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Resonant averaging for weakly nonlinear Schrödinger equations

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1 Introduction: weak turbulence (WT)

(One of) origines: Rudolf Peierls, in *Annalen der Physik* **3** (1929)
(this is his thesis with W.Pauli). Modern state of affairs see in
[Naz] S. Nazarenko, *Wave Turbulence*, Springer 2011.

The method of WT applies to various equations. E.g., to NLS:

A) *Deterministic setting.* Consider NLS equation:

$$\dot u - i\Delta u + i|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d).$$

WT deals with small solutions. So let us better consider

(NLS) $$\dot u - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L; \quad \rho = \text{const.}$$

Take the exponential basis $\{e_{\mathbf{k}} = e^{i\mathbf{k}\cdot(x/L)}, \mathbf{k} \in \mathbb{Z}^d\}$. Then

$$-\Delta e_{\mathbf{k}} = \lambda_{\mathbf{k}} e_{\mathbf{k}}; \quad \lambda_{\mathbf{k}} = L^{-2}|\mathbf{k}|^2.$$

So there is plenty of exact resonances in the spectrum of Δ. – This is a prerequisite for WT.

We decompose u in the Fourier series, $u(t,x)=\sum u_{\mathbf{k}}(t)e_{\mathbf{k}}(x)$, and write (NLS) as

(*) $$\dot u_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar u_{\mathbf{k}_3}, \qquad \mathbf{k}\in\mathbb{Z}^d.$$

In WT they do the following:

◇ Study solutions for $(*)$ with a "typical" initial data $u(0)=u^0$, during "long" time. Time is so long that "solutions approach an invariant measure of $(*)$". They make

Claim: *For large values of time only resonant terms in $(*)$ are important.*

◇ For $t \gg 1$, they decompose solutions in asymptotical series in ε. Find first non-trivial term of this decomposition. It is given by a complicated explicit formula (or by an equation).

◇ Study that formula when $L\to\infty$. Go to a limit, by replacing sums $\sum_{\mathbf{k}\in L^{-1}\mathbb{Z}^d}$ by integrals $\int_{\mathbf{k}\in\mathbb{R}^d}$. In particular, study under that limit properly scaled quantities $|u_{\mathbf{k}}(t)|^2$, and prove that

(KZ spectrum) $$\langle |u_{\mathbf{k}}(t)|^2\rangle \sim |\mathbf{k}|^{-\varkappa}, \qquad \varkappa>0\,,$$

if $|\mathbf{k}|$ "belongs to the inertial range". Here "$\langle\cdot\rangle$" indicates certain averaging.

(KZ spectrum) $$\langle |u_{\mathbf{k}}(t)|^2\rangle \sim |\mathbf{k}|^{-\varkappa}, \qquad \varkappa>0.$$

Remarks. 1) Certainly (KZ spectrum) cannot be true for all solution of (NLS). Say, because of KAM. So we must assume that u_0 is random, and then try to prove (KZ spectrum) for typical u_0, or to incorporate in the averaging $\langle\cdot\rangle$ the ensemble-averaging.

2) It is not quite clear in what order we send $\varepsilon\to 0$ and $L\to\infty$. It may be better to talk not about the limit of WT, but about WT limits.

B) *Stochastic setting.* Following

V. Zakharov, V.L'vov, in *Radiophys. Quant. Electronics(1975),*

Cardy, Falcovich, Gawedzki "Non-Equilibrium Stat. Phys. and Turbulence", CUP 2008.

consider small solutions of NLS equation with small damping and small random force:

$$\text{(ZL)} \qquad \dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = -\nu(-\Delta+1)^p u + \sqrt{\nu}\,\langle\text{rand. force}\rangle, \quad x \in \mathbb{T}^d_L,$$

where $\varepsilon, \nu \ll 1$. Here ν – inverse time-scale of the forced oscillations; ε - amplitude of small oscillations. They impose some relation between ν and ε.

Random Force is

$$\sum b_{\mathbf{k}} \frac{d}{dt} \boldsymbol{\beta}^{\mathbf{k}}(t) e^{i\mathbf{k}\cdot(x/L)}, \quad b_{\mathbf{k}} > 0 \text{ and } b_{\mathbf{k}} \to 0 \text{ fast,}$$

where $\{\boldsymbol{\beta}^{\mathbf{k}}(t)\}$ – indep. standard complex Wiener processes.

Fact: As $t \to \infty$, solution of (ZL) converges in distribution to a stationary measure $\mu_{\varepsilon,\nu}$ of the equation (which is a "statistical equilibrium of the equation"):

$$\mathcal{D}u(t) \rightharpoonup \mu_{\varepsilon,\nu} \qquad \text{as } t \to \infty.$$

Similar to the deterministic case, Zakharov - L'vov do the following:

◇ Write the equation in Fourier:

$$\dot{u}_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho\, i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} - \nu(\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + \sqrt{\nu}\, b_{\mathbf{k}} \dot{\boldsymbol{\beta}}^{\mathbf{k}}(t)$$

The term $i\rho \sum u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3}$ is hamiltonian, with the Hamiltonian

$$\mathcal{H}^4 = \frac{\rho}{4} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} \bar{u}_{\mathbf{k}_4}.$$

◇ They decompose solutions of (ZL) and/or the stationary measure in series in a suitable small parameter and find the first nontrivial term of this decomposition. Again, for that only resonant terms of the equation are important.

◇ Study that term when $L \to \infty$ to calculate the corresponding (KZ) spectrum.

Same remark as before has to be made concerning the two limits $\langle\text{small parameter}\rangle \to 0$ and $L \to \infty$.

We choose $\varepsilon^2 = \nu$ – this is within the bounds, usually imposed in physics. It is illuminating to pass to the slow time $\tau = \nu t$:

(ZL) $$u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta+1)^p u + \langle \text{rand. force}\rangle', \quad x \in \mathbb{T}^d_L.$$

This is the equation I will discuss, mostly following my work with Alberto Maiocchi, who is now a post-doc in Paris.

We suggest to study the WT limits (at least, some of them) by splitting the limiting process in two steps:

I) prove that when $\nu \to 0$, main characteristics of solutions u^ν have limits of order one, described by certain *effective equation*.

II) Show that main characteristics of solutions for the effective equation have non-trivial limits of order one, when $L \to \infty$ and $\rho = \rho(L)$ is a suitable function of L.

Step I has been done rigorously, and I discuss it in this talk. I stress that the results of Step I along cannot justify the predictions of WT since the (KZ spectrum) cannot hold when the period L is fixed and finite.

At the end of my talk I will show that a heuristic argument *a-la* WT with a suitable choice of the function $\rho(L)$ leads in the limit of $L \to \infty$ to a Kolmogorov-Zakharov type equation and to a (KZ spectrum).

2 Averaging for PDEs without resonances

In my works $[1]$ SK $\&$ A.Piatnitski, JMPA (2008); $[2]$ SK, GAFA (2010); $[3]$ SK, Ann. Inst. Fourier - PR, 2013.

I studied the long-time behaviour of solutions for perturbed hamiltonian PDE without strong resonances. Namely, in [1,2] I considered equations like

$$\dot{u} - iu_{xx} + i|u|^2 u = \nu(u_{xx} - u) + \sqrt{\nu}\,\langle \text{rand. force} \rangle, \quad x \in S^1,$$

and in [3] – equations like

$$(*) \quad \dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu(-\Delta+1)^p u + \sqrt{\nu}\,\langle \text{rand. force} \rangle, \quad x \in \mathbb{T}^d,$$

where $p \in \mathbb{N}$ and $V(x)$ is such that there are no resonances in the spectrum of $-\Delta + V(x)$. The key idea was suggested in [2] – describe the long-time behaviour of the actions in the perturbed equations, using certain auxiliary Effective Equation. This is a well posed quasilinear SPDE with a non-local nonlinearity. For eq. $(*)$ without resonances, the Effective Equation is linear and does not depend on the Hamiltonian term $\nu i|u|^2 u$.

Situation changes if we add a non-linear dissipation and consider the equation

$$\dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu C|u|^{2q} u - \nu(-\Delta+1)^p u + \sqrt{\nu}\,\langle \text{rand. force} \rangle.$$

Now the effective equation is non-linear.

Main Results:

1) Actions of solutions for the initial-value problem for the equation, for $t \leq$ const ν^{-1} converge in distribution (as $\nu \to 0$) to those of the Eff. Eq.

2) Consider stationary solutions of the equation. They converge in distribution to stationary solutions of Eff. Eq.

3 Averaging for PDEs with resonances

Now the new results.

[KM1] SK & A. Maiocchi, Preprint, arXiv 1309.5022

[KM2] SK & A. Maiocchi, paper under preparation.

We apply the method of [1-3] to the equation of Zakharov-L'vov with $\varepsilon^2 = \nu$, written using the slow time $\tau = \nu t$:

(ZL) $$u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta+1)^p u + \langle \text{rand. force}\rangle',$$

We write $u(\tau,x) = \sum u_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot(x/L)}$, and re-write the equation in Fourier:

$$\frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

where $\mathbf{k}\in\mathbb{Z}^d$. We wish to control the asymptotic behaviour of the actions $\frac{1}{2}|u_{\mathbf{k}}|^2(\tau)$ and other characteristics of solutions via suitable effective equation. The Effective Equation for (ZL) may be derived through the *interaction representation*, i.e. by transition to the fast

rotating variables a:

$$a_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}v_{\mathbf{k}}(\tau), \qquad \mathbf{k}\in\mathbb{Z}^d$$

(*the variation of constant*). Note that

(*) $$|a_{\mathbf{k}}(\tau)| \equiv |v_{\mathbf{k}}(\tau)|.$$

In these variables the (ZL) equation reeds

$$\begin{aligned}\frac{d}{d\tau}a_{\mathbf{k}} = &-(\lambda_{\mathbf{k}}+1)^p a_{\mathbf{k}} + b_{\mathbf{k}}\, e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}\frac{d}{d\tau}\beta^{\mathbf{k}}(\tau)\\ &-i\rho\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} a_{\mathbf{k}_1}a_{\mathbf{k}_2}\bar{a}_{\mathbf{k}_3}\exp\left(-i\nu^{-1}\tau(\lambda_{\mathbf{k}_1}+\lambda_{\mathbf{k}_2}-\lambda_{\mathbf{k}_3}-\lambda_{\mathbf{k}})\right)\,.\end{aligned}$$

The terms, constituting the nonlinearity, oscillate fast as ν goes to zero, unless the sum of the eigenvalues in the second line vanishes. So only the terms for which this sum equals zero contribute to the limiting dynamics. The processes $\{\tilde{\boldsymbol{\beta}}^{\mathbf{k}}(\tau),\ \mathbf{k}\in\mathbb{Z}^d\}$ such that $\frac{d}{d\tau}\tilde{\boldsymbol{\beta}}^{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}\frac{d}{d\tau}\beta^{\mathbf{k}}(\tau)$ also are stand. independent complex Wiener processes. Accordingly, the effective equation should be the following damped/driven hamiltonian system

(Eff.Eq.) $$\frac{d}{d\tau}v_{\mathbf{k}} = -(\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} - R_{\mathbf{k}}(v) + b_{\mathbf{k}}\frac{d}{d\tau}\tilde{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k}\in\mathbb{Z}^d,$$

where $R_{\mathbf{k}}(v)$ is the resonant part of the hamiltonian nonlinearity:

$$R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}\\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar{v}_{\mathbf{k}_3}.$$

It is easy to see that $R(v)$ is the hamiltonian vector field $R = i\nabla\mathcal{H}^4_{\text{res}}$, where $\mathcal{H}^4_{\text{res}}$ is the resonant part of the Hamiltonian $\mathcal{H}^4$:

$$\mathcal{H}^4_{\text{res}} = \frac{\rho}{4} \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4\\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}_4|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar{v}_{\mathbf{k}_3}\bar{v}_{\mathbf{k}_4}.$$

◇ We have to impose some restrictions on p and d to make (ZL) well posed. E.g., $p=1, \quad d\le 3$ (if $p>1$, then d may be bigger than 3).

Properties of $\mathcal{H}^4_{\text{res}}$ and of Eff. Eq.:

Lemma. 1) $\mathcal{H}^4_{\text{res}}$ has two convex quadratic integrals of motion, $H_0 = \sum|v_{\mathbf{k}}|^2$ and $H_1 = \sum(|v_{\mathbf{k}}|^2|\mathbf{k}|^2)$.

2) The hamiltonian vector-field $i\nabla\mathcal{H}^4_{\text{res}}(v)$ is Lipschitz in sufficiently smooth Sobolev spaces.

3) (EffEq) is well posed in sufficiently smooth Sobolev spaces.

(ZL)

$$\frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

$$(EffEq) \qquad \frac{d}{d\tau}v_{\mathbf{k}} = -R_{\mathbf{k}}(v) - (\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}^d.$$

Actions of a solution $u^\nu(\tau)$ are

$$I^\nu_{\mathbf{k}}(\tau) = \tfrac{1}{2}|u^\nu_{\mathbf{k}}(\tau)|^2, \qquad \mathbf{k} \in \mathbb{Z}^d.$$

Theorem 1. Let $u^\nu_{\mathbf{k}}(\tau)$ and $v(\tau)$ be solutions of (ZL) and (EffEq) with same initial data. Then, for each $\mathbf{k}$ and for $0 \le \tau \le 1$,

$$\mathcal{D}I^\nu_{\mathbf{k}}(\tau) \rightharpoonup \mathcal{D}|v_{\mathbf{k}}(\tau)|^2 \text{ as } \nu \to 0.$$

Does the effective equation control the angles $\varphi_{\mathbf{k}} = \arg u_{\mathbf{k}} =: \varphi(u_{\mathbf{k}})$? No, instead it controls the angles of of the a-variables, $a^\nu_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}v^\nu_{\mathbf{k}}(\tau)$. Let $(s_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d)$, be a resonant vector, i.e. an integer vector of finite length such that

$$\sum s_{\mathbf{k}}\lambda_{\mathbf{k}} = 0.$$

Then $\sum s_{\mathbf{k}}\varphi(u_{\mathbf{k}}) = \sum s_{\mathbf{k}}\varphi(a_{\mathbf{k}})$. Therefore we have

Theorem $1'$. Under assumptions of Theorem 1, let s be a resonant vector. Then

$$\sum s_{\mathbf{k}}\varphi(u_{\mathbf{k}}(\tau)) \longrightarrow \sum s_{\mathbf{k}}\varphi(v_{\mathbf{k}}(\tau)) \text{ as } \nu \to 0, \quad \text{in probability,}$$

after mollification in τ.

Now consider a stationary measure μ^ν for (ZL). Let $u^\nu(\tau) = (u^\nu_{\mathbf{k}}(\tau), \mathbf{k} \in \mathbb{Z}^d)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}(u^\nu(\tau)) \equiv \mu^\nu .$$

Theorem 2. Every sequence $\nu'_j \to 0$ has a subsequence $\nu_j \to 0$ such that

$$\mathcal{D}(I_{\mathbf{k}}(u^{\nu_j})) \rightharpoonup \mathcal{D}(I_{\mathbf{k}}(v(\tau)) \quad \text{as} \quad \nu_j \to 0\,,$$

for each $\mathbf{k}$, where $v(\tau)$ is a stationary solution for the Eff. Eq.

What about phases of the stationary solutions, $\varphi_{\mathbf{k}}(u^\nu(\tau))$?

Assume that Eff. Eq. has a unique stationary measure. Let $v(\tau)$ be a corresponding stationary solution.

For any $N \geq 1$ consider

$$\varphi^{\nu N}(u^\nu(\tau)) = (\varphi^\nu_{\mathbf{k}}(u^\nu(\tau)), |\mathbf{k}| \leq N) \in \mathbb{T}^{D_N},$$
$$\varphi^{N}(v(\tau)) = (\varphi_{\mathbf{k}}(v(\tau)), |\mathbf{k}| \leq N) \in \mathbb{T}^{D_N}\,.$$

I will decompose $\mathbb{T}^{D_N} = \{\theta\}$ as $\mathbb{T}^{D_N} = \mathbb{T}_1 \times \mathbb{T}_2 = \{(\theta^1, \theta^2)\}$, and will denote $\pi_1 : \mathbb{T}^{D_N} \to \mathbb{T}_1$, $\pi_2 : \mathbb{T}^{D_N} \to \mathbb{T}_2$.

Theorem 3. Let Eff. Eq. has a unique stationary measure. Then
1) for each $\mathbf{k}$,

$$\mathcal{D}(I_{\mathbf{k}}(u^{\nu})) \rightharpoonup \mathcal{D}(I_{\mathbf{k}}(v(\tau)) \quad \text{as} \quad \nu \to 0\,,$$

and
2) $\forall\, N$, there is an unimodular linear isomorphism

$$L : \mathbb{T}^{D_N} \to \mathbb{T}^{D_N} \simeq \mathbb{T}_1 \times \mathbb{T}_2 = \{(\theta^1, \theta^2)\},$$

such that

$$\mathcal{D}(L(\varphi^{\nu N}(u^\nu(\tau))) \rightharpoonup \mathcal{D}(\pi_1 \circ L(\varphi^N(v(\tau)) \times d\theta^2\,.$$

So, if in addition the (ZL) equation has a unique stationary measure, then for ANY its solution $u^\nu(\tau)$ we have

$$\lim_{\nu\to 0}\lim_{\tau\to\infty}\mathcal{D}\big(I(u^\nu(\tau))\big)=\mathcal{D}(I(v(\tau)),$$

where $I=(I_{\mathbf{k}},\mathbf{k}\in\mathbb{Z}^d)$ and $v(\tau)$ is a stat. solution of Eff. Eq. And similar limit holds for the distribution of the angles of $u^\nu(\tau)$.

But when Eff. Eq. has a unique stat. measure? I note before hand that *existing technique allows to prove the uniqueness only for equations which have at most cubic growth.*

(Eff.Eq.)

$$\frac{d}{d\tau}v_{\mathbf{k}}=-(\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}}-R_{\mathbf{k}}(v)+b_{\mathbf{k}}\frac{d}{d\tau}\beta^{\mathbf{k}}(\tau),\quad \mathbf{k}\in\mathbb{Z}^d\,,$$
$$R_{\mathbf{k}}(v)=i\rho\sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}\\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar v_{\mathbf{k}_3}\,.$$

Theorem 4. 1) Let $p=1$. Then Eff. Eq. has a unique stat. measure if $d\le 3$.

2) Take any d. Then Eff. Eq. has a unique stat. measure if $p\ge p_d$ for a suitable $p_d\ge 0$.

4 Limit $L \to \infty$ (on the physical level of accuracy).

Now let us parametrise the Fourier modes by

$$\mathbf{k} \in \mathbb{Z}_L^d := \mathbb{Z}^d / L \,,$$

and write the Eff. Eq. as

$$(EffEq) \qquad \begin{aligned} &\frac{d}{d\tau} v_{\mathbf{k}} = -R_{\mathbf{k}}(v) - \gamma_{\mathbf{k}} v_{\mathbf{k}} + b_{\mathbf{k}} \frac{d}{d\tau} \boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d, \\ &R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k} \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3} . \end{aligned}$$

Here $\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$.

Consider the moments

$$M^{\mathbf{k}_1,\dots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1},\dots\mathbf{k}_{n_1+n_2}}(\tau) = \mathbf{E}(v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{n_1}} \bar{v}_{\mathbf{k}_{n_1+1}} \dots \bar{v}_{\mathbf{k}_{n_1+n_2}}).$$

Physical Assumptions:

i) *Quasi-Gaussian approximation:*

$$M^{\mathbf{l}_1,\mathbf{l}_2}_{\mathbf{l}_3,\mathbf{l}_4} \sim M^{\mathbf{l}_1}_{\mathbf{l}_1} M^{\mathbf{l}_2}_{\mathbf{l}_2} (\delta^{\mathbf{l}_3}_{\mathbf{l}_1} + \delta^{\mathbf{l}_4}_{\mathbf{l}_1})(\delta^{\mathbf{l}_3}_{\mathbf{l}_2} + \delta^{\mathbf{l}_4}_{\mathbf{l}_2}) \,,$$

and similar for higher order moments.

ii) *Quasi stationary approximation for equations in the chain of moment equations.*

Denote

$$n_{\mathbf{k}} = L^d M^{\mathbf{k}}_{\mathbf{k}} / 2 \,, \qquad \tilde{b}_{\mathbf{k}} = L^{d/2} b_{\mathbf{k}}.$$

$n_{\mathbf{k}}$ - normalised energy of the wave-vector $\mathbf{k}$.

Accepting the two hypotheses above we get the KZ kinetic equation:

Theorem 5. When $L \to \infty$ we have

$$(KZ)\qquad \begin{aligned} \frac{d}{d\tau} n_{\mathbf{k}} &= -2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \tilde{b}_{\mathbf{k}}^2 \\ &+4\,\frac{\rho^2}{L}\int_\Gamma d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{f_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}+\gamma_{\mathbf{k}_4}} \\ &\qquad\times\left(n_{\mathbf{k}_1}n_{\mathbf{k}_2}n_{\mathbf{k}_3}+n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_2}-n_{\mathbf{k}}n_{\mathbf{k}_2}n_{\mathbf{k}_3}-n_{\mathbf{k}}n_{\mathbf{k}_1}n_{\mathbf{k}_3}\right). \end{aligned}$$

Here Γ is the resonant surface,

$$\Gamma = \{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\in\mathbb{R}^{3d} : \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}+\mathbf{k}_3,\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}|^2+|\mathbf{k}_3|^2\},$$

$\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m+b)$, $f_{\mathbf{k}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ - some bounded smooth function, constructed in terms of the normal frame to Γ at $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$.

Because of the dissipation in the Eff. Eq., our (KZ) equation is “better” then usually the (KZ) equations are: the divisor in the integrand has no zeroes.

The KZ spectra.

I recall that $\gamma_k = a|k|^m + b$. Looking for solutions of (KZ), where $n_{\mathbf{k}}$ depends only on $|\mathbf{k}|$ and arguing *a-la* Zakharov, we find the following:

i) if $0 < a \ll b \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-d+2/3}, \quad \text{or} \quad n_{\mathbf{k}} \sim |\mathbf{k}|^{-d}.$$

ii) if $0 < b \ll a \ll 1$, then

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d-2}{3}}, \quad \text{or} \quad n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d}{3}}.$$