Probabilistic Abstraction for Model Checking: an Approach Based on Property Testing

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Abstract

The goal of model checking is to verify the correctness of a given program, on all its inputs. The main obstacle, in many cases, is the intractably large size of the program's transition system. Property testing is a randomized method to verify whether some fixed property holds on individual inputs, by looking at a small random part of that input. We join the strengths of both approaches by introducing a new notion of probabilistic abstraction, and by extending the framework of model checking to include the use of these abstractions.

Our abstractions map transition systems associated with large graphs to small transition systems associated with small random subgraphs. This reduces the original transition system to a family of small, even constant-size, transition systems. We prove that with high probability, "sufficiently" incorrect programs will be rejected (ε-robustness). We also prove that under a certain condition (exactness), correct programs will never be rejected (soundness).

Our work applies to programs for graph properties such as bipartiteness, k-colorability, or any $\exists\forall$ first order graph properties. Our main contribution is to show how to apply the ideas of property testing to syntactic programs for such properties. We give a concrete example of an abstraction for a program for bipartiteness. Finally, we show that the relaxation of the test alone does not yield transition systems small enough to use the standard model checking method. More specifically, we prove, using methods from communication complexity, that the OBDD size remains exponential for approximate bipartiteness.

1. Introduction

The verification of programs is a fundamental problem in computer science, where logic, complexity and combinatorics have brought new ideas which have been influential in practical applications. We put two general methods together: model checking, where one formally proves that a program is correct for all its inputs, up to a given length, and property testing, where a randomized algorithm makes random local checks within a particular input to decide if this input has a given property. Our approach brings the notions of sampling and approximation from property testing to model checking.

Model checking is an algorithmic method for deciding if a program with bounded inputs, modeled as a transition system, satisfies a specification, expressed as a formula of a temporal logic such as CTL or CTL* [9]. This verification can be carried out, for CTL, in time linear in both the size of the transition system and of the specification. For CTL* it is still linear in the size of the transition system, but exponential in the size of the specification [7]. However, a program given in a classical programming language, like C, converted to a transition system, typically undergoes an exponential blowup in the size of the input. Symbolic model checking [15, 9] addresses this problem by using ordered binary decision diagrams [5, 6] (OBDDs, or equivalently read-once branching programs with an ordering restriction on the variables), which in many practical cases provide a compact representation of the transition system. Nevertheless, in some cases, such as programs for integer multiplication or bipartiteness, the OBDD size remains exponential.

The abstraction method [8] provides a solution in some cases when the OBDD size is intractable. By way of an abstraction, a large transition system is approximated by a smaller one, on which the specification can be efficiently verified. A classical example is multiplication, where modular arithmetic is the basis of the abstraction. Our goal is to extend the range of abstractions to programs for a large family of graphs properties using randomized methods.

In the late eighties the theory of *program checking* and *self-testing/correcting* was pioneered by the work of Blum and Kannan [3], and Blum, Luby and Rubinfeld [4]. This theory addresses the problem of program correctness by verifying a coherence property (such as linearity) between

the outputs of the program on randomly selected inputs. Rubinfeld and Sudan [16] formulated the notion of *property testing* which arises in every such tester. One is interested in deciding whether an object has a global property ϕ by performing random local checks, or queries, on it. The goal is to distinguish with sufficient confidence between objects that satisfy ϕ and those that are ε-far from any objects that satisfy ϕ, for some confidence parameter $\varepsilon > 0$, and some distance measure. The surprising result is that when ε is fixed, this relaxation is sufficient to decide many properties with a sub-linear or even a constant number of queries.

Goldreich, Goldwasser, and Ron [10, 12, 11] investigated property testing for several graph properties such as k-colorability. Alon, Fischer, Krivelevich, and Szegedy [2] showed a general result for all first order graph properties of type $\exists\forall$.

We identify a notion which is implicit in many graph property testers: a graph property ϕ is *ε-reducible to ψ* if testing ψ on small random subgraphs suffices to distinguish between graphs which satisfy ϕ, and those that are ε-far from satisfying ϕ. Our goal will be to distinguish with sufficient confidence between programs that accept only graphs that satisfy ϕ and those which accept some graph that is ε-far from any graph that satisfies ϕ. We introduce *probabilistic abstractions* which to a program associate small random transition systems. We show that for probabilistic abstractions based on ε-reducibility this goal can be achieved.

In Section 2 we review basic notions of model checking and property testing, and define ε-reducibility (**Definition 5**). In Section 3, we introduce the notion of probabilistic abstraction (**Definition 6**). To be useful, an abstraction must preserve the behavior of the program. An abstraction is *sound* (**Definition 8**) if it preserves program correctness. It is *ε-robust* (**Definition 7**) if correctness on the abstraction implies that the original program does not accept any graph that is ε-far from any graph that satisfies ϕ. The latter is an extension to abstraction of robustness introduced in [16]. We show how to derive a probabilistic abstraction using ε-reducibility (Section 3.4). We give a generic proof of ε-robustness for a large class of specifications (**Theorem 4**). Moreover, we give a sufficient condition for soundness to hold (**Theorem 5**). We establish the applicability of our method by applying it to a program for bipartiteness. On the one hand, we show how to construct a robust and sound abstraction on a specific program for testing bipartiteness (**Corollary 1**). On the other hand, in Section 4 we show that abstraction is necessary, in the sense that the relaxation of the test alone does not yield OBDDs small enough to use the standard model checking method. More specifically, we prove, using methods from communication complexity [13], that the OBDD size remains exponential for approximate bipartiteness (**Theorem 6**). This lower bound may also be of independent interest.

2. Framework and preliminaries

2.1. Programs and transition systems

We will use transition systems to represent all possible executions of a given program.

Definition 1. *A* transition system *is a triple $M = \langle S, I, R\rangle$ where S is the* set of states, *$I \subseteq S$ is the* set of initial states *and $R \subseteq S \times S$ is the* transition relation.

To simplify the discussion we only consider programs which implement boolean functions. They will be written in a simple language that manipulates bit, bounded integer and finite array variables, using basic instructions: while statements, conditionals, assignments and a `get` instruction which allows the user to interact with the program. The *input variables* correspond to the function's input. An implicit variable `ack` is set to $false$ at the beginning of the program and is set to $true$ at the end of the computation. An implicit variable `ret` is defined together with the instruction `RETURN` where `RETURN b` sets `ret` to `b` and `ack` to $true$. To verify properties on the behavior of a program, we must know values of the variables at certain points of the program, called *control points*. The control points are at the beginning of lines labeled by integers. An implicit variable `PC` contains the label value of the last visited control point. Appendix A gives a sample toy program, with its transition system and a specification of its behavior, as we define in the following paragraphs.

Let P be such a program with a finite set of variables $\{v_1, \ldots, v_n\}$ including the implicit variables `PC`, `ack`, and `ret`. Each variable v_i ranges over a (finite) domain D_i. We define the *transition system of P*. A *state of P* is an n-tuple $s{=}(s_1, \ldots, s_n)$ corresponding to an assignment of variables $v_1, \ldots, v_n$ at a control point during a computation. The *set of states of P* is $S{=}D_1{\times}\ldots{\times}D_n$. The *initial states of P* are all the possible states before any computation starts, and the *transition relation of P* is the set of all possible transitions of the program between two control points. When the program terminates, the transition system loops with an infinite sequence of transitions on the final state.

Model checking does not manipulate transition system directly; it manipulates a logical representation of the transition system, expressed as a set of relational expressions. A *relational expression* is a formula of the first-order logic built up from the programming language's constants and basic operators (such as $+$, $-$, and $=$). We always assume that relational expressions are in negation normal form, that is, negations pushed down to the atomic level.

Definition 2. *Let $\mathcal{I}$ (resp. $\mathcal{R}$) be a relational expression on S (resp. $S \times S$). Then $(\mathcal{I}, \mathcal{R})$ is a* representation *of a transition system $\langle S, I, R\rangle$ if and only if $I = \{s \in S : \mathcal{I}(s){=}true\}$ and $R = \{(s, s') \in S \times S : \mathcal{R}(s, s'){=}true\}$.*

2.2. Temporal logic and model checking

To express the desired behavior of a transition system (associated with a program), we use a branching-time temporal logic. All our results will be stated for the temporal logic CTL^*. We refer the reader to [9] for more details on CTL^*, but briefly, formulas of CTL^* are defined inductively from a set of atomic propositions and built up by boolean connectives ($\neg$, $\wedge$, $\vee$), path quantifiers $\forall$ ("for all paths") and $\exists$ ("for some path"), temporal operators $\mathbf{X}$ ("next") and $\mathbf{U}$ ("until"). In our framework, the *atomic propositions* are $(v_i = d)$ where v_i is a variable which corresponds to the ith coordinate of a state, and d is any constant.

Let M be a transition system with representation $(\mathcal{I}, \mathcal{R})$ and let Θ be a CTL^* formula. Let us briefly review the symbolic model checking method [9] for $M \models \Theta$ (note that in model checking, the notation $M \models \Theta$ is shorthand for $M, s \models \Theta$ for every initial state s). The verification proceeds in three steps. This process is fully algorithmic, in contrast with methods which require human assistance. First, OBDD representations [5, 6] are constructed for $\mathcal{R}$ and $\mathcal{I}$. Then, an OBDD $\mathrm{check}(\mathcal{R}, \Theta)$ is constructed, whose entries are states of S, such that for every $s \in S$, $(\mathrm{check}(\mathcal{R}, \Theta)(s) = \mathit{true}) \iff (M, s \models \Theta)$. Finally, the verification of $M \models \Theta$ is achieved by checking the validity of the OBDD of $\big(\neg\mathcal{I} \vee \mathrm{check}(\mathcal{R}, \Theta)\big)$.

When the resulting OBDD is polynomial in size, the verification can be carried out in polynomial time. A typical example where this is not the case is multiplication, since any OBDD for multiplication has exponential size [5]. In the next section, we will see how abstraction can overcome this problem.

2.3. Abstractions

The use of an *abstraction* [8] helps in some cases to overcome the problem of intractably large OBDDs. The objective of abstractions is to replace the transition system with an abstract version which is smaller, but sufficient for verifying the specification on original system. For each variable, a surjection is used to reduce the size of the domain, and transitions are made between the resulting equivalence classes, as we define below.

Definition 3. *([8]) Let $M = \langle S, I, R\rangle$ be a transition system, where $S = D_1 \times \ldots \times D_n$. An* abstraction for M *is a surjection $h : S \to \widehat{S}$, such that h can be decomposed into an n-tuple $h = (h_1, \ldots, h_n)$, where $h_i : D_i \to \widehat{D}_i$ is any surjection, and $\widehat{D}_i$ is any set. The* minimal transition system of M with respect to h *is the transition system $\widehat{M}_{\min} = \langle \widehat{S}, \widehat{I}_{\min}, \widehat{R}_{\min}\rangle$ such that $\widehat{S} = \widehat{D}_1 \times \ldots \times \widehat{D}_n$, $\widehat{I}_{\min} = h(I)$, and $(\widehat{s}, \widehat{s}') \in \widehat{R}_{\min} \iff \exists (s, s') \in S^2,\ (h(s) = \widehat{s}) \wedge (h(s') = \widehat{s}') \wedge \big((s, s') \in R\big)$.*

Note that minimal transition systems and all the notions that follow are defined with respect to a fixed abstraction h. When it is not clear from the context, we will specify the abstraction h as a superscript. When h is applied to a variable, it is understood that the corresponding h_i is applied.

For every abstraction h, define the operator $[\cdot]$ so that for any first order formula ϕ:

$$[\phi](\widehat{v}_1, \ldots, \widehat{v}_k) \stackrel{\mathrm{def}}{=} \exists v_1 \ldots \exists v_k \ \big(\textstyle\bigwedge_{i=1}^{k} \widehat{v}_i = h(v_i)\big) \wedge \phi(v_1, \ldots, v_k).$$

Then observe that $\widehat{R}_{\min} = [R]$. In general, it is very difficult to construct $\widehat{M}_{\min}$ because the full description of the transition system M is needed in order to carry out the construction. Nevertheless, one can produce an approximation directly from its representation. Let us first define the notion of approximation.

Definition 4. *([8]) Let $M = \langle S, I, R\rangle$ be a transition system, and let $h : S \to \widehat{S}$ be an abstraction for M. A transition system $\widehat{M} = \langle \widehat{S}, \widehat{I}, \widehat{R}\rangle$* approximates *$M$ with respect to h ($M \sqsubseteq_h \widehat{M}$ for short) if and only if $\widehat{I}_{\min} \subseteq \widehat{I}$ and $\widehat{R}_{\min} \subseteq \widehat{R}$.*

The approximation operator, which is denoted by $\mathcal{A}$, is inductively defined on formulas that are in negation normal form by applying $[\cdot]$ only at the atomic level (including negations). For every transition system $M = \langle S, I, R\rangle$ with the representation $(\mathcal{I}, \mathcal{R})$, we denote by $\mathcal{A}(M)$ the transition system with the set of states $\widehat{S} = h(S)$ and representation $(\mathcal{A}(\mathcal{I}), \mathcal{A}(\mathcal{R}))$. Clarke, Grumberg and Long [8] show that the approximation operator $\mathcal{A}$ gives an approximation for any M, h, that is, $M \sqsubseteq_h \mathcal{A}(M)$.

Let $\widehat{M}$ be an approximation of M. Suppose that $\widehat{M} \models \Theta$. What can we conclude on the concrete model M? To answer, let us first consider the following transformations $\mathcal{C}$ and $\mathcal{D}$ between CTL^* formulas on M and their approximation on $\widehat{M}$. These transformations preserve boolean connectives, path quantifiers, and temporal operators, and act on atomic propositions as follows:

$$\mathcal{C}(\widehat{v}_i = \widehat{d}_i) \stackrel{\mathrm{def}}{=} \textstyle\bigvee_{d_i : h_i(d_i) = \widehat{d}_i} (v_i = d_i),$$
$$\mathcal{D}(v_i = d_i) \stackrel{\mathrm{def}}{=} (\widehat{v}_i = h_i(d_i)).$$

Denote by $\forall\mathrm{CTL}^*$ and $\exists\mathrm{CTL}^*$ the universal fragment and the existential fragment of CTL^*. The following theorem gives correspondences between concrete models and their approximations.

Theorem 1 ([8]). *Let $M = \langle S, I, R\rangle$ be a transition system. Let $h : S \to \widehat{S}$ be an abstraction for M, and let $\widehat{M}$ be such that $M \sqsubseteq_h \widehat{M}$. Let Θ be a $\forall\mathrm{CTL}^*$ formula on $\widehat{M}$, and Θ' be a $\exists\mathrm{CTL}^*$ formula on M. Then*

$$\widehat{M} \models \Theta \implies M \models \mathcal{C}(\Theta),$$
$$\text{and} \quad M \models \Theta' \implies \widehat{M} \models \mathcal{D}(\Theta').$$

The second implication of the theorem is only implicit in [8]. Notice that the two statements are not reciprocals of one another. In both cases, reciprocals can be shown

under certain conditions [8]. The first result validates the usefulness of abstractions in practical model checking. The second will be used in our proof of Theorem 4.

2.4. Property testing

We consider only undirected, simple graphs (no multiple edges or self-loops). For a graph G, we denote by V_G its vertex set, by E_G its edge set, and by n the cardinality $|V_G|$ of V_G. When there is no ambiguity, we will simply write V and E instead of V_G and E_G. In the remainder of the paper we will use the following distance measure: for any two graphs G and G' on the same n-vertex set, $\mathsf{Dist}(G, G')$ is the number of edges on which the graphs disagree, divided by n^2. Our theory and our main results (Theorems 4 and 5) hold for any distance measure.

Let ϕ be a graph property. We use the notation $G \models \phi$ as a shorthand to mean G has the property ϕ. An *ε-test* for ϕ is a probabilistic algorithm that accepts every graph with property ϕ, and rejects with probability $2/3$ every graph which has distance more than ε from any graph having the property.[1] Moreover, an ε-test can only access the input graph by querying whether or not any chosen pair of vertices are adjacent. The property ϕ is called *testable* if for every $\varepsilon > 0$, there exists an ε-test for ϕ whose total number of queries depends on ε, but does not depend on the size of the graph. In several cases, the proof of testability is based on a reduction between two properties. The notion of ε-reducibility highlights this idea. This notion is central to the design of our abstractions. Denote by G_π the vertex-induced subgraph of G on the vertex set $\pi \subseteq V_G$. For any graph property ϕ and any graph G, we denote by $G \models \phi_\varepsilon$ the property:

$$\exists H,\ V_H = V_G,\ \mathsf{Dist}(G, H) \leq \varepsilon,\ \text{and}\ H \models \phi.$$

For any vertex set V and integer $k \geq 1$, let Π denote the set of all $\pi \subseteq V$ such that $|\pi| = k$, where it is understood that Π depends on both k and V. For convenience, we will always assume that $|V| \geq k$.

Definition 5. *Let $\varepsilon > 0$ be a real, $k \geq 1$ an integer, and ϕ, ψ two graph properties. Then ϕ is* (ε, k)-reducible to ψ *if and only if for every graph G,*

$$\begin{aligned} G \models \phi \quad &\Longrightarrow \quad \forall \pi \in \Pi,\ G_\pi \models \psi, \\ G \not\models \phi_\varepsilon \quad &\Longrightarrow \quad \mathop{\mathbf{Pr}}_{\pi \in \Pi}[G_\pi \models \psi] \leq 1/3, \end{aligned}$$

where $\Pi = \{\pi \subseteq V_G : |\pi| = k\}$.

We say that ϕ is *ε-reducible to* ψ if there exists a constant k such that ϕ is (ε, k)-reducible to ψ.

We can recast the testability of c-colorability and bipartiteness [10, 1] in terms of ε-reducibility.

Theorem 2 ([1]). *For all $c \geq 3$, $\varepsilon > 0$,*

1. *c-colorability is $(\varepsilon, O((c \ln c)/\varepsilon^2))$-reducible to c-colorability;*
2. *bipartiteness is $(\varepsilon, O((\ln^4(\frac{1}{\varepsilon}) \ln\ln(\frac{1}{\varepsilon}))/\varepsilon))$-reducible to bipartiteness.*

Recently, Alon, Fischer, Krivelevich, and Szegedy [2] showed that all first order graph properties of type $\exists\forall$ have an ε-tester. Their results can also be recast in terms of ε-reducibility, as follows. Note, however, that in this result, the function f is a tower of towers.

Theorem 3 ([2]). *There exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, such that every first order graph property of type $\exists\forall$ with t bound variables is $(\varepsilon, O(f(t + 1/\varepsilon)))$-reducible to some graph property.*

3. Verification of graph properties

3.1. Context and objectives

In order to extend the framework of model checking to include the use of probabilistic abstractions, we would like to prove an analogue of Theorem 1. We prove that with high probability, "sufficiently" incorrect programs (in a sense to be defined below) will be rejected (*ε-robustness*). We also prove a reciprocal, which states that under a certain condition (*exactness*), correct programs will never be rejected (*soundness*).

A second goal is to extend the framework of model checking to include the verification of programs purportedly deciding graph properties. The standard model checking method is not adapted to programs on inputs that are first-order structures such as graphs. We overcome this by dealing with the specification of the program, and the property of the graph, separately. The former is handled with standard tools of model checking. The latter will reduce, as a result of the ε-reduction, to verifying a property on constant size graphs, which can be carried out in constant time.

We give an example of a very simple program for bipartiteness, together with an abstraction, and show that the approximation operator $\mathcal{A}$ results in an exact approximation of the transition system. Hence, this abstraction can be used to verify the program. One might ask whether the relaxation brought about by the use of property testing is in itself enough to beat the exponential lower bounds on the original problem. We will show in Section 4 that this is not the case, by giving a lower bound on the relaxed version of bipartiteness.

3.2. Handling first order structures

Consider the following example of a formula we would like to verify, where P is a program which is supposed to compute some boolean function on bounded size graphs,

[1] We may also consider two-sided error, and the choice of $2/3$ as the success probability is of course arbitrary.

and ϕ is a graph property:

The program P accepts only graphs which satisfy ϕ. Suppose that G is an input variable of P, such that G is interpreted as a graph G (with respect to some fixed encoding): this will be written as $\mathsf{G} = G$. A state s of the transition system $M = \langle S, I, R\rangle$ of P is a finite sequence of variables $(\ldots, \mathsf{G}, \ldots)$. For every graph G, we then define $I_G = \{s \in I : \mathsf{G} = G\}$. Formally, what we would like to check is the following:

$$\forall G\Big(\big(\forall s \in I_G \quad M, s \models \exists\,((\neg \mathtt{ack})\,\mathbf{U}(\mathtt{ack} \wedge \mathtt{ret}))\big) \quad \Longrightarrow \quad G \models \phi\Big).$$

Note that on the right hand side of the implication, ϕ is interpreted in a structure for G which does not include the transition system. This is because the standard model checking algorithms are not suited for programs with inputs that are first order structures. When there is no ambiguity, we will write $M, G \models \Theta$ instead of $\forall s \in I_G, \quad M, s \models \Theta$.

More generally, our framework applies to the following type of formulas:

$$\forall G \quad (M, G \models \Theta \quad \Longrightarrow \quad G \models \phi), \tag{1}$$

where the input includes the graph G and may also include auxiliary data, Θ is a CTL^* formula, and ϕ is a graph property. Henceforth, we always assume a graph G to be an input variable in the program. Since G is a bounded size graph and ϕ is a formula expressing a property on G, we can determine whether $G \models \phi$ using an OBDD. Let $\mathrm{sat}(\phi, G)$ be such an OBDD. Then verifying (1) can be achieved by checking the validity of $\Big((\neg \mathcal{I}_G \vee \mathrm{check}(\mathcal{R}, \Theta)) \implies \mathrm{sat}(\phi, G)\Big)$, where $\mathcal{I}_G = \mathcal{I}(G/\mathsf{G})$ (i.e., all occurrences of the variable G are substituted for G).

For the graph properties that we consider, such as bipartiteness, the OBDDs for $G \models \phi$ have exponential size. As we show in Section 4, the relaxation brought about by property testing is not sufficient to reduce the OBDD size of bipartiteness. We use ε-reducibility to construct probabilistic abstractions, yielding smaller, even constant-size, OBDDs. Using such OBDDs, we are able to guarantee that P approximately decides ϕ on all its inputs.

3.3. Probabilistic abstractions

Definition 6. *Let M be a transition system. A* probabilistic abstraction of M *is a triple $(\mathcal{H}, \mathcal{M}, \mu)$, where $\mathcal{H}$ is a set of abstractions for M, $\mathcal{M}$ is a functional which maps every $h \in \mathcal{H}$ to a transition system $\widehat{M}^h = \mathcal{M}(h)$ such that $M \sqsubseteq_h \widehat{M}^h$, and μ is a probability distribution over $\mathcal{H}$.*

Let Θ be a CTL^* formula on M, and ψ be a graph property. Then any probabilistic abstraction of M induces the following probabilistic test, where we require that $\widehat{G}^h$ be interpreted as a graph, and the operator $\mathcal{D}$ (see Section 2.3) is applied with respect to the chosen abstraction h.

Generic Test $\big((\mathcal{H}, \mathcal{M}, \mu), \Theta, \psi\big)$
1. Choose an element $h \in \mathcal{H}$ according to μ.
2. Accept if (and only if)
$\forall \widehat{G}^h \quad (\widehat{M}^h, \widehat{G}^h \models \mathcal{D}(\Theta) \quad \Longrightarrow \quad \widehat{G}^h \models \psi)$.

The probability that the test rejects will be denoted by $\mathsf{Rej}\big((\mathcal{H}, \mathcal{M}, \mu), \Theta, \psi\big)$. The distribution μ will be omitted when it denotes the uniform probability distribution. To be useful in practice, a probabilistic abstraction should be both ε-robust (programs are rejected with probability $2/3$ if the relaxed specification is false for some input) and sound (no correct programs are rejected), in which case we say that it is an ε-abstraction. When this is the case, checking the correctness of a program can be easily done on the abstracted model with high confidence using **Generic Test**. Fix a confidence parameter $0<\gamma<1$, and iterate **Generic Test** $O(\ln 1/\gamma)$ times. If the program is correct, **Generic Test** always accepts; and if there is an instance on which the program is not correct with respect to the relaxed specification, **Generic Test** rejects at least once with probability at least $(1-\gamma)$.

Definition 7. *Let M be a transition system, $\varepsilon > 0$, Θ be a CTL^* formula, and let ϕ, ψ be two graph properties. A probabilistic abstraction $(\mathcal{H}, \mathcal{M}, \mu)$ of M is* ε-robust with respect to (Θ, ϕ, ψ) *if*

$$\big(\exists G \quad (M, G \models \Theta \quad \text{and} \quad G \not\models \phi_\varepsilon)\big) \quad \Longrightarrow \quad \mathsf{Rej}\big((\mathcal{H}, \mathcal{M}, \mu), \Theta, \psi\big) \geq \tfrac{2}{3}.$$

Definition 8. *Let M be a transition system, Θ be a CTL^* formula, and let ϕ, ψ be two graph properties. A probabilistic abstraction $(\mathcal{H}, \mathcal{M}, \mu)$ of M is* sound with respect to (Θ, ϕ, ψ) *if*

$$\big(\forall G \quad (M, G \models \Theta \quad \Longrightarrow \quad G \models \phi)\big) \quad \Longrightarrow \quad \mathsf{Rej}\big((\mathcal{H}, \mathcal{M}, \mu), \Theta, \psi\big) = 0.$$

Definition 9. *Let M be a transition system, $\varepsilon > 0$, Θ be a CTL^* formula, and let ϕ, ψ be two graph properties. A probabilistic abstraction $(\mathcal{H}, \mathcal{M}, \mu)$ of M is an* ε-abstraction for (Θ, ϕ, ψ) *if it is both ε-robust and sound with respect to (Θ, ϕ, ψ).*

3.4. Constructing ε-abstractions

We now explain how to construct ε-abstractions based on ε-reducibility. Fix $\varepsilon > 0$, and assume that ϕ is (ε, k)-reducible to ψ, for some $k \geq 1$. We give a generic proof of robustness of our probabilistic abstraction, and we isolate a sufficient condition which implies soundness. Under this condition, we obtain an ε-abstraction. As in Section 2.4, for any fixed k and any fixed vertex set V, we let Π be the set of all subsets π of V with $|\pi| = k$, and for any graph G with $V_G = V$, the vertex-induced subgraph on the vertex set π is

denoted by G_π.

Since we relax ϕ with respect to ε, we can decompose our initial specification (1) into the following family of reduced specifications:

$$\{\forall G \quad (M, G \models \Theta \quad \Longrightarrow \quad G_\pi \models \psi) : \pi \in \Pi\}.$$

For every π, the corresponding reduced specification can now be subject to an abstraction h_π. Every corresponding abstracted variable v and constant d will be denoted respectively by $\widehat{v}^\pi$ and $\widehat{d}^\pi$. We require that the abstraction of G be exactly G_π, that is, $\widehat{G}^\pi = G_\pi$. Let $\widehat{M}^\pi$ be such that $M \sqsubseteq_{h_\pi} \widehat{M}^\pi$. We define the (uniform) probabilistic abstraction $(\mathcal{H}, \mathcal{M})$ (also denoted by $(\Pi, \mathcal{M})$) as $\mathcal{H} = \{h_\pi : \pi \in \Pi\}$ and $\mathcal{M}(h_\pi) = \widehat{M}^\pi$, for every $\pi \in \Pi$. This leads to the following test, derived from **Generic Test** for this family of abstractions:

Graph Test$\big((\Pi, \mathcal{M}), \Theta, \psi\big)$
1. Randomly choose a subset of vertices $\pi \in \Pi$.
2. Accept if (and only if)
$\forall \widehat{G}^\pi \quad (\widehat{M}^\pi, \widehat{G}^\pi \models \mathcal{D}(\Theta) \quad \Longrightarrow \quad \widehat{G}^\pi \models \psi)$.

We show that if Θ is an $\exists\mathrm{CTL}^*$ formula and ϕ is ε-reducible to ψ, then our probabilistic abstraction is ε-robust. This, together with its conditional reciprocal in Theorem 5, establishes the validity of the method.

Theorem 4. *Let Θ be a $\exists\mathrm{CTL}^*$ formula. Let $\varepsilon > 0$ be a real, $k \geq 1$ an integer, and let ϕ be (ε, k)-reducible to ψ. Let $(\Pi, \mathcal{M})$ be a probabilistic abstraction such that $\widehat{G}^\pi = G_\pi$, for every $\pi \in \Pi$. Then $(\Pi, \mathcal{M})$ is ε-robust with respect to (Θ, ϕ, ψ).*

Proof. Let G be such that $M, G \models \Theta$ and $G \not\models \phi_\varepsilon$. By Theorem 1, $\widehat{M}^\pi, \widehat{G}^\pi \models \mathcal{D}(\Theta)$, for every $\pi \in \Pi$. Moreover, by definition of (ε, k)-reducibility we know that $\mathbf{Pr}_{\pi\in\Pi}\left[\widehat{G}^\pi \models \psi\right] \leq 1/3$. Therefore,

$$\mathbf{Pr}_{\pi\in\Pi}\left[\widehat{M}^\pi, \widehat{G}^\pi \models \mathcal{D}(\Theta) \quad \Longrightarrow \quad \widehat{G}^\pi \models \psi\right] \leq \tfrac{1}{3}.$$

We conclude by observing that $1-\mathsf{Rej}(M)$ (in **Graph Test**) is bounded above by the term on the left hand side of the inequality. □

Having shown that the abstraction is ε-robust, we give a sufficient condition for soundness: exactness.

Definition 10. *Let M be a transition system, Θ be a* CTL^* *formula, h be an abstraction, and let $\widehat{M}$ be such that $M \sqsubseteq_h \widehat{M}$. Then the approximation $\widehat{M}$ is* exact with respect to Θ *if and only if for every graph $\widehat{G}$:*

$$\widehat{M}, \widehat{G} \models \mathcal{D}(\Theta) \quad \Longrightarrow \quad \exists H\,, \widehat{H} = \widehat{G} \quad \text{and} \quad M, H \models \Theta.$$

Theorem 5. *Let Θ be a $\exists\mathrm{CTL}^*$ formula. Let $\varepsilon > 0$ be a real, $k \geq 1$ an integer, and let ϕ be (ε, k)-reducible to ψ. Let $(\Pi, \mathcal{M})$ be a probabilistic abstraction such that $\widehat{G}^\pi = G_\pi$ and $\widehat{M}^\pi$ is an exact approximation with respect to Θ, for every $\pi \in \Pi$. Then $(\Pi, \mathcal{M})$ is sound with respect to (Θ, ϕ, ψ).*

Proof. Fix $\pi \in \Pi$. Let $\widehat{G}^\pi$ be a k-vertex graph such that $\widehat{M}^\pi, \widehat{G}^\pi \models \mathcal{D}(\Theta)$. From the exactness of $\widehat{M}^\pi$, there exists a graph H such that $\widehat{H}^\pi = \widehat{G}^\pi$ and $M, H \models \Theta$. Therefore, from the hypotheses we get $H \models \phi$. The (ε, k)-reducibility of ϕ to ψ implies that $H_\pi \models \psi$, that is, $\widehat{G}^\pi \models \psi$. Thus, for all $\pi \in \Pi$ and $\widehat{G}^\pi$: $\widehat{M}^\pi, \widehat{G}^\pi \models \mathcal{D}(\Theta) \quad \Longrightarrow \quad \widehat{G}^\pi \models \psi$. □

3.5. An ε-abstraction for bipartiteness

In this section, we give a short program for bipartiteness, and an ε-abstraction for this program. We consider a function which decides, given a graph G and a coloring $Color$ (entered by the user), if $Color$ is a bipartition for G. The graph G is represented in the program in the natural way by the upper triangular entries of a boolean matrix variable `G` and $Color$ by a boolean array variable `Color`.

```
FUNCTION CHECK-PARTITION
  CONSTANT INTEGER n=10000
  INPUT G : ARRAY[n,n] of BOOLEAN
  VAR Color : ARRAY[n] of BOOLEAN
  VAR u,v : INTEGER 1..n+1
1: get(Color)
2: u=2
3: WHILE u<=n DO {
    v=1
4:  WHILE v<=u-1 DO {
5:   IF G[u,v]&&(Color[u]=Color[v]) RETURN false
     v=v+1         }
    u=u+1       }
6: RETURN true
```

We want to verify that, for every input `G`, if there exists an input value for `Color` for which the program accepts, then `G` represents a bipartite graph. More formally, we want to verify the following property:

$$\forall G \quad \Big(M, G \models \exists\big((\neg \mathtt{ack})\,\mathbf{U}(\mathtt{ack} \wedge \mathtt{ret})\big) \Longrightarrow G \text{ is bipartite}\Big). \quad (2)$$

Note that $\exists$ ranges over all the possible initial values of `Color` which the user can enter with the instruction `get(Color)`. For each π we define the abstraction which maps G to the subgraph G_π, $Color$ to the coloring on the subset of vertices induced by π, and `u`, `v` refer to the vertices as follows: $\widehat{u}^\pi$ is u if $u \in \pi$, and is $\min\{w : \ \forall t\ (w \leq t \leq u \implies t \notin \pi)\}$ otherwise.

For this abstraction, the following lemma holds.

Lemma 1. *For every $\pi \in \Pi$, $\mathcal{A}^\pi(M)$ is exact with respect to the $\exists\mathrm{CTL}^*$ formula of Equation* (2).

Proof. The proof is in two parts. First, we will prove that the abstraction which maps $\mathtt{G} \mapsto \mathtt{G}_\pi$, and preserves the other variables, is sound for every π. This is the most difficult part. Then we show how it can be extended to the complete abstraction.

For convenience, we use a compact representation of relational expressions representing the transition relation.

Each line corresponds to a transition between two control points. The transition relation is represented by the disjunction of these lines. We use '$i \mapsto j$:' as an abbreviation for $(\mathtt{PC} = i) \wedge (\mathtt{PC}' = j)$. On any given line, for any pair $(\mathtt{v}, \mathtt{v}')$ of program variables, if $\mathtt{v}'$ does not occur in the relational expression, then the atomic proposition $(\mathtt{v}' = \mathtt{v})$ is understood, but omitted from the compact form. Furthermore, the expression $(\mathtt{v}' = *)$ is used when the value of $\mathtt{v}'$ is unspecified. This typically occurs after a `get` instruction, and corresponds to a nondeterministic transition.

The initial states of the transition system of `CHECK-PARTITION` are $(\mathtt{ack}{=}\mathit{false}) \wedge (\mathtt{PC}{=}1)$. The relational expression of the transition system of `CHECK-PARTITION` is given in compact form by the disjunction of the following boolean formulas.

$1 \mapsto 2 : (\mathtt{Color'}{=}*)$
$2 \mapsto 3 : (\mathtt{u'}{=}2)$
$3 \mapsto 4 : (\mathtt{u} \leq \mathtt{n}) \wedge (\mathtt{v'}{=}1)$
$3 \mapsto 6 : (\mathtt{u}{=}\mathtt{n}+1)$
$4 \mapsto 3 : (\mathtt{v}{=}\mathtt{u}) \wedge (\mathtt{u'}{=}\mathtt{u}+1)$
$4 \mapsto 5 : (\mathtt{v} \leq \mathtt{u}-1)$
$5 \mapsto 5 : \mathtt{G(u,v)} \wedge (\mathtt{Color[u]}{=}\mathtt{Color[v]})$
$\wedge\, (\mathtt{ack'}{=}\mathit{true}) \wedge (\mathtt{ret'}{=}\mathit{false})$
$5 \mapsto 4 : ((\neg \mathtt{G(u,v)}) \vee (\mathtt{Color[u]} \neq \mathtt{Color[v]}))$
$\wedge\, (\mathtt{v'}{=}\mathtt{v}+1)$
$6 \mapsto 6 : (\mathtt{ack'}{=}\mathit{true}) \wedge (\mathtt{ret'}{=}\mathit{true})$

We first suppose that only `G` is abstracted. Then the operator $\mathcal{A}$ transforms only the relational expression part of transitions $5 \mapsto 5$ and $5 \mapsto 4$ into:

$5 \overset{\pi}{\mapsto} 5 : \big(\exists H\ (H_\pi{=}\mathtt{G}_\pi) \wedge H(\mathtt{u},\mathtt{v})\big) \wedge (\mathtt{ack'}{=}\mathit{true})$
$\wedge\, (\mathtt{Color[u]}{=}\mathtt{Color[v]}) \wedge (\mathtt{ret'}{=}\mathit{false})$
$5 \overset{\pi}{\mapsto} 4 : \big(\exists H\ (H_\pi{=}\mathtt{G}_\pi) \wedge ((\neg H(\mathtt{u},\mathtt{v}))$
$\vee\, (\mathtt{Color[u]} \neq \mathtt{Color[v]}))\big) \wedge (\mathtt{v'}{=}\mathtt{v}+1)$

Fix some n-vertex graph $G = (V, E)$. Suppose there is an accepting path in the abstracted transition system when the graph input is set to G, that is, the program variable `G` takes the value G. Along the path, when the program counter is 5, there always exists a graph H such that $H_\pi = G_\pi$, so the system makes the transition $5 \overset{\pi}{\mapsto} 4$. Let $G_0 = (V, E_0)$ be the n-vertex graph whose vertices are defined by

$$(u, v) \in E_0 \quad \Longleftrightarrow \quad (u, v) \in E \text{ and } u, v \in \pi.$$

Observe that the transition is also made when the input is G_0. Thus there is an accepting path in the concrete system when the input graph is G_0.

In the general case, consider an accepting path in the completely abstracted transition system. Again, the abstracted transition $5 \overset{\pi}{\mapsto} 4$ is still made when the input is G_0. Then observe that only the indices of `Color` in π are relevant for this transition. Therefore, one can fix any value Color, such that $\widehat{\mathtt{Color}}^{\pi} = \widehat{\mathit{Color}}^{\pi}$. Thus the path of the concrete model which starts from the initial state $\mathtt{G} = G_0$ and $\mathtt{Color} = \mathit{Color}$, is again an accepting path. □

Since the size of π is fixed, our abstraction induces a constant size OBDD. By Lemma 1, Theorems 2, 4, and 5, we know that our probabilistic abstraction is an ε-abstraction, so **Graph Test** can be used for checking the validity of (2).

Corollary 1. *Let $\varepsilon > 0$. Using previous probabilistic abstraction,* **Graph Test** *on* `CHECK-PARTITION` *satisfies:*

1. *If* `CHECK-PARTITION` *satisfies specification* (2), **Graph Test** *always accepts;*
2. *If there exists a graph G which has distance more than ε from any bipartite graph, but which is accepted by* `CHECK-PARTITION` *for some coloring Color, then* **Graph Test** *rejects with probability at least $2/3$;*
3. *The time complexity of* **Graph Test** *is exponential in $\mathrm{poly}(1/\varepsilon)$ and does not depend on* `n`*, the input size.*

4. Lower bound for approximate bipartiteness

In this section, we show how the communication complexity lower bound of Hajnal, Maass, and Turán [13] can be extended to yield a lower bound on OBDD size for the relaxed version of bipartiteness. This establishes that in the case of bipartiteness, reducing the size of the OBDD cannot be achieved solely by relaxing the exactness of the result. A lower bound for the relaxed version of connectivity can also be obtained using similar arguments.

A graph $G = (V, E)$ is *k-bipartite* if there is a set of edges $F \subseteq E$ with $|F| \leq k$ such that $G' = (V, E \setminus F)$ is bipartite. In particular, a graph is 0-bipartite if and only if it is bipartite. The k-bipartiteness is the following partially defined problem.

Definition 11 (k-bipartiteness). *Let $k \in \mathbb{N}$. The k-bipartiteness problem is a partial function f on graphs G:*

$$f(G) = \begin{cases} 1 & \text{if } G \text{ is bipartite,} \\ 0 & \text{if } G \text{ is not } k\text{-bipartite.} \end{cases}$$

An OBDD *solves* the k-bipartiteness problem for n-vertex graphs if its output agrees with f whenever f is defined. The rest of this section is devoted to proving the following theorem.

Theorem 6. *Any OBDD solving the k-bipartiteness problem for n-vertex graphs has width $2^{\Omega((n-2\sqrt{k+1})\log(n-2\sqrt{k+1}))}$. When $k = \varepsilon n^2$, the width is $2^{\Omega((1-2\sqrt{\varepsilon})n\log((1-2\sqrt{\varepsilon})n))}$.*

4.1. Preliminaries

We denote by $S \dot\cup T$ the disjoint union of sets S and T. A *partition* of a finite set S is a (finite) set of non-empty parts $S_1, S_2, \ldots$ whose disjoint union $S_1 \dot\cup S_2 \dot\cup \cdots$ equals S. The number of partitions of a set S containing n elements is B_n, the nth Bell number, where B_n is $2^{\Omega(n \log n)}$.

Two kinds of partitions will be considered: partitions of a subset of vertices in the HMT graphs (defined below), and

partitions of the edge variables of the graph (also explained below). To avoid confusion we call the latter a coloring instead of a partition, and use the letters R (red variables given to Player I) and Y (yellow variables given to Player II) to denote the color sets. In the remainder, we only consider colorings which divide the edge variables into two sets of equal size (plus or minus one).

Let $f : \{0,1\}^N \to \{0,1\}$ be a boolean function which two players wish to compute. Let $R\dot{\cup}Y$ be a coloring of the N input variables. Player I's input x corresponds to the variables of R, Player II's input y corresponds to the variables of Y. In a *one-way communication protocol for* f, Player I sends one message to Player II, who outputs the value of $f(x,y)$, where it is understood that the variables are reordered appropriately according to R, Y. The *communication* $\kappa_{\rightarrow}^{R:Y}(\mathcal{P}; x, y)$ *incurred by a one way-communication protocol* $\mathcal{P}$ on input x, y for the coloring R, Y is the number of bits sent by Player I.

For a fixed input length N, the *one-way communication complexity of* f for coloring R, Y is denoted $\kappa_{\rightarrow}^{R:Y}(f)$, and is the minimum of $\max_{x,y}\{\kappa_{\rightarrow}^{R:Y}(\mathcal{P}; x, y)\}$, over all one-way communication protocols $\mathcal{P}$ for f. The *one-way communication complexity for the best-case coloring of variables* is $\kappa_{\rightarrow}^{best}(f) = \min_{R\dot{\cup}Y}\{\kappa_{\rightarrow}^{R:Y}(f)\}$.

Let f be a boolean function whose variables are colored by R, Y. The *communication matrix associated with* f is the matrix representation M_f of f, that is $M_{f,x,y} = f(x,y)$. The lower bound on the width of OBDDs that compute f is related to the communication matrix of f by the following proposition. We state the result for one-way communication complexity, which can be easily derived from [14].

Proposition 1 (follows from [14, Page 144]).

1. *If f has an OBDD of width at most w, then $\kappa_{\rightarrow}^{best}(f) \leq \log w$.*
2. *Let M_f be the communication matrix associated with the boolean function f whose variables are colored by R, Y. Then $\kappa_{\rightarrow}^{R:Y}(f) \geq log(l)$, where l is the number of distinct lines in M_f.*

Since we will study partially defined problems the communication matrices we consider will have entries 0 and 1 when the problem is defined, and $\star$ when the computation can output either 0 or 1. Therefore, when we prove that there is a large number of distinct lines, we only consider two lines to be distinct if on some column, one line contains a 0 and the other contains a 1.

4.2. The k-partition problem

We introduce the k-partition problem. In Section 4.3 we will show that its communication matrix appears as a sub-matrix of the k-bipartiteness communication matrix. Hence, it will suffice to show a lower bound for the k-partition problem.

Figure 1. Three examples of H_P^k graphs.

We denote by $[m]$ the set $\{1, \ldots, m\}$. In the rest of the paper, for every integer $k \geq 1$, we fix in some canonical way a set E_k of $k+1$ bipartite edges from vertices of $[2\lceil\sqrt{k+1}\rceil]$ in such a way that these new edges only go between even and odd vertices.

Definition 12 (Figure 1). *Let $k, m \geq 0$ be two integers. For any partition P of $[m]$, H_P^k denotes the multigraph (where multiple edges are allowed) such that*

1. *vertices are the parts of P,*
2. *there is an edge in H_P^k between two parts $Q, Q' \in P$ if and only if some edge $e \in E_k$ "crosses over" Q, Q', that is, $e = \{a, b\}$, with $a \in Q$ and $b \in Q'$.*

These graphs lead to the following partially defined problem.

Definition 13 (k-partition problem). *Let $k, m \geq 0$ be two integers. The k-partition problem is a partial function g on partitions P of $[m]$:*

$$g(P) = \begin{cases} 1 & \text{if } H^k(P) \text{ contains } k+1 \text{ edges} \\ & \text{and no odd cycle,} \\ 0 & \text{if } H^k(P) \text{ is empty.} \end{cases}$$

For any two partitions P, P' of a set X, $P \vee P'$ is the finest partition which is refined both by P and P'. For the k-partition problem, the input of Player I is (an encoding of) some partition P, and the input of Player II is (an encoding of) a partition P'. The goal of the communication game is to compute $g(P \vee P')$. The corresponding communication matrix is M, whose rows and columns are labeled by partitions P and P' of $[m]$. The number of rows and columns is B_m, the mth Bell number ($B_m = 2^{\Omega(m \log m)}$). We show that M has an exponential number of distinct lines.

Lemma 2. *The communication matrix for the k-partition problem has $2^{\Omega((m-2\sqrt{k+1})\log(m-2\sqrt{k+1}))}$ distinct lines.*

Proof. Let P be a partition of the set S. The expression $P + l$ denotes the partition of $\{x + l | x \in S\}$ obtained from P by adding l to each element in all parts of P.

We show that the partitions of the form $\{\{1\}, \ldots, \{2\sqrt{k+1}\}\} \cup (P^* + 2\sqrt{k+1})$, where P^* range over the partitions of $[m-2\sqrt{k+1}]$, correspond to a large set of pairwise distinct lines in M.

Consider $P_1 = \{\{1\}, \ldots, \{2\sqrt{k+1}\}\} \cup (P_1^* + 2\sqrt{k+1})$

and P_2=$\{\{1\},\ldots,\{2\sqrt{k+1}\}\} \cup (P_2^* + 2\sqrt{k+1})$ two such distinct partitions. Since $P_1 \neq P_2$, it must be the case that $P_1^* \neq P_2^*$. Therefore, there must be some pair x, y such that (without loss of generality) x, y are in the same part of P_1^*, but are in different parts in P_2^*. We will use the notation $Q_{x,y}$ to mean the part of P_1^* that contains x and y, and Q_x (resp. Q_y) to mean the part of P_2^* that contains x (resp. y).

We exhibit a partition P' such that $M_{P_1,P'}$ differs from $M_{P_2,P'}$. Recall that the new edges E_k connect only odd vertices to even vertices. Let $\text{EVEN}_k = \cup_{1 \leq i < \sqrt{k+1}}\{2i\}$ and $\text{ODD}_k = \cup_{1 \leq i < \sqrt{k+1}}\{2i-1\}$. Let P' contain the parts $\text{EVEN}_k \cup \{x\}, \text{ODD}_k \cup \{y\}$ plus all the remaining vertices in singletons. Then,
$P_1 \vee P'$=EVEN$\cup$ODD$\cup Q_{x,y}, \{2\sqrt{k+1}+1, ..., m\} \setminus Q_{x,y}$
$P_2 \vee P'$=EVEN$\cup\, Q_x$, ODD$\cup\, Q_y, \{2\sqrt{k+1}+1, ..., m\} \setminus (Q_x \cup Q_y)$.

Now $H^k_{P_1 \vee P'}$ contains no edges, because all the new edges go between odd and even vertices. Likewise, $H^k_{P_2 \vee P'}$ contains a single edge of multiplicity $k+1$. Therefore the lines indexed by P_1 and P_2 are different at column P'.

The number of distinct lines must be at least the number of partitions P^* of $[m - 2\sqrt{k+1}]$. This concludes the proof of the lemma. □

4.3. Reduction to k-bipartiteness

In this section, we show that the communication matrix associated with the k-partition problem appears as a submatrix of the communication matrix of the k-bipartiteness problem (Lemma 4).

For the k-bipartiteness problem, the variables in the communication problem are pairs of vertices: the variable is 1 whenever the corresponding edge is in the graph, and 0 otherwise. Each player is given half of the variables partitioned according to some coloring. The input of the communication protocol is the graph formed by the union of edges from the players' variables.

Hajnal, Maass and Turán [13] give a reduction from bipartiteness to a property on partitions. We show how this reduction can be extended to show a reduction from k-bipartiteness to the k-partition problem. We re-use the main technical component of [13], namely, HMT graphs.

4.3.1. HMT graphs

The construction of [13] produces a large family of graphs (parameterized by P, P') that can be embedded into a coloring of the edge variables, in such a way that Player I's (red) edges represent a partition P of a set of vertices of size $\Omega(n)$, and Player II's (yellow) edges represent a partition P' of the same set.

Definition 14 ([13], Figure 2). *Let $n \geq 1$ be an integer, and $A \dot\cup B \dot\cup C \subseteq [n]$ with $|B| = \Omega(n)$. Let $\{G_{P,P'}\}$ be a graph family with vertices $[n]$, where P, P' ranges over the pairs of partitions of B. $\{G_{P,P'}\}$ is an* HMT graph family on A, B, C for a coloring R, Y *if for any pair of vertices $x, y \in B$ included in some part of P (resp. P'), the only edges of the graph $G_{P,P'}$ are vertex-disjoint red (resp. yellow) paths in $A \cup \{x, y\}$ (resp. $C \cup \{x, y\}$) of length 4.*

Figure 2. An HMT graph: $Q \in P$, $Q', Q'' \in P'$.

Lemma 3 ([13]). *Fix any coloring of the edge variable of graphs with vertices $[n]$, with half the edges colored red and half the edges colored yellow. Then if n is large enough, there exists sets A, B, C such that there is an HMT graph family on A, B, C for this coloring of the edge variables.*

4.3.2. Bipartiteness and partitions

For a coloring R, Y of the edge variables, we show how to construct a large family of instances of the k-partition problem from a family of HMT graphs for this coloring.

Proposition 2 ([13]). *Let $\{G_{P,P'}\}$ be an HMT graph family on A, B, C for a coloring R, Y. Let $v_1, v_2 \in B$ be two distinct vertices. For any pair of partitions P, P' of B, let $G^0_{P,P'}$ be $G_{P,P'}$ to which the single edge $\{v_1, v_2\}$ has been added. Then $G^0_{P,P'}$ is bipartite if and only if $\{v_1, v_2\}$ is not included in any part $Q \in P \vee P'$.*

A key observation is that the connected components of an HMT graph $G_{P,P'}$ correspond exactly to the parts of $P \vee P'$. The proposition follows because the addition of an edge within a part $Q \in P \vee P'$ creates an odd cycle in the graph, and furthermore no cycle is created if the edge "crosses over" two parts of $P \vee P'$.

Without loss of generality, we now renumber the vertices so that $B = \{1, 2, \ldots\}$. For the k-bipartiteness problem, instead of adding a single, fixed, edge of $B \times B$ to each graph $G_{P,P'}$, we add the fixed bipartite set E_k of $k+1$ edges (see Section 4.2). We call the resulting graph $G^k_{P,P'}$.

There are two ways in which an odd cycle can be created in $G_{P,P'}$ when adding the $k+1$ new edges: either by adding an edge within a part of $P \vee P'$, or by creating an odd cycle in $H^k_{P \vee P'}$ (see Definition 12).

Lemma 4. *Let $\{G^k_{P,P'}\}$ be an HMT graph family on A, B, C for a coloring R, Y.*

1. $G^k_{P,P'}$ is bipartite if and only if $H^k_{P \vee P'}$ has $k+1$

edges and no odd cycle.

2. *$G^k_{P,P'}$ is not k-bipartite if and only if $H^k_{P\vee P'}$ has no edges.*

Proof. For the implication of Part 1, we show the contrapositive. First, if $H^k_{P\vee P'}$ has fewer than $k+1$ edges, then some edge in E_k lies within a part of $P\vee P'$. By definition of $G^k_{P,P'}$, this creates a cycle of length 5. Second, if $H^k_{P\vee P'}$ contains an odd cycle of length t, then this forms a cycle in $G^k_{P,P'}$ of length t plus a multiple of 4. For the converse, notice that the new edges form a bipartite subgraph so they cannot form an odd cycle on their own.

For the implication of Part 2, we show the contrapositive. Assume that some edge $e \in E_k$ gave rise to an edge in $H^k_{P\vee P'}$. Removing k edges suffices to make $G^k_{P,P'}$ bipartite, because it is enough to remove all the new edges except e. For the converse, assume $H^k_{P\vee P'}$ is empty. By definition the paths in $G^k_{P,P'}$ are vertex disjoint. Furthermore, the $k+1$ new edges form a bipartite graph. Therefore $k+1$ edges must be removed from $G^k_{P,P'}$ to remove the $k+1$ odd cycles. □

Lemma 4 establishes that the k-partition communication matrix appears as a sub-matrix of the k-bipartiteness communication matrix. Theorem 6 therefore follows, by Proposition 1, from the lower bound on the number of distinct lines, proven in Lemma 2.

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A. Programs and transitions systems

```
FUNCTION GUESS
  INPUT a : BOOLEAN
  VAR b : BOOLEAN
1: get(b)
2: IF (a=b) RETURN true
   ELSE     RETURN false
```

The program variables are a and b with the implicit variables PC, ack, and ret. A state of the program is a 5-tuple $(\mathtt{PC}, \mathtt{ack}, \mathtt{ret}, \mathtt{a}, \mathtt{b})$. A transition of the program is a pair of states $\big((\mathtt{PC}, \mathtt{ack}, \mathtt{ret}, \mathtt{a}, \mathtt{b}), (\mathtt{PC}', \mathtt{ack}', \mathtt{ret}', \mathtt{a}', \mathtt{b}')\big)$. The relational expression for the initial states of the program is $(\mathtt{PC} = 1) \wedge (\mathtt{ack} = \mathit{false})$. The relational expression for the transition relation of the program is defined as the disjunction of the following three formulas:

- $(\mathtt{PC} = 1) \wedge (\mathtt{PC}' = 2) \wedge (\mathtt{ack}' = \mathtt{ack})$
  $\wedge\, (\mathtt{ret}' = \mathtt{ret}) \wedge (\mathtt{a}' = a)$,
- $(\mathtt{PC} = 2) \wedge (\mathtt{PC}' = 2) \wedge (\mathtt{ack}' = \mathit{true})$
  $\wedge\, (\mathtt{ret}' = \mathit{true}) \wedge (a = b) \wedge (\mathtt{a}' = \mathtt{a}) \wedge (\mathtt{b}' = \mathtt{b})$,
- $(\mathtt{PC} = 2) \wedge (\mathtt{PC}' = 2) \wedge (\mathtt{ack}' = \mathit{true})$
  $\wedge\, (\mathtt{ret}' = \mathit{false}) \wedge (a \neq b) \wedge (\mathtt{a}' = \mathtt{a}) \wedge (\mathtt{b}' = \mathtt{b})$.

Due to user interaction, $\mathtt{b}'$ does not appear in the first formula, and the first transition is therefore nondeterministic. The following CTL* formula is a specification of the behavior of the program GUESS:

$$\forall\Big(\neg\mathtt{ack}\,\mathbf{U}\,\mathtt{ack}\wedge\Big(\big(\mathtt{ret}\wedge(\mathtt{a}{=}\mathtt{b})\big)\vee\big(\neg\mathtt{ret}\wedge(\mathtt{a}{\neq}\mathtt{b})\big)\Big)\Big)$$