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Damped and driven Hamiltonian PDE

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§1. Damped-driven Hamiltonian PDE

We are interested in the following class of equations:

$$\langle \text{Hamiltonian PDE} \rangle = \nu\text{-small damping} + \kappa_\nu \langle \text{force} \rangle, \qquad (*)$$

where $\nu \ll 1$ and the scaling constant κ_ν is such that solutions stay of order one as $\nu \to 0$ and $t \gg 1$. The constant κ_ν is unknown, to find it is a part of the problem. Equations $(*)$ are important for physics. They describe turbulence in various physical media.

♢ All equations will be considered in the finite-volume case. The force will be random. The objects and the constructions make sense in the deterministic case as well, but then we can prove less.

♢ The damping usually is the Laplacian. But it may be another operator, linear or nonlinear.

♢ If in $(*)$ Hamiltonian PDE is replaced by a finite-dimensional Hamiltonian system, then $\kappa_\nu = \sqrt{\nu}$.

Examples. 1) 2d Navier-Stokes:

$$\dot{u} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \sqrt{\nu}\,(\text{random force}), \quad \text{div}\, u = 0, \qquad \dim x = 2.$$

Now $\kappa_\nu = \sqrt{\nu}$ (I will discuss this later).

2) Burgers equation on a circle:

$$\dot{u} + u u_x = \nu u_{xx} + (\text{random force}), \qquad x \in S^1, \ \int u\, dx \equiv 0.$$

It is well known that now $\kappa_\nu = 1$.

3) 3d Navier-Stokes:

$$\dot{u} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \kappa_\nu\,(\text{random force}), \quad \text{div}\, u = 0, \qquad \dim x = 3.$$

Very complicated equation. Right κ_ν is unknown. Kolmogorov believed that $\kappa_\nu = 1$.

4) CGL:

$$\dot{u} + i|u|^2 u = \nu(\Delta - 1)u + \kappa_\nu\,(\text{random force}), \quad x \in \mathbb{T}^d.$$

Right κ_ν is unknown.

§2. Navier-Stokes equations

Consider 2d NSE under periodic boundary conditions:

$$\dot{u} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \sqrt{\nu}\,(\text{random force}), \quad \text{div}\, u = 0, \qquad (NSE)$$

$$x \in \mathbb{T}^2, \qquad \int u\,dx = \int \text{force}\,dx = 0.$$

Most of results below hold true if $x \in \Gamma^2$, Γ^2 – a compact Riemann surface. Majority of them hold if $x \in \Omega \Subset \mathbb{R}^2$ and $u\,|_{\partial\Omega} = 0$.

The random force is smooth and homogeneous in x. As a function of time this is

- either a nondegenerate white noise,
- or a nondegenerate kick-process.

Below I assume that the force is a white noise.

Let $\mathcal{H}$ be the L_2-space of divergence-free vector fields $u(x), x \in \mathbb{T}^2$, such that $\int u\,dx = 0$. Then:

- (NSE) has a unique statistical equilibrium (= a stationary measure). This is a space-homogeneous measure μ_ν in the space $\mathcal{H}$ such that the distribution $\mathcal{D}(u(t))$ of any solution $u(t,x)$ of (NSE) converges to μ_ν as $t \to \infty$.

- The set of stationary measures $\{\mu_\nu, 0 < \nu \le 1\}$ is pre-compact in the space of measures in $\mathcal{H}$, and every limiting measure $\mu_0 = \lim_{\nu_j \to 0} \mu_{\nu_j}$ is an invariant measure of the 2d Euler equation (i.e. of $(NSE)_{\nu=0}$). It also is space-homogeneous.

- This limiting measures μ_0 are "genially infinite-dimensional": μ_0-measure of any finite-dimensional subset of $\mathcal{H}$ equals zero.

- Denote $\mathcal{H}^s = \mathcal{H} \cap \langle$Sobolev space of order $s\rangle$. Then $\mu_0(\mathcal{H}^2) = 1$.

  Conjecture 1. $\mu_0(\mathcal{H}^{2+\nu}) = 0$ if $\nu > 0$.

  Conjecture 2. The energy spectrum of the measure μ_0 behaves as $E_k \sim k^{-5}$.

These two conjectures are closely related.

The limiting invariant measure(s) μ_0 describes the space-periodic 2d turbulence. But it is unknown if μ_0 is unique, i.e. if it depends on the sequence $\nu_j \to 0$, and how to calculate it.

To study the limiting measure(s) μ_0 means to study the periodic 2d turbulence.

See

[SK, A. Shirikyan] *"Mathematics of 2d Statistical Hydrodynamics"* CUP (2012). See my web-page

For the inviscid limit in the 1d Burgers equation we now know a lot. The best results (in a sense, they are final) are obtained by Alexander Boritchev in his thesis, see

[Boritchev] *Sharp estimates for turbulence in white-forced generalised Burgers equation*, arXiv 2011.

Below I will discuss the case when the Hamiltonian PDE is either an integrable PDE, or a linear Hamiltonian PDE in a general position. Then a progress may be achieved using a form of averaging.

§3. Damped-driven linear Schrödinger equation.

Consider small oscillations in a damped-driven NLS in the presence of an external electric field. Then the corresponding Hamiltonian PDE is a linear Schrödinger equation with a potential. We assume that the damping is nonlinear and write the equation using the slow time $\tau = \nu t$:

$$\text{(1)} \quad \frac{\partial u}{\partial \tau} + \nu^{-1} i \left(-\Delta u + V(x)u\right) = \Delta u - \gamma_R |u|^{2p} u - i\gamma_I |u|^{2q} u + (\text{random force}),$$

$$x \in \mathbb{T}^d, \;\; \gamma_R, \gamma_I \geq 0, \; \gamma_R + \gamma_I = 1; \;\; V(x) \in C^\infty(\mathbb{T}^d; \mathbb{R}).$$

Some restrictions should be imposed on d, p and q. The random force is smooth, white in time and non-degenerate.

$$(1)\qquad \frac{\partial u}{\partial \tau} + \frac{i}{\nu}(-\Delta u + V(x)u) = \Delta u - \gamma_R |u|^{2p} u - i\gamma_I |u|^{2q} u + (\text{random force}),$$

Denote $\mathcal{H}^r = H^r(\mathbb{T}^d; \mathbb{C})$. Eq. (1) defines in $\mathcal{H}^r$ a Markov process.

Applying the methods, developed in the last 10 years to study the randomly forced 2d NSE, one can show that eq. (1) has a unique stationary measure:

Theorem (SK-Shirikyan, Harier, Odasso, Shirikyan). Under more restrictions on d, p, q eq. (1) has a unique stationary measure μ^ν. For any solution $u(\tau)$ of (1) we have

$$\text{dist}\,(\mathcal{D}(u(\tau)), \mu^\nu) \to 0 \quad \text{as} \quad \tau \to \infty.$$

$\diamondsuit$ This theorem is not at all a finite result since some of the restrictions ARE NOT TECHNICAL. E.g., if $\gamma_R = 0$, then we must have $p \leq 1$. On the contrary, its counterpart for equations with kick-forces IS a final result.

MAIN PROBLEMS. a) What happens to solutions $u^\nu(\tau, x)$ as $\nu \to 0$?
b) What happens to the stationary measure μ^ν as $\nu \to 0$?

It is easy to see that the set of stationary measures $\{\mu^\nu, 0 < \nu \leq 1\}$ is pre-compact in the space of measure. So $\mu^{\nu_j} \rightharpoonup \mu^0$ as $\nu_j \to 0$. As before, one can check that μ^0 is an invariant measure for the corresponding Hamiltonian PDE, which is now the linear Schrödinger equation

(Lin.Schrödinger) $$\dot{u} + i(-\Delta u + V(x)u) = 0.$$

But which one? - This equation has plenty of invariant measures! And does the limit μ^0 depend on the sequence $\nu_j \to 0$??

I recall that in the case of NSE we do not have answers to these questions.

v-coordinates.

Consider $A = -\Delta + V(x)$. Assume that $V \geq 1$. Let $\xi_1, \xi_2, \dots$ be its L_2-normalised real eigenfunctions and $1 \leq \lambda_1 \leq \lambda_2 \leq \dots$ – corresponding eigenvalues. Assume that the spectrum $\{\lambda_1, \lambda_2, \dots\}$ is nonresonant in the sense that

$$\sum \lambda_j \cdot s_j \neq 0 \quad \forall\, s \in \mathbb{Z}^\infty, \;\; 0 < |s| < \infty.$$

This is a mild restriction: majority of potentials V are non-resonant.

FOURIER TRANSFORM. For any $u(x) \in \mathcal{H} = L_2(\mathbb{T}^d, \mathbb{C})$ decompose it in the ξ-basis:

$$u(x) = v_1 \xi_1 + v_2 \xi_2 + \dots, \quad v_j \in \mathbb{C}.$$

Denote $\mathbf{v} = (v_1, v_2, \dots)$. These are the (complex) v*-coordinates*. Consider the map

$$\Psi : u(\cdot) \mapsto \mathbf{v}.$$

This an unitary isomorphism $\mathcal{H} \to l_2$.

In the v-coordinates (Lin.Schrödinger) reeds

$$\frac{\partial}{\partial \tau} v_j + i\nu^{-1}\lambda_j v_j = 0 \qquad \forall\, j.$$

Consider the action-angle variables for these equations

$$I_j = \frac{1}{2}|v_j|^2, \;\; \varphi_j = \text{Arg}\, v_j \in S^1; \quad I = (I_1, \dots) \in \mathbb{R}^\infty_+, \;\; \varphi = (\varphi_1, \dots) \in \mathbb{T}^\infty.$$

In these coordinates the equations become

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j + \nu^{-1}\lambda_j = 0 \qquad \forall j\,.$$

Let us write eq. (1) in the v-variables:

$$\frac{\partial}{\partial\tau}\mathbf{v} + i\nu^{-1}\text{diag}\,\{\lambda_j\}\mathbf{v} = \Psi\Big(\Delta u - \gamma_R|u|^{2p}u - \gamma_I|u|^{2q}u\Big) + \Psi\big((\text{random force})\big),$$

where $\mathbf{v} = (v_1, v_2, \dots)$ and $u := \Psi^{-1}(\mathbf{v})$. Pass to the action-angles:

$$\frac{\partial}{\partial\tau}I_j + 0 = F_j(I,\varphi) + (\text{random force})_j, \quad j \geq 1,$$
$$\frac{\partial}{\partial\tau}\varphi_j + \nu^{-1}\lambda_j = \dots, \qquad j \geq 1.$$

Now we are in the setting of the averaging theory. Accordingly we expect that the limit $\lim_{\nu\to 0} I^\nu(\tau)$ $(0 \leq \tau \leq T)$ exists and satisfies the averaged I-equations:

$$\frac{\partial}{\partial\tau}I_j(\tau) = \langle F_j\rangle(I) + \langle(\text{random force})_j\rangle, \quad j \geq 1.$$

Here $\langle F_j\rangle(I) = \int_{\mathbb{T}^\infty} F_j(I,\varphi)\,d\varphi$ and $\langle(\text{random force})_j\rangle$ is defined by similar stochastic rules. In infinite dimensions this idea does not work well since $I(\tau) \in \mathbb{R}^\infty_+$ – this is a very bad phase-space, and the averaged I-equations are singular! To study the limiting dynamics of $I(\tau)$ we consider other - non-singular - limiting equations.

Effective equations.

Denote $\hat{A} =$ diag $\{\lambda_j, j \geq 1\}$ and consider new equation, constructed by a kind of averaging:

$$\frac{\partial}{\partial \tau}\mathbf{v} = \big(-\hat{A} + \mathcal{L}\big)\mathbf{v} + R(\mathbf{v}) + (\text{random force}). \tag{2}$$

Here $\mathcal{L}, R$ and the force are constructed by some explicit rools, and

- $\mathcal{L}$ =diag $\{l_k\}$ – bounded linear operator, constructed in terms of the Fourier transform of the potential $V(x)$, such that $-\hat{A} + \mathcal{L} \leq -\frac{1}{2}\hat{A}$.
- The nonlinearity $R(v)$ is constructed from the dissipative term of eq. (1) $-\gamma_R|u|^{2p}u$. This is a nice locally Lipschitz analytic mapping.
- random force in (2) is explicit: $(\text{random force})_k = \frac{\partial}{\partial \tau} Y_k \boldsymbol{\beta}_k(\tau)$, where $Y_k = \left(\sum_l b_l^2 |\Psi_{kl}|^2\right)^{1/2}$.

Eq. (2) is called the *Effective Equation*. This is a semilinear stochastic heat equation with a non-local nonlinearity, written in terms of Fourier coefficients of a solution.

The Effective Equation is independent from the Hamiltonian part $-i\gamma_I|u|^{2q}u$ of the perturbation in eq. (1). If we add to eq. (1) any other Hamiltonian term $i\nabla_u \tilde{h}(u)$, this *will not change* the Effective Equation. It depends only on the damping.

Examples. 1) If $\gamma_R = 0$, then effective equation is linear.

2) If $p = 1$, i.e. the dissipative nonlinearity in eq. (1) is $-\gamma_R|u|^2u$, then

$$R(\mathbf{v})_k = -\gamma_R v_k \sum_l |v_l|^2 L_{kl},$$

where L_{kl} is some explicit tensor. So Effective Equations take the form

$$\frac{\partial}{\partial\tau}v_k = (-\lambda_k + l_k)v_k - \gamma_R v_k \sum_l |v_l|^2 L_{kl} + Y_k \frac{\partial}{\partial\tau}\boldsymbol{\beta}_k(\tau), \qquad k \geq 1.$$

Equation (1):

$$\frac{\partial u}{\partial \tau} + \frac{i}{\nu}\left(-\Delta u + V(x)u\right) = \Delta u - \gamma_R |u|^{2p} u - i\gamma_I |u|^{2q} u + (\text{random force}).$$

Effective Equation (2):

$$\frac{\partial}{\partial \tau}\mathbf{v} = -\hat{A}\mathbf{v} + \mathcal{L}\mathbf{v} + R(\mathbf{v}) + (\text{random force}).$$

Theorem 1. Let $u^\nu(\tau), 0 \le \tau \le T$, be a solution of eq. (1), $u^\nu(0) = u_0$, and $\mathbf{v}^\nu(\tau) = \Psi(u^\nu(\tau))$. Let $\mathbf{v}^0(\tau)$ be a solution of Effective Equation (2) such that $\mathbf{v}^0(0) = \mathbf{v_0} = \Psi(u_0)$. Then

$$I(\mathbf{v}^\nu(\tau)) \to I(\mathbf{v}^0(\tau)), \quad 0 \le \tau \le T,$$

as $\nu \to 0$, in probability.

Let μ^ν be a unique stationary measure for (1) and $\Psi \circ \mu^\nu$ – this measure, written in the v-variables.

Lemma. Effective Equation (2) has a unique stationary measure m_0.

Theorem 2. $\Psi \circ \mu^\nu \rightharpoonup m_0$ as $\nu \to 0$. I.e., the unique stationary measure of (1), written in the v-variables, converges to the unique stationary measure of the Effective Equation (2).

That is, for ANY solution $u^\nu(\tau)$ of (1) we have

$$\boxed{\lim_{\nu\to 0} \lim_{\tau\to\infty} \mathcal{D}(\Psi(u^\nu(\tau))) = m_0.}$$

Example. If $\gamma_R = 0$, then the Effective Equation is linear. Its unique stationary measure m_0 is the Gaussian measure which is the law of the Ornstein-Uhlenbeck process

$$\int_{-\infty}^{0} e^{s(-\hat{A}+\mathcal{L})} \cdot (\text{diag}\,\{Y_j\})\; d\boldsymbol{\beta}(s).$$

Equations without dissipation.

Following Debussche–Odasso (2005), consider similar 1d-equations without dissipation but with friction :

$$\frac{\partial u}{\partial \tau} + \nu^{-1} i\,(-u_{xx} + V(x)u) = -\gamma_R |u|^2 u - i\gamma_I |u|^{2q} u + (\text{random force}), \quad \gamma_R > 0.$$

Now the effective equation is the system

$$\frac{\partial}{\partial \tau} v_k = -\gamma_R v_k \sum_l |v_l|^2 L_{kl} + Y_k \frac{\partial}{\partial \tau} \bar{\beta}_k(\tau), \quad k \geq 1.$$

This is a stochastic ODE in the space l_2.

Theorems 1, 2 remain true.

Big Problem. Study for these equations the limit

$$\langle \text{space-period} \rangle \to \infty.$$

I believe that this corresponds to the Weak Turbulence.

For all these results see

SK *“Weakly nonlinear stochastic CGL equations”*. ArXiv 2011, to appear in Ann. IHP_PS.

§4. Damped-driven KdV

(See SK, GAFA **20** (2010), 1431-1463.) Consider the damped-driven KdV:

(3) $\dot{u}+u_{xxx}-uu_x = \nu u_{xx}+\sqrt{\nu}\,(\text{random force}), \quad x \in \mathbb{T}^1 = \mathbb{R}/2\pi, \int u\,dx \equiv 0.$

KdV equation $(3)_{\nu=0}$ admits a nonlinear Fourier Transform (NFT) which is an analytic mapping

$$\Psi : u(x) \mapsto \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots), \qquad \mathbf{v}_j \in \mathbb{R}^2,$$

such that

a) $d\Psi(0)(u(x)) = \hat{u}$ is the Fourier transform of $u(x)$;
b) (Quasilinearity): the nonlinear part of Ψ, i.e., the mapping $\Psi - d\Psi(0)$, is 1-smoother than Ψ;
c) Ψ transforms KdV to the equation

$$\dot{\mathbf{v}}_j + \Phi_j(\mathbf{v}) = 0, \qquad j = 1, 2, \dots,$$

where $\mathbf{v}_j \cdot \Phi_j(\mathbf{v}) = 0$ for each j. So, each $I_j(\mathbf{v}) = \frac{1}{2}|\mathbf{v}_j|^2$ is an integral of motion.

See:

$[1]$ T. Kappeler, J. Pöshel "KdV $\&$ and KAM", Springer 2003.

$[2]$ S.K, Galina Perelman, DCDS-A **27** (2010), 1-24.

The perturbed KdV (3), written in the $\mathbf{v}$-variables, is similar to the damped-driven linear Schrödinger eq. (1), written in its own $\mathbf{v}$-variables:

$$\frac{d}{d\tau}\mathbf{v}(\tau) + \nu^{-1}\Phi(\mathbf{v}) = P(\mathbf{v}) + B(\mathbf{v})\frac{d}{d\tau}\beta; \qquad 0 \le \tau \le T.$$

Here $\beta(\tau) = (\beta_1(\tau), \beta_2(\tau), \dots)^t$ and τ is the slow time.

We can write down the corresponding effective equation:

$$\frac{d}{d\tau}\mathbf{v}(\tau) = \langle P\rangle(\mathbf{v}) + \gamma\langle\langle B\rangle\rangle(\mathbf{v})\frac{d}{d\tau}\beta; \quad \mathbf{v}(0) = \mathbf{v}_0. \tag{4}$$

Here $\langle P\rangle(\mathbf{v})$ – effective drift; $\langle\langle B\rangle\rangle(\mathbf{v})$ – effective dispersion operator. They are defined as follows.

For any vector $\theta = (\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$ consider the rotation Π_θ of the l_2 space of vectors v which rotates the component $\mathbf{v}_j$ by the angle θ_j. Then $\langle P\rangle$ is

$$\langle P\rangle(\mathbf{v}) = \int_{\mathbb{T}^\infty} \Pi_\theta^{-1} \circ P(\Pi_\theta \mathbf{v})\, d\theta$$

(note that the integrand is the vectorfield $P(v)$ on the space l_2, transformed by the linear transformation Π_θ of that space). $\langle\langle B\rangle\rangle(\mathbf{v})$ is defined in a similar way.

Eq. (4) turns out to be a quasilinear stochastic heat equation, written in the Fourier variables. It is more complicated than the previous effective equation (2) since now the noise is non-additive. Still it allows to study (3) in the same way as we studied (1).

The approach also applies to other perturbations of KdV. For example, to

(5) $$\dot{u} + u_{xxx} - uu_x = -\nu u + \nu\,(\text{Hamiltonian term}) + \sqrt{\nu}\,(\text{random force}).$$

Now the effective equation is an SDE in l_2 (not an SPDE !). It is independent from the Hamiltonian part of the equation.

§5. Remarks on non-equilibrium statistical physics.

(jointly with Armen Shirikyan and Andrey Dymov).

Let us apply to the perturbed KdV equation above the NFT (i.e. write it in the v-variables):

$$\dot{\mathbf{v}}_j + \Phi_j(\mathbf{v}) = \nu\big((-\mathbf{v}_j + Q_j(\mathbf{v}) + H_j(\mathbf{v})\big) + \sqrt{\nu}\, B_j(\mathbf{v})\frac{d}{d\tau}\beta(\tau), \quad j \geq 1;$$

$v_j \in \mathbb{R}^2$. The system in the l.h.s. is an integrable chain of coupled rotators;

Q – a HS operator, $H(v)$ – a Hamiltonian vector field, $B(v)$ – the dispersion operator.

Eq. has a unique stationary measure μ_ν , and we use an effective equation to study its behaviour as $\nu \to 0$.

By analogy, consider a system of $N \gg 1$ free nonlinear rotators with a small Hamiltonian coupling and damp-drive it:

$$\dot{\mathbf{v}}_j + if_j(|\mathbf{v}_j|^2)\mathbf{v}_j + \gamma H_j(\mathbf{v}_j, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}) = -\nu\mathbf{v}_j + \sqrt{\nu T_j}\, \dot{\beta}_j(t); \;\; v_j \in \mathbb{C}.$$

Here $1 \leq j \leq N, 0 < \nu < \gamma < 1, f_j \in \mathbb{R}, T_j \geq 0, \beta_j(t)$ – standard complex Wiener process and H – a Hamiltonian vectorfield. Let $\mu_{\nu,\gamma,N}$ be the unique stationary measure.

$$\dot{\mathbf{v}}_j + i f_j(|\mathbf{v}_j|^2)\mathbf{v}_j + \gamma H_j(\mathbf{v}_j, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}) = -\nu \mathbf{v}_j + \sqrt{\nu T_j}\,\dot{\beta}_j(t);$$

$1 \le j \le N, 0 < \nu < \gamma < 1$, $f_j \in \mathbb{R}$, H – a Hamiltonian vectorfield;

$\mu = \mu_{\nu,\gamma,N}$ – the unique stationary measure. E.g., $f_j = f$ for all j.

Question. How behaves μ when $N \gg 1, \nu \le \gamma$ and $\gamma, \nu \to 0$?

Proposition. Let $\gamma = \nu$ and $T_j > 0$ for all j. Then for $\nu \to 0$ and $N \to \infty$

(or $\nu \to 0$, while $N \gg 1$ is fixed)

we have $\mu_{\nu,\gamma,N} \to \mu_0$, where $\mu_0 = \mu_{T_1} \oplus \mu_{T_2} \oplus \dots$ and μ_T is the stationary

(Gaussian) measure for the process

$$\dot{v} = -v + \sqrt{T}\,\dot{\beta}_j(t), \qquad v \in \mathbb{C}.$$

If $\gamma \gg \nu$, the system may start to feel resonances. For $\gamma = \sqrt{\nu}$ we can control them for SOME systems as above and show that Proposition remains true (but not if $\gamma \gg \sqrt{\nu}$).

By analogy with the Weak Turbulence me make

Conjecture. If $\gamma = \sqrt{\nu}$ and $N = N(\nu) \to \infty$, then generically the limit $\lim_{\nu \to 0} \mu_{\nu,\gamma,N}$ depends on the Hamiltonian part H.