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Mathematics of 2d Turbulence

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§1. Introduction

Statistical 2D turbulence is described by small-viscosity 2D Navier-Stokes equation (NSE), perturbed by a random force:

$$u'_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \eta(t,x),$$

$$x\in\Gamma,\quad \operatorname{div} u = 0\,;\quad 0<\nu\le 1.$$

Here $u(t,x)\in\mathbb{R}^2$ – velocity, $p(t,x)\in\mathbb{R}$ – pressure, $\eta(t,x)\in\mathbb{R}^2$ – random force.
Γ is either the sphere, $\Gamma = S^2$, or a torus
$\Gamma = \mathbb{T}^2 = \{(x_1,x_2)\mid 0\le x_1\le a,\ 0\le x_2\le b\}$. Or

$$\Gamma \Subset \mathbb{R}^2 \quad \text{and} \quad u\mid_{\partial\Gamma} = 0.$$

Below, for simplicity of notation, $\Gamma = \mathbb{T}^2$ or $\Gamma \Subset \mathbb{R}^2$. So

$$(u\cdot\nabla)u = u_1\frac{\partial}{\partial x_1}u + u_2\frac{\partial}{\partial x_2}u.$$

Remarks

1) Another way to introduce randomness in Navier-Stokes equations would be through random initial data. For that model it is more difficult to get physically interesting results. I will not discuss it.

2) The random force in Navier-Stokes equations may be of the form

$$\text{deterministic force} \quad + \quad \text{small random perturbation}$$

Leray Projection. Denote

$$\mathcal{H} = \{u(x) \in L^2(\Gamma, \mathbb{R}^2), \;\; \mathrm{div}\, u = 0\}, \quad \|\cdot\| - \text{the } L^2\text{-norm.}$$

Then

$$L^2 = \mathcal{H} \oplus \{u(x) = \nabla p\}.$$

Denote $\Pi : L^2 \to \mathcal{H}$. Apply Π to the equation. Since $\Pi \nabla p = 0$ and $\Pi u = u$, then

$$u_t' - \nu A u + B(u) = \Pi\, \eta(t) =: \eta, \qquad (NSE)$$

where $Au = \Pi \Delta u$ and $B(u) = \Pi(u \cdot \nabla)u$.

Let $e_1, e_2, \dots \in \mathcal{H}$ be the basis of $\mathcal{H}$, formed by eigen-functions of A,

$$Ae_j = \lambda_j e_j \quad \forall\, j \geq 1$$

(if $\Gamma = \mathbb{T}^2$, this the sin/cos basis).

Random Force. The force η has the form $\eta(t,x)=\sum_j b_j\beta_j(t)e_j(x)$, where $\{\beta_j(t)\}$ are i.i.d. random processes and the constants $b_j\geq 0$ fast enough decay to zero. We can handle 3 classes of forces:

1) (random kicks). Fix any $T>0$ (period between the kicks)

$$\eta^\omega(t,x)=\sum_{l=1}^{\infty}(\mathrm{kick}_l)\cdot\delta(t-lT),\qquad (\mathrm{kick}_l)=\sum_{j=1}^{\infty}b_j\xi_{jl}^\omega e_j(x)$$

Here ξ_{jl}^ω are i.i.d. random variables with some mild restrictions on their distribution; not necessarily $\mathbf{E}\xi_{jl}=0$. How to obtain a solution $u(t,x)$ (NSE) with such a force? For $0<t<T$, $u(t,x)$ is a solution of the free equation. At $t=T$ the first kick comes and u instantly changes from $u(T,x)$ to $u(T,x)+$ kick $_1$, for $T<t<2T$ again u is a solution of the free equation, at $t=2T$ the second kick kick $_2$ comes, etc.

2) (a Levi process)

$$\eta^\omega(t,x)=\sum_{l=1}^{\infty}(\mathrm{kick}_l)\delta(t-T_l^\omega).$$

Here $T_1^\omega,T_2^\omega,\dots$ is a Poisson sequence.

3) (white noise).

$$\beta_j^\omega(t) = \frac{d}{dt} w_j(t) + f_j,$$

where $f_1, f_2, \ldots$ are constants and $w_1(t), w_2(t), \ldots$ are standard independent Wiener processes.

Solutions. A solution u is a random process $u^\omega(t) \in \mathcal{H}$. We are interested NOT in individual trajectories $t \mapsto u^\omega(t)$, but in distribution (=the law) of a solution u, $\mathcal{D}u(t) =: \mu_t$. This is a probability measure in the function space $\mathcal{H}$:

$$\mu_t(Q) = \mathbf{P}(u(t) \in Q), \quad Q \subset \mathcal{H}; \qquad \int_{\mathcal{H}} f(u)\, \mu_t(du) = \mathbf{E} f(u(t)).$$

For all 3 types of forces, considered above, $u(t)$ is a Markov process. Therefore

$$\mu_t = S_t^*(\mu_0),$$

where the operators $S_t^*(\mu_0)$ form a semi-group of linear operators in the space of measures in $\mathcal{H}$.

Task: Study qualitative properties of distributions of solutions, i.e. of the measures $\mu_t = \mathcal{D}(u(t)),\ t \geq 0$.

Definition: a measure μ on $\mathcal{H}$ is called a stationary measure for (NSE) if

$$S_t^* \mu \equiv \mu.$$

If $u(t)$ is a solution such that $\mathcal{D}u(0) = \mu$, then $\mathcal{D}u(t) \equiv \mu$. It is called a stationary solution.

Existence of a stationary measure is an easy fact (it follows from the Bogoliubov-Krylov compactness arguments). Not its uniqueness! – That is complicated.

§2. Limit "time to infinity" (mixing).

[1] "Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions", Europ. Math. Soc. Publ. House, 2006

Recall that we consider

$$u'_t - \nu A u + B(u) = \eta, \qquad \eta(t,x) = \sum b_j \beta_j(t) e_j(x), \qquad (NSE)$$

$x \in \Gamma$, Γ as above. Coefficients b_j fast decay with j: $B_0 = \sum b_j^2 < \infty$.

Condition (C). $b_j \neq 0$ for each $j \leq N$, where $N = N(B_0, \nu, \Gamma)$.
For example, (C) holds if $b_j \neq 0$ for all j.

THEOREM 1. If (C) holds, then: 1) there exists a unique stationary measure μ.
2) For any solution $u(t)$ of (NSE) we have

$$\mathrm{dist}(\mathcal{D}u(t), \mu) \leq Ce^{-ct}, \qquad c, C > 0.$$

Here dist is one of the 'usual' distances in the space of measures (e.g., Prokhorov's or Wasserstein's).
3) If force $\eta(t,x)$ is smooth in x, then μ is supported by smooth functions.

So, “statistical properties of solutions for $t \gg 1$ are universal and are described by a unique stationary measure μ.”

History of the result’s proof:

(2000), Kuksin-Shirikyan in CMP 213. For any Γ and for kick-forces η assertions 1) and 2) are established, without proving that the rate of convergence is exponentional.

(2001), E-Mattingly-Sinai in CMP 224. For $\Gamma = \mathbb{T}^2$ and white forces η such that (C) holds and also $b_j = 0$ for very large j, proved 1) (but not 2)) .

(2002), Bricmont-Kupiainen-Lefevere in CMP 230. For $\Gamma = \mathbb{T}^2$ and white forces η such that (C) holds and also $b_j = 0$ for very large j, proved 1) and 2).

Etc, see in [1].

§3. Consequence of the mixing.

Ergodicity:

THEOREM 2 (SLLN). For any solution $u(t)$ of (NSE) and any 'good' $f(u)$ we have

$$\frac{1}{T}\int_0^T f(u(s))\,ds \to \langle \mu, f\rangle, \quad a.s.$$

All important functionals $f(u)$ are good.

So "for a $2d$ turbulent flow time-average equals ensemble-average". See in [1].

THEOREM 3 (CLT). Let $\langle \mu, f \rangle = 0$. Then

$$\mathcal{D}\Big(\frac{1}{\sqrt{T}}\int_0^T f(u(s))\,ds\Big) \rightharpoonup N(0, \sigma_f),$$

for some $\sigma_f \geq 0$.

EXAMPLE: Fix any point x_0 and take $f(u) = u(x_0)$. If the force is homogeneous, then σ is independent from x_0. It would be very good to calculate it.

So “on large time-scales a turbulent flow is Gaussian”.

§4. Eulerian limit.

Below $\Gamma = \mathbb{T}^2 = \{1 \le x_1 \le a,\ 0 \le x_2 \le b\}$ and the force η is white in time.

Statistical 2d turbulence is described by solutions of (NSE) with $\nu \ll 1$. Consider the equation with small ν and with the force, multiplied by some degree of ν:

$$u'_t - \nu A u + B(u) = \nu^a \eta, \qquad a \in \mathbb{R} \qquad (\text{maybe } \ a = 0). \qquad\qquad (*)$$

Proposition (see [1]). Solutions of $(*)$ remain ~ 1 as $\nu \to 0$ if and only if $a = \frac{1}{2}$.

Accordingly, below we discuss equation

$$u'_t - \nu A u + B(u) = \sqrt{\nu}\, \eta, \qquad 0 < \nu \le 1. \qquad\qquad (NSE_\nu)$$

Remark. By suitable scaling of u, t and ν, for any a we can reduce eq. $(*)$ to eq. (NSE_ν).

Let (C) holds. Then eq. (NSE_ν) has a unique stationary measure μ_ν, and

- $\mathcal{D}u(t) \rightharpoonup \mu_\nu$ as $t \to \infty$ exponentially fast, for any solution $u(t)$.
- There is a solution $u_\nu(t,x)$ s.t. $\mathcal{D}u_\nu(t) \equiv \mu_\nu$.
- $u_\nu(t,x)$ is stationary in t. It is homogeneous in x if the force $\eta(t,x)$ is and is smooth in x if the force η is.
- Reynolds number of u_ν is $Re(u_\nu) \sim \nu^{-1}$.

Task: study μ_ν and u_ν as $\nu \to 0$.

Fact: $\mathbf{E}\,\|\nabla u_\nu(t)\|^2 = B_0, \quad \mathbf{E}\,\|\Delta u_\nu(t)\|^2 = B_1,$ where

$$B_0 = \sum b_j^2, \quad B_1 = \sum b_j^2 \lambda_j$$

.

λ_j's – eigenvalues of the Stokes operator.

Below I always assume that the force $\eta(t,x)$ is homogeneous in x.

Theorem 4 (Eulerian Limit), see [1]. Every sequence $\nu'_j \longrightarrow 0$ has a subsequence $\nu_j \longrightarrow 0$ such that the process $u_{\nu_j}(t,x)$ converges in distribution to a limiting process $U(t,x)$. The process $U(t,x)$ is stationary in t and homogeneous in x. Moreover,

a) every its trajectory $U(t,x)$ is such that $U(\cdot) \in L_{2\,loc}(0,\infty;\mathcal{H}\cap H^2)$, and satisfies the free Euler equation

$$\dot{u} + (u\cdot\nabla)u + \nabla p = 0,\ \operatorname{div} u = 0. \qquad \text{(Eu)}$$

b) The energy $E(U) = \frac{1}{2}\|U(t)\|^2 = \frac{1}{2}\int |U(t,x)|^2\,dx$ is time-independent. If $g(\cdot)$ is a bounded continuous function, then $\int g(\operatorname{rot} U(t,x))\,dx$ also is time-independent.

c) $\mu_0 = \lim \mu_{\nu_j} = \mathcal{D}U(t)$ is an invariant measure for (Eu).

d) $\int_{\mathcal{H}} \|\nabla u\|^2\,\mu_0(du) = B_0,\ \int_{\mathcal{H}} \|\Delta u\|^2\,\mu_0(du) \le B_1,\ \int_{\mathcal{H}} e^{\sigma\|\nabla u\|^2}\,\mu_0(du) < \infty.$

✠ μ_0 and $\mathcal{D}U(\cdot)$ are called the Eulerian limit (for eq. (NSE_ν)). They describe the 2D turbulence since they describe solutions of (NSE) with $\nu \ll 1$ and Re $\gg 1$.

Measure μ_0 is supported by Sobolev space H^2. If we write

$$u(x) = \sum_{s \in \mathbb{Z}^2} u_s \frac{s^\perp}{|s|} e^{is\cdot x}, \qquad E_s = \int |u_s|^2 \mu_0(du), \quad s \in \mathbb{Z}^2.$$

then

$$\sum_{s \in \mathbb{Z}^2} |s|^2 E_s = \int \|u\|_{H^2}^2 \, \mu_0(du) < \infty.$$

I cannot prove a better estimate. Numerics show that the relation above gives the right level of decay of E_s and

$$\sum |s|^{2+\varepsilon} E_s = \infty \quad \text{for} \quad \varepsilon > 0.$$

If so, then it is likely that $E_s \sim |s|^{-4} \cdot (log-$ correction$)$. Then $\sum_{|s|\sim r} E_s \sim r^{-3} \cdot (log-$ correction$)$, as the Kreichnan theory predicts.

But I cannot prove that.

Task: a) Check if the Eulerian limits μ_0 and $\mathcal{D}U(\cdot)$ depend on the sequence $\nu_j \to 0$.

b) Study the measure μ_0 and the distribution of the process U.

$\S$5. Properties of the Eulerian limit.

We have $\mu_{\nu_j} \rightharpoonup \mu_0$, where μ_ν – the stationary measure for (NSE_ν) and μ_0 – invariant measure for (Eu). By the theorem on the Eulerian limit, some exponential moments of the energy $E(u) = \frac{1}{2}\|u\|^2$ are finite. Hence, for some $C > 0, \sigma > 0$ we have

$$\mu_0\{E(u) > K\} \le Ce^{-\sigma K^2} \quad \text{for any} \quad K \ge 1.$$

So the energy of turbulent flow is big with small probability. Can the energy be small? This also is unlikely:

Theorem 5. $\mu_0\{E(u) < \delta\} \le \text{const}\,\sqrt{\delta}$ for any $\delta > 0$.

This result is important both physically and mathematically. In particular, it shows that $\mu_0\{E(u) = 0\} = 0$. This is crucial for further study of the measure μ_0 and the Eulerian limit U.

Theorem 5_1. Distribution of energy, corresponding to μ_0, has an (integrable) density with respect to the Lebesgue measure, i.e $dE = p_E(e)\,de$.

Recall that measure μ_0 is supported by $H^2(\mathbb{T}^2, \mathbb{R}^2)$. Consider functionals

$$F_j : \; u(x) \mapsto \int_{\mathbb{T}^2} (\operatorname{rot} u(x))^{2j} \, dx, \quad j \geq 1.$$

They are integrals of motion for (Eu). Consider the mapping

$$\Pi_N : \mathcal{H} \to \mathbb{R}^N, \quad u(x) \mapsto (F_1, \ldots, F_N).$$

Consider $\Pi_N \circ \mu_0$ – the image of measure μ_0 under this map.

Theorem 6. The measure $\Pi_N \circ \mu_0$ has an (integrable) density with respect to the N-dimensional Lebesgue measure, $\Pi_N \circ \mu_0 = p_N(y)\, dy, \; y \in \mathbb{R}^N$.

Corollary. $\dim_{\mathcal{H}} \text{supp}\,(\mu_0) = \infty$.

That is, *a) Distribution of the integrals of motion is non-singular.*

b) The measure μ_0 is genuinely infinite-dimensional.

See SK, CMP 284 (2008).

Main Problem. Study the limiting measure(s) μ_0. In particular:

1) is the limit μ_0 unique?

2) Study the correlation

$$K^{ij}(y) = \int_{\mathcal{H}} u^i(x)u^j(x+y)\,\mu_0(du).$$

§6. The balance relations (universality of 2d turbulence).

These are my results, obtained jointly with O. Penrose, see in [1].
Let the force $\eta(t,x)$ be homogeneous in x. Then the stationary solution of (NSE_ν) $u_\nu(t,x)$ is homogeneous in x, and the stationary measure $\mathcal{D}u_\nu(t)=\mu_\nu$ is a homogeneous measure in space $\mathcal{H}$. We have $\mu_{\nu_j} \rightharpoonup \mu_0$ (Eulerian limit).

Fix any t. Denote $\xi_\nu(x) = \text{rot}\, u_\nu(t,x)$ and set

$$\Gamma_\nu(\tau) = \{x \in \mathbb{T}^2 \mid \xi_\nu(x) = \tau\}, \ \tau \in \mathbb{R}\,.$$

Theorem 7_1. For any $\nu > 0$ and $\tau \in \mathbb{R}$

$$\mathbf{E} \int_{\Gamma_\nu(\tau)} |\nabla \xi_\nu|\, d\gamma = \frac{1}{2}\, \frac{B_1}{\text{Area of } \mathbb{T}^2}\, \mathbf{E} \int_{\Gamma_\nu(\tau)} |\nabla \xi_\nu|^{-1}\, d\gamma;$$

$d\gamma$ – the length element on $\Gamma_\nu(\tau)$.

These are infinitely-many relations, satisfied by measures rot $\circ\mu_\nu$, $\nu > 0$. We call them balance relations.

The balance relations admit an equivalent form. Recall that $\xi_\nu(x) = \text{rot}\, u_\nu(t,x)$.

Theorem 7_2. For any $x \in \mathbb{T}^2$,

$$\mathbf{E}\big(|\nabla\xi(x)|^2 \mid \mathcal{F}_{\xi(x)}\big) = \mathbf{E}|\nabla\xi(x)|^2 = \frac{1}{2}\,\frac{B_1}{\text{Area of } \mathbb{T}^2}\,. \qquad (*)$$

Relation $(*)$ means that for any point x and any function f, the random variables $|\nabla\xi(x)|^2$ and $f(\xi(x))$ are non-correlated:

$$\mathbf{E}\big(|\nabla\xi(x)|^2\, f(\xi(x))\big) = \mathbf{E}|\nabla\xi(x)|^2 \cdot \mathbf{E}\big(f(\xi(x))\big)\,.$$

Corollary. The stationary measures μ_ν, $0 < \nu \le 1$, and the Eulerian limit μ_0 satisfy the exponential estimates

$$\int_{\mathcal{H}} e^{\sigma|\,\mathrm{rot}\, u(x)|}\,\mu_\nu(du) \le C, \quad \int_{\mathcal{H}} e^{\sigma|u(x)|}\,\mu_\nu(du) \le C, \quad \int_{\mathcal{H}} e^{\sigma|\nabla u(x)|^{1/2}}\,\mu_\nu(du) \le C$$

for any x, with some $\sigma > 0$, $C \ge 1$.

§7. Remark on anisotropic 3d turbulence.

Consider 3d NSE in the thin domain $(x_1, x_2, x_3) \in \Gamma \times (0, \varepsilon)$, perturbed by a random force. Assume free boundary conditions in the thin direction x_3:

$$u_3 \mid_{x_3=0,\,\varepsilon} = 0, \quad \partial_3 u_{1,2} \mid_{x_3=0,\,\varepsilon} = 0.$$

Then the law of $(u_1, u_2)(t, x_1, x_2, x_3)$ converges, as $\varepsilon \to 0$, to the law of a solution of randomly forced 2d NSE in Γ and we have

$$\mathbf{E}\,\langle \text{normalised energy of 3d flow} \rangle \longrightarrow \mathbf{E}\,\langle \text{ energy of 2d flow} \rangle \qquad (*)$$

(so $\varepsilon^{-1} \int |u_3|^2\, dx \to 0$). It seems that (in non-trivial situations) $(*)$ does not hold for enstrophy, and that $\varepsilon^{-1} \int |\nabla u_3|^2\, dx$ does not converge to zero.

So randomly forced 2d NSE describe a class of anisotropic 3d turbulence.

For these results for randomly forced NSE see

Chuyeshov and Kuksin, ARMA 188 (2008) and Physica D 237 (2008).

Cf. well known related results for deterministic 3d NSE in thin domains.