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Space-multidimensional hamiltonian PDE and KAM

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joint work with Hakan Eliasson and Benoit Grebert

1 Small oscillations in nonlinear hamiltonian PDEs

EXAMPLE: Consider the NLS equation:

(NLS) $u_t + i\Delta u - imu - ig(x, |u|^2)u = 0, \qquad u = u(t,x), \quad x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d;$

$g(x,v) = v + O(v^2)$, g – real analytic function. I wish to study **small** solutions for **all** values of t. NOTE THAT if $d > 3$, then even for small and smooth initial data it is unknown if a solution exists forever. A-priori, for this equation we have only a local flow.

WHAT IS KNOWN: i) sufficient conditions in terms of the function g and the dimension d so that for smooth initial data solutions exist for all values of time. For example, for equation

$$u_t + i\Delta u - i|u|^{2q}u = 0, \quad q \in \mathbb{N}, \qquad x \in \mathbb{T}^d,$$

this is true if $d \le 2$ and q is any, or $d = 3$ and $q \le 2$. In this case non-trivial UPPER bounds on the growth of the Sobolev norms of solutions as $t \to \infty$ are obtained by J. Bourgain and others.

ii) For the cubic NLS equation $u_t + i\Delta u - i|u|^2 u = 0$ in $\mathbb{T}^2$, Colliander -Keel - Staffilani - Takaoka - Tao obtained LOWER bounds for growth of SOME solutions on long (not infinite) time intervals.

iii) The heuristic theory of wave turbulence studies behaviour of small solutions for NLS equations when $t \gg 1$ and the space-period is not 2π, but L, where $L \gg 1$.

Naturally, to study small solutions, eq. (NLS) should be regarded as a perturbation of the linear Schrödinger equation

(S) $$u_t + i\Delta u - imu = 0, \qquad x \in \mathbb{T}^d\,.$$

Solutions for (S) may be written by the Fourier method:

$$u(t,x) = \sum_{s\in\mathbb{Z}^d} u_s e^{i\lambda_s t} e^{is\cdot x}, \qquad \lambda_s = |s|^2 + m\,.$$

This is a superposition of linear waves with integer wave-vectors s. The function $s \mapsto \lambda_s$ is very important, and is called the dispersion relation.

These solutions u are almost-periodic functions of t. I regard them as almost-periodic curves in a functional space, and write $u(t,\cdot)=u(t)\in\{\text{function space}\}$.

An important special case is given by time quasiperiodic (or QP) solutions, which are superpositions of finitely-many linear waves. Namely, let $\mathcal{A}\subset\mathbb{Z}^d$, $|\mathcal{A}|=n<\infty$. Consider a superposition of linear waves with the wave-vectors in $\mathcal{A}$:

$$u_{\mathcal{A}}(t,x)=\sum_{s\in\mathcal{A}}u_s e^{i(|s|^2+m)t}e^{is\cdot x}.$$

I will call the finite set $\mathcal{A}$ the set of linearly excited modes. The solution $u_{\mathcal{A}}$ defines a QP curve $u_{\mathcal{A}}(t)$ in the function space. Let us write it as

$$u_{\mathcal{A}}(t)=\sum_{s}u_s(t)e^{is\cdot x},\qquad u_s(t)=u_s e^{i(|s|^2+m)t}\,.$$

Then $u_s(t)=0$ if $s\notin\mathcal{A}$, and

$$\rho_s:=\tfrac{1}{2}\,|u_s(t)|^2=\text{const}\,,\qquad s\in\mathcal{A}\,.$$

Denote by $\rho = (\rho_s, s \in \mathcal{A}) \subset \mathbb{R}^n_+$ the *vector of actions* of the solution $u_{\mathcal{A}}$. Then

$$u_{\mathcal{A}}(t) \in T^n_\rho = \{\sum u_s e^{is\cdot x},\ \tfrac{1}{2}|u_s|^2 = \rho_s \text{ if } s \in \mathcal{A};\ u_s = 0 \text{ otherwise}\}\,.$$

This set T^n_ρ is an n-torus in the function space. It is invariant for the linear equation (S) and is filled in with its QP solutions $u_{\mathcal{A}}(t)$.

PROBLEMS. Study how

1) small almost-periodic solutions $u(t,x)$ of (S) are perturbed in (NLS), for all t.

2) Study how small QP solutions $u_{\mathcal{A}}$ and the corresponding small invariant tori T^n_ρ are perturbed in (NLS).

Question 1) is hopelessly complicated, even for $d=1$, and question 2) is what the KAM for PDE theory studies. For $d=1$ the question 2) was resolved in [SK, J.Pöschel] Ann. Math. 143 (1996). This was done in two steps:

STEP 1. Put the equation to a normal form in the vicinity of a torus T^n_ρ.

STEP 2. After a proper scaling, the obtained normal form hamiltonian becomes a perturbation of the hamiltonian of certain parameter-depending linear system. Apply to it a KAM-theorem for perturbations of parameter-depending linear systems.

But for $d > 1$ the task turned out to be much more complicated since,

Secondly, at the Step 2 the required KAM-theorem for perturbations of multi-dimensional parameter-depending linear equations is significantly more complicated than its 1d analogy. – This is an analytic difficulty.

Firstly, at Step 1 to put the equation to a normal form in the vicinity of a torus T^n_ρ, one has to verify a number of EXTREMELY complicated algebraical non-degeneracy relations. – This is an algebraical difficulty. See in

$\big[$Cl. Procesi & M. Procesi$\big]$ – a paper in CMP, and preprints. Also see a MS by W.-M.Wang.

For the moment we see no way to handle the algebraical difficulty.

We encounter the same two problems when study small solutions of other space- -multidimensional hamiltonian PDEs.

I will present a way to overcome the two difficulties, suggested in

[EGK1] *"KAM for nonlinear beam eq. 1: small-amplitude solutions"* (arXiv 2014)

[EGK2] *"KAM for nonlinear beam eq. 2: a normal form theorem"* (arXiv 2015).

The idea to handle the crucial algebraical difficulty is the following: the non-degeneracy relations which have to be checked crucially depend on the dispersion function λ_s. So let us consider Hamiltonian PDEs which involve a mass-parameter m, and try to prove that the non-degeneracy relations hold for a.a. values of m, for the reason of analyticity of these relations in m. Next do the "KAM-job" for those a.a. typical values of m.

For the (NLS) the dispersion function is $\lambda_s = |s|^2 + m$. It depends on m linearly, i.e. in a degenerate way, and the idea does not work.

MAIN EXAMPLE. The Klein-Gordon equation.

$$(KG) \qquad\qquad u_{tt} - \Delta u + mu + g(x,u) = 0, \quad x \in \mathbb{T}^d,\ m \in [1,2]\,.$$

Dispersion relation $\lambda_s = \sqrt{|s|^2 + m}$ is nice nonlinear function of m. But asymptotically $\lambda_s \sim |s| + O(s^{-1})$. If $d > 1$, then asymptotically $\lambda_s = |s|$ is a $\sqrt{\text{integer}}$, so it has complicated Diophantine properties. For the moment (KG) is a bit too complicated for us.

2 Beam equation.

Following Geng $\&$ J. You (Nanjing), we consider instead the beam equation:

$$u_{tt} + \Delta^2 u + mu + g(x,u) = 0, \qquad x \in \mathbb{T}^d\,,\ g = u^3 + O(u^4)\,,\ m \in [1,2].$$

Now $\lambda_s = \lambda_s(m) = \sqrt{|s|^4 + m} \sim |s|^2 + O(|s|^{-2})$. This is better than in the case of (KG) equation since $|s|^2 \in \mathbb{Z}$. Preliminary transformation of the equation: denote

$$\Lambda = \sqrt{\Delta^2 + m}, \quad \psi = 2^{-1/2}(\Lambda^{1/2}u + i\Lambda^{-1/2}\dot{u})\,.$$

Then the complex curve $\psi(t)$ satisfies:

$$\dot{\psi} = i\Big(\Lambda\psi - 2^{-1/2}\Lambda^{-1/2}g(x, 2^{-1/2}\Lambda^{-1/2}(\psi + \bar{\psi}))\Big)\,.$$

This is a hamiltonian system. It may be written as

$$\dot{\psi} = i\nabla_\psi H_{beam}\,, \quad H_{beam} = \int \Big[(\Lambda\psi)\bar{\psi} + G(x, \Lambda^{-1/2}\Big(\frac{\psi + \bar{\psi}}{\sqrt{2}}\Big))\Big]dx\,,$$

where $G_v(x,v) = g(x,v)$. This equation is similar to NLS.

Write $\psi(x)$ as Fourier series

$$\psi(x) = \sum_{s\in\mathbb{Z}^d} \xi_s e^{is\cdot x}, \qquad \bar\psi(x) = \sum_{s\in\mathbb{Z}^d} \bar\xi_s e^{-is\cdot x}.$$

Denote $\eta_s = \bar\xi_s$. Then $\bar\psi(x) = \sum_s \eta_s e^{-is\cdot x}$ and we may write the beam equation as the system

(Beam) $$\dot\xi_s = i\frac{\partial H}{\partial \eta_s}, \qquad \dot\eta_s = -i\frac{\partial H}{\partial \xi_s}, \quad s\in\mathbb{Z}^d,$$

where

$$H_{beam} = \sum \lambda_s\xi_s\eta_s + \int G\Big(x, \sum_s \frac{\xi_s e^{is\cdot x} + \eta_s e^{-is\cdot x}}{\sqrt{2\lambda_s}}\Big).$$

Consider the infinite complex vectors $\xi = (\xi_s, s\in\mathbb{Z}^d), \eta = (\eta_s, s\in\mathbb{Z}^d)$.
For a solution ψ to be real we must have the reality condition $\eta = \bar\xi$.

I recall that $G_u(x,u) = g(x,u),\ G = \frac{1}{4}u^4 + O(u^5)$.

As before, we fix a finite set of linearly excited modes

$$\mathcal{A} \subset \mathbb{Z}^d, \quad |\mathcal{A}| = n < \infty, \quad \text{denote } \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A},$$

choose a small *vector of amplitudes* $\rho \in \mathbb{R}^n_+,\ |\rho| \sim \varepsilon \ll 1$, and study (Beam) in the vicinity of the n-torus

$$T^n_\rho = \{(\xi,\eta) : \xi_a = \bar\eta_a, \tfrac{1}{2}|\xi_a|^2 = \tfrac{1}{2}|\eta_a|^2 = \rho_a \text{ if } a \in \mathcal{A};\ \xi_s = \eta_s = 0 \text{ if } a \in \mathcal{L}\}$$

This torus is invariant for the linear beam equation, and QP solutions $(\xi,\eta)_{\mathcal{A}}(t)$ of the equation wind on it.

GOAL: Study solutions of (Beam) near T^n_ρ.

3 Small divisors and the mass parameter

Definition. A finite set of linearly excited modes $\mathcal{A} \subset \mathbb{Z}^d$ is called admissible if

$$a, b \in \mathcal{A}, a \neq b \;\Rightarrow\; |a| \neq |b|\,.$$

Lemma. *Admissible sets are typical*: take at random n points $a_1, \ldots a_n$ in the large cube

$$K^d = \{s \in \mathbb{Z}^d : |s_j| \leq N, j = 1, \ldots, d\}.$$

Set $\mathcal{A} = \{a_1, \ldots, a_n\}$. Then $\mathbf{P}\{\mathcal{A}$ is admissible $\} = 1 - O(N^{-1})$.

Everywhere below I assume that $\mathcal{A}$ is admissible.

The frequencies of the linearised at zero equation are
$\lambda_a = \lambda_a(m) = \sqrt{|a|^4 + m} = |a|^2 + O(|a|^{-2})$.
Denote by ω the frequency-vector of the linearly excited modes:

$$\omega = \omega^{\mathcal{A}}(m) = (\omega_a, a \in \mathcal{A}) \in \mathbb{R}^n, \qquad \omega_a = \lambda_a\,.$$

Then $\omega_{a_1} \neq \omega_{a_2}$ if $a_1 \neq a_2$ since the set $\mathcal{A}$ is admissible.

FACT: small divisors for the KAM-constructions are

$$\mathcal{R}(m) = k\cdot\omega(m) + l_a\lambda_a(m) \pm l_b\lambda_b(m), \quad k \in \mathbb{Z}^n,\ l_a, l_b \in \mathbb{Z},\ 0 \le |l_a| + |l_b| \le 2\,.$$

I recall that $\omega(m) = (\lambda_a(m), a \in \mathcal{A}) \in \mathbb{R}^n$, and that $\lambda_s(m) = \sqrt{|s|^4 + m}$, where $m \in [1, 2]$.

Main Lemma (based on results of Bourgain and Bambusi). There is a "bad" zero-measure subset $\mathcal{C} \subset [1, 2]$ such that for each divisor $\mathcal{R}(m)$ we have:

- either $\mathcal{R}(m) \equiv 0$,

- or for each $m \notin \mathcal{C}$ we have $|\mathcal{R}(m)| \ge \varkappa(m)|k|^{-c_n}$, where $\varkappa(m) > 0$, and c_n is some (fixed) polynomial of n.

Remark. How it may be that $\mathcal{R} \equiv 0$? – Take $|k| = 1, l_a = 0, l_b = 1$ and choose the sign "-". Then $\mathcal{R}(m) = \omega_j(m) - \lambda_b(m)$, where $\omega_j = \lambda_a$ for some $a \in \mathcal{A}$. But this is dead-zero if b is such that $|a| = |b|$.

If $\mathcal{R} \equiv 0$, then ether $k = 0$ and $\mathcal{R} = \lambda_a - \lambda_b$, where $|a| = |b|$, or $\mathcal{R}$ is as above, or $\mathcal{R}$ is the sum or the difference of two divisors as above.

4 The normal form

I recall that in terms of the Fourier coefficients $\xi = (\xi_s, s \in \mathbb{Z}^d)$, $\eta = (\eta_s, s \in \mathbb{Z}^d)$, the Hamiltonian of (Beam) is

$$H_{beam} = \sum \lambda_s \xi_s \eta_s + \int G\Big(x, \sum_s \frac{\xi_s e^{is\cdot x} + \eta_s e^{-is\cdot x}}{\sqrt{2\lambda_s}}\Big) =: H^2_{beam} + \dots\,,$$

and that we wish to study (Beam) near the n-torus

$$T^n_\rho = \{(\xi,\eta) : \xi_a = \bar\eta_a, \tfrac12|\xi_a|^2 = \tfrac12|\eta_a|^2 = \rho_a \text{ if } \ a \in \mathcal{A};\ \xi_s = \eta_s = 0 \text{ if } \ a \in \mathcal{L}\}$$

The vector $\rho = (\rho_a, a \in \mathcal{A})$ is small, or order $\varepsilon \ll 1$. I write it as $\rho = \varepsilon\tilde\rho$,

$$\tilde\rho \in [1,2]^n\,,$$

and regard ε as the size of the perturbation, and $\tilde\rho$ – as a parameter.

Near T^n_ρ I make the usual elementary change of coordinate: I keep the coordinates ξ_s, η_s with $s \in \mathcal{L}$ without change, and pass from ξ_a, η_a with $a \in \mathcal{A}$ to the action-angles (r, θ):

$$r \in \mathbb{R}^n, \quad \theta \in \mathbb{T}^n : \xi_a = \sqrt{2(\tilde{\rho}_a + r_a)}\, e^{i\theta_a}, \quad \eta_a = \sqrt{2(\tilde{\rho}_a + r_a)}\, e^{-i\theta_a}, \quad a \in \mathcal{A}\,.$$

Now the torus T^n_ρ reeds

$$T^n_\rho = \{r = 0, \theta \in \mathbb{T}^n, \xi_s = \eta_s = 0 \,\forall\, s \in \mathcal{L}\}\,.$$

In the new variables the quadratic part $\sum_{s\in\mathbb{Z}^d} \lambda_s(m)\xi_s\eta_s$ of the hamiltonian becomes

$$H^2_{beam} = \text{Const} + \omega(m)\cdot r + \sum_{s\in\mathcal{L}} \lambda_s(m)\xi_s\eta_s\,.$$

I recall that $\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$.

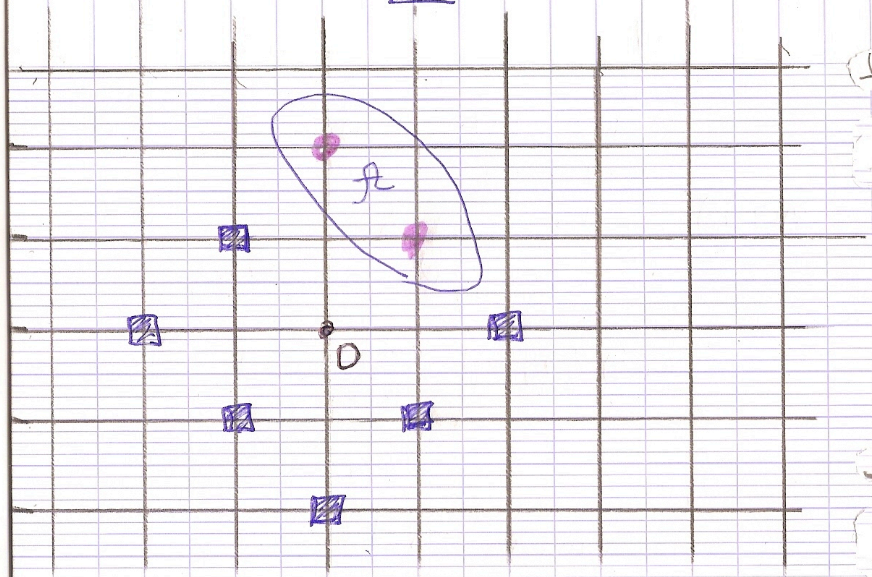
$DENOTE$ $\quad \mathcal{L}_f = \{s \in \mathcal{L} : |s| = |a| \text{ for some } a \in \mathcal{A}\}\,, \quad \mathcal{L}_\infty = \mathcal{L}\setminus\mathcal{L}_f\,.$

$$\xi^f = (\xi_s, s \in \mathcal{L}_f)\,, \qquad \xi^\infty = (\xi_s, s \in \mathcal{L}_\infty)\,.$$

Define η^f and η^∞ similarly.

$\mathcal{L}_f$ is the shade of the set $\mathcal{A}$ on $\mathcal{L}$.

$\mathbb{Z}^2$



$d = 2, \quad |\mathcal{A}| = 2$

$|\mathcal{L}_{\mathcal{A}}| = 6$

I recall that

$$H_{beam} = \text{Const} + \omega(m) \cdot r + \sum_{s \in \mathcal{L}} \lambda_s(m) \xi_s \eta_s + \dots .$$

Assume that $\mathcal{A}$ is admissible. Then for $m \in [1, 2]$ outside the zero-measure bad set $\mathcal{C}$, in [EGK1] we obtain the following

Normal Form Theorem:

THEOREM 1. For any action-vector $\tilde\rho \in [1,2]^n$ there exists a canonical transformation from new variables $\left(\tilde r, \tilde\theta, (\tilde\xi_s, \tilde\eta_s, s \in \mathcal{L})\right)$ to the old variables $\left(r, \theta, (\xi_s, \eta_s, s \in \mathcal{L})\right)$, such that in the new variables the Hamiltonian $H_b = H_b(\tilde r, \tilde\theta, \tilde\xi, \tilde\eta;\ \tilde\rho)$ reeds:

$$H_b = \Omega(\tilde\rho)\cdot\tilde r + \sum_{a\in\mathcal{L}_\infty} \Lambda_a(\tilde\rho)\tilde\xi_a\tilde\eta_a + \varepsilon\Big\langle K(\tilde\rho)\left(\tilde\xi^f, \tilde\eta^f\right)^t, \left(\tilde\xi^f, \tilde\eta^f\right)^t \Big\rangle + H_3.$$

Here:

$\Omega(\tilde\rho) = \omega + O(\varepsilon), \quad \Lambda_a(\tilde\rho) = \lambda_a + O(\varepsilon), \quad H_3$ – higher-order term;

$K(\tilde\rho)$ – complex symmetric matrix, $K(\tilde\rho) = O(1)$.

All these objects are explicit. Moreover,

i) the hamiltonian operator $iJK(\tilde\rho)$, corresponding to $K(\tilde\rho)$, for some choices of the set $\mathcal{A}$ is stable $\forall\,\tilde\rho$, but for some other its choices it is unstable $\forall\,\tilde\rho$.

ii) The vector-function $\Omega(\tilde\rho)$ is affine, $\Omega(\tilde\rho) = \omega(m) + \varepsilon L\tilde\rho$, where the linear operators $L, L^{-1} \sim 1$.

iii) The Hamiltonian vector-field $iJ\nabla H_3$ is 1-smoothing. Estimates on H_3 depend on the mass parameter $m \in [1,2]\setminus\mathcal{C}$.

The normal form Hamiltonian:

$$H_b = \Omega(\tilde\rho)\cdot\tilde r + \sum_{a\in\mathcal{L}_\infty}\Lambda_a(\tilde\rho)\tilde\xi_a\tilde\eta_a + \varepsilon\Big\langle K(\tilde\rho)\begin{pmatrix}\tilde\xi^f\\ \tilde\eta^f\end{pmatrix}, \begin{pmatrix}\tilde\xi^f\\ \tilde\eta^f\end{pmatrix}\Big\rangle + H_3.$$

The difficulty here is the matrix $K(\tilde\rho)$ since its Hamiltonian operator $iJK(\tilde\rho)$ IS NOT Hermitian or anti-Hermitian. What do we know about $iJK(\tilde\rho)$?

i) for most values of the action $\tilde\rho$, the operator $iJK(\tilde\rho)$ is invertible;

ii) if $d=1$, the eigenvalues of $iJK(\tilde\rho)$ are elliptic. If $d\geq 2$, some of them may be hyperbolic;

iii) if $d=2$, then for typical $\tilde\rho$ the hyperbolic eigenvalues are simple. But for $d\geq 3$ it seems that for some sets $\mathcal{A}$ the hyperbolic spectrum of $iJK(\tilde\rho)$ may be multiple for all $\tilde\rho$ (but the operator $iJK(\tilde\rho)$ has no Jordan cells).

So, to treat H_b for $d\geq 3$ we need a really good KAM theorem! For the finite-dimension case such KAM-theorem was proved by You with collaborators. For the PDE case at hand a needed theorem is proven in [EGK2]. Applying it to the NF we get:

5 KAM theorem

For each $m \notin \mathcal{C}$ we have:

THEOREM 2. There is a bad set of action-vectors $[1,2]^n_{bad} \subset [1,2]^n = \{\tilde\rho\}$, small in the sense of measure: meas $[1,2]^n_{bad} \le C\varepsilon^a$, $a > 0$, such that if $\tilde\rho \notin [1,2]^n_{bad}$, then the time-QP solution $(\xi,\eta)_{\mathcal{A}}(t)$ of the linear beam equation persists as a time-QP solution of (Beam). The persisted solution is linearly stable if and only if the hamiltonian matrix $iJK(\tilde\rho)$ is stable. If $d = 1$, then $iJK(\tilde\rho)$ always is stable. If $d \ge 2$, then for some admissible sets $\mathcal{A}$ it is unstable (for all values of $\tilde\rho$).

Remark. If $d \ge 2$, then the constructed linearly unstable KAM-solutions $(\xi,\eta)_{\mathcal{A}}(t)$ of (Beam) create around them certain zones of instability. So the KAM theory implies some instability results for small-amplitude solutions of (Beam)!