Int}(\gamma)}\,[n]) &\simeq \mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{0\}}, \mathrm{D}'\mathbf{k}_{\gamma})[n] \\ &\simeq \mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{0\}} \otimes \mathbf{k}_{\gamma}, \mathbf{k}_{\mathbb{V}})[n] \\ &\simeq \mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{0\}}, \mathbf{k}_{\mathbb{V}})[n] \simeq \mathbf{k}_{\{0\}}.\end{aligned}$$

∎

Proposition 3.14. *Assume that γ is polyhedral. Let Ω be a γ-flat open set and let $Z = (\Omega + \gamma) \cap \overline{\Omega + \gamma^a}$, a γ-locally closed subset. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}c,\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and assume that $F|_{\Omega}$ is locally constant. Then $F|_Z$ is locally constant.*

Proof. Let $x \in Z$ and let U be an open convex neighborhood of x such that $F(U) \xrightarrow{\sim} F_x$ and such that, applying Proposition 3.8, F is constant on $(x + \gamma^a) \cap U$. We choose a vector $v \in \mathrm{Int}(\gamma^a)$ and $\varepsilon > 0$ such that $[x, x + \varepsilon v] \subset (x + \gamma^a) \cap U \subset Z$. Let $y = x + \varepsilon v$. Then $y \in \Omega$ and F is constant on $[x, y]$. Set $W := U \cap (y + \mathrm{Int}(\gamma))$ so that $x \in W$. We shall show that F is constant on $W \cap Z$.

Let $z \in W \cap Z$. Then $[z, y] \subset \Omega + \gamma$, $y - z \in \gamma$ and $]z, y] \subset \overline{(\Omega + \gamma^a)} + \mathrm{Int}(\gamma^a) = \Omega + \gamma^a$. The last equality follows from Lemma 3.1. Hence we have $]z, y] \subset \Omega$ and $F|_{[z,y]}$ is constant by Proposition 3.8. Consider the commutative diagram:



The horizontal arrows are isomorphism since $F|_{[x,y]}$ and $F|_{[z,y]}$ are constant. It follows that the map $F(U) \to F_z$ is an isomorphism. ∎

3.3 Piecewise linear γ-sheaves

Let γ be a cone as in (1.3).

Definition 3.15. A *γ-stratification* of a closed subset S is the data of a set of indices A and a family $Z = \{Z_\alpha\}_{\alpha\in A}$ of subsets of $\mathbb{V}$, these data satisfying conditions (i)–(iv) below.

$$
(3.2)\quad \begin{cases} \text{(i) the } Z_\alpha\text{'s are subanalytic and } \gamma\text{-locally closed subsets of } \mathbb{V}, \\ \text{(ii) the family } Z = \{Z_\alpha\}_{\alpha\in A} \text{ is locally finite in } \mathbb{V}, \\ \text{(iii) } Z_\alpha \cap Z_\beta = \varnothing \text{ for } \alpha \neq \beta, \\ \text{(iv) } S = \bigcup_{\alpha\in A} \overline{Z_\alpha}, \\ \text{(v) the } Z_\alpha\text{'s are convex polytopes.} \end{cases}
$$

If moreover condition (v) is satisfied, then one says that the γ-stratification is PL.

Lemma 3.16. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_\mathbb{V})$, and let $\{Z_\alpha\}_{\alpha\in A}$ be a γ-stratification of* $\mathrm{supp}(F)$*. Then $F_x \simeq 0$ for any $x \notin \bigcup_{\alpha\in A} Z_\alpha$.*

Proof. We can assume that γ is polyhedral. Assuming that $F_x \not\simeq 0$, let us show that $x \in \bigcup_{\alpha\in A} Z_\alpha$. By Proposition 3.8, there exists an open neighborhood U of x such that $F|_{(x+\gamma^a)\cap U}$ is constant. Then one has $(x+\gamma^a)\cap U \subset \mathrm{supp}(F) \subset \bigcup_{\alpha\in A}\overline{Z_\alpha}$. Hence we obtain

$$
\begin{aligned}
x \in \overline{\big(x+\mathrm{Int}(\gamma^a)\big)\cap U} \subset \overline{\big(x+\mathrm{Int}(\gamma^a)\big)\cap \big(\bigcup_{\alpha\in A}\overline{Z_\alpha}\big)} &= \bigcup_{\alpha\in A} \overline{\big(x+\mathrm{Int}(\gamma^a)\big)\cap \overline{Z_\alpha}} \\
&= \bigcup_{\alpha\in A} \overline{\big(x+\mathrm{Int}(\gamma^a)\big)\cap Z_\alpha}\,.
\end{aligned}
$$

Hence there exists $\alpha \in A$ such that $x \in \overline{\big(x+\mathrm{Int}(\gamma^a)\big)\cap Z_\alpha}$. Then Lemma 3.3 (b) implies that $x \in Z_\alpha$. ∎

Conjecture 3.17. Let λ be a closed convex proper cone with non empty interior and contained in $\mathrm{Int}(\gamma)\cup\{0\}$. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_\mathbb{V})$. Then, there exists a λ-stratification (A, Z) in $\mathbb{V}$ such that $\mathrm{supp}(F) = \mathrm{supp}(A, Z)$ and for each $\alpha \in A$ and $j \in \mathbb{Z}$, $H^j(F)|_{Z_\alpha}$ is constant.

We shall solve this conjecture in the particular case of PL-sheaves.

Recall that for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_\mathbb{V})$, $\mathrm{Sing}(F)$ denotes its singular locus.

Theorem 3.18. *Assume that γ is polyhedral and $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_\mathbb{V})$. Then there exists a* PL-*γ-stratification $\{Z_\alpha\}_{\alpha\in A}$ of* $\mathrm{supp}(F)$ *such that $F|_{Z_\alpha}$ is constant for each α.*

Proof. (i) Assume first that $\mathrm{supp}(F)$ is compact.

(a) Since F is PL, there exists a finite family of affine hyperplanes $\{H_a\}_{a\in A}$ such that $\mathrm{Sing}(F) \subset \bigcup_{a\in A} H_a$. On the other hand, $\mathrm{Sing}(F)$ has pure codimension 1, thanks to Proposition 3.10. Let $B = \{a \in A; \mathrm{Int}_{H_a}\big(\mathrm{Sing}(F)\cap H_a\big) \neq \varnothing\}$. Then, $\mathrm{Sing}(F)\cap\big(\bigcup_{a\in B} H_a\big)$ is dense in $\mathrm{Sing}(F)$. Hence we obtain $\mathrm{Sing}(F) \subset \bigcup_{a\in B} H_a$. Set

$\Omega = \mathrm{supp}(F) \setminus \left(\bigcup_{a\in B} H_a\right)$. Then Ω is an open subset of $\mathbb{V}$ and dense in $\mathrm{supp}(F)$ by Corollary 3.9.

(b) Let $\Omega = \bigsqcup_{i\in I} \Omega_i$ be the decomposition of Ω into connected components. Then each Ω_i is an open convex polytope.

At generic points of H_a ($a \in B$), one has $\dot{T}^*_{H_a}\mathbb{V} \cap \mu\mathrm{supp}(F) \neq \varnothing$ by Proposition 3.10. If $H_a = \{\langle x, \xi\rangle = c\}$, one has $\pm\xi \in \gamma^{\circ a}$ and thus $H_a \cap (x + \mathrm{Int}(\gamma)) = \varnothing$ for any $x \in H_a$. Denote by $H_a^{\pm}$ the two open half-spaces with boundary H_a. These are γ-flat open sets and it follows that any connected component Ω_i of Ω, which is a finite intersection of such half-spaces, is also γ-flat.

Set $Z_i = (\Omega_i + \gamma) \cap \overline{\Omega_i + \gamma^a}$. Then each Z_i is γ-locally closed and $\mathrm{supp}(F) = \bigcup_{i\in I} \overline{Z_i}$. By Lemma 3.5, $Z_i \cap Z_j = \varnothing$ if $i \neq j$. Hence $\{Z_i\}_{i\in I}$ is a γ-stratification of $\mathrm{supp}(F)$. By Proposition 3.14, $F|_{Z_i}$ is locally constant. Since Z_i is convex, $F|_{Z_i}$ is constant.

(ii) Now we consider the general case where $\mathrm{supp}(F)$ is not necessarily compact.

Taking $v \in \mathrm{Int}(\gamma)$, one sets $U_n = -nv + \mathrm{Int}(\gamma)$ and $S_n = nv + \gamma^a$ for $n \in \mathbb{Z}$. Then $\{U_n\}_{n\in\mathbb{Z}}$ is an increasing family of γ-open subsets, and $\{S_n\}_{n\in\mathbb{Z}}$ is an increasing family of γ-closed subsets. Moreover, one has $\mathbb{V} = \bigcup_{n\in\mathbb{Z}} U_n = \bigcup_{n\in\mathbb{Z}} S_n$ and $\bigcap_{n\in\mathbb{Z}} U_n = \bigcap_{n\in\mathbb{Z}} S_n = \varnothing$. Set $I = \mathbb{Z}\times\mathbb{Z}$ and, for $i = (m,n) \in I$, set $K_i := (U_m \setminus U_{m-1}) \cap (S_n \setminus S_{n-1})$. Then $\{K_i\}_{i\in I}$ is a locally finite family of γ-locally closed subsets such that $\mathbb{V} = \bigsqcup_{i\in I} K_i$.

Set $F_i = F \otimes \mathbf{k}_{K_i}$. Then $F_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and $\mathrm{supp}(F_i)$ is compact. Hence by Step (i) there exists a finite PL-γ-stratification $\{Z_\alpha\}_{\alpha\in A_i}$ of $\mathrm{supp}(F_i)$ such that $F_i|_{Z_\alpha}$ is constant for each $\alpha \in A_i$. Then, setting $A = \bigsqcup A_i$, we obtain a desired PL-γ-stratification $\{Z_\alpha\}_{\alpha\in A}$ of $\mathrm{supp}(F)$. Note that $Z_\alpha \cap Z_{\alpha'} = \varnothing$ for $\alpha \neq \alpha'$ follows again from Lemma 3.5. Indeed, assume that $\alpha \in A_i$ and $\alpha' \in A_{i'}$. If $i = i'$, one has obviously $Z_\alpha \cap Z_{\alpha'} = \varnothing$. If $i \neq i'$, then $\mathrm{Int}(Z_\alpha) \cap \mathrm{Int}(Z_{\alpha'}) \subset K_i \cap K_{i'} = \varnothing$, and Lemma 3.5 implies that $Z_\alpha \cap Z_{\alpha'} = \varnothing$. □

Remark 3.19. In the course of the proof of Theorem 3.18, we have also obtained the following result.

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$, and let $\{H_a\}_{a\in A}$ be a locally finite family of affine hyperplanes such that $\mathrm{Sing}(F) \subset \bigcup_{a\in A} H_a$. Then, one has $\mathrm{Sing}(F) \subset \bigcup_{a\in B} H_a$ where $B = \left\{a \in A\,;\, \mathrm{Int}_{H_a}\left(\mathrm{Sing}(F) \cap H_a\right) \neq \varnothing\right\}$. Moreover, F is PL by Lemma 3.16.

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