Sergei Kuksin

Non-Autonomous Schrödinger Equation on d-torus

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§1. Introduction

Consider non-autonomous Schrödinger equation on $\mathbb{T}^d$, $d \geq 1$:

(1) $$\dot{u} = -i\big(\Delta u + V(t,x)u\big), \quad u = u(t,x) \in \mathbb{C}, \quad x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d.$$

Here $V(t,x) \in \mathbb{R}$ is sufficiently smooth. Multiplying (1) by $\bar{u}$ and integrating over $\mathbb{T}^d$ we get that

(2) $$|u(t)|^2_{L_2} = \text{const} .$$

Write $u(t,x) = \sum_s u_s(t)e^{is\cdot x}$. Relation (2) does not rule out that

(3) $$|u_s(t)| \text{ decays when } t \to \infty \text{ if } |s| \sim 1,$$
$$|u_s(t)| \text{ grows when } t \to \infty \text{ if } |s| \gg 1.$$

That is, that 'energy goes to high modes'. If (3) holds, then

(4) $$\|u(t)\|_s := \|u(t)\|_{H^s} \quad \text{grows with } t \text{ for } s \geq 1.$$

Question: Is (4) possible? Is it typical?

◇ This question is physically relevant.

◇ It models the same question for solutions of NLS, which now is a popular open problem.

Physical predictions:

T. Spencer (early 90's): If $V(t,x)$ is smooth and time-periodic, then

$$\|u(t)\|_s \le (\ln t)^{a_s} \|u(0)\|_s \quad \text{for} \quad t \gg 1, \tag{5}$$

for each s with a suitable $a_s > 0$.

V. Zakharov (early 2000's ?): If $V(t,x)$ is random and mixing, then

$$\|u(t)\|_1 \sim \text{const}\ \sqrt{t}. \tag{6}$$

No proof for (5) or (6) was given.

Inspired by T. Spencer's prediction, in late 90's J. Bourgain started to work on Question. He proved that

1) if $V(t,x)$ is smooth and all its C^k-norms are bounded uniformly in t, x, then for any $s \in \mathbb{N}$ and any $a > 0$

$$\|u(t)\|_s \le t^a \|u(0)\|_s \quad \text{for} \quad t \gg 1$$

(cf. the talk by J.-M.Delort at this workshop).

2) if $V(t,x)$ is smooth and time-quasiperiodic, then for any s there exists $c(s)$ such that

$$\|u(t)\|_s \le (\ln t)^{c(s)} \|u(0)\|_s \quad \text{for} \quad t \gg 1.$$

W.-M. Wang: same is true if V is analytic in t and x, "uniformly in t".

§2. Schrödinger equations with time-quasiperiodic potentials.

See $[EK1]$ H. Eliasson and S. Kuksin, CMP 286 (2009), 125-136.
1d case is due to Bambusi-Graffi (2001).

Consider eq. (1) with a time-quasiperiodic potential:

$$\text{(7)} \qquad \dot{u} = -i\big(\Delta u + \varepsilon V(\phi_0 + t\omega, x)u\big), \quad V = V(\phi, x),\ \phi \in \mathbb{T}^n,\ x \in \mathbb{T}^d.$$

Potential $V(\phi, x) \in \mathbb{R}$ is real-analytic. Vector ω is a parameter of the problem,

$$\omega \in U \Subset \mathbb{R}^n.$$

Denote $L^2 = L^2(\mathbb{T}^d, \mathbb{C})$. Provide it with the exponential basis $\{e^{is\cdot x}, s \in \mathbb{Z}^d\}$. For any linear operator $B : L^2 \to L^2$ let $(B_{ab}, a, b \in \mathbb{Z}^d)$ be its matrix in this basis.

We speak about eq. (7): $\dot u = -i\big(\Delta u + \varepsilon V(\phi_0 + t\omega, x)u\big)$.

Main Theorem. If $\varepsilon \ll 1$, then for most ω we can find an ϕ-dependent complex-linear isomorphism $\Psi(\phi) = \Psi_{\varepsilon,\omega}(\phi)$

$$\Psi(\phi) : L^2 \to L^2, \quad u(x) \mapsto \Psi(\phi)u(x),$$

and a bounded Hermitian operator $Q = Q_{\varepsilon,\omega}$ such that a curve $u(t) \in L^2$ solves (7) if and only if $v(t) = \Psi(\phi_0 + t\omega)u(t)$ satisfies

$$\dot v = -i\big(\Delta v + \varepsilon Q v\big).$$

The matrix (Q_{ab}) is block-diagonal, i.e. $Q_{ab} = 0$ if $|a| \neq |b|$, and it satisfies

$$Q_{ab} = (2\pi)^{-n-d} \int\int V(\phi, x) e^{1(a-b)}\, dx d\phi + O(\varepsilon^\gamma), \quad \gamma > 0$$

("averaging"). Moreover, for any $p \in \mathbb{N}$ we have $\|Q\|_{H^p,H^p} \le C_1$ and $\|\Psi(\phi) - \mathrm{id}\,\|_{H^p,H^p} \le \varepsilon C_2$.

"For most" means "for all $\omega \in U_\varepsilon \subset U$, where mes $(U \setminus U_\varepsilon) \le \varepsilon^\kappa$ for some $\kappa > 0$".

Corollary. For ω as in the theorem and for any p solutions of (7) satisfy

$$(1 - C\varepsilon)\|u(0)\|_p \le \|u(t)\|_p \le (1 + C\varepsilon)\|u(0)\|_p, \quad \forall t \ge 0$$

(this is the “dynamical localisaton”).

Proof. Since Q is block-diagonal, then $\|v(t)\|_p =$const. Since $v(t) = \Psi(t)u(t)$ and $\|\Psi - \mathrm{id}\|_{H^p,H^p} \le \varepsilon C_2$, then the estimate follows. □

Remarks. 1) Let $n = 0$. Then (7) is $\dot{u} = -i\big(\Delta u + \varepsilon V(x)u\big)$. Theorem states that this equation may be reduced to a block-diagonal equation $\dot{u} = -iAu$, where $A_{ab} = 0$ if $|a| \ne |b|$. This is a well known fact.

2) For $n = 1$ the theorem's assertion is the Floquet theorem for the time-periodic equation (7). In difference with the finite-dimensional case, this is a perturbative result, valid only for ‘typical’ frequencies $\omega \in \mathbb{R}$ and small ε.

3) Theorem's claim is not true for all frequencies ω. For exceptional ω's we expect indefinite growth of Sobolev norms with time.

4) Proof uses essentially that V is analytic both in t and x.

Proof. Eq. (7) is a non-autonomous linear Hamiltonian system in L^2:

$\dot{u} = i\frac{\delta}{\delta \bar{u}} H_\varepsilon(u), \quad H_\varepsilon(u) = \frac{1}{2}\langle \nabla u, \nabla \bar{u} \rangle + \frac{1}{2}\varepsilon \langle V(\phi_0 + t\omega, x)u, \bar{u}\rangle.$

Consider the extended phase-space $L^2 \times \mathbb{T}^n \times \mathbb{R}^n = \{(u, \phi, r)\}$.

In this space the equation above can be written as the autonomous system

$$\dot{u} = i\frac{\delta}{\delta \bar{u}} h_\varepsilon(u, \phi, r),$$
$$\dot{\phi} = \nabla_r h_\varepsilon = \omega,$$
$$\dot{r} = -\nabla_\phi h_\varepsilon,$$

where $h_\varepsilon(u, \phi, r, \varepsilon) = \omega \cdot r + \frac{1}{2}\langle \nabla u, \nabla \bar{u} \rangle + \frac{1}{2}\varepsilon \langle V(\phi, x)u, \bar{u}\rangle.$

h_ε is a small perturbation of the integrable quadratical hamiltonian $h_0 = \omega \cdot r + \frac{1}{2}\langle \nabla u, \nabla \bar{u} \rangle$. To perturbations of h_0 applies the KAM-theorem from $[EK2]$ Eliasson-Kuksin, “KAM for nonlinear Schrödinger equation”.

How the theorem from [EK2] implies Main Theorem? Let us write h_ε as

$$h_\varepsilon(u,\phi,r,\varepsilon)=\omega\cdot r+\frac{1}{2}\langle\nabla u,\nabla\bar u\rangle+\varepsilon f(u,\phi,r).$$

In our case $f=\frac{1}{2}\langle V(\phi,x)u,\bar u\rangle$.

KAM Theorem from [EK2]: There exists domain $\mathcal{O}=\{\|u\|<\delta\}\times\mathbb{T}^n\times\{|r|<\delta\}$ and symplectic transformation $\Phi:\mathcal{O}\to L^2\times\mathbb{T}^n\times\mathbb{R}^n$ which transforms h_ε to

$$h_0=\omega'\cdot r+\frac{1}{2}\langle\nabla u,\nabla\bar u\rangle+\varepsilon\langle Qu,\bar u\rangle+f'(u,\phi,r),$$

where $f'=O(|u|^3)+O(|r|^2)$.

Torus $T_0=0\times\mathbb{T}^n\times 0$ is invariant for the transformed system, so $\Phi(T_0)$ is invariant for the original equation. This is the usual KAM statement. NOW it is trivial: it simply states that $u(t)\equiv 0$ is a solution on the original equation.

But KAM theorem above tells more! Simple analysis of the proof (see a Remark in [EK2]) shows that if the perturbation εf is quadratic in u and r-independent, then

the KAM-transformations are linear in u and do not change ω.

So the transformed hamiltonians stay quadratic in u. Hence, the transformed hamiltonian h_0 is such that $f' = 0$. That is,

$$h_0 = \omega' \cdot r + \frac{1}{2}\langle \nabla u, \nabla \bar{u} \rangle + \varepsilon \langle Qu, \bar{u} \rangle.$$

This proves Main theorem.

§3. Digression on classical systems.

The Hamiltonian operator $\Delta + \varepsilon V(\phi + t\omega, x)$ is a quantisation of the classical hamiltonian $|\xi|^2 + \varepsilon V(\phi + t\omega, x)$, $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$. It is quasiperiodic in time. Fix any initial data (x_0, ξ_0). Then

i) by KAM, for a typical ω a solution such that $(x(0), \xi(0)) = (x_0, \xi_0)$ is time-quasiperiodic;

ii) for exceptional ω we "should" have diffusion: solution $(x(t), \xi(t))$ grows to infinity with t (reference ???).

Accordingly, for solutions of (7) with exceptional ω we expect indefinite growth of high Sobolev norms as $t \to \infty$.

Main Theorem and Remark 3 are the quantum versions of i) and ii).

§4. Perturbed unharmonic oscillator.

⋇ Consider Schrödinger equation in $\mathbb{R}^1$:

$$\dot{u} = -i\big(- u_{xx} + (x^2 + \mu x^{2m})u + \varepsilon V(\phi_0 + t\omega, x)u\big),$$

where $\mu > 0,\ m \in \mathbb{N},\ m \geq 2$; $V(\phi, x)$ is C^2-smooth in ϕ, x and analytic in ϕ, bounded uniformly in ϕ, x. An analogy of Main Theorem holds. See [SK] LNM 1556 (Section 2.5) for the needed KAM-theorem.

⋇ Due to Bambusi-Graffi (CMP **219** (2001), 465-480) the result holds for non-integer m. That is, for equations

$$\dot{u} = -i\big(- u_{xx} + Q(x)u + \varepsilon V(\phi_0 + t\omega, x)u\big),$$

where $Q(x) \sim |x|^\alpha, \alpha > 2$ as $|x| \to \infty$. Moreover, they allow $V \to \infty$ as $|x| \to \infty$.

⋇ Liu-Yuan (CPAM, to appear) allow faster growth of V in x. They proved reducibility of the *quantum Duffing oscillator*

$$\dot{u} = -i\big(- u_{xx} + x^4 u + \varepsilon x V(\phi_0 + t\omega, x)u\big).$$

Problem. The assertion holds for the perturbed harmonic oscillator

$$\dot{u} = -i\big(-u_{xx} + x^2 u + \varepsilon V(\phi_0 + t\omega, x)u\big).$$

This may follow from [EK2].

What happens in higher dimensions? – Nobody knows.

§5. Other equations on d-tori, $d \geq 2$.

The approach of [EK1] applies to non-linear wave equations. The corresponding KAM-theorem (under preparation) implies reducibility to constant coefficients for non-autonomous wave equation

$$\ddot{u} = \Delta u + \varepsilon V(\phi_0 + t\omega, x)u, \quad x \in \mathbb{T}^d, \;\; \phi \in \mathbb{T}^n.$$