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Damped and driven KdV equation

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§1. KdV equation

Consider KdV equation under periodic boundary conditions with zero mean value:

$$\text{(KdV)} \qquad \dot{u} + u_{xxx} - uu_x = 0, \qquad x \in \mathbb{T}^1 = \mathbb{R}/2\pi, \quad \int u\,dx \equiv 0.$$

It may be written in the Hamiltonian form

$$\dot{u} - \frac{\partial}{\partial x}\frac{\delta}{\delta u(x)} H_{KdV} = 0, \quad H_{KdV} = \int \left(\frac{1}{2}u_x^2 + \frac{1}{6}u^3\right) dx,$$

and defines a Hamiltonian dynamical system in the L^2 function space

$$Z = \{u(x) \in L^2(\mathbb{T}^1) \mid \int u\,dx = 0\}, \qquad \|\cdot\| \text{ – the } L^2\text{-norm in } Z.$$

I provide Z with the usual trigonometric basis $\{e_s(x), s \in \mathbb{Z} \setminus \{0\}\}$:

$$e_s(x) = \cos sx, \; s > 0, \qquad e_s(x) = \sin sx, \; s < 0.$$

In Z KdV has ∞-many analytic integrals of motion $I_1(u), I_2(u), \dots$ and since the works of Novikov and Lax is known to be integrable. These issues may be conveniently described in terms of the Nonlinear Fourier Transfrorm:

Nonlinear Fourier Transform (NFT)

NFT is an analytic mapping

$$\Psi : u(\cdot) \mapsto v = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots), \quad \mathbf{v}_j = (v_j^+, v_j^-)^t \in \mathbb{R}^2,$$

such that:

1) for any $m \geq 0$ Ψ defines an analytic diffeomorphism $\Psi : \dot{H}^m \to h^m,$ where $\dot{H}^m$ is the Sobolev space of zero-meanvalue functions and $h^m = l_2^m$ is the weighted l_2-space;
2) Ψ is a symplectomorphism, if the v-space h^m is given the symplectic form
$\omega_2 = \sum j^{-1} dv_j^+ \wedge dv_j^-$;
3) $d\Psi(0)$ is the linear Fourier transform $d\Psi(0)u = ((u_1^+, u_1^-)^t, ((u_2^+, u_2^-)^t, \dots),$ where $\{u_j^\pm\}$ are the $(\sin / \cos)$ Fourier coefficients.

Denote $I_j = \frac{1}{2}|v_j|^2$ and $q_j = \text{Arg}(v_j)$; then $\{(I_j, q_j), j \geq 1\}$ also are symplectic coordinates, i.e. $\omega_2 = \sum j^{-1} dI_j \wedge dq_j$. We have
4) $H_{KdV} = h(I_1, I_2, \dots)$. Accordingly,

$$\frac{d}{dt} I_j = 0, \qquad \frac{d}{dt} q_j = j \frac{\partial}{\partial I_j} h(I)$$

and in the v-variables KdV takes the form

$$\dot{v} + \Phi(v) = 0, \quad \Phi_j(v) = j \mathbf{v}_j^\perp \frac{\partial}{\partial I_j} h(I), \quad \mathbf{v}_j^\perp = (-v_j^-, v_j^+)^t.$$

So I_j's are integrals of motion (or actions), q_j's are the angles, and the v-variables provide global Birkhoff coordinates for KdV. All solutions of KdV are almost-periodic functions of time. In the v-variable they may be written explicitly.

NFT has a long story. See the book Kappeler-Pöschel (Springer 2003).

Also see

SK, Galina Perelman *“Vey theorem in infinite dimension and...”* DCDS-A, **27** (2010), 1-24.

(arXiv 2009)

§2. Perturbations of KdV

I will discuss small perturbations of KdV

$$\dot{u} + u_{xxx} - uu_x = \varepsilon F(u, Du, D^2 u), \qquad D = D_x = \frac{\partial}{\partial x},$$

and small perturbations with randomness:

$$\dot{u} + u_{xxx} - uu_x = \varepsilon F(u, Du, D^2 u) + \sqrt{\varepsilon}\,\gamma\,\eta(t,x), \quad u = u^{\varepsilon,\omega};$$

(1) $$\eta(t,x) = \frac{\partial}{\partial t} \sum_j b_j \beta_j(t) e_j(x), \quad \eta = \eta^\omega.$$

Here $0 \le \gamma \le 1$, ω is a random parameter, $\omega \in (\Omega, \mathcal{F}, \mathbf{P})$, and

- all $b_j \neq 0$ and $b_j \equiv b_{-j}$;
- for any $N > 0$ there is C_N such that $|b_j| \le C_N j^{-N}$ for every j. So the force $\eta(t,x)$ is smooth in x.
- $\{\beta_j(t) = \beta_j^\omega(t)\}$ are independent standard Wiener processes.

TASK:

Study solutions for the perturbed KdV equation (1)

$$\dot{u} + u_{xxx} - uu_x = \varepsilon F(u, Du, D^2u) + \sqrt{\varepsilon}\,\gamma\,\eta(t,x)$$

with given initial data

$$u(0,x) = u_0(x),$$

for long time

$$0 \le t \le \varepsilon^{-1}T,$$

or for even longer times.

KAM

If the perturbation is Hamiltonian, i.e. the perturbed equation is

$$(*) \qquad \dot{u} + u_{xxx} - u u_x = \varepsilon \frac{\partial}{\partial x} f(u, x),$$

then the KAM-theory applies.

See my book (SK, OUP, 2000) and the book by Kappeler-Pöschel (Springer, 2003).

Let us apply the NFT and write perturbed KdV (1) in the v-variables:

$$\dot{\mathbf{v}}_k + \Phi_k(v) = \varepsilon P_k(v) + \sqrt{\varepsilon}\,\gamma \sum_j B_{kj}(v) \frac{d}{dt}\beta_j(t), \quad 0 \le t \le \varepsilon^{-1} T.$$

It is convenient to go to the slow time $\tau = \varepsilon t$:

(2) $$\frac{d}{d\tau}\mathbf{v}_k + \varepsilon^{-1}\Phi_k(v) = P_k(v) + \gamma \sum_j B_{kj}(v) \frac{d}{d\tau}\beta_j(\tau), \quad 0 \le \tau \le T.$$

TASK: Study solutions of (2) for $0 \le \tau \le T$.

Denote

$I(v(\tau)) = (I_1(v), I_2(v), \dots)(\tau) \in \mathbb{R}_+^\infty$,

$q(v(\tau)) = (q_1(v), q_2(v), \dots)(\tau) \in \mathbb{T}^\infty$.

Then we have to

a) study the actions $I(v(\tau)), \ \ 0 \le \tau \le T$ – the main task,

b) study the angle $q(v(\tau)), \ \ 0 \le \tau \le T$.

Usually (since Laplace and Lagrange) to do this people write equations for $I(\tau)$

$$\frac{d}{d\tau} I_k = G_k(I,q) + \gamma \sum_j H_{kj}(I,q) \frac{d}{d\tau}\beta_j(\tau), \quad 0 \le \tau \le T$$

(we have no terms of order ε^{-1} since for $\varepsilon = 0$ the functionals I_k are integrals of motion for KdV). Next they average these equations in q to get a system of equations on the I-vector:

$$(AvEq) \qquad \frac{d}{d\tau} I_k = \langle G_k\rangle(I) + \gamma \sum_j \langle H_{kj}\rangle(I) \frac{d}{d\tau}\beta_j(\tau),$$

where

$$\langle G_k\rangle(I) = \int G_k(I,q)\, dq, \quad \langle H_{kj}\rangle(I) = \dots.$$

They clame that solution of the averaged system well approximates the actions $I(v(\tau))$ for $0 \le \tau \le T$. This is the classical averaging. For the non-random case see the books by Arnold - Kozlov - Neshtadt (Springer) and by Lochak - Meunier (Springer).

Unforunately, the (AvEq), corresponding to the perturbed KdV (2) are VERY singular. So I will proceed similar, but differently:

§3. Effective equations for $I(v(\tau))$.

We study eq. (2) (the perturbed KdV, written in the v-variables):

$$\frac{d}{d\tau}v(\tau)+\varepsilon^{-1}\Phi(v)=P(v)+\gamma B(v)\frac{d}{d\tau}\beta;\quad v(0)=v_0=\Psi(u_0),\quad 0\le\tau\le T.$$

Now $\beta(\tau)=(\beta_1(\tau),\beta_2(\tau),\dots)^t$ and $B(v)$ is an infinite matrix.

$P(v)$ is called the *drift* and $B(v)$ – the *dispersion operator*. Its square $B(v)B^t(v)$ is the *diffusion operator*. This equation is singular when $\varepsilon\to 0$.

Effective equation for (2)**:**

(3)
$$\frac{d}{d\tau}v(\tau)=\langle P\rangle(v)+\gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta;\quad v(0)=v_0.$$

$\langle P\rangle(v)$ – effective drift; $\langle\langle B\rangle\rangle(v)$ – effective dispersion operator. They are defined as follows:

Effective drift $\langle P\rangle(v)$. For any vector $\theta = (\theta_1, \theta_2, \dots) \in \mathbb{T}^\infty$ consider the corresponding rotation in the space of vectors v:

$$\Pi_\theta : v = (I, q) \mapsto v' = (I, q'), \qquad q' = q + \theta$$

(this is the rotation by the angle θ_j in each $\mathbf{v}_j$-plane).

Then $\langle P\rangle$ is defined as follows: $\langle P\rangle(v) = \int_{\mathbb{T}^\infty} \Pi_\theta^{-1} \circ P(\Pi_\theta v)\, d\theta.$

Effective dispersion operator $\langle\langle B(v)\rangle\rangle$. This is a non-symmetric 'square root' of the averaged diffusion:

$$\langle\langle B(v)\rangle\rangle \cdot \langle\langle B(v)\rangle\rangle^t = \int_{\mathbb{T}^\infty} \left(\Pi_{-\theta} \cdot B(\Pi_\theta v) \cdot B^t(\Pi_\theta v) \cdot \Pi_\theta\right)\, d\theta.$$

It is constructed to be analytic in v.

By definition of the effective objects $\langle P\rangle$ and $\langle\langle B\rangle\rangle$, the effective equation

$$\frac{d}{d\tau} v(\tau) = \langle P\rangle(v) + \gamma \langle\langle B\rangle\rangle(v) \frac{d}{d\tau}\beta; \quad v(0) = v_0.$$

is invariant under the rotations Π_θ. That is, if $v(\tau)$ solves (3), then $\Pi_\theta v(\tau)$ also does.

Why the effective equations (3) are important?

$$\frac{d}{d\tau}v(\tau) = \langle P\rangle(v)+\gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta. \tag{3}$$

– Because they "covers" the averaged equation (AvEq):

Lemma. Let $0 \le \gamma \le 1$. If $v(\tau)$ solves (3), then $I(v(\tau))$ solves (AvEq). If $I(\tau)$ is a solution of (AvEq), then there exists a solution $v(\tau)$ of (3) such that $I(v(\tau)) = I(\tau)$. Solution of (3) exists and is unique.

Averaging Principle: Let $v^\varepsilon(\tau)$ be a solution of the perturbed KdV (2) and $v^0(\tau)$ be a solution of the effective equation (3). Then

$$\sup_{0\le\tau\le T} |I(v^\varepsilon(\tau)) - I(v^0(\tau))| = o(1) \quad \text{as} \quad \varepsilon \to 0.$$

TASKS: 1) Prove the Averaging Principle for various classes of perturbed KdV equations.

2) Study properties of the effective equation (3)

DIGRESSION: Whitham averaging

I am discussing perturbations of *fast in time oscillating solutions* of KdV, written in slow time $\tau = \varepsilon t$:

$$\frac{d}{d\tau} u + \varepsilon^{-1}(u_{xxx} - u u_x) = F(u, D_x u, D_x^2 u), \quad 0 \leq \tau \leq T.$$

Similar one can consider perturbations of *fast IN SPACE AND TIME oscillating solutions*, written in slow time τ and slow space $X = \varepsilon x \in \mathbb{R}$:

$$\frac{d}{d\tau} u + \varepsilon^2 u_{XXX} - u u_X = F(u, \varepsilon D_X u, \varepsilon^2 D_X^2 u), \; u(0, X) = v(x, X); \; 0 \leq \tau \leq T.$$

Corresponding averaging principle is due to Whitham. It was not justified completely for any class of perturbations of KdV (still, see works of Lax, Levermore and Venakides). But corresponding averaged equations are being intensively studied (cf. Task 2 – *study properties of the effective equations*.

To justify the Whitham averaging is a very hard open problem.

Examples of effective equations. 1) Let the perturbation be $\varepsilon\partial^2/\partial x^2 + \sqrt{\varepsilon}\gamma\eta(t,x)$. I.e. $F(u, D_x u, D_x^2 u) = \Delta u$. Denote by $\hat\Delta$ the Fourier-image of Laplacian: $\hat\Delta : v \mapsto v', \quad \mathbf{v}'_k = -k^2\mathbf{v}_k \quad \forall\, k.$ **Claim:** now $\langle P\rangle = \hat\Delta + \langle lower\ order\ term\rangle.$ So effective equation (3) is a quasilinear (stochastic) heat equation

$$\frac{d}{d\tau}v(\tau) = \hat\Delta v + \langle\ldots\rangle + \gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta.$$

2) Consider non-stochastic Hamiltonian perturbation $F(u) = \frac{\partial}{\partial x}f(u,x)$ (now $\gamma = 0$). Then $\langle P\rangle(v)$ is an integrable hamiltonian vector field, and the averaged equation for $I(\tau)$ is $\frac{d}{d\tau}I(\tau) = 0$. So

averaging principle $\Rightarrow$ adiabatic invariance of actions $I(\tau)$ for solutions of the perturbed KdV.

CONJECTURE. The actions $I_j(u(t)$ of solutions $u(t)$ for KdV under hamiltonian perturbations indeed are adiabatic invariants. I.e. they change by $o(1)$ of time-intervals $t \le C\varepsilon^{-1}$.

§4. Damped-driven KdV

[1] SK, A. Piatnitski "Khasminsii–Whitham averaging for randomly perturbed KdV equation" J. Math. Pures Appl. **89**, 400-428 (2008).

[2] SK "Damped-driven KdV and effective equations…" . To appear in GAFA, see in ArXiv.

Consider the original randomly perturbed and damped KdV, using the slow time $\tau = \varepsilon t$:

$$(4) \quad u'_\tau + \varepsilon^{-1}(u_{xxx} - uu_x) = u_{xx} + \gamma\eta(\tau, x), \quad 0 < \gamma \leq 1; \quad u(0, x) = u_0(x).$$

Due to Example 1 above the corresponding effective equation is

$$(5) \qquad v'_\tau = \hat{\Delta} v + F'(v) + \gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta(\tau), \quad v(0) = v_0 = \Psi(u_0).$$

Here F' is analytic operator of order one. So (5) is quasilinear stochastic heat equation. It turns out to be well posed.

Theorem. Let $u^\varepsilon(\tau, x), 0 \leq \tau \leq T$, be a solution of (4) and $v(\tau)$ be a solution of the effective equation (5). Let $v^\varepsilon(\tau) = \Psi(u^\varepsilon(\tau))$. Then $I(v^\varepsilon(\cdot)) \rightharpoonup I(v(\cdot))$ as $\varepsilon \to 0$, in distribution.

Under the limit $\varepsilon \to 0$ the phase $q(v^\varepsilon(\tau))$ becomes uniformly distributed on $\mathbb{T}^\infty$:

Theorem . For any continuous function $f(\tau)$ we have

$$\int_0^T f(\tau)\,\mathcal{D}\big(q(v^\varepsilon(\tau))\big)\,d\tau \to dq \cdot \int f(\tau)\,d\tau.$$

Here $\mathcal{D}\big(q(v^\varepsilon(\tau))\big)$ is the law of $q(v^\varepsilon(\tau)) \in \mathbb{T}^\infty$ and dq is the Haar measure on $\mathbb{T}^\infty$.

Problem. Is this convergence true when $\gamma = 0$, for a typical initial data? Actually, what happens to solutions of the perturbed equation with $\gamma = 0$ when time is very large???

§5 . The double limit for damped-driven KdV

(work in progress)

Existing techniques and some reasonable conjectures on properties of the NFT (which have to be verified) allow to prove that the effective eq. (5)

$$\frac{d}{d\tau}v(\tau) = \hat{\Delta}v + F'(v) + \gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta(\tau)$$

has a unique stationary measure μ.

Consider the original randomly perturbed KdV equation:

(6)
$$\dot{u} + u_{xxx} - uu_x = \varepsilon u_{xx} + \gamma\sqrt{\varepsilon}\,\eta(t,x).$$

It has a unique stationary measure. Denote it μ^ε. Then

$$\mathcal{D}u(t) \rightharpoonup \mu^\varepsilon \quad \text{as} \quad t\to\infty$$

for any solution $u(t)$.

It is known that $\{\mu^\varepsilon\}$ is compact and all limiting measures (as $\varepsilon\to 0$) are invariant for KdV. But how can we identify them? Is there one limiting measure, or there are many?

Theorem. 1) The limit $\mu^0 = \lim_{\varepsilon\to 0}\mu^\varepsilon$ exists and is an invariant measure for KdV;
2) Write μ^0 in the (I,q)-variables, i.e. consider $(I\times q)\circ\mu^0$. Then

$$(I\times q)\circ\mu^0 = (I\circ\mu)\times dq.$$

In particular, $I\circ\mu^0 = I\circ\mu.$

Accordingly we have

Theorem (the Double Limit). For any smooth function $u_0(x)$ let $u^\varepsilon(t,x)$ be a solution of the damped-driven KdV (6), equal u_0 at $t=0$. Then

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\mathcal{D}\left[(I\times q)(u^\varepsilon(t))\right] = (I\circ\mu)\times dq$$

So, the unique stationary measure μ of the effective equation (5) is the key to study perturbed equations!

Example (linear approximation). Consider the random force $\gamma(d/d\tau)\beta(\tau)$, where

$$\beta(\tau) = \sum_j b_j(\tau) e_j(x), \quad \text{with} \quad b_{m_0} = b_{-m_0} = 1 \quad \text{and} \quad b_l \ll 1 \quad \text{if} \quad |l| \neq m_0.$$

Denote $\mu = \mu_\gamma$. Send γ to zero.

Claim. We have $\mu_\gamma = \mu_\gamma^1 + O(\gamma^2)$, where μ_γ^1 is the Gaussian measure in the plane $\mathbb{R}e_{m_0} \oplus \mathbb{R}e_{-m_0} \subset Z$ with zero meanvalue and the dispersion matrix $\sigma^2 I$,
$\sigma^2 = \frac{1}{2}\gamma^2 j^{-2}$.

§6 . Summary.

Consider the randomly perturbed and damped KdV equation

$$\dot{u} + u_{xxx} - uu_x = \varepsilon u_{xx} + \gamma\sqrt{\varepsilon}\,\eta(t,x), \quad u(0,x) = u_0(x) \in C^\infty.$$

For $0 < \varepsilon \ll 1$ its solution u^ε may be well approximated in terms of the solution for the effective equation

$$\frac{d}{d\tau}v(\tau) = \hat{\Delta}v + F'(v) + \gamma\langle\langle B\rangle\rangle(v)\frac{d}{d\tau}\beta(\tau), \quad v(0) = v_0.$$

That is, for $t = \varepsilon^{-1}\tau$, $\tau \sim 1$, we have

$$\mathcal{D}\big(I(u^\varepsilon(t))\big) \rightharpoonup \mathcal{D}\big(I(v(\tau))\big) \quad \text{as} \quad \varepsilon \to 0.$$

While for $t \gg \varepsilon^{-1}$ we have

$$\mathcal{D}\big(I(u^\varepsilon(t)), q(u^\varepsilon(t)\big) \rightharpoonup (I \circ \mu) \times dq \quad \text{as}\ \varepsilon \to 0,$$

where μ is the unique stationary measure for the effective equation.

When $\gamma \to 0$ the measure $\mu = \mu_\gamma$ may be decomposed in asymptotical series in γ.