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# Eulerian limit for randomly forced 2D Navier-Stokes equation

Thesis: Stationary space-periodic 2D turbulence is described by a measure in the space of divergence-free vector-field on $\mathbb{T}^2$. This measure is an invariant measure for the (free) 2D Euler equation. It comes from the small-viscosity 2D Navier-Stokes equation under the inviscid limit.

The measure is called *the Eulerian limit*.

## §1. The equation

Space-periodic 2D turbulence is described by the small-viscosity 2D Navier-Stokes equation (NSE) under periodic boundary conditions, perturbed by stationary random force:

$$v'_\tau - \varepsilon\Delta v + (v\cdot\nabla)v + \nabla\tilde p = \varepsilon^a\,\tilde\eta(\tau,x),$$

$$x\in\mathbb{T}^2,\quad \operatorname{div} v = 0,\quad \int v\,dx \equiv \int \tilde\eta\,dx \equiv 0.$$

- $0<\varepsilon\ll 1$ and the scaling exponent $a$ is $a<\frac{3}{2}$ (e.g., $a=0$),
- $\tilde\eta$ is a divergence-free Gaussian random field, white in time, smooth and stationary in $x$, and “sufficiently non-degenerate”.

NSE defines Markov process in space $\mathcal{H}$,

$$\mathcal{H}=\{u(x)\in L^2 \mid \operatorname{div} u = 0, \int u\,dx = 0\},\quad \|\cdot\| \text{ – the } L^2\text{-norm in } \mathcal{H}.$$

Let $v(\tau)$ be a solution of the 2D NSE above.

*Notation.* $\mathcal{D}v(\tau)$ – law of $v(\tau)$ (measure in $\mathcal{H}$).

*Postulate 1.* Only $\mathcal{D}v(\tau)$ matters.

*Scaling*

$v = \varepsilon^b u,\ \tau = \varepsilon^{-b} t,\ \nu = \varepsilon^{3/2-a},\ b = a - 1/2$

transforms the eq. above to

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\nu}\,\eta(t,x), \quad \text{(NSE)}$$

where $\eta(t) = \varepsilon^{b/2}\tilde{\eta}(\varepsilon^{-b}t)$ is distributed as $\tilde{\eta}(t)$. Below I discuss (NSE).

*Postulate 2.* Only limiting properties as $t \to \infty$ matter.

*Facts:* $\exists!$ probability measure $\mu_\nu$ in $\mathcal{H}$ such that

- $\mathcal{D}u(t) \rightharpoonup \mu_\nu$ as $t \to \infty$ exponentially fast, for any solution $u(t)$.
- There is solution $u_\nu(t,x)$ s.t. $\mathcal{D}u_\nu(t) \equiv \mu_\nu$.
- $u_\nu(t,x)$ is stationary in $t$ and $x$.
- Re$(u_\nu) \sim \nu^{-1}$.

$\mu_\nu$ – *stationary measure*; $u_\nu(t)$ – *stationary solution*.

See SK "Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions", Europ. Math. Soc. Publ. House, 2006

*Postulate 3.* 2D turbulence is described by $\mu_\nu$ and the distribution of $u_\nu(t,x)$ as $\nu \to 0$.

## §2. The Eulerian limit.

*Fact:* $\mathbf{E}\,\|\nabla u_\nu(t)\|^2 = B_0, \quad \mathbf{E}\,\|\Delta u_\nu(t)\|^2 = B_1,$ where $B_0, B_1$ - explicit constants.

**Theorem.** Along sequences $\nu_j \to 0$ we have

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}U(\cdot) \quad \text{in} \quad \mathcal{P}\big(C(0,\infty;\mathcal{H})\big)\,.$$

$U(t,x)$ is stationary in $t$ and $x$. Moreover,

a) every its trajectory $U(t,x)$ is such that $U(\cdot) \in L_{2\,loc}(0,\infty;\mathcal{H}\cap H^2)$,

b) it satisfies the free Euler equation

$$\dot{u} + (u\cdot\nabla)u + \nabla p = 0, \ \operatorname{div} u = 0. \qquad \text{(Eu)}$$

c) $\|U(t)\|^2$ is time-independent. If $g(\cdot)$ is bounded continuous function, then $\int g(\operatorname{rot} U(t,x))\,dx$ is time-independent.

d) $\mu_0 = \lim \mu_{\nu_j} = \mathcal{D}U(t)$ is invariant measure for (Eu).

e) $\int_{\mathcal{H}} \|\nabla u\|^2\,\mu_0(du) = B_0, \ \int_{\mathcal{H}} \|\Delta u\|^2\,\mu_0(du) \le B_1, \ \int_{\mathcal{H}} \|e^{\sigma \nabla u}\|^2\,\mu_0(du) < \infty.$

$\mu_0$ and $\mathcal{D}U(\cdot)$ describe the space-periodic 2D turbulence.

**Task:** Study $\mu_0$ and the distribution of $U$.

## §3. Disintegration of $\mu_0$.

$L$ – set of bounded Lipschitz functions $\mathbb{R} \to \mathbb{R}$, and $\{g_1, g_2, \dots\}$ – dense in $L$ sequence. Consider map $G : \mathcal{H} \cap H^1 \to \mathbb{R}^\infty \times \mathbb{R}$:

$$u(x) \mapsto \Big( \{ \int g_j(\mathrm{rot}(u(x))\, dx,\ j \geq 1\},\ \ \|u\|^2 \Big).$$

This is the set of all integrals of motion for (Eu). For any $b \in \mathbb{R}^\infty \times \mathbb{R}$ denote $G_b = G^{-1}(b) \subset \mathcal{H} \cap H^1$ the iso-integral set. Each $G_b$ is invariant for (Eu).

Denote $\lambda = G \circ \mu_0$ – measure on $\mathbb{R}^\infty \times \mathbb{R}$.

**Theorem 1.** $\forall\, b$ there is measure $\theta_b$ on $G_b$ such that

i) $\theta_b$ is invariant for (Eu), restricted to $G_b$,

ii) $\mu_0 = \int_{\mathbb{R}^\infty \times \mathbb{R}} \theta_b \, \lambda(db)$.

This is a *disintegration of measure* $\mu_0$. It is unique.

**Task:** Study measures $\theta_b$ and measure $\lambda$.

Definition. $\theta_b$'s – the iso-integral measures,

$\lambda$ – the distribution of integrals of motion.

## Iso-integral measures $\theta_b$.

Nothing is known about them.

**Conjecture:** *For a.e. $b$, (Eu) restricted to $G_b$ is uniquely ergodic.*

(Wait till the end of the lecture for the motivation.)

If so, then $\theta_b$ is this ergodic measure.

It is plausible that measures $\theta_b$'s are "rather singular".

## §4. $\lambda$ (distribution of integrals of motion).

$\mu_0$ – Eulerian limit; $\lambda = G \circ \mu_0$, where $G(u) \in \mathbb{R}^\infty \times \mathbb{R}$ – vector of all Euler's integrals of motion, evaluated for $u(x)$.

**Theorem 2.** $\lambda(\{u \mid \|u\| > K\}) \leq Ce^{-\sigma K^2}$ for any $K \geq 1$,
$\lambda(\{u \mid \|u\| < \delta\}) \leq C\sqrt{\delta}$ for any $\delta > 0$.

That is, *Energy of turbulent flow is big (is small) with small probability*.

For any $N$ consider projection of vectors in $\mathbb{R}^\infty$ on the first $N$ components:

$$\pi_N : \mathbb{R}^\infty \times \mathbb{R} \to \mathbb{R}^N \times \{0\}.$$

**Theorem 3.** The measure $\pi_N \circ \lambda$ is absolutely continuous with respect to $N$-dimensional Lebesgue measure.

**Corollary.** If $K \subset \mathcal{H}$ is a compact of finite Hausdorff dimension, then $\mu_0(K) = 0$.

That is, *The distribution of integrals of motion is non-singular.*

## §5. The balance relations.

*REMINDING.* $u_\nu(t)$ – stationary solution of (NSE); $\mathcal{D}u_\nu(t) = \mu_\nu$ – stationary measure.

$\mu_{\nu_j} \rightharpoonup \mu_0$ – Eulerian limit.

Denote $\xi_\nu(t,x) =$ rot $u_\nu(t,x)$ and

$$\Gamma_\nu(\tau) = \{x \in \mathbb{T}^2 \mid \xi_\nu(t,x) = \tau\}, \ \tau \in \mathbb{R}.$$

**Theorem 4.** For any $\nu > 0$, $\tau \in \mathbb{R}$

$$\mathbf{E} \int_{\Gamma_\nu(\tau)} |\nabla \xi_\nu| \, d\gamma = B \mathbf{E} \int_{\Gamma_\nu(\tau)} |\nabla \xi_\nu|^{-1} \, d\gamma.$$

$B$ – an explicit constant, $d\gamma$ – length element on $\Gamma_\nu(\tau)$.

These are infinitely-many relations, satisfied by measures rot $\circ \mu_\nu$, $\nu > 0$.

**Corollary.** The Eulerian limit $\mu_0$ satisfies

$$\int_{\mathcal{H}} e^{\sigma|\,\mathrm{rot}\, u(x)|}\,\mu_0(du) \leq C, \quad \int_{\mathcal{H}} e^{\sigma|\,u(x)|}\,\mu_0(du) \leq C,$$

for any $x$, with some $\sigma > 0$, $C \geq 1$.

## §6. Two models for 2D NSE.

**A.** Replace in (NSE) 2D Euler eq. by KdV eq.

$$\dot{u} + u_{xxx} - uu_x = 0.$$

We get the damped-driven KdV equation:

$$\dot{v} - \nu v_{xx} + v_{xxx} - 6vv_x = \sqrt{\nu}\,\eta(t,x)\,, \tag{1}$$

$$x \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}\,, \quad \int v\,dx \equiv \int \eta\,dx \equiv 0\,.$$

Eq. (1) has a unique stat. measure $\mu_\nu$. The limit $\mu_{\nu_j} \to \mu_0$ as $\nu_j \to 0$ exists, and $\mu_0$ is an invariant measure for the KdV. Now the corresponding iso-integral sets $G_b$ are diffeomorphic to $\mathbb{T}^\infty$, and the iso-integral measures $\theta_b$ are isomorphic to the Haar measure on $\mathbb{T}^\infty$. They *are* uniquely ergodic. Measure $\lambda$ (the distribution of integrals of motion) now is a stationary measure for the corresponding Whitham-averaged equation.

SK, A. Piatnitski, JMPA (2008), to appear

## Whitham-averaged equation.

$(I, \varphi)$, $I \in \mathbb{R}_+^\infty$, $\varphi \in \mathbb{T}^\infty$ – action-angle variables for KdV. KdV becomes

$$\dot{I} = 0, \qquad \dot{\varphi} = W(I).$$

Eq. (1) becomes :

$$dI = \nu\, F(I,\varphi)\, dt + \sqrt{\nu}\, G(I,\varphi)\, d\beta_t, \quad d\varphi = \dots$$

Averaged equation:

$$dI = \langle F\rangle(I)\, d\tau + \langle\langle G\rangle\rangle(I)\, d\beta_\tau\,, \tag{2}$$

where $\tau = \nu t$ and

$$\langle F\rangle(I) = \int_{\mathbb{T}^\infty} F(I,\varphi)\, d\varphi, \quad \langle\langle G\rangle\rangle(I) = \dots$$

**Theorem.** In the $(I, \varphi)$–variables the limiting measure $\mu_0$ is $\mu_0(dI\, d\varphi) = \nu(dI) \times d\varphi$, where $\nu$ is a stationary measure for eq (2).

**B** (finite-dimensional model). Replace in (NSE) the 2D Euler eq. by the Euler equation for rotating solid body:

$$\dot{M} + [M, A^{-1}M] + \nu M = \sqrt{\nu}\,\eta(t)\,.$$

$M \in \mathbb{R}^3$ – momentum, $A$ – operator of inertia. Iso-integral sets are formed by 1 or 2 circles. Iso-integral measures $\theta_b$ may be written explicitly. $\lambda$ – unique stationary measure for the averaged equation.

For the results in **A, B** see my paper

SK, *"Rigorous results and conjectures on stationary space-periodic 2D turbulence"* (mp_arc 07-134 or my web-page)