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PROPAGATION AT THE BOUNDARY OF ANALYTIC SINGULARITIES

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We study non elliptic boundary value problems in the framework of microfunctions. We introduce the concept of N-regularity and apply it to the problem of propagation of analytic singularities at the boundary with some possible diffraction.

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1. VARIOUS SHEAVES OF MICROFUNCTIONS.

Let X be a complex analytic manifold, T^*X its cotangent bundle, $T^*_X X \simeq X$ the zero section of T^*X, $\dot{T}^*X = T^*X - T^*_X X$, π the projection of T^*X on X, $\mathcal{O}_X$ the sheaf of holomorphic functions on X.

The space T^*X is endowed with the sheaf $\mathcal{E}_X$ of microdifferential operators (of finite orders) constructed in(20). Let us recall that if X is open in a complex vector space E, and U is open in $X \times E^*$, $\mathcal{E}_X(U)$ is isomorphic to the set of series $\{\sum_{j \in \mathbb{Z}} p_j(z,\zeta)\}$ with $p_j = 0$ for $j \gg 0$, p_j is holomorphic on U, homogeneous of degree j on ζ and such that for any compact set $K \subset U$ there exists $t > 0$ whith

$$\sum_{j \leq 0} |p_j|_K \, t^{-j}/(-j)! < \infty$$

The composite $P \circ Q$ of two microdifferential operators P and Q is given in a system of local coordinates, by the usual Leibnitz formula, and makes $\mathcal{E}_X$ a unitary non commutative sheaf of algebra, whose restriction to $T^*_X X$ is isomorphic with $\mathcal{D}_X$, the sheaf of holomorphic differential operators on X, (cf. (17) for a quick expository on this subject). If P is a section of $\mathcal{E}_X$, we denote by $\sigma(P)$ its principal symbol.

Let now M be a real analytic manifold of dimension n, X a complexification of M, N a real analytic submanifold of M, Y the complexification of N in X. Let $T^*_N X$ be the conormal bundle of N in X that is the kernel of the map $T^*_X N \longrightarrow T^*N$

The sheaf $C_{N|X}$ on $T^*_N X$ is constructed in(20) in a intrinsic way but to understand it, let us assume that M is a real vector space, N a linear subspace of M, F a supplementary of N in M, so that $M = N \oplus F$, $X = N^{\mathbb{C}} \oplus F^{\mathbb{C}}$,

$$T^*_N X = (N \times \{0\}) \quad \times \quad (i\, N^* \times (F^{\mathbb{C}})^*)$$

Let ω be an open set in N, I a conic open set in $i\,N^* \times (F^{\mathbb{C}})^*$. The sheaf $C_{N|X}$ is associated to the presheaf

$$\check{C}_{N|X}(\omega \times I) = \varinjlim_{U} H^n_{(\omega \times G) \cap U}(U, \mathcal{O}_X)$$

where G is the closed cone of $i\,N \times F^{\mathbb{C}}$ polar of I, and U runs over the neighborhoods of ω in X.

We also define the sheaf $\Gamma_N(\mathcal{B}_M)$ on N (the sheaf of hyperfunctions on M supported by N):

$$\Gamma_N(\mathcal{B}_M) = \mathcal{H}^n_N(\mathcal{O}_X) .$$

The sheaf $C_{N|X}$ is locally constant on the orbits of the action of $\mathbb{R}^+$ on $T^*_N X$, and the natural morphism

$$\Gamma_N(\mathcal{B}_M) \longrightarrow \pi_* C_{N|X} = C_{N|X} \,|\, T^*_N N$$

associated to the inclusions $\omega \times \{0\} \subset \omega \times G$, is an isomorphism. When $N = M$, the sheaf $C_{M|X}$, just denoted C_M, is the sheaf of microfunctions introduced by M. Sato in 1969, and $\Gamma_M(\mathcal{B}_M) = \mathcal{B}_M$ the sheaf of hyperfunctions of M. Sato (introduced in 1959). The sheaves $C_{N|X}$ are naturally endowed with a structure of $\mathcal{E}_X | T^*_N X$-modules. (c.f.(3), (8) for an explicit representation of the action of $\mathcal{E}_X$ on C_M).

<u>Remark</u> For the sake of simplicity we have systematically forgotten the various sheaves of relative orientation that should appear in the definitions of the sheaves $C_{N|X}$. If u is a microfuntion (that is a section of C_M), its support is also called its singular support and is denoted $SS(u)$. If u is an hyperfunction, $SS(u)$ is a closed conic set of $T^*_M X$, and $SS(u)$ is contained in $T^*_M M$ iff u is real analytic, and one says sometime that u is micro-analytic at $x^* \in T^*_M X$ if $x^* \notin SS(u)$. The sheaves $C_{N|X}$ and C_M are related by the

following theorem of Kashiwara (5), to appear in (10)(cf. also (9)).

Theorem 1.1. We have at, least outside of $T^*_Y X$, natural *injective* morphisms of sheaves on $T^*_N X \cap T^*_M X$:

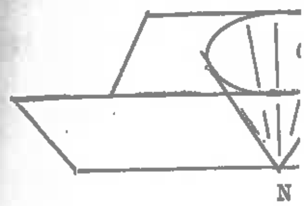
$$C_{N|X} | T^*_N X \cap T^*_M X \rightarrowtail \Gamma_{T^*_N X \cap T^*_M X}(C_M)$$

$$\rightarrowtail C_M | T^*_N X \cap T^*_M X \rightarrowtail \mathcal{H}^p_{T^*_N X \cap T^*_M X}(C_{N|X})$$

where p is the codimension of N in M.

Let us assume that N is an hypersurface of M, M_+ an open half space of M with N as boundary, and let Q_+ be the half space of $T^*_N X$ with $T^*_N X \cap T^*_M X$ as boundary such that in a choice of local coordinates, if $(z^o, \zeta^o) \in Q_+$, the polar of a neighborhood of ζ^o contains $\overline{M}_+$ (cf. figure below). Then we have a natural morphism from $\Gamma_{\overline{M}_+}(M, B_M)$ to $\Gamma(Q_+, C_{N|X})$ and this morphism is injective.

We can describe the preceding morphisms by the following pictures, where C_M and $C_{N|X}$ are associated to the cohomology groups with support in the closed sets $M \times \Gamma$ and $N \times G$.



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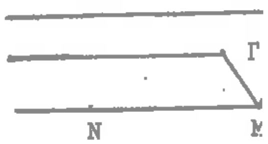

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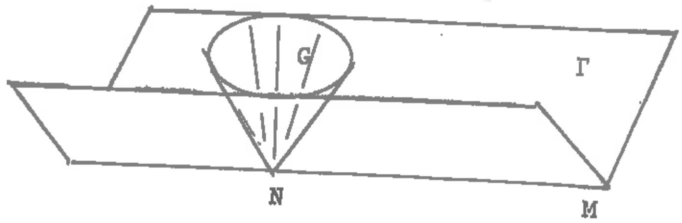
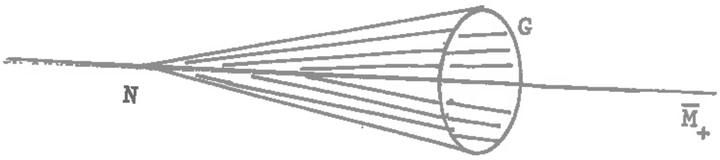
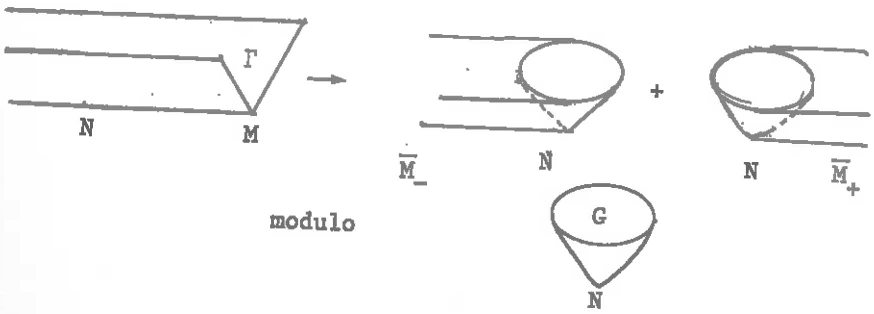


$$C_{N|X}|_{T^*_N X \cap T^*_M X} \longrightarrow \Gamma_{T^*_N X \cap T^*_N X}(C_M)$$



$$\Gamma_{\overline{M}_+}(\mathcal{B})|_N \longrightarrow C_{N|X}|_{Q_+}$$



$$C_M|_{T^*_M X \cap T^*_N X} \longrightarrow \mathcal{H}^p_{T^*_M X \cap T^*_N X}(C_{N|X}) \text{ when } p = 1 .$$

As these cohomology groups are rather non intuitive objects, it is satisfactory to give a new interpretation of those sheaves, by the aid of a quantized complex contact transform. Let $M = \mathbb{R}^n$, $X = \mathbb{C}^n$, $z = (z_1,\dots z_n) \in X$, $(z,\zeta) = (z_1,\dots z_n,\zeta_1,\dots,\zeta_n) \in T^*X$.

Let Φ be the complex homogeneous canonical transform, defined for $\zeta_n \neq 0$ by :

$$\varphi : T^*\mathbb{C}^n - \{\zeta_n = 0\} \longrightarrow T^*\mathbb{C}^n$$

$$(z,\zeta) \longrightarrow (z + i\,\varphi'(\zeta),\zeta)$$

$$\text{where } \varphi(\zeta) = \frac{\zeta_1^2 + \dots + \zeta_{n-1}^2}{4\,\zeta_n}$$

It is easy to see that Φ exchanges $T^*_{\mathbb{R}^n}\mathbb{C}^n$ with $(T^*_{\partial\Omega_o}X)^+$, the exterior conormal bundle to the boundary of the tube

$$\Omega_o = \{z \in \mathbb{C}^n \ ; \ y_n > \sum_{i=1}^{n-1} y_i^2\} .$$

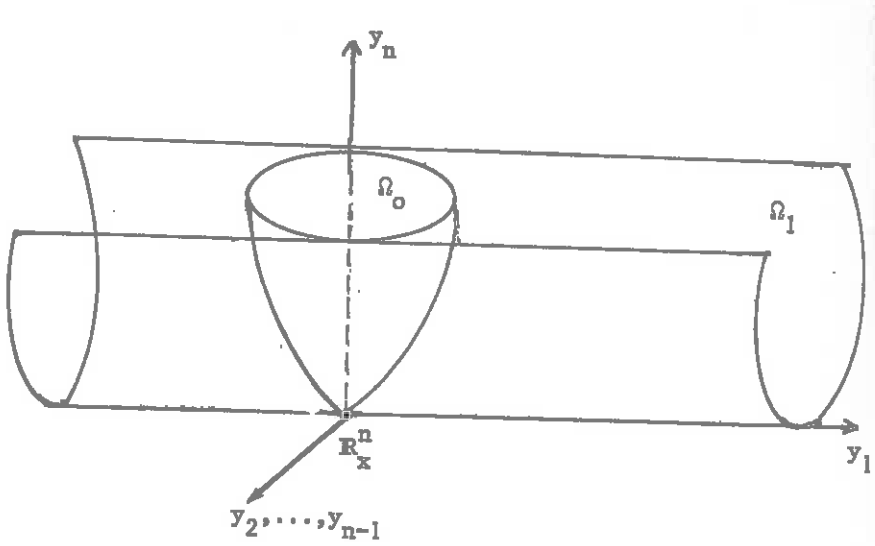
Let $p < n$, and let $N = \{x \in M \ ; \ x_1 = \dots = x_p = 0\}$, $Y = N^{\mathbb{C}} = \{z \in X \ ; \ z_1 = \dots = z_p = 0\}$ and $L = \{z \in X ; y_{p+1} = \dots = y_n = 0\}$. We may exchange $T^*_N X$ and $T^*_L X$ for $\zeta_n \neq 0$ by a partial Legendre contact transform Ψ. As Ψ is real, Ψ exchanges $T^*_M X$ with itself, hence $\Phi \circ \Psi$ exchanges $T^*_M X$ and $(T^*_{\partial\Omega_o}X)^+$ (for $\zeta_n \neq 0$) and $T^*_N X$ and $(T^*_{\partial\Omega_1}X)^+$, where Ω_1 denotes the tube :

$$\Omega_1 = \{z \in \mathbb{C}^n \ ; \ y_n > \sum_{i=p+1}^{n-1} y_i^2\} .$$

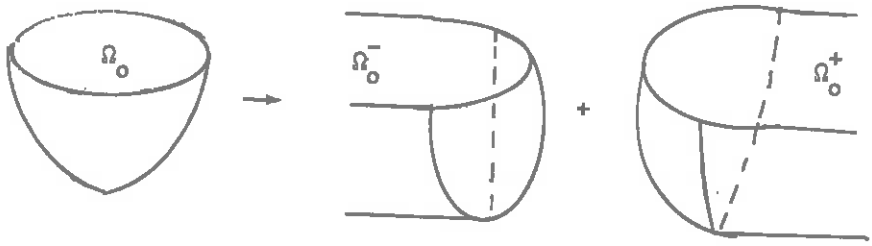
Let $\mathcal{O}^+_{\partial\Omega_j}$ $(j = 0,1)$ be the sheaf on $\partial\Omega_j$ of boundary values of holomorphic functions on Ω_j :

$$\mathcal{O}^+_{\partial\Omega_j} = i_*(\mathcal{O}_{\Omega_j})/\mathcal{O}_X$$

where $i_*(O_{\Omega_j})(U) = O(U \cap \Omega_j)$ for any open set U in X. We denote by $C^+_{\partial\Omega_j}$ the inverse image of $O^+_{\partial\Omega_j}$ by the projection $\pi : (T^*_{\partial\Omega_j} X)^+ \to \partial\Omega_j$. The sheaf $C^+_{\partial\Omega_j}$ is naturally endowed with a structure of $\mathcal{E}_X$ - module (cf. (11)) and it can be proved that the contact transform $\Phi \circ \Psi$ can be extended (locally) as an isomorphism of $\mathcal{E}_X$ onto itself, and an isomorphism of C_M on $C^+_{\partial\Omega_o}$ and $C_{N|X}$ on $C^+_{\partial\Omega_1}$, these last two isomorphism being compatible with the action of $\mathcal{E}_X$. Then theorem 11 can be re-interpreted with the sheaves $O^+_{\partial\Omega_o}$ and $O^+_{\partial\Omega_1}$, as shown in the following pictures, (Ω_o and Ω_1 are open, and $p = 1$) (cf. (16) for a construction analogous to the isomorphism of C_M onto $C^+_{\partial\Omega_o}$).



$\mathcal{O}^+_{\partial\Omega_1}|_{\partial\Omega_o \cap \partial\Omega_1} \longrightarrow \Gamma_{\partial\Omega_o \cap \partial\Omega_1}(\mathcal{O}^+_{\partial\Omega_o})$



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$\mathcal{O}^+_{\partial\Omega_o}|_{\partial\Omega_o \cap \partial\Omega_1} \longrightarrow \mathcal{H}^1_{\partial\Omega_o \cap \partial\Omega_1}(\mathcal{O}^+_{\partial\Omega_1})$

$\forall\, f \in \mathcal{O}(\Omega_o)$, $f = f_o^- + f_o^+ \mod h$

where $f_o^{\pm} \in \mathcal{O}(\Omega_o^{\pm})$, $h \in \mathcal{O}(\Omega_1)$

2. MICRO LOCAL HOLMGRE

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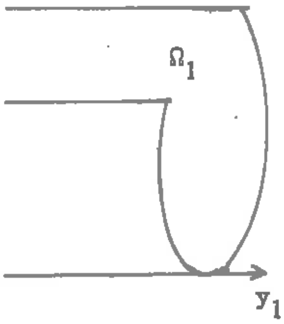

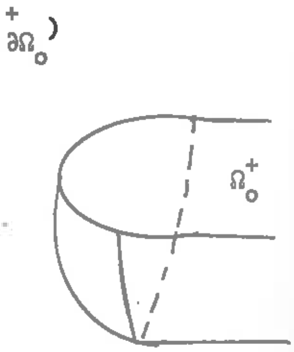

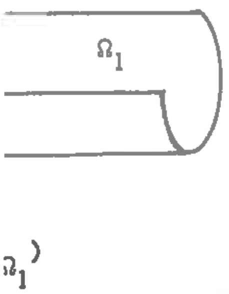

2. MICRO LOCAL HOLMGREN THEOREM

Let $N \subset M \subset X$ be as in the preceding section, and Y the complexification of N in X.
The manifold $T^*_N X$ admits a natural foliation by complex manifolds of dimension p, where p is the codimension of N in M: these are the bicharacteristic leaves of the complex involutive manifold $Y \times_X T^*X \subset T^*X$, $T^*_N X$ being invariant by the Hamiltonian flow of $Y \times_X T^*X$.

The following theorem was announced by Kashiwara and Kawai in (7).

<u>Theorem 2.1</u> Let u be a section of $C_{N|X}$ on an open set $U \subset T^*_N X$. Then the support of u is a union of bicharacteristic leaves of $Y \times_X T^*X$.

<u>Proof</u>

a) We first reduce the problem to the case where $U \cap T^*_Y X = \emptyset$, by a trick introduced by M. Kashiwara. We set $\tilde{X} = X \times \mathbb{C}$, $\tilde{N} = N \times \mathbb{R}$, $\tilde{U} = \{ z,t \, ; \zeta, \tau) \in T^*_{\tilde{N}}\tilde{X} \, ; \tau \neq 0 , \zeta/\tau \in U \}$ where t is a coordinate on $\mathbb{C}$.
If u belongs to $C_{N|X}$, $u \otimes \delta(t)$ belongs to $C_{\tilde{N}|\tilde{X}}$, and :

$$(z,\zeta) \in SS(u) \iff (z,o \, ; \zeta , i) \in SS(u \otimes \delta(t))$$

it is thus enough to prove the theorem for $C_{\tilde{N}|\tilde{X}}$, at points $(z,t \, ; \zeta , \tau)$ where $\tau \neq 0$.

b) We can exchange, by a quantized complex transform, the problem to the following.
Let $\Omega_1 = \{ z \in \mathbb{C}^n \, ; y_n > \sum_{j=p+1}^{n-1} y_j^2 \}$, let W be an open set of $\mathbb{C}^n$ with $W \cap \partial\Omega_1 = U$, and set $W \cap \Omega_1 = W^+$. Let f be holomorphic in W^+, which extends holomorphically in a neighbourhood of the point $z^o = (z'^o_1 , z''^o) \in U$ where $z' = (z_1,\ldots z_p)$ (that is : f gives 0 in $C^+_{\Omega_1}$ near that point).

Then f is holomorphic near any point (z'_1, z''^o) which belongs to the connected component of (z'^o, z''^o) in U. The proof is an immediat consequence of the well known Bochner local tube theorem cf. (6) or (14).

We assume now that N is an hypersurface of the real analytic manifold M, and let $\overline{M}_+$ be one of the closed half-space of M with N as boundary.

The manifold $T^*_M X \underset{M}{\times} N$ is involutive in $T^*_M X \simeq i\, T^* M$, and admits a foliation by the so-called bicharacteristic curves.

<u>Theorem 2.2</u> (Holmgren-Kashiwara) (5)

Let u be an hyperfunction on M supported by $\overline{M}_+$. Then $SS(u) \cap (T^*_M X \underset{M}{\times} N)$ is a union of bicharacteristic curves of $T^*_M X \underset{M}{\times} N$.

If we take a system of local coordinates $(x_1, \dots, x_n)$ on M, such that $N = \{x \in M \,;\, x_1 = 0\}$, and if we set $z = x + iy$, $\zeta = \xi + i\eta$, where (z, ζ) are local coordinates on $T^* X$, the theorem asserts that if $(0, x'^o ; i\eta_1^o, i\eta'^o) \in SS(u)$ then $(0, x'^o ; i\eta_1, i\eta'^o) \in SS(u)$ for any $\eta_1 \in \mathbb{R}$.

If we take $\eta_1^o = 0$, $\eta'^o = 0$, we find in particular that $(0, x'^o) \in \operatorname{supp}(u)$ implies that $(0, x'^o ; +i, 0 \dots 0)$ and $(0, x'^o ; -i, 0 \dots 0)$ belong to $SS(u)$: this is the "classical" Holmgren theorem (20).

<u>Proof of theorem 2.2.</u>

a) By the same trick as in the proof of theorem 2.1 it is enough to prove the theorem for $i\eta'^o \neq 0$.

b) Let Q_+ be the open half space of $T^*_N X$ associated to $\overline{M}_+$ (cf. theorem 1.2) and let us consider the diagram :

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$$\begin{array}{ccc} \Gamma_{\overline{M}_+}(M, B_M) & \longrightarrow & \Gamma(T^{*}_{M}X , C_M) \\ \downarrow & & \downarrow \\ \Gamma(Q_+ , C_{N|X}) & \longrightarrow & H^{1}_{T^{*}_{M}X \cap T^{*}_{N}X}(T^{*}_{N}X , C_{N|X}) \end{array}$$

Let $u \in \Gamma_{\overline{M}_+}(B_M)$ and assume that $x^{*} = (0, x'^{o} ; i\eta^{o}_{1}, \eta'^{o})$ does not belong to SS(u) . Then $u = 0$ near x^{*} in Q_+ as a section of $C_{N|X}$, and theorem 2.1. implies that $u = 0$ near any point $(0 , x'^{o} ; \zeta_1 , i\,\eta'^{o})$ of Q^{+} . Then the image of u in $H^{1}_{T^{*}_{M}X \cap T^{*}_{N}X}(T^{*}_{N}X , C_{N|X})$ will be 0 at any point $(0, x'^{o} ; i\,\eta_1 , i\,\eta'_{o})$, and this achieves the proof, because of the injectivity of the second vertical arrow (theorem 1.1).

3. DIVISION THEOREM FOR THE SHEAF $C_{N|X}$

We choose local coordinates $z = (z_1, \dots, z_n)$ on X , with $z = x + iy$, $M = \{z \in X ; y = 0\}$, $N = \{x \in M ; x_1 = 0\}$. We say that a microdifferential operator P of order m defined in a neighborhood of $(z^{o}, \zeta^{o}) \in T^{*}X$ if of Weierstrass type in D_{z_1} if P is written :

$$P(z , D_z) = \sum_{j=o}^{m} A_j(z, D_{z'})\, D^{j}_{z_1}$$

where $D_{z'} = (D_{z_2}, \dots, D_{z_n})$, and A_j are microdifferential operators of order $\leq m - j$, with $A_m \equiv 1$.

Let $Y = \{z \in X ; z_1 = 0\}$ and let ρ denotes the projection $T^{*}X \times_X Y \longrightarrow T^{*}Y$.

Theorem 3.1 (7). Let P be a microdifferential operator defined in a neighborhood of $x^\star = (0, x'^o ; \tau^o, i\eta'^o) \in T^\star_N X$. Let μ be the multiplicity, that we assume to be finite, of the root at $\tau = \tau^o$ of the equation $\sigma(P)\,(0, x'^o ; \tau, i\eta'^o) = 0$. Then the map from $(C_{N|X})_{x^\star} \times (C_N^{\mu})_{\rho(x^\star)}$ to $(C_{N|X})_{x^\star}$:

$$(v, (w_j)_{j=o}^{\mu-1}) \longrightarrow P v + \sum_{j=o}^{\mu-1} w_j \otimes \delta_1^j$$

is an isomorphism.

For a proof we refer to (11 § 6). We can get a semi-global version of the preceding theorem, by purely algebraic considerations (cf. (22) or (12)).

Theorem 3.2 - Let $y^\star = (x'^o ; i\eta'^o) \in T^\star_N Y$, and let P be a microdifferential operator of Weierstrass type in D_{z_1}, defined in a neighborhood of $\rho^{-1}(y^\star)$. Let U be an open set in $T^\star_N X$ which contains exactly m' roots (counted with multiplicities) of the equation $\sigma(P)\,(0, x'^o ; \tau, i\eta'^o) = 0$. Then the map from $(C_{N|X})_{\rho^{-1}(y^\star) \cap U} \times (C_N^{m'})_{y^\star}$ to $(C_{N|X})_{\rho^{-1}(y^\star) \cap U}$:

$$(v, (w_j)_{j=o}^{m'-1}) \longrightarrow P v + \sum_{j=o}^{m'-1} w_j \otimes \delta_1^j$$

is an isomorphism.

This theorem is in particular true all over $T^\star_N X$ and implies :

Corrollaire 3.3 (21) - Let P be a differential operator of order m for which N is non characteristic. Then the map from $\Gamma_N(B_M) \times B_N^m$ to $\Gamma_N(B_M)$:

$$(v, (w_j)_{j=o}^{m-1}) \longrightarrow P v + \sum_{j=o}^{m-1} w_j \otimes \delta_{x_1}^{(j)}$$

is an isomorphism.

Remark that this co
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4. PROPAGATION AND

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Remark that this corollary is in fact equivalent to the Cauchy-Kowalewska theorem (21). We recall now the concept of N-regularity introduced in (22) which will allow us to study propagation of singularities at the boundary.

Definition 3.4. (22)- Let P be a microdifferential operator defined near $x^* \in T_M^* X \cap T_N^* X$. We say that P is N-regular at x^* if for any $u \in \Gamma_{T_M^* X \cap T_N^* X}(C_M)_{x^*}$, $P u \in (C_{N|X})_{x^*}$ implies that $u \in (C_{N|X})_{x^*}$.

This definition does not make any difference between $\bar{M}_+$ and $\bar{M}_-$ and is not precise enough for diffrative problems. It has been refined by K . Kataoka as follow. First, for a section $u \in C_M \mid T_M^* X \cap T_N^* X$, we will say that $SS_N^2(u)$ is contained in Q_+ if the image of u in $\mathcal{H}^1_{T_M^* X \cap T_N^* X}(C_{N|X})$ belongs to the image of the sheaf $C_{N|X}|Q_+$ in $\mathcal{H}^1_{T_M^* X \cap T_N^* X}(C_{N|X})$ (cf. theorem 1.1).
For example, if $u \in \Gamma_{\bar{M}_+}(B_M)$, $SS_N^2(u)$ is contained in Q_+ .

Definition 3.5 (13) - We say that P is N^+-regular at x^* , if for any $u \in \Gamma_{T_M^* X \cap T_N^* X}(C_M)$, such that $SS_N^2(u)$ is contained in Q_+, $P u \in (C_{N|X})_{x^*}$ implies $u \in (C_{N|X})_{x^*}$.

4. PROPAGATION AND REFLECTION

Let N, Y, M, X be as in the preceding section, and let P be a differential operator of order m for which N is non characteristic. Let M_+ be an open half space of M , with N as boundary (we assume of course that M_+ is locally on one side of N). If u is an hyperfunction on M_+ solution of the equation $P u = 0$, we have constructed in (21)(cf. also (15)) the m traces of u on N as follows : let $\bar{u} \in \Gamma_{\bar{M}_+}(M, B_M)$ be an extension of u (which always exists the sheaf B_M being flabby)

Then $P\,\overline{u} \in \Gamma_N(M,\mathcal{B}_M)$ and $P\,\overline{u}$ can be written for a choice of coordinates $(x_1,\dots,x_n)$:

$$P\,\overline{u} = P\,v + \sum_{j=o}^{m-1} w_j \otimes \delta_{x_1}^{(m-1-j)}$$

with $v \in \Gamma_N(M,\mathcal{B}_M)$, $w_j \in \mathcal{B}_N(N)$ (cf. corollary 3.3). It is clear that $(w_j)_{j=o}^{m-1}$ only depend on u , and not on the choice of $\overline{u}$. We say that $(w_o,\dots,w_{m-1})$ are the traces of u on N . The traces depend on the choice of the coordinates, but to say for example, that the p-first traces, $(w_o,\dots,w_{p-1})$, are zero at some point $y^\star \in T_N^\star Y$, does not depend of this choice.

<u>Theorem 4.1</u> (22) – In the preceding situation let u be an hyperfunction on M_+ , solution of $P\,u = 0$. Let $y^\star \in T_N^\star Y$, and let $Z = \rho^{-1}(y^\star) \cap \{\sigma^{-1}(P)\ (0)\}$, where ρ denotes the canonical projection of $T_N^\star X$ on $T_N^\star Y$. Let $Z^+ = Z \cap Q^+$, and $Z^o = Z \cap T_M^\star X$. Let $Z^o = Z^{o,1} \sqcup Z^{o,2}$ be a partition of Z^o.
We assume :

a) at any point of $Z^{o,1}$, P is N^+-regular ;

b) the closure in $T_M^\star X$ of $SS(u)$ does not intersect $Z^{o,1}$;

c) there are $m-p$ points (counted with their multiplicities) in $Z^+ \sqcup Z^{o,1}$;

d) the p-first traces of u are zero near $y^\star$.

Then :

- the m traces of u are zero near $y^\star$
- the closure in $T_M^\star X$ of $SS(u)$ does not intersect $\rho^{-1}(y^\star)$.

<u>Proof</u>

There are $m-p$ points in $Z^+ \sqcup Z^{o,1}$, and p points in $Z^- \sqcup Z^{o,2}$, if we set $Z^- = Z \setminus (Z^o \sqcup Z^+)$. Let $\overline{u} \in \Gamma_{\overline{M}_+}(M,\mathcal{B}_M)$ be an extension of u , and let us consider $P\,\overline{u} \in \Gamma_N(M,\mathcal{B}_M)$

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as an element of $\Gamma(T^*_N X, C_{N|X})$. By hypothesis d) we have, on $\rho^{-1}(y^*)$:

$$P\,\bar{u} = P\,v + \sum_{j=o}^{m-l-p} w_j \otimes \delta_l^{(j)}$$

where $v \in (C_{N|X})_{\rho^{-1}(y^*)}$, $w_j \in (C_N)_{y^*}$.

At any point $x^* \in Z^+ \sqcup Z^{o,1}$, we can find $\tilde{v} \in (C_{N|X})_{x^*}$ with $\bar{u} = \tilde{v}$: if $x^* \in Z^+$ this is because $\Gamma_{\overline{M}_+}(M,B_M)$ is sent into $\Gamma(Q^+, C_{N|X})$, and if $x^* \in Z^{o,1}$ this follows from hypothesis a) and b) . Let U be an open set of $T^*_Z X$, with

$$U \supset Z^+ \sqcup Z^{o,1} , \quad U \cap (Z^- \sqcup Z^{o,2}) = \emptyset .$$

Then $\bar{u} = \tilde{v}$ for $\tilde{v} \in (C_{N|X})_{\rho^{-1}(y^*) \cap U}$. If we replace $\tilde{v}$ by $\tilde{v} - v$, we obtain :

$$P\,(\tilde{v} - v) = \sum_{j=o}^{m-l-p} w_j \otimes \delta_l^{(j)}$$

with $(\tilde{v} - v) \in (C_{N|X})_{\rho^{-1}(y^*) \cap U}$, $w_j \in (C_N)_{y^*}$.

We conclude by theorem 3.2. that all w_j, $j = 0,\dots,m-l-p$, are zero near y^* . Then u admits an extension $\bar{u} \in \Gamma_{\overline{M}_+}(M,B_M)$ with $P\,\bar{u} = 0$ on $\rho^{-1}(y^*) \cap T^*_M X$. As P is invertible on $\rho^{-1}(y^*) \cap T^*_M X - Z^o$, $SS(\bar{u}) \cap \rho^{-1}(y^*) = \emptyset$ by the Holmgren-Kashiwara theorem (theorem 2.2.).

We shall explain how to apply this theorem.

<u>Example 1</u> - Assume $Z^o = \emptyset$. Then we get a micro-local version of the Morrey-Niremberg theorem (18) : if p traces of u are zero near y^* , where $m-p = \# Z^+$, then the m traces of u are zero near y^* .

Example 2 - Assume for simplicity that P has real simple characteristics, the bicharacteristic curves being transversal to N (that is the roots of the equation $\sigma(P)(0,x';\tau,i\eta')=0$ are all purely imaginary, and simple). Let $(b_i)_{i=1}^{m}$ be the bicharacteristic curves issued from $\rho^{-1}(y^{\star})$, where $y^{\star} = (x'^{o}, i\,\eta'^{o})$. Then if the p-first traces of u are zero at $y^{\star}$, and if u is zero along $m-p$ bicharacteristic curves, u will be zero along all the bicharacteristic curves. To apply theorem 4.1. one has to prove the N-regularity of P, but this will be a particular case of theorem 5.1. below. Thus we obtain the Lax - Nirenberg theorem on reflection on singularities, in the analytic case.

In fact theorem 5.1. allows to treat propagation along leaves of any dimension, as shown in the following.

Example 3 - Let us take $M = \mathbb{R}^n$,

$N = \{x \in M \,;\, x_1 = 0\}$, $M_+ = \{x \in M \,;\, x_1 > 0\}$,

$P = \sum_{j=1}^{n-1} D_{x_j}^2 - D_{x_n}$. Let $u \in B(M_+)$ be an hyperfunction solution of $P\,u = 0$. Then if u is micro-analytic in the $+i\,dx_n$ direction (resp. $-i\,dx_n$ direction), the same will be true for the traces of u on N.

More precisely let $x_n^o \in \mathbb{R}$ and let $b^+(x_n^o)$ be the leaf :

$b^+(x_n^o) = \{(x,i\eta) \in T^{\star}_{M_+}X \,;\, x_n = x_n^o \,,\, i\eta = 0,\dots,0,+i)\}$.

We know by the main theorem of (3) that $b^+(x_n^o)$ is contained or disjoined from SS(u). As the operator P is N-regular at any point $(x\,;0,\dots,0,\pm i)$ by theorem 5.1. we conclude that the two traces of u will be micro-analytic at any point

$y^{\star} = (x_2,\dots,x_{n-1},x_n^o\,;\,0,\dots,0,+i)$ of $T^{\star}_N Y$.

We now discuss some situations with diffraction.

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Example 4 - Assume for simplicity $Z = Z^o$ reduced to a single point $x^\star \in T^\star_M X \underset{M}{\times} N$ (with multiplicity m). Then if P is N^+-regular at $x^\star$, and if at least one of the m traces of u is non zero at $\rho(x^\star)$, $x^\star$ belongs to the closure of SS(u) in $T^\star_M X$.

This will be the case if P is micro-hyperbolic for N , or even N^+-semi-hyperbolic (cf. § 6 below), as for the operator $D_1^2 - x_1^k D_2^2$, where $k \in \mathbb{N}$, $M_+ = \{x \in M \,,\, x_1 > 0\}$ (cf. (2),(22), (13), (4)). An other very interesting situation is the following. P is of order two, with simple and real characteristics, the equation $\sigma(P)\ (0,x'^o;\tau,i\,\eta'^o) = 0$ has a double root at $\tau = \tau^o \in i\,\mathbb{R}$, and the bicharacteristic b through $x^\star = (0,\,x'^o;\,\tau^o,\,i\,\eta'^o)$ is contained in $\pi^{-1}(M_+) \cup \{x^\star\}$, and tangent at the first order to $\pi^{-1}(N)$ at $x^\star$. Then P is N^+-regular at $x^\star$ by a theorem of K. Kataoka (13) (cf. also J. Sjöstrand (24) who obtain the same conclusions by a very different approach). This means that if the traces are not both zero at $\rho(x^\star)$, at least one of the half bicharacteristic b^+ or b^- issued from $(0,x'^o\,;\,\tau^o,\,i\,\eta'^o)$ is contained in SS(u).

5. A FIRST CLASS OF N-REGULAR OPERATORS

Let Λ be a conic involutive complex submanifold of $T^\star X$. Let us recall (cf. (1), (3)), that a vector $(x^\star,\theta) \in (T_\Lambda T^\star X)_{x^\star}$ is non micro-characteristic on Λ for a microdifferential operator P , if $\sigma(P)$ being zero to the order r on Λ , $\sigma(P)$ is not zero to an order $r' > r$ on Λ in the θ-direction at $x^\star$. If S is an hypersurface of $T^\star X$ and $f = 0$ is an equation of S , we say that S is non micro-characteristic for P at $x^\star$ if it is the case for $\theta = H_f$, where H_f denotes the Hamiltonian field of f .

We have proved in (23) the following.

Theorem 5.1. (23) Let N be an hypersurface of M, Y the complexification of N in X, and let P be a microdifferential operator defined near $x^* \in T^*_N X \cap T^*_M X$. We assume that there exists an involutive conic analytic manifold $\Lambda^{\mathbb{R}}$ of $T^*_M X$ with $x^* \in \Lambda$, such that $T^*X \times_X Y$ is non micro-characteristic at x^* for P on Λ, the complexification of $\Lambda^{\mathbb{R}}$ in T^*X. Then P is N-regular at x^*.

Sketch of the proof

a) By the trick of the adjunction of an auxillary variable, we may assume that Λ is involutive and regular (the fundamental 1-form ω on T^*X does not vanish on Λ). If $\sigma(P)(x^*) = 0$, Λ and $T^*X \times_X Y$ are orthogonal (there exists f which is zero on $T^*X \times_X Y$ and g which is zero on Λ such that the Poisson bracket $\{f,g\}$ is not zero at x^*) and we may find a real contact transform such that for new coordinates (z,ζ) on T^*X, we have :

$$T^*X \times_X Y = \{(z,\zeta) \in T^*X \,;\; z_1 = 0\}$$

$$\Lambda = \{(z,\zeta) \in T^*X \,;\; \zeta_1 = \ldots = \zeta_p = 0\}$$

As the hypothesis of the theorem remains true if we replace Λ by another submanifold $\Lambda' \subset \Lambda$, we may even assume :

$\Lambda = \{(z,\zeta) \in T^*X \,;\; \zeta_1 = \ldots = \zeta_{n-1} = 0\}$. Let

$\tilde{\Lambda} = \{(z,\zeta) \in T^*X \,;\; \zeta_1 = \ldots = \zeta_{n-1} = y_n = \zeta_n = 0\}$ be the complexification of the bicharacteristic leaves of $\Lambda \cap T^*_M X$, and let C_Λ be the sheaf on $\tilde{\Lambda}$ of microfunction in the $(z_1,\ldots,z_{n-1}, x_n)$ variables, holomorphic in $z_1,\ldots,z_{n-1}$. If we restrict our study to the set of $\tilde{\Lambda}$ where η_n is positive, C_Λ is isomorphic to the inverse image by the projection π of T^*X on X of the sheaf $\mathcal{O}_+$ on $\mathbb{C}^{n-1} \times \mathbb{R}$ of boundary values of holomorphic functions of $\mathbb{C}^{n-1} \times \mathbb{C}_+$, where $\mathbb{C}_+ = \{z_n \in \mathbb{C} \; ; \; y_n > 0\}$.

Then one can developp a theory analogous to the theory of hyperfunctions but with the sheaf C_Λ instead of $\mathcal{O}_X$, ((5), (10), (9)), and we get :

$$\mathcal{H}^p_{T^*_M X \cap \Lambda}(C_\Lambda) = 0 \qquad p \neq n-1 .$$

The sheaf $B_\Lambda \underset{\text{def.}}{=} \mathcal{H}^{n-1}_{T_M X \cap \Lambda}(C_\Lambda)$ is flabby.

There exists a natural injective morphism of $\mathcal{E}_X$-modules

$$C_M \mid T_M X \cap \Lambda \longrightarrow \check{B}_\Lambda$$

(this last point is a particular case of theorem 1.1.).

If we replace Λ by $\Lambda_Y = \rho(T^*X \underset{X}{\times} Y \cap \Lambda)$ and if we denote by C_{Λ_Y} the sheaf on $\tilde{\Lambda}_Y$ of microfunctions of the $(z_2, \ldots, z_{n-1}, x_n)$ variables, holomorphic in the $z_2, \ldots, z_{n-1}$ variables, we construct in the same way the sheaf $\check{B}_{\Lambda_Y}$ on $T^*_N Y \cap \Lambda_Y$.

b) Let $x^* = (0, x'^0 \; ; \; 0, \ldots, 0, i) \in T^*_M X \cap T^*_N X \cap \Lambda$ and let m be the order of the root of the equation $\sigma(P)(0, x'^0, \tau, 0, \ldots, 0, i)$ at $\tau = 0$. By the hypothesis that $T^*X \underset{X}{\times} Y$ is non micro-characteristic for P on Λ we may write, after dividing P by an invertible operator :

$$P(z, D_z) = \sum_{|\alpha| \leq m} A_\alpha(z, D_{z'}) D^\alpha_z$$

where $A_\alpha(z, D_{z'})$ are micro-differential operators not depending on D_{z_1}, of order 0 defined near x^*, and where :

$A_{(m,0,\dots,0)} \equiv 1$, and $\alpha = (\alpha_1,\dots,\alpha_{n-1},0)$ when $|\alpha| = m$.

c) Let us write N for $T^*_M X \times_N N$. We want to prove that :

$$u \in \Gamma_N (C_M)_{x^*},\ P u = \sum_{j=o}^{m-1} w_j \otimes \delta_1^{(j)}$$

where $w_j \in (C_N)_{\rho(x^*)}$, implies $u = 0$. By the injectivity of the morphism from $C_M | \Lambda \cap T^*_M X$ to B_Λ, it is enough to prove the following stronger result :

Lemma 5.2 Under the previous hypothesis on the operator P, the map from $\Gamma_N(\check{B}_\Lambda) \times \check{B}^m_{\Lambda_Y}$ to $\Gamma_N(\check{B}_\Lambda)$:

$$(v,(w_j)_{j=o}^{m-1}) \longrightarrow P v + \sum_{j=o}^{m-1} w_j \otimes \delta_1^{(j)} \quad \text{is an}$$

isomorphism.

(Compare with corollary 3.3.).

Proof of the lemma

We identify C_Λ with the sheaf $\mathcal{O}_+$ on $\mathbb{C}^{n-1} \times \mathbb{R}$, and C_{Λ_Y} with the corresponding sheaf on $(\mathbb{C}^{n-1} \times \mathbb{R}) \cap Y$. At this stage of the proof it is convenient to use the language of derived category.

Let $\mathcal{M}$ be the coherent $\mathcal{E}_X$-module $\mathcal{E}_X/\mathcal{E}_X P$, and $\mathcal{M}_Y$ the induced system by $\mathcal{M}$ on Y (then $\mathcal{M}_Y$ is locally isomorphic to $\mathcal{E}^m_Y$). We have the two following isomorphisms which are easy consequences of the results of (3) or (11), which traduce the precise Cauchy-Kowalewska theorem for P in C_Λ :

$$(*) \quad \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C_\Lambda)|Y \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, C_{\Lambda_Y})$$

$$(**) \quad \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C_\Lambda)|Y \simeq \mathbb{R}\Gamma_Y \; \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C_\Lambda) \; [+2] \; .$$

Then we get

$$(***) \quad \mathbb{R}\Gamma_Y \; \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, C_\Lambda) \; [+2] \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, C_{\Lambda_Y})$$

and one apply the functor

$\mathbb{R}\Gamma_N \circ [n-1]$ to this isomorphism.

As $\mathbb{R}\Gamma_N = \mathbb{R}\Gamma_N \circ \mathbb{R}\Gamma_M$, we get :

$$\mathbb{R}\Gamma_N \; \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathbb{R}\Gamma_M(C_\Lambda)) \; [n+1]$$

$$\simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \mathbb{R}\Gamma_N(C_{\Lambda_Y})) \; [n-1] \; .$$

But we have :

$$\mathbb{R}\Gamma_N \; \mathbb{R}\Gamma_M(C_\Lambda) \; [n] = \Gamma_N(\check{B}_\Lambda)$$

$$\mathbb{R}\Gamma_N(C_{\Lambda_Y}) \; [n-1] = \check{B}_{\Lambda_Y}$$

we obtain :

$$\mathcal{E}xt^1_{\mathcal{E}_X}(\mathcal{M}, \Gamma_N(\check{B}_\Lambda)) \simeq \mathcal{H}om_{\mathcal{E}_Y}(\mathcal{M}_Y, \check{B}_{\Lambda_Y})$$

or $\Gamma_N(\check{B}_\Lambda) / P\Gamma_N(\check{B}_\Lambda) \simeq B^m_{\Lambda_Y}$

Of course one should verify that this last isomorphism is equivalent to that of the lemma, but there is no difficulty in it.

Corollary 5.2 In the situation of theorem 5.1 we assume moreover that P is a differential operator of order m, and that any $\theta \in T_\Lambda \; T^\star X$ is non micro-characteristic for P on Λ. Let $\Lambda_Y = \rho(\Lambda \cap T^\star X \underset{X}{\times} Y)$ and $\Lambda^{\mathbb{R}}_{\dot{Y}} = \Lambda_Y \cap T^\star_N Y$, and assume that $\rho^{-1}(\Lambda_Y) \cap \sigma^{-1}(P)\,(0) = \Lambda$. Then if u is an hyperfunction on M^+ solution of $Pu = 0$, and if $\gamma(u)$ denotes the m traces of u, we have: $SS(\gamma(u))$ is a union of bicharacteristic leaves of $\Lambda^{\mathbb{R}}_Y$.

Proof

Let $y^\star \in \Lambda^{\mathbb{R}}_Y$, $\{x^\star_i, i \in I\} = \rho^{-1}(y^\star) \cap \sigma^{-1}(P)(0)$. Let $b(y^\star)$ in the bicharacteristic leaf through $y^\star$ in $\Lambda^{\mathbb{R}}_Y$ and $b(x^\star_i)$ the bicharacteristic leaves through $x^\star_i$ in $\Lambda^{\mathbb{R}}$.
Assume that $y^\star \notin SS(\gamma(u))$. Then $\rho^{-1}(y^\star) \cap \overline{SS\;(u)} = \emptyset$ by the Holmgren-Kashiwara theorem (or by theorem 4.1 where we take $g^{0,1} = \emptyset$, $m = p$). It follows from the main theorem of [3] that $b(x^\star_i) \cap SS(u) = \emptyset \;\; \forall i \in I$. As the operator P is N-regular at each point of $\Lambda^{\mathbb{R}} \cap T^\star_M X \underset{M}{\times} N$ by theorem 5.1 we get by theorem 4.1 that $b(y^\star) \cap SS(\gamma(u)) = \emptyset$.

Example Let $f(x,t)$ be an hyperfunction on $\mathbb{R}^2$ which is the boundary value of $g(x + iy, t)$, hyperfunction defined for $y > 0$, solution of $(D_x + i\,D_y)g = 0$. Then if $(x_o, t_o\,;\,0, i\,dt) \notin SS(f)$, we have : $(x, t_o\,;\,0, i\,d\,t) \notin SS(f) \;\; \forall x \in \mathbb{R}$.

6. HYPERBOLICITY AN

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6. HYPERBOLICITY AND N-REGULARITY

Let M be a real analytic manifold, X a complexification of M, P a microdifferential operator defined in a neighborhood of $x^* \in T^*_M X$. Let $\theta \in T_{T^*_M X} T^*X$ be a vector of the normal bundle to $T^*_M X$ in T^*X at x^*. Recall the definition of Kashiwara - Kawai (8) : P is micro-hyperbolic at x^* in the θ-direction if, for a choice of local coordinates $z = (z_1,\dots,z_n)$ on X, (z,ζ) on T^*X, with $z = x+iy$, $\zeta = \xi + i\eta$, $T^*_M X = \{ (z,\zeta) \in T^*X \,;\, y = \xi = 0 \}$, $x^* = (x^o, i\eta^o)$, we have for an $\varepsilon_o > 0$:

$$\begin{cases} \sigma(P)\,(\,(x,i\eta) + \varepsilon\,\theta) \neq 0 & \text{for} \\ |(x,i\eta) - (x^o,i\eta^o)| < \varepsilon_o\,, & 0 < \varepsilon < \varepsilon_o\,. \end{cases}$$

Thanks to the local Bochner tube theorem this definition is free from coordinates (and can be extended to overdetermined systems (11)).

We denote by H the symplectic isomorphism from T^*T^*X to TT^*X defined for $v \in TT^*X$, $\theta \in T^*T^*X$ by :

$$<\theta, v> = <d\omega\,,\, v \wedge H(\theta)>$$

where ω is the canonical 1-form on T^*X. For local coordinates (z,ζ) on T^*X, with $\omega = \sum_j \zeta_j\, dz_j$,

$$H(dz_j) = -\frac{\partial}{\partial\zeta}\,, \quad H(d\,\zeta_j) = \frac{\partial}{\partial z_j}\,.$$

Let $H^{\mathbb{R}}$ be the real symplectic isomorphism associated to $\mathrm{Re}\,d\omega$ on the real analytic manifold $(T^*X)^{\mathbb{R}}$. Then $H^{\mathbb{R}}$ defines an isomorphism from $T^*(T^*_M X)$ on $T_{T^*_M X}(T^*X)$. Moreover, if f is an holomorphic function on T^*X, we have $\mathrm{Re}\,H_f = H^{\mathbb{R}}_{\mathrm{Re} f}$.

If f is a real analytic function on M, we still write f to denote the complexification of f in X, or even to denote $\check{f} \circ \pi$, the inverse image of f on $T^*_M X$, or on T^*X. Let now N be an analytic hypersurface of M defined by $f = 0$, with $df \neq 0$ on N, and let $M_+ = \{x \in M ; f(x) > 0\}$. If $x \in N$, $df(x)$ is called the conormal to $\overline{M}_+$ at x and we say (cf. (11)) that $df(x)$ is micro-hyperbolic for P at $x^* \in T^*_M X \times_M N$, with $\pi(x^*) = x$, if $H(df(x))$ is micro-hyperbolic for P at that point.

<u>Theorem 6.1 (22)</u> In the preceding situation, assume that the conormal to $\overline{M}_+$ at x^* is micro-hyperbolic for P at x^*. Then P is N^+-regular at x^*.

<u>Proof</u>

Let $\Omega_o = \{z \in \mathbb{C}^n ; y_n > \sum_{j=1}^{n-1} y_j^2\}$,

an $\Omega_1 = \{z \in \mathbb{C}^n ; y_n > \sum_{j=2}^{n-1} y_j^2\}$,

be the open sets of $\mathbb{C}^n$ already considered in § 1, $\mathcal{O}^+_{\partial\Omega j}$ the sheaves on $\partial\Omega_j$ $(j = 0,1)$ of boundary values of holomorphic functions of Ω_j, and $\mathcal{C}^+_{\partial\Omega_j}$ the inverse image by π on $(T^*_{\partial\Omega_j} X)^+$, the exterior conormal bundles to $\partial\Omega_j$, of the preceding sheaves (cf. (11)). A quantized complex transform Φ allows us to replace $\mathcal{C}_M$ by $\mathcal{C}^+_{\partial\Omega_o}$ and $\mathcal{C}_{N|X}$ by $\mathcal{C}^+_{\partial\Omega_1}$. We introduce also the sets :

$$\Omega_o^+ = \Omega_o \cup \{ z \in \Omega_1 , x_1 > 0 \}$$

$$\Omega_o^- = \Omega_o \cup \{ z \in \Omega_1 , x_1 > 0 \}$$

and the corresponding sheaves $\mathcal{C}^+_{\partial\Omega_o^+}$ and $\mathcal{C}^+_{\partial\Omega_o^-}$. Then we have

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exact sequence of sheaves on $(T^{*}_{\partial\Omega_o})^{+} \cap (T^{*}_{\partial\Omega_1}X)^{+}$

$$0 \longrightarrow C^{+}_{\partial\Omega_1} \longrightarrow C^{+}_{\partial\Omega_o^+} \oplus C^{+}_{\partial\Omega_o^-} \longrightarrow C^{+}_{\partial\Omega_o} \longrightarrow 0$$

we still denote by P the image of P by the quantized contact transform Φ. The hypothesis of micro-hyperbolicity implies that P is an isomorphism from $\Gamma_{\{x_1 \leq o\}}(C^{+}_{\partial\Omega_o})$ onto itsef at $\varphi(x^{*})$ ((11), theorem 5.1.2), and an isomorphism from $C^{+}_{\partial\Omega_o^+}$ onto itself at $\varphi(x^{*})$ ((11), proposition 5.5.1). Let u be a section of $C^{+}_{\partial\Omega_o^-} \cap \Gamma_{\{x_1 \leq o\}}(C^{+}_{\partial\Omega_o})$ at $\Phi(x^{*})$ with $Pu \in C^{+}_{\partial\Omega_o^+}$. Let $v \in C^{+}_{\partial\Omega_o^+}$ be a solution of $Pv = Pu$. Then $u - v = 0$ in $C^{+}_{\partial\Omega_o}$ because $u - v \in \Gamma_{\{x_1 \leq o\}}(C^{+}_{\partial\Omega_o})$ and u comes from a section of $C^{+}_{\partial\Omega_1}$, which achieves the proof.

<u>Remark</u>

In fact, we had only proved theorem 6.1. in (22) under the stronger assumption that the conormal to $\overline{M}_+$ and its opposite, are micro-hyperbolic.

It is clear that theorem 6.1. extends to (overdetermined) systems.

Theorem 6.1 as been refined by K. Kataoka as follow (13).

We choose local coordinates $z = (z_1,\dots,z_n)$ on X, with $z = x+iy$, $M = \{z \in X ; y = 0\}$, $M_+ = \{x \in M ; x_1 > 0\}$.

Let P be a microdifferential operator defined near $(x^o ; i\eta^o) \in T^{*}_{M}X \underset{M}{\times} N$. One says, with A. Kaneko (4) and K. Kataoka (13) that P is semi-hyperbolic with respect to dx_1 at $(x^o, i\eta^o)$ if there exists $\varepsilon_o > 0$ such that :

$$\begin{cases} |x - x^o| < \varepsilon_o \ , \ |\eta - \eta_o| < \varepsilon_o \ , \ x_1 \geq 0 \ , \ 0 < \varepsilon < \varepsilon_o \\ \text{implies} \quad \sigma(P)\,(x, i\eta + \varepsilon\theta) \neq 0 \quad \text{where} \quad \theta = (-1, 0, \ldots, 0). \end{cases}$$

Then Kataoka has proved :

Theorem 6.2. (13) If P is semi-hyperbolic with respect to dx_1 at $(x^o, i\eta^o)$, P is N_+-regular at that point.

We do not give the proof here wich is based on an inequality obtained by applying the local Bochner tube theorem to $\sigma(P)\ (z_1^2, z' ; \zeta)$ (cf. also (2) where the same method was already used to obtain a local (non micro-local) version of theorem 6.2).

Let Γ be an open convex cone of $\pi^{-1}(M_+)$ in $T^*_M X$, with vertex at x^* . We may extend the definition of semi-hyperbolicity and say that P is Γ-hyperbolic with respect to dx_1 at x^* if $\sigma(P)(x, i\eta + \varepsilon\theta) \neq 0$ for $|x - x^o| < \varepsilon_o$, $|\eta - \eta_o| < \varepsilon_o$, $0 < \varepsilon < \varepsilon_o$, $(x, i\eta) \in \overline{\Gamma}$, with $\theta = (-1, 0 \ldots 0)$.

It would be very interesting to prove the N_+-regularity of P under the assumption of Γ-hyperbolicity for a non void convex cone Γ such that $\langle dx_1 , v \rangle \geq 0 \ \ \forall v \in \Gamma$, to cover the following example to which generic diffraction problems can be reduced.

Example Assume that

$\sigma(P)\ (x, i\eta) = (i\eta_1)^2 - (x_1 - x_2)^k \ \ a(x, i\eta')$ where a is an homogeneous symbol of order two not depending on η_1 , real and positive on $T^*_M X$, defined near $x^* = (0 ; i\, dx_n)$, with $n \geq 3$.

For $k = 1$, K. Kat
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The case where k
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To end this sectio
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$= (0 ; i\, d x_n)$,

For $k = 1$, K. Kataoka (13) has proved the N_+-regularity of P at x^* , and J. Sjöstrand (24) has obtained in this case similar results.

The case where k is odd, $k > 1$, seems to remain open, but we remark that P is "Γ-hyperbolic" at x^* in the $d x_1$-codirection for $\Gamma = \{ (x,i\eta) \in T^*_M X ; x_1 > |x_2| \}$.

To end this section, let us mention the very last work of J. Sjöstrand (25) who studies various situations where propagation of analytic singularities at the boundary occurs.

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LOWER BOUNDS AT INFINI
WITH CONSTANT COEFFICI

Y. Shibata

Department
Ibaraki, Ja

1. INTRODUCTION

Let Ω be an unbounde
In this note, for th

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Problem 1. Determ
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Problem 2. What c
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