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Nonlinear PDE with random force and 2d Turbulence

(Kharkov, 19 April 2013)

References:

[1] S. Kuksin, “Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions”, Europ. Math. Soc. Publ. House, 2006

[2] S. Kuksin, A. Shirikyan ”Mathematics of 2d Statistical Hydrodynamics”'. CUP 2012

See my web-page.

§1. Introduction: Statistical Hydrodynamic approach to turbulence

Space-periodic 2D turbulence is described by small-viscosity 2D Navier-Stokes equation (NSE), perturbed by a random force:

$$u_t' - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \eta(t,x),$$

$$x \in \mathbb{T}^2, \quad \operatorname{div} u = 0\,; \quad 0 < \nu \le 1.$$

Here $u(t,x) \in \mathbb{R}^2$ – velocity, $p(t,x) \in \mathbb{R}$ – pressure, $\eta(t,x) \in \mathbb{R}^2$ – random force, ν – viscosity,

$$(u \cdot \nabla)u = u_1 \frac{\partial}{\partial x_1} u + u_2 \frac{\partial}{\partial x_2} u.$$

I normalise the problem and assume that $\int u\,dx = \int \eta\,dx = 0$.

Leray Projection. Denote

$$\mathcal{H} = \{u(x) \in L^2(\mathbb{T}^2, \mathbb{R}^2), \ \ \mathrm{div}\, u = 0, \int u\, dx = 0\}, \quad \|\cdot\| - \text{the } L^2\text{-norm.}$$

Then

$$L^2(\mathbb{T}^2, \mathbb{R}^2) = \mathcal{H} \oplus \{u(x) = \nabla p\} \oplus \mathbb{R}^2.$$

Denote $\Pi : L^2 \to \mathcal{H}$. Apply Π to the equation. Since $\Pi \Delta u = \Delta u$ for $u \in \mathcal{H}$, $\Pi \nabla p = 0$ and $\Pi u = u$, then

$$u_t' - \nu \Delta u + B(u) = \Pi\, \eta(t) =: \eta. \qquad (NSE)$$

Here $B(u) = \Pi(u \cdot \nabla)u$ – a complicated quadratic nonlinearity. So (NSE) is a nonlinear parabolic equation. Many of the results I will talk below apply to various classes of nonlinear parabolic equation with random force.

Let $e_1, e_2, \cdots \in \mathcal{H}$ be the basis of $\mathcal{H}$, formed by eigen-functions of Δ,

$$\Delta e_j = -\lambda_j e_j \quad \forall\, j \geq 1$$

(this is the sin/cos basis, parameterised by natural numbers).

Random Force. The force η has the form

$$\eta(t,x) = \sum_j b_j \beta_j(t) e_j(x),$$

where $\{\beta_j(t) = \beta_j^\omega(t)\}$ are i.i.d. random processes and the constants $b_j \geq 0$ fast enough decay to zero. We can handle 3 classes of forces, corresponding to three classes of random processes $\beta(t)$. I will discuss only one of them:

$$\beta_j^\omega(t) = \tfrac{\partial}{\partial t} w_j(t) + f_j,$$

where $f_1, f_2, \dots$ are non-random vectors in $\mathcal{H}$ (i.e. non-random vector-fields) and $w_1(t), w_2(t), \dots$ are standard independent Wiener processes.

$\eta(t) = \eta^\omega(t)$ is a white noise in the function space $\mathcal{H}$.

Solutions. We regard a solution $u(t,x)$ as a random process $u^\omega(t) \in \mathcal{H}$. We are interested NOT in individual trajectories $t \mapsto u^\omega(t)$, but in distribution (=the law) of a solution u, $\mathcal{D}u(t) =: \mu_t$. This is a probability measure in $\mathcal{H}$:

$$\mu_t(Q) = \mathbf{P}(u(t) \in Q), \ \ Q \subset \mathcal{H}; \qquad \int_{\mathcal{H}} f(u)\, \mu_t(du) = \mathbf{E} f(u(t)).$$

A solution $u(t)$ is a Markov process. Therefore

$$\mu_t = S_t^*(\mu_0),$$

where the operators $\{S_t^*,\ t \geq 0\}$ extend to a semi-group of linear operators in the space of signed measures in $\mathcal{H}$.

Task: Study qualitative properties of distributions of solutions, i.e. of the measures $\mu_t = \mathcal{D}(u(t)),\ t \geq 0$.

§2. Limit "time to infinity" (mixing).

Definition: a measure μ in $\mathcal{H}$ is called a stationary measure for (NSE) if

$$S_t^* \mu \equiv \mu \qquad \forall\, t.$$

If $u(t)$ is a solution such that $\mathcal{D}u(0) = \mu$, then $\mathcal{D}u(t) \equiv \mu$. It is called a stationary solution.

Existence of a stationary measure is an easy fact which follows from the compactness argument due to Bogolyubov-Krylov. Not its uniqueness! – This is complicated.

Recall that we consider

$$u'_t - \nu\Delta u + B(u) = \eta, \qquad \eta(t,x) = \sum b_j \beta_j(t) e_j(x), \qquad\qquad (NSE)$$

$x \in \mathbb{T}^2$. Coefficients b_j decay with j: $B_0 = \sum b_j^2 < \infty.$

Condition (C). $b_j > 0$ for each $j \le N$, where $N = N(B_0, \nu)$ is sufficiently big.

For example, (C) holds if $b_j \neq 0$ for all j.

THEOREM 1. If (C) holds, then: 1) there exists a unique stationary measure μ.

2) For any solution $u(t)$ of (NSE) we have

$$\mathrm{dist}(\mathcal{D}u(t), \mu) \le Ce^{-ct}, \qquad c, C > 0. \qquad\qquad \text{(mixing)}$$

Here dist is one of the 'usual' distances in the space of measures (e.g., Prokhorov's or Wasserstein's).

3) If force $\eta(t,x)$ is smooth in x, then μ is supported by smooth functions. I.e., $\mu(\mathcal{H} \cap C^\infty) = 1$.

So, “statistical properties of solutions $u(t,x)$ for $t \gg 1$ are universal and are described by a unique stationary measure μ.” This result always was postulated by physicists as an axiom:

”... we put our faith in the tendency for dynamical systems with a large number of degrees of freedom, and the coupling between these degrees of freedom, to approach a statistical state which is *independent* (partially, if not wholly) of the initial condition”.
G. K. Batchelor ”The Theory of Homogeneous Turbulence”, p.6. Also see
U. Frisch ”Turbulence”.

Stationary measures μ interest physicists the most.

For deterministic Navier-Stokes equations, where $\eta(t,x) = \eta(x)$ is non-random, an analogy of the unique stationary measure is the equation’s attractor.

Dependence on Parameters:

Let the force $\eta(t,x)=\sum b_j\beta_j(t)e_j(x)$ continuously depends on a parameter, i.e.

$$b_j = b_j(a), \qquad 0 \le a \le 1,$$

and the condition (C) holds for each a. Let $u_a(t)$ be a solution of (NSE) with this force and initial data $u_0(a)$, continuous in a. Then law $\mathcal{D}(u_a(t))$ continuously depends on a UNIFORMLY IN TIME:

$$\sup_{t\ge 0}\Big\{\operatorname{dist}(\mathcal{D}u_a(t),\mathcal{D}u_{a_0}(t))\Big\}\to 0 \qquad \text{as} \quad a\to a_0.$$

§3. Consequences of the mixing.

Ergodicity:

THEOREM 2 (SLLN). If (C) holds, then for any solution $u(t)$ of (NSE) and any 'good' $f(u)$ we have

$$\frac{1}{T}\int_0^T f(u(s))\,ds \to \langle \mu, f\rangle := \int f(u)\,\mu(du), \quad a.s.$$

Remark. The rate of convergence is $T^{-\gamma},\ \gamma < 1/2$.

So "for a turbulent flow time-average equals ensemble-average". This is another postulate of the theory of turbulence:

"...we can anticipate, assuming applicability of ergodic theory..., that a time average is identical with a probability average for the experimental fields".
(G. K. Batchelor "The Theory of Homogeneous Turbulence", p.17)

THEOREM 3 (CLT). Let $\langle \mu, f \rangle = 0$. Then

$$\mathcal{D}\Big(\frac{1}{\sqrt{T}}\int_0^T f(u(s))\,ds\Big) \rightharpoonup N(0,\sigma),$$

for some $\sigma > 0$ (depending on f).

So "on large time-scales a turbulent flow is Gaussian". Cf. the book of Batchelor, p.174.

§4. Inviscid limit ($\nu \to 0$)

Consider (NSE) with small ν and with the force, multiplied by some degree of ν:

$$u'_t - \nu\Delta u + B(u) = \nu^a \eta, \qquad a \in \mathbb{R}. \qquad (*)$$

Proposition. Solutions of $(*)$ remain ~ 1 as $\nu \to 0$ if and only if $a = \frac{1}{2}$.

Accordingly, below we discuss the scaled NSE

$$u'_t - \nu\Delta u + B(u) = \sqrt{\nu}\,\eta, \qquad 0 < \nu \leq 1. \qquad (NSE_\nu)$$

Let all $b_j \neq 0$, i.e. the force η is non-degenerate. Then for each ν eq. (NSE_ν) has a unique stationary measure μ_ν, and

- $\mathcal{D}u(t) \rightharpoonup \mu_\nu$ as $t \to \infty$ exponentially fast, for any solution $u(t)$.
- There is a solution $u_\nu(t, x)$ s.t. $\mathcal{D}u_\nu(t) \equiv \mu_\nu$; u_ν is stationary in t.
- $u_\nu(t, x)$ is smooth in x if the force η is.
- Reynolds number of u_ν is $Re(u_\nu) \sim \nu^{-1}$.

Physicists are interested the most in properties of μ_ν and u_ν when $\nu \to 0$.

Fact: $\mathbf{E}\,\|\nabla u_\nu(t)\|^2 = B_0, \quad \mathbf{E}\,\|\Delta u_\nu(t)\|^2 = B_1$, where

$$B_0 = \sum b_j^2, \ \ B_1 = \sum b_j^2 \lambda_j$$

.

Theorem 4. Every sequence $\nu_j' \to 0$ has a subsequence $\nu_j \to 0$ such that the limit $\lim \mu_{\nu_j} = \mu_0$ exists, and is an invariant measure for the 2d Euler equation

$$\dot{u} + (u \cdot \nabla) u + \nabla p = 0, \; \operatorname{div} u = 0. \qquad \text{(Eu)}$$

We know that

a) $\int_{\mathcal{H}} \|\nabla u\|^2 \, \mu_0(du) = B_0, \; \int_{\mathcal{H}} \|\Delta u\|^2 \, \mu_0(du) \leq B_1, \; \int_{\mathcal{H}} e^{\sigma \|\nabla u\|^2} \, \mu_0(du) < \infty.$

b) The measure μ_0 is ”genially infinite-dimensional”: if $K \subset \mathcal{H}$ and $\dim_H K < \infty$, then $\mu_0(K) = 0$.

✠ μ_0 is called the Eulerian limit (for (NSE_ν)). It describes the space- periodic 2D turbulence since it describes solutions of (NSE) with $\nu \ll 1$ and Re $\gg 1$.

✠ "Universality of 2d turbulence". I recall that

$$B_0 = \sum b_j^2, \qquad B_1 = \sum b_j^2 \lambda_j$$

Fact: The measure μ_ν, $1 \gg \nu > 0$, satisfies infinitely-many explicit algebraical relations which are independent from ν and depend only on B_0 and B_1.

✠ Measure μ_0 is supported by Sobolev space H^2: $\mu_0(H^2) = 1$. It is plausible that

$$\mu_0(H^{2+\varepsilon}) = 0 \qquad \text{if} \qquad \varepsilon > 0 \qquad\qquad (*)$$

But I cannot prove that.

Task: a) Check if the Eulerian limit μ_0 depends on the sequence $\nu_j \to 0$.

b) Study the measure μ_0. E.g., prove $(*)$.

§5. Anisotropic 3d turbulence in thin domains.

Consider 3d NSE in the thin domain $(x_1, x_2, x_3) \in \mathbb{T}^2 \times (0, \varepsilon)$ with free boundary conditions in the thin direction x_3:

$$u_3 \mid_{x_3=0,\,\varepsilon} = 0, \quad \partial_3 u_{1,2} \mid_{x_3=0,\,\varepsilon} = 0.$$

Perturb it by a random force of the form

$$\eta(t,x) = \sum b_j \beta_j(t) e_j(x),$$

where now $\{e_j(x), j \geq 1\}$, are eigen-functions of the 3d Stokes operator, and $\beta_j(t)$ are kick-processes ("discrete-time white processes"). These processes have bounded integrals:

$$\int_0^t \beta_j(s)\,ds \leq C \qquad \forall t,\ \forall\,\omega,$$

which is a technical advantage.

Theorem 5. The law of $(u_1, u_2)(t, x_1, x_2, x_3)$ converges, as $\varepsilon \to 0$, *uniformly in time* t, to the law of a solution $(v_1, v_2)(t, x_1, x_2)$ of randomly forced 2d NSE in $\mathbb{T}^2$, and we have

$$\mathbf{E}\,\langle\text{normalised energy of 3d flow}\rangle \to \mathbf{E}\,\langle\text{energy of 2d flow}\rangle \qquad (*)$$

(so $\varepsilon^{-1} \int |u_3|^2\, dx \to 0$).

It seems that (in non-trivial situations) $(*)$ does not hold for enstrophy, and that $\varepsilon^{-1} \int |\nabla u_3|^2\, dx$ does not converge to zero.

So randomly forced 2d NSE describe a class of anisotropic 3d turbulence.

For these results for randomly forced 3d NSE see
I. Chuyeshov and S. Kuksin, ARMA 188 (2008) and Physica D 237 (2008).
See [2] for discussion.

Cf. well known related results for deterministic 3d NSE in thin domains by G. Raugel, G. Sell (and many people after them).

These are the rigorous results on turbulence in 2d and 3d. They form tiny isles of rigorous knowledge in the ocean of unknown.

Still, it is not any more true that

”Nothing can be proven in the theory of turbulence” (G. K. Batchelor, 1998).

REFERENCES:

[1] S. Kuksin, “Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions”, Europ. Math. Soc. Publ. House, 2006

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