Abstracts

Wave turbulence: the conjecture, approaches and rigorous results

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Consider the following nonlinear PDE on the torus $\mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d)$, $L \geq 1$:

$$\dot{u} + i\Delta u + \varepsilon\rho i\,|u|^2 u = -\nu(-\Delta u + 1)^p u + \sqrt{\nu}\,\langle\text{random force}\rangle\,. \tag{1}$$

Here $u = u(t,x)$ $(t \geq 0,\ x \in \mathbb{T}^d)$ is an unknown complex function, the parameters ε, ν, ρ satisfy $0 < \varepsilon \leq 1$, $0 \leq \nu \leq 1$, $\rho \geq 1$, and p is an integer, "not too small in terms of d" (e.g. if $d \leq 3$, then it suffices to assume that $p \geq 1$). The random force is smooth in x and is a while noise in t; it is specified below. We regard (1) as a dynamical system in a suitable function space of complex functions on $\mathbb{T}^d$. This is a random dynamical system if $\nu > 0$.

The wave turbulence (WT) studies solutions of this equation for large t when $\varepsilon, \nu \ll 1$ and $L \gg 1$ (ρ and the random force for a while are assumed to be constant). Classically $\nu = 0$, then (1) becomes the defocusing NLS equation; see [1, 4]. The stochastic model $(1)_{\nu>0}$ was suggested in [3], also see [2] and [4]. Relation between the small parameters ε and ν is crucial. If $\nu > 0$ is "much smaller than ε", then the stochastic model becomes similar to the deterministic case, while if ν is "much bigger than ε", then (1) becomes similar to the non-interesting linear stochastic system $(1)_{\varepsilon=0}$. Natural relation between ν and ε is

$$\nu = \varepsilon^2 \tag{2}$$

(cf. the references above and [5, 6]). We assume it everywhere below when talking about eq. (1) with positive ν.

In the classical setting (when $\nu = 0$) the WT is concerned with the behaviour of solutions for (1) with "typical initial data" when

$$t \to \infty\,, \quad \varepsilon \to 0\,, \quad L \to \infty \tag{3}$$

(the relation between the three parameters in unclear and has to be specified). Usually people, working in WT, decompose solutions u to Fourier series

$$u(t,x) = \sum_{k\in\mathbb{Z}^d_L} v_k(t) e^{2\pi i k\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d\,.$$

One of their prime interests is the behaviour under the limit (3) of the averaged actions

$$n_k(t) = \tfrac{1}{2}\big\langle |v_k|^2(t)\big\rangle, \qquad k \in \mathbb{Z}^d_L\,,$$

where $\langle\cdot\rangle$ signifies a suitable averaging. When $L \to \infty$, the function $n_k(t) = n^L_k(t)$ on the lattice $\mathbb{Z}^d_L$ asymptotically becomes a function $n^0_k(t)$ on $\mathbb{R}^d$. The main conjecture of the WT, made in 1960's (and going back to an earlier work of R. Peierls on the heat conduction in crystals) says that, under a suitable scaling of time $\tau = \varepsilon^{a_1} t$ and of the constant $\rho = L^{a_2}$, where $a_1, a_2 > 0$, the limit $n_k^0(\tau)$, $k \in \mathbb{R}^d$, exists and satisfies the *wave kinetic equation*

$$\frac{d}{d\tau} n_k(\tau) = \text{Const} \int_{\Gamma_k \subset \mathbb{R}^{3d}} n_{k_1} n_{k_2} n_{k_3} n_k \Big(\frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \Big) dk_1 dk_2 dk_3 \,,$$
$$\Gamma_k = \{ (k_1, k_2, k_3) : k_1 + k_2 = k_3 + k, \ |k_1|^2 + |k_2|^2 = |k_3|^2 + |k|^2 \} \,. \tag{4}$$

The celebrated Zakharov ansatz (see [1, 4]) applies to this equation and implies that it has autonomous solutions of the form $n_k(\tau) = |k|^\gamma$, $\gamma < 0$, intensively discussed in the physical literature as the *Kolmogorov-Zakharov energy spectra.*

Progress in the rigorous study of the deterministic equation $(1)_{\nu=0}$ was achieved in two very different works [7, 8], but it seems that both approaches do not allow to derive rigorously the kinetic equation (4).

In our talk we report recent progress in deriving a wave kinetic equation for the limiting behaviour of averaged actions for solutions of the stochastic equation (1), (2). Let us pass in this equation to the slow time $\tau = \nu t = \varepsilon^2 t$ and denote $\lambda_k = (|k|^2 + 1)^p$. Then the equation, written in terms of the Fourier coefficients v_k, reeds

$$\frac{d}{d\tau} v_k(\tau) + i|k|^2 \nu^{-1} v_k = -i\rho \sum_{k_1+k_2=k_3+k} v_{k_1} v_{k_2} \bar{v}_{k_3} - \lambda_k v_k + b_k \frac{d}{d\tau} \beta_k(\tau) \,. \tag{5}$$

Here $\{\beta_k, k \in \mathbb{Z}_L^d\}$ are independent standard complex Wiener processes and the numbers b_k are real non-zero, fast converging to zero when $|k| \to \infty$ (the random force in eq. (5) specifies that in (1)). This equation is well posed and mixing. The latter means that in a suitable function space of complex sequences $\{v_k, k \in \mathbb{Z}_L^d\}$ there exists a unique measure $\mu_{\nu,L}$, called a *stationary measure* for the equation, such that the law $\mathcal{D}(v(\tau))$ of any solution $v(\tau)$ for (5) weakly converges to $\mu_{\nu,L}$ when $\tau \to \infty$; see in [5]. So if t is much bigger than L and ν^{-1}, then the problem of studying the behaviour of solutions for (5) under the limit (3) may be recast as the problem of studying the measure $\mu_{\nu,L}$ when $L \to \infty$ and $\nu \to 0$.

Consider the following ν-independent *effective equation* for (5):

$$\frac{d}{d\tau} a_k = -i\rho \sum_{\substack{k_1+k_2=k_3+k \\ |k_1|^2+|k_2|^2=|k_3|^2+|k|^2}} a_{k_1} a_{k_2} \bar{a}_{k_3} - \lambda_k a_k + b_k \frac{d}{d\tau} \beta_k(\tau) \,. \tag{6}$$

This equation also is well posed and mixing, see [5]. Denote by m_L its unique stationary measure. It is proved in [5] that eq. (6) comprises asymptotical properties of solutions for (5) as $\nu \to 0$. Namely, that

i) $\mu_{\nu,L} \rightharpoonup m_L$ as $\nu \to 0$;

ii) if $v^{\nu,L}(\tau)$ is a solution for (5) with some ν-independent initial data at $\tau = 0$ and $a^L(\tau)$ is a solution for (6) with the same initial data, then

$$\mathcal{D}\Big(\tfrac{1}{2} |v_k^{\nu,L}(\tau)|^2 \Big) \rightharpoonup \mathcal{D}\Big(\tfrac{1}{2} |a_k^L(\tau)|^2 \Big) \quad \text{as} \ \ \nu \to 0 \quad \text{for} \quad 0 \le \tau \le T \,,$$

for each fixed $T > 0$ and every $k \in \mathbb{Z}_L^d$.

Denote $n_k^L(\tau) = \frac{1}{2}\,\mathbb{E}|a_k^L(\tau)|^2$ and choose in (6)

$$\rho = L^{1/2}\rho_0, \qquad b_k = L^{-d/2} b_k^0\,. \tag{7}$$

Using certain heuristic tools from the arsenal of WT, in [6] we proved, on the physical level of rigour, that under the limit $L \to \infty$ the function $\mathbb{Z}_L^d \ni k \mapsto n_k^L(\tau)$ weakly converges to a function $\mathbb{R}^d \ni k \mapsto n_k(\tau)$, which is a solution of the damped/driven wave kinetic equation

$$\begin{aligned}\frac{d}{d\tau} n_k(\tau) = -2\lambda_k n_k + (b_k^0)^2 + \int_{\Gamma_k} \frac{f_k(k_1,k_2,k_3)}{\lambda_k + \lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3}}\, n_{k_1} n_{k_2} n_{k_3} n_k \\ \times \Big(\frac{1}{n_k} + \frac{1}{n_{k_3}} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \Big)\, dk_1 dk_2 dk_3\,.\end{aligned} \tag{8}$$

Here the surface $\Gamma_k \subset \mathbb{R}^{3d}$ is the same as in (4), and the function f_k is constructed in terms of Γ_k. It is positive, smooth outside the origin, and such that $C^{-1} \le f \le C$ for all k, k_1, k_2, k_3 and a suitable constant C. Moreover, the Zakharov ansatz applies to (8) under a certain natural limit and allows to construct its homogeneous solutions of the Kolmogorov-Zakharov form. Based on this heuristics result and the rigorous assertions i), ii) above, we conjectured in [6] that, under the double limit $\lim_{L\to\infty}\lim_{\nu\to 0}$, the function $\mathbb{Z}_L^d \ni k \mapsto n_k^{\nu,L}(\tau) = \frac{1}{2}\,\mathbb{E}|v_k^{\nu,L}(\tau)|^2$ weakly converges to a function $n_k^0(\tau)$, $k \in \mathbb{R}^d$, which is a solution of eq. (8). The main goal of this talk is to announce the following result, which is a modification of that conjecture:

Theorem (SK, a work under preparation). Let $\mathbb{R}^d \ni k \mapsto v_k^0$ be a smooth function with compact support, let $v_k^{\nu,L}(\tau)$ be a solution of (5), (7) such that $v_k^{\nu,L}(0) = v_k^0$ for $k \in \mathbb{Z}_L^d$, and let $n_k^{\nu,L} = \frac{1}{2}\mathbb{E}|v_k^{\nu,L}|^2$. Let $\nu \to 0$ and $L \to \infty$ in such a way that $\nu L \to 0$ sufficiently fast. Then there exists $\tau_0 > 0$ such that

$$\lim_{\nu\to 0,\, L\to\infty} n_k^{\nu,L}(\tau) = n_k^0(\tau) \quad \text{for } 0 \le \tau \le \tau_0\,,$$

where $n_k^0(\tau)$, $k \in \mathbb{R}^d$, is a solution for (8), and the limit holds with respect to a suitable weak convergence of functions.

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