Around the Cauchy-Kowalewski and Ovsyannikov-Nirenberg theorems

Sergei Kuksin

(based on a joint work with Nikolai Nadirashvili

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§1. Introduction

Consider the Cauchy problem for a quasilinear wave equation:

$$(1) \qquad \Box u + f(t, x, u, \nabla u, \dot{u}) = 0, \qquad \dim x = d; \ \ t \geq 0;$$

$$(2) \qquad u_{t=0} = u_0, \qquad \dot{u}_{t=0} = u_1.$$

We can study it globally, when

x belongs to a compact Riemann manifold M and $\Box u = \ddot{u} - \Delta u$, where

Δ – Laplace - Beltrami oparator.

Or locally, when

x belongs to a characteristic cone for $\Box u = \ddot{u} - \Delta u$, where Δ – Laplace - Beltrami

oparator. w.r.t. some Riemann metric on $\mathbb{R}^d$

To begin I will assume that

$$x \in \mathbb{T}^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d.$$

Let f and u_0, u_1 be analytic in all its arguments. What do we know about the solutions?

I) The Cauchy-Kowalewski theorem: There exists $\varepsilon_1 > 0$ such that for $0 \le t \le \varepsilon_1$ and $x \in \mathbb{T}^d$ problem (1), (2) has a unique analytic solution.

S. V. Kowalewski, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math. 80 (1875).

In the proof ε_1 is the radius of analyticity, so this is a small number. What happens for $t > \varepsilon_1$?

II) The Ovsiannikov-Nirenberg theorem: Let f be continuous in t and analytic in all other variables, as well as u_0 and u_1. Then there exists $\varepsilon_2 > 0$ such that for $0 \le t \le \varepsilon_2$ and $x \in \mathbb{T}^d$ problem (1), (2) has a unique solution, analytic in x and C^1 in t.

T. Nishida, *A note on a theorem of Nirenberg*, J. Diff. Geom. 12 (1977).

It is also known that

III) Theorem: If f is sufficiently smooth in x, then there exists $T > 0$ such that for $0 \le t \le T$ and $x \in \mathbb{T}^d$ the problem has a classical solution.

Sometimes T is fairly large, e.g. $T = \infty$.

If both theorems I) and III) apply, then, clearly, $T \ge \varepsilon_1$. But is $T > \varepsilon_1$, or $T = \varepsilon_1$?

§2. Main Result.

Choose $m > d/2,$ assume that the nonlinearity $f(t, x, u, \nabla u, \dot{u})$ is continuous in t, H^m-smooth in x and analytic in $u, \nabla u, \dot{u}$. Let $u_0 \in H^{m+1}, u_1 \in H^m$.

Main Theorem (propagation of analyticity): Let $u(t, x),\ 0 \le t \le T, x \in \mathbb{T}^d$, be a solution of the Cauchy problem (1), (2) which is H^{m+1}-smooth in x. Then:

i) If u_0, u_1 and f are real-analytic in $(x_1, \ldots, x_k), 1 \le k \le d$, then u also is analytic in these variables.

ii) If u_0, u_1 and f are real-analytic in all their arguments, then u also is.

Assertion ii) was known, see:

S. Alihnac and G. Metivier, *Propagation de l'analyticité . . .*, Invent. Math. 75 (1984).
 The proof of this work uses heavy tools of paradifferential calculus (and their result applies to strongly nonlinear hyperbolic equations). Also see

C. Bardos and S. Benachour, *Domaine d 'analyticite des solutions de l'equation d'Euler dans un ouvert de R^n.* Ann. Scu. Norm. di Pisa, 4(1977)   where similar result is obtained for solutions of the Euler equation (using its hyperbolic features).

Assertion i) and Theorem III) (on local in time existence of a classical solution) imply a generalisation of the Ovsiannikov-Nirenberg theorem:

If the Cauchy data and the r.h.s. are sufficiently smooth in x and are analytic in x_j for some j, then locally in time there exists a a unique classical solution, which is analytical in that x_j.

§3. Discussion of the proof.

I am speaking about the Cauchy problem (1)-(2):

$$\Box u + f(t, x, u, \nabla u, \dot{u}) = 0, \qquad u(0, x) = u_0(x), \quad \dot{u}(0, x) = u_1(x).$$

Denote $\mathcal{H}^m = H^{m+1} \times H^m$, $m > d/2$. This will be the phase space;
$(u_0, u_1) \in \mathcal{H}^m$. Consider the Cauchy operator for the linear wave equation:

$$\widetilde{\Box} : u \mapsto (u_{t=0}, \dot{u}_{t=0}, \Box u).$$

This is an embedding. For any $T > 0$ consider the spaces

$$X_m^T = C(0, T; H^{m+1}) \cap C^1(0, T; H^m), \qquad Y_m^T = \mathcal{H}^m \times C(0, T; H^m).$$

X_m^T - space of solutions, Y_m^T - space of Cauchy data and the r.h.s.'s.
I will call solutions $u(t, x) \in X_m^T$ classical solutions. We have

$$\widetilde{\Box}^{-1} : Y_m^T \to X_m^T, \quad \text{but} \quad \widetilde{\Box} : X_m^T \to \mathcal{H}^m \times C(0, T; H^{m-1}).$$

In this scales of function spaces we lose 1 unit of smoothness, when apply $\widetilde{\Box}$ after $\widetilde{\Box}^{-1}$.
Not good! Let us use a simple trick:

Denote $\widetilde{\Box}^{-1} Y_m^T = Z_m^T$ and provide the space Z_m^T with a norm, induces from Y_m^T. This is a Banach space such that

$$1)\ \ \widetilde{\Box} : Z_m^T \to Y_m^T \qquad \text{is an isomorphism,}$$

$$2)\ \ Z_m^T \subset X_m^T \qquad \text{continuously,} \quad \text{and}\ \ X_{m+1}^T \subset Z_m^T,$$

since $\widetilde{\Box}^{-1} : Y_m^T \to X_m^T$. Denote by Φ the operator of nonlinear Cauchy problem:

$$\Phi(u) = (u_{t=0}, \dot{u}_{t=0}, \Box u + f(t, x, u, \nabla u, \dot{u})).$$

Since $m > d/2$, then the space $C(0, T; H^m)$ is a Banach algebra. As for each $u \in Z_m^T$ we have $u, \nabla u, \dot{u} \in C(0, T; H^m)$, then the mapping

$$\Phi : Z_m^T \to Y_m^T \qquad \text{is analytic.}$$

Problem (1), (2) with 0 in the r.h.s. replaced by a function $g(t,x)$ has a unique solution. So

$$\Phi : Z_m^T \to Y_m^T \qquad \text{is an analytic embedding.}$$

Consider differential of Φ at any point $u \in Z_m^T$:

$$d\Phi(u)(v) = \left(v_{t=0}, \dot{v}_{t=0}, \Box v + d_3 f[u] v + d_4 f[u] \nabla v + d_5 f[u] \dot{v}\right).$$

Here $f[u] = f(x, u, \nabla u, \dot{u})$. Easy to see that this also is an isomorphism $Z_m^T \overrightarrow{\sim} Y_m^T$. Since Φ is an embedding, then the inverse function theorem implies

Lemma: In the space of Cauchy data $\mathcal{H}^m$ there is a domain $\mathcal{O}^0 = \mathcal{O}^0([0,T])$ such that the problem (1), (2)

$$\Box u + f(t, x, u, \nabla u, \dot{u}) = 0, \ 0 \le t \le T, \qquad u(0, x) = u_0(x), \ \dot{u}(0, x) = u_1(x).$$

has a classical solution $u \in X_m^T$ iff $(u_0, u_1) \in \mathcal{O}^0$. This solution is unique and analytically depends on (u_0, u_1). If f analytically depends on some extra parameter ξ, then $\mathcal{O}^0 = \mathcal{O}_\xi^0$ and the solution u also analytically depends on ξ.

So for $0 \le t \le T$ the flow-maps

$$S_0^t : \mathcal{O}^0 \to \mathcal{H}^m, \qquad (u_0, u_1) \to (u(t), \dot{u}(t)),$$

are well defined, analytic and analytically depend on the extra parameters.

Introducing the parameters.

For simplicity let $k = d$. Then u_0, u_1 and f are analytic in x. The space $\mathbb{R}^d = \{\theta = (\theta_1, \ldots, \theta_d)\}$ acts on $\mathbb{T}^d$ by the shifts $_\theta R$,

$$_\theta R(x) = (x + \theta)$$

Accordingly $\mathbb{R}^d$ acts on the nonlinear operators $f(t, x, u, \nabla u, \dot{u})$ by shifting their coefficients: $\qquad (_\theta R f)(t, x, u, \nabla u, \dot{u}) = f(t, _\theta R x, u, \nabla u, \dot{u}).$

Since the shifts commute with $\square$, then we have

$$(\square + _\theta R f(t, x, u, \nabla u, \dot{u}))(_\theta R u) = _\theta R\big((\square u + f(t, x, u, \nabla u, \dot{u}))\big).$$

Consider the operator of the shifted Cauchy problem $_\theta \Phi(u) = (u_{t=0}, \dot{u}_{t=0}, \square u + _\theta R f(u))$. It defines an analytic mapping

$$\bar{\Phi} : \mathbb{R}^d \times Z_m^T \to Y_m^T, \qquad (\theta, u) \to {_\theta}\Phi(u).$$

For any θ denote by $_\theta u(t)$ solutions of the shifted equation $\square + _\theta R f$, and by $_\theta S_0^t$, $0 \le t \le T$, flow-maps of that equation.

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Consider $\bar{\Phi}$ for θ in a small ball $B_\varepsilon = \{|\theta| \le \varepsilon\}$. These are the external parameters for the operator of the Cauchy problem. Applying Lemma above we get that for $\theta \in B_\varepsilon$ and (u_0', u_1') close to (u_0, u_1) the flow-maps for the shifted equation

$$_\theta S_0^t : (u_0', u_1') \mapsto (_\theta u(t), _\theta \dot{u}(t)), \quad 0 \le t \le T,$$

are well defined, analytic in θ and in (u_0', u_1') . We have:

$$_\theta R \circ S_0^t(u_0, u_1) = {}_\theta S_0^t \circ {}_\theta R(u_0, u_1) \quad \text{if } \theta \in B_\varepsilon$$

Let $u(t, x)$ be a solution of (1), (2). Then

$$\boxed{u(t, x + \theta) = {}_\theta R \circ S_0^t(u_0, u_1) = {}_\theta S_0^t \circ {}_\theta R(u_0, u_1) = {}_\theta S_0^t\big((u_0, u_1)(x + \theta)\big)}$$

if $\theta \in B_\varepsilon$. We assumed that u_0 and u_1 are analytic. Then $(u_0, u_1)(x + \theta)$ is analytic in θ. As the operator $_\theta S_0^t(u_0', u_1')$ analytically depends on $\theta \in B_\varepsilon$, then $_\theta S_0^t\big((u_0, u_1)(x + \theta)\big)$ also is. So $u(t, x + \theta)$ is analytic in $\theta \in B_\varepsilon$!

We have proved the first assertion of the Main Theorem:

i) If u_0, u_1 and f are real-analytic in $(x_1, \ldots, x_k)$, $1 \le k \le d$, then u also is analytic in these variables.

To prove the second assertion we have to show that u is analytic in t. By analogy, we have to shift the Cauchy data u_0, u_1 not in an x-direction, but in the t-direction. How to do this? – Apply the Cauchy-Kowalevski theorem to find the solution $u(\theta, x)$, $|\theta| < \varepsilon$! It is analytic, so it gives the needed time-θ shifts of the Cauchy data, analytic in θ. Now we argue as before to prove that $u(t, x)$ is analytic in t, till it exist as a classical solution. We assumed that the Cauchy-Kowalevski theorem is applicable, i.e. that all the data are analytic.

§4. Related results.

i) *Equations in homogeneous spaces.*

The proof applies to quasilinear wave equations in a compact Riemann homogeneous space. In this case $\Box = \partial^2/\partial t^2 - \Delta$, where Δ is the corresponding Laplace-Beltrami operator. Now the translations $_\theta R$ should be replaced by the local isometies. For example, the theorem remans true for quasilinear wave equations on the standard sphere S^d.

ii) *A local version of the result.* Consider the problem

$$\Box u + f(t, x, u, \nabla u, \dot u) = 0, \ 0 \le t \le T, \qquad u(0, x) = u_0(x), \ \dot u(0, x) = u_1(x).$$

in the characteristic cone

$$\{(t, x) \in [0, T] \times \mathbb{R}^d : |x| < T - t\},$$

when the Cauchy data are given at the ball $\{|x| < T\}$. Then a natural version of the results hold true with the same proof.

iii) *Quasilinear parabolic equations.*

The approach to study analyticity and partial analyticity of solutions in x (but not in t) applies to other equations. For example, to quasilinear parabolic equations

(3) $\dot{u} - \Delta u + f(t, x, u, \nabla u) = 0, \quad x \in \mathbb{T}^d, \ t \geq 0, \qquad u_{t=0} = u_0,$

where f is sufficiently smooth in t, x and is analytic in u and ∇u. One can find suitable space Z_m^T and Y_m^T such that the operator Φ of the Cauchy problem (3) defines an analytic diffeomorphism between Z_m^T and a subdomain of the space Y_m^T, see [SK82], *Diffeomorphisms of functional spaces that correspond to quasilinear parabolic equations*, Math. USSR Sbornik 117 (1982).

In the same way as before we prove that if f is analytical in all its variables, then classical solutions of (3) with analytical initial data are analytical in x. This is a well known result. But we also can prove that if f is analytic in $u, \nabla u$ and in a part of the space-variables, as well as the function u_0, then the solution $u(t, x)$ also is analytic in these space variables. This result seems new.

This approach applies to the Navier-Stokes system on the d-torus with $d = 2$ or $d = 3$, perturbed by a sufficiently smooth force $h(t, x)$, see [SK82]. It implies that if the initial data and the force h are analytical in space-variables $x_1, \ldots, x_k$, where $1 \leq k \leq d$, then a corresponding strong solution $u(t, x)$ remains analytic in this space-variables till it exists.

Example. Consider the 3d NSE in the spherical layer $S^2 \times (0, \varepsilon) = \{(\varphi, r\})$. Let the force and initial data are

i) analytic in φ,

ii) bounded uniformly in $t \geq 0$, uniformly in $\varepsilon \in (0, 1)$.

Due to Raugel-Sell, if positive ε is sufficiently small, then there exists a unique strong solution $u(t, \varphi, r), \ t \geq 0$. By our results this solution is analytic in φ.

iv) *NLS equation*

The results about analyticity and partial analyticity in x (not in t!) remain true for the nonlinear Schrödinger equation

$$(4) \qquad \dot{u} - i\Delta u + f(t, x, \operatorname{Re} u, \operatorname{Im} u) = 0, \qquad u_{t=0} = u_0, \ \ x \in \mathbb{T}^d.$$

Function $f \in \mathbb{C}$ is continuous in t, H^m-smooth in x $(m > d/2)$, analytic in $\operatorname{Re} u$, $\operatorname{Im} u$. We can replace $\mathbb{T}^d$ by any homogeneous Riemann space, analytic and compact.

§5. Energy Transfer to High Frequencies.

Consider eq. (4), where u_0 is analytic, f is continuous in t, analytic in x, $\operatorname{Re} u$, $\operatorname{Im} u$. Solution $u(t, x)$ is analytic in x, continuous in t. Denote

$$\rho(t) = \text{ radius of analyticity of } u(t,x) \text{ in } x = \min \big\{ \operatorname{Im} z \mid z- \text{ singular point of } u(t,z) \big\}.$$

Set $C(t) = \sup \big\{ |u(t,z)| \mid |\operatorname{Im} z| \leq \tfrac{1}{2}\rho(t) \big\}$ and write $u(t,x) = \sum_s u_s(t) e^{is \cdot x}$.

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Then

$$|u_s(t)| \le C(t) e^{-\frac{1}{2}\rho(t)|s|} \quad \forall\, s,$$

and

$$\rho(t) = -\limsup_{s\to\infty} \big(\ln |u_s(t)| \big)\, |s|^{-1}.$$

So, *the Fourier coefficients $u_s(t)$ exponentially in $|s|$ decay to zero iff $\rho(t) > 0$.*

eq. (4) exhibits the energy transfer to high modes iff $\liminf_{t\to\infty} \rho(t) = 0$.

Example. Let (4) be the 1d defocusing Zakharov-Shabat equation. Its integrable solutions $u(t,x)$ are given by the Its-Matveev-McKean-Trubowitz formula. Each such solution $u(t,x)$ is a double-periodic meromorphic function of x and

$$\rho(t) = \min\{|\mathrm{Im}\, z| \mid z - \text{pole of } u(t,x) \text{ in } x\}.$$

Now the function $\rho(t)$ is almost-periodic and $\rho \ge \rho_0 > 0$.

No energy transfer!

Example. Consider

(5) $$\dot{u} - \delta i \Delta u + i|u|^{2p} u = 0, \qquad u_{t=0} = u_0, \;\; x \in \mathbb{T}^d,$$

where $p > 0,\ \delta > 0,\ u_0(x)$ is analytic and $|u_0| \sim L$. Low estimates on Sobolev norms of $u(t)$ from SK, GAFA 9 (1999), 141-184 imply that

$$\liminf_{t \to \infty} \rho(t) \leq C\big(\delta/L^{2p}\big)^{1/3}.$$

$\diamond$ I cannot prove this estimate for $\limsup_{t \to \infty} \rho(t)$!

Conjecture. If in (5) $d \geq 2$, then for a typical solution u we have $\liminf_{t \to \infty} \rho(t) = 0$.

1) Let in (5) the Cauchy data $u_0(x_1, \ldots, x_d)$ as a function of each x_j is double-periodic and meromorphic. Is it true that in this case a solution $u(t, x)$ for each t also is a double-periodic meromorphic function of each x_j (with poles, depending on t)? If so then $\rho(t)$ is the minimal real part of these poles.

2) If not, do solutions $u(t, x)$ as above exist and are they typical?

3) In any case, what kind of analytic in x functions are solutions $u(t, x)$ of (5)?

4) Study question 1) numerically.

§6. Kolmogorov-Zakharov spectra.

Sometime $\rho(t) \geq \rho_0 > 0$ for all t, and we look for more delicate information. E.g. consider a parabolic version of eq. (5):
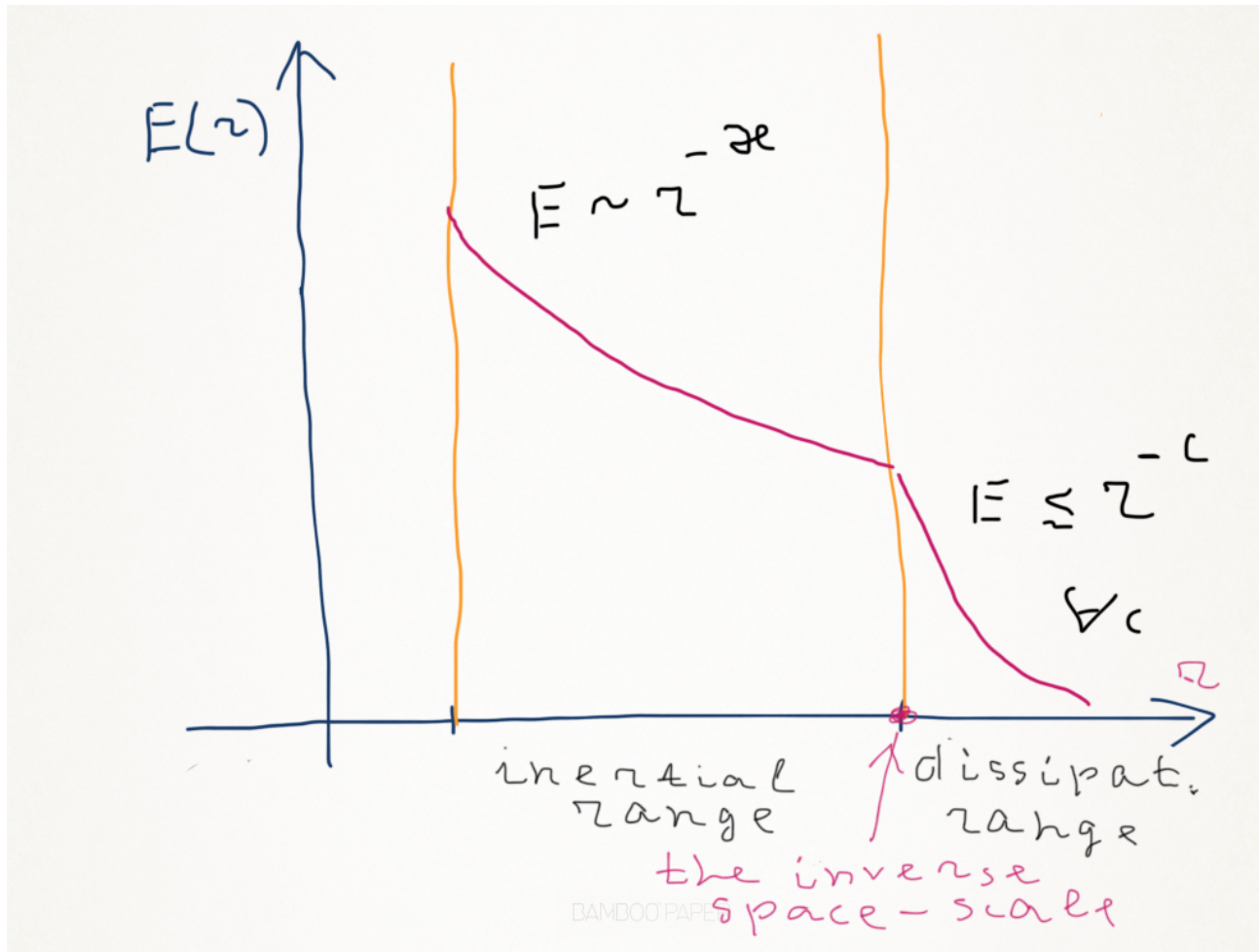
$$(6) \qquad \dot{u} + \delta(1 - \Delta)u + i|u|^{2p}u = 0, \qquad u_{t=0} = u_0, \ \ x \in \mathbb{T}^d,$$

where u_0 is analytic. Now $\rho(t) \geq \rho_0 > 0$ for all t – the solution is "well analytic". Denote

$$E(r) = \langle |u_s(t) \rangle,$$

where $\langle \cdot \rangle$ is the averaging in time $0 \leq t \leq \mathsf{Const}\, \delta^{-a}$ and in s such that $|s| \sim r$.

The physical conjecture is that:

Here the the space-scale depends on δ, but the estimate in the dissipation range holds for each c uniformly in δ.

The problem is to justify the conjecture. Now this is done for the 1d Burgers equation, see

[1] E. Aurell, U. Frisch, J. Lutsko, M. Vergassola "On the Multifractal Properties of the Energy Dissipation Derived from Turbulence Data", J. Fluid Mech. 238, 1992, 467-486.

[2] A. Biryuk "Spectral Properties of Solutions of the Burgers Equation with Small Dissipation", Funct. Anal. Appl. 35:1 (2001), 1-12.

[3] A. Boritchev "Sharp Estimates for Turbulence in White-Forced Generalised Burgers Equation" (2012), arXiv:1201.5567v4

They consider the Burgers equation

$$\dot{u} - \delta u_{xx} + u u_x = \dots, \qquad u_{t=0} = u_0 \in C^\infty, \ \ x \in \mathbb{T}^1,$$

where $\dots$ is either zero, or a random force, smooth in x and white in t.

The argument in [1-3] works in the x-presentation. Accordingly:

Problem. Describe analytical functions with K-Z spectra in x-presentation.

iv) *Strongly nonlinear equations.*

Consider a "strongly quasilinear" wave equation:

$$\ddot{u} + \frac{\partial}{\partial x_j} A_{jk}(t, x, u, \nabla u, \dot{u}) \frac{\partial u}{\partial x_k} + f(t, x, u, \nabla u, \dot{u}) = 0,$$

The Cauchy-Kowalewski theorem still applies to the corresponding Cauchy problem. Does the principe of propagation of analyticity holds? Yes it does, but some extra ideas have to be used for a proof.

REFERENCE

SK, N. Nadirashvili "Analyticity of solutions for quasilinear wave equations and other systems" (12 p), preprint 2012, (arXiv 1205.5926)