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Navier-Stokes system with random force

What are the reasons to to consider nonlinear PDE with random force?

1) For some problems from physics (and not only from physics) they are more adequate then deterministic PDE.

2) There are some natural and important objects, related to a nonlinear PDE with a random force, which have no direct analogy for the non-random PDE. They give insights on the former which do not exits for the latter.

3) For some natural questions concerning PDE and their solutions, answers in the stochastic case are better than in the deterministic one.

4) There are some questions to which we cannot answer in the deterministic setting, but can answer in the stochastic one.

References:

[1] S. Kuksin, A. Shirikyan ”Mathematics of Two-Dimemsional Turbulence”', CUP 2012.

A shorter version of the book:

$[2]$ S. Kuksin, “Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions”, Europ. Math. Soc. Publ. House, 2006

§1. Introduction: randomly forced NSE versus deterministic NSE.

Consider NSE on $\mathbb{T}^d$, $d \le 3$, and write it a-la Leray:

$$(NSE) \qquad \dot u(t) + \nu A u(t) + B(u(t)) = \eta(t,x); \qquad \operatorname{div} u = 0, \quad \int_{\mathbb{T}^d} u\,dx = 0.$$

Denote

$$\mathcal{H} = \{u(x) \in L^2(\mathbb{T}^2, \mathbb{R}^2), \ \operatorname{div} u = 0, \int u\,dx = 0\}, \quad \|\cdot\| - \text{the } L^2\text{-norm;}$$

$$\mathcal{H}^m = \mathcal{H} \cap H^m(\mathbb{T}^d; \mathbb{R}^d).$$

$$\{e_j, j \ge 1\} - \text{orthobasis in } \mathcal{H}, \qquad Ae_j = \lambda_j e_j \quad \forall\, j.$$

Three classes of random forces. Write force η as Fourier series:

$$\eta = \eta^\omega(t,x) = \sum_{s\ge1} f_s e_s(x) + \frac{d}{dt}\sum_{s\ge1} b_s w_s^\omega(t) e_s(x), \qquad \omega\in(\Omega,\mathcal{F},\mathbf{P}).$$

Here real numbers b_s and f_s decay fast when s grows, and $\{w_s^\omega(t), s\ge1\}$ are i.i.d. random process with zero mean-value and "with independent increments":

a) (white in time forces) $\{w_s^\omega(t)\}$ – standard Wiener processes.

b) (random kicks). Each process w_s is such that

$$w_s(t) = \text{const} \quad \text{for } n-1 < t \le n, n\in\mathbb{N} \quad \text{and} \quad w_s(n+0) - w_s(n) = \xi^\omega_{s,n},$$
$$\{\xi^\omega_{s,n}\} - \text{i.i.d. random variables ("kicks").}$$

c) (Levi processes). Processes $\{w_s(t)\}$ are as above, but the kicks ξ come at random instants of time, which form a Poisson process.

Below I will mostly speak about NSE with a white in time force η.

3D NSE. What do we know about stochastic 3D NSE?

a) A global weak solution exists, but it is unknown if it is unique. See works of Da Prato-Debussche-Odasso and Flandoli-Romito, see [1] for references.

b) In difference with the deterministic case, we cannot study special strong solutions (since we must study the equation for a.e. value of the random parameter).

c) Local in time theory is equivalent to the global theory:

Theorem (Flandoli-Romito, see in [1]). Consider 3D NSE with $u(0) = 0$. Assume that $\exists\varepsilon > 0$ such that with probability 1, for $0 \le t \le \varepsilon$ the equation has a strong solution. Then for any given $u(0)$ a strong solution exists for all $t \ge 0$, with probability 1.

So we cannot study local in time strong solutions.

That is, the modern theory of stochastic 3D NSE is not rich.

2D NSE with white in time force $\eta(t,x)$.

We regard a solution $u(t,x)$ as a random process $u^\omega(t) \in \mathcal{H}$. We are interested NOT in individual trajectories $t \mapsto u^\omega(t)$, but in distribution (=the law) of a solution u, $\mathcal{D}u(t) =: \mu_t$. This is a probability measure in $\mathcal{H}$:

$$\mu_t(Q) = \mathbf{P}(u(t) \in Q), \quad Q \subset \mathcal{H}; \qquad \int_{\mathcal{H}} f(u)\, \mu_t(du) = \mathbf{E} f(u(t)).$$

A solution $u(t)$ is a Markov process. Therefore

$$\mu_t = S_t^*(\mu_0),$$

where the operators $\{S_t^*,\ t \geq 0\}$ extend to a semi-group of LINEAR operators in the space of signed measures in $\mathcal{H}$.

Task: Study qualitative properties of distributions of solutions, i.e. of the measures $\mu_t = \mathcal{D}(u(t)),\ t \geq 0$.

§2. Limit "time to infinity" (the mixing).

Definition: a measure μ in $\mathcal{H}$ is called a stationary measure for (NSE) if

$$S_t^* \mu \equiv \mu \qquad \forall\, t.$$

If $u(t)$ is a solution such that $\mathcal{D}u(0) = \mu$, then $\mathcal{D}u(t) \equiv \mu$. This $u(t)$ is called a stationary solution.

Existence of a stationary measure is an easy fact which follows from the compactness argument due to Bogolyubov-Krylov. Not its uniqueness! – This is complicated.

Recall that we consider 2D NSE:

$$\dot{u}(t) - \nu A u + B(u) = \eta, \qquad \eta(t,x) = \sum \Big(b_j \frac{d}{dt} w_j(t) + f_j \Big) e_j(x), \qquad (NSE)$$

$x \in \mathbb{T}^2$. Coefficients b_j and f_j decay with j sufficiently fast.

Condition (C). $b_j \neq 0$ for each $j \leq N$, where N depends on ν and the rate of decay of $\{b_j\}$ and $\{f_j\}$.

For example, (C) holds for each $\nu > 0$ if $b_j \neq 0$ for all j.

THEOREM 1 (first proved by SK and A. Shirikyan in 2000 for kick-forces, see in [1,2]).

If (C) holds, then: 1) there exists a unique stationary measure μ.

2) For any solution $u(t)$ of (NSE) we have

$$\mathrm{dist}(\mathcal{D}u(t), \mu) \leq Ce^{-ct}, \qquad c, C > 0. \qquad \text{(mixing)}$$

Here dist is one of the 'usual' distances in the space of measures (e.g., Prokhorov's or Wasserstein's).

3) If force $\eta(t,x)$ is smooth in x, then μ is supported by smooth functions. I.e., $\mu(\mathcal{H} \cap C^\infty) = 1$.

So, “statistical properties of solutions $u(t, x)$ for $t \gg 1$ are universal and are described by a unique stationary measure μ.” This result always was postulated by physicists as an axiom:

”... we put our faith in the tendency for dynamical systems with a large number of degrees of freedom, and the coupling between these degrees of freedom, to approach a statistical state which is *independent* (partially, if not wholly) of the initial condition”.
(G. K. Batchelor ”The Theory of Homogeneous Turbulence”, p.6)

An analogy of the unique stationary measure μ for the case of 2D NSE with a time-independent non-random force $\eta(x)$ is an attractor of the equation.

Stationary measures μ interest physicists the most.

§3. Consequences of the mixing.

Ergodicity:

THEOREM 2 (SLLN). If (C) holds, then for any solution $u(t)$ of (NSE) and any 'good' $f(u)$ we have

$$\frac{1}{T}\int_0^T f(u(s))\,ds \to \langle \mu, f\rangle := \int f(u)\,\mu(du), \quad a.s.$$

Remark. The rate of convergence is $T^{-\gamma},\ \gamma < 1/2$.

So "for a turbulent flow time-average equals ensemble-average". This is another postulate of the theory of turbulence:

"...we can anticipate, assuming applicability of ergodic theory..., that a time average is identical with a probability average for the experimental fields".
(G. K. Batchelor "The Theory of Homogeneous Turbulence", p.17)

THEOREM 3 (CLT). Let $\langle \mu, f \rangle = 0$. Then

$$\mathcal{D}\Big(\frac{1}{\sqrt{T}}\int_0^T f(u(s))\,ds\Big) \rightharpoonup N(0,\sigma),$$

for some $\sigma > 0$ (depending on f).

So “on large time-scales a turbulent flow is Gaussian”. Cf. the book of Batchelor, p.174.

Dependence on Parameters:

Let the force $\eta(t,x) = \sum b_j \Big(f_j + \frac{d}{dt} w_j^\omega(t) \Big) e_j(x)$ continuously depends on a parameter, i.e.

$$f_j = f_j(a), \qquad b_j = b_j(a), \qquad 0 \le a \le 1,$$

and the condition (C) holds for each a. Let $u_a(t)$ be a solution of (NSE) with this force and some fixed (a-independent) initial data u_0. Then its law continuously depends on a UNIFORMLY IN TIME:

THEOREM 4

$$\sup_{t \ge 0} \big\{ \operatorname{dist}(\mathcal{D}u_a(t), \mathcal{D}u_0(t)) \big\} \to 0 \qquad \text{as} \quad a \to 0.$$

§4. Inviscid limit $\nu \to 0$ (the 2D turbulence)

Consider (NSE) with small ν and with the random force η, scaled by some degree of ν:

$$u_t' - \nu Au + B(u) = \nu^a \eta, \quad a \in \mathbb{R}; \quad \eta(t,x) = \sum \Big(b_j \frac{d}{dt} w_j(t) \Big) e_j(x), \qquad (*)$$

Proposition. Solutions of $(*)$ remain ~ 1 as $\nu \to 0$ and $t \gg 1$ if and only if $a = \frac{1}{2}$.

Accordingly, below we discuss the scaled NSE

$$u_t' - \nu Au + B(u) = \sqrt{\nu}\, \eta, \qquad 0 < \nu \le 1. \qquad (NSE_\nu)$$

Let all $b_j \neq 0$, i.e. the force η is non-degenerate. Then for each ν eq. (NSE_ν) has a unique stationary measure μ_ν, and

- $\mathcal{D}u(t) \rightharpoonup \mu_\nu$ as $t \to \infty$ exponentially fast, for any solution $u(t)$.
- There is a solution $u_\nu(t,x)$ s.t. $\mathcal{D}u_\nu(t) \equiv \mu_\nu$; u_ν is stationary in t.
- $u_\nu(t,x)$ is smooth in x if the force η is.
- Reynolds number of u_ν is $Re(u_\nu) \sim \nu^{-1}$.

Physicists are interested the most in properties of μ_ν and u_ν when $\nu \to 0$. – This is the 2d turbulence.

Fact: $\mathbf{E}\,\|\nabla u_\nu(t)\|^2 = B_0, \quad \mathbf{E}\,\|\Delta u_\nu(t)\|^2 = B_1$, where

$$B_0 = \sum b_j^2, \;\; B_1 = \sum b_j^2 \lambda_j$$

.

Theorem 5. Every sequence $\nu_j' \to 0$ has a subsequence $\nu_j \to 0$ such that

$$u_{\nu_j}(\cdot) \to U(\cdot) \quad \text{as} \quad \nu_j \to 0,$$

in distribution, where the random field $U = U(t,x)$ is stationary in t. Moreover,

a) every its trajectory $U(t,x) = U^\omega(t,x)$ satisfies the free Euler equation

$$\dot{u} + (u \cdot \nabla)u + \nabla p = 0, \ \operatorname{div} u = 0. \qquad \text{(Eu)}$$

b) The energy $E(U) = \frac{1}{2}\|U(t)\|^2 = \frac{1}{2}\int |U(t,x)|^2\, dx$ is time-independent $\forall \omega$. If $g(\cdot)$ is a bounded continuous function, then $\int g(\operatorname{rot} U(t,x))\, dx$ also is time-independent.

c) $\lim \mu_{\nu_j} = \mu_0 = \mathcal{D}U(t)$ is an invariant measure for (Eu).

d) $\int_{\mathcal{H}} \|\nabla u\|^2\, \mu_0(du) = B_0, \quad \int_{\mathcal{H}} \|\Delta u\|^2\, \mu_0(du) \le B_1.$

e) The measure μ_0 is ”genially infinite-dimensional”: if $K \subset \mathcal{H}$ and $\dim_H K < \infty$, then $\mu_0(K) = 0$.

✠ μ_0 (and $\mathcal{D}U(\cdot)$) describe the space-periodic 2D turbulence since it describes solutions of (NSE) with $\nu \ll 1$ and Re $\gg 1$.

✠ "Universality of 2d turbulence". I recall that $B_0 = \sum b_j^2, \qquad B_1 = \sum b_j^2 \lambda_j.$

Fact: The measure μ_ν, $\nu > 0$, satisfies infinitely-many explicit algebraical relations which are independent from ν and depend only on B_0 and B_1.

✠ μ_0 depends on the damping: if in the equation we replace the viscosity $-\nu\Delta u$ by the hyperviscosity $\nu\big(-\Delta u\big)^a$, $a > 1$, then the limiting measure μ_0 will change.

✠ Measure μ_0 is supported by the Sobolev space H^2: $\mu_0(H^2) = 1$. I conjecture that

$$\mu_0(H^{2+\varepsilon}) = 0 \qquad \text{if} \qquad \varepsilon > 0.$$

But I cannot prove this.

Rigorous study of μ_0 is very hard and very important.

Energy spectrum of measure μ_0. For any $u(x) \in \mathcal{H}$ and $s \in \mathbb{Z}^2$ denote

$$|u_s|^2 = \Big| \int u(x) e^{-is\cdot x}\, dx \Big|^2,$$

and for $k \geq 1$ and a suitable $C > 0$ define

$$E_k = \frac{1}{2C} \sum_{k-C\leq|s|\leq k+C} \mathbf{E}^{\mu_0} |u_s|^2.$$

The function $k \mapsto E_k$ is called *the energy spectrum* of μ_0. Clearly we have

$$\mathbf{E}^{\mu_0}(\|u\|_m^2) \sim \sum_k E_k k^{2m}.$$

Assume that $E_k \sim k^{-r}$ for some r. Since $\mathbf{E}^{\mu_0}\|u\|_2^2 < \infty$ by Theorem 5, then $r \geq -5$ (I neglect the logarithmic divergence). If the conjecture above holds, i.e. if $\mathbf{E}^{\mu_0}\|u\|_{2+\varepsilon}^2 = \infty$ for $\varepsilon > 0$, then $r = 5$. That is,

$$E_k \sim k^{-5}.$$

§5. Anisotropic 3d turbulence in thin domains.

Consider 3d NSE in the thin domain $(x_1, x_2, x_3) \in M^2 \times (0, \varepsilon)$, where $M^2 = \mathbb{T}^2$ or $M^2 = S^2$, with free boundary conditions in the thin direction x_3:

$$u_3 \,|_{x_3=0,\,\varepsilon} = 0, \quad \partial_3 u_{1,2} \,|_{x_3=0,\,\varepsilon} = 0.$$

Perturb the equation by a random kick-force

$$\eta(t, x) = \sum b_j \beta_j(t) e_j(x),$$

where $\{e_j(x), j \geq 1\}$, are eigen-functions of the 3d Stokes operator, and $\beta_j(t)$ are kick-processes.

Theorem 6. The law of the horisontal component of solution $(u_1, u_2)(t, x_1, x_2, x_3)$ converges, as $\varepsilon \to 0$, *uniformly in time* t, to the law of a solution $(v_1, v_2)(t, x_1, x_2)$ of randomly forced 2d NSE in M^2, and we have

$$\mathbf{E}\,\langle \text{normalised energy of 3d flow } u\rangle \to \mathbf{E}\,\langle \text{energy of 2d flow } v\rangle \qquad (*)$$

(so $\varepsilon^{-1}\int |u_3|^2\,dx \to 0$).

It seems that (in non-trivial situations) $(*)$ does not hold for enstrophy, and that $\varepsilon^{-1}\int |\nabla u_3|^2\,dx$ does not converge to zero.

So randomly forced 2d NSE describe a class of anisotropic 3d turbulence.

For these results for randomly forced 3d NSE see

Chuyeshov and Kuksin, ARMA 188 (2008) and Physica D 237 (2008).

See [1] for discussion.

Cf. well known related results for deterministic 3d NSE in thin domains by G. Raugel, G. Sell (and by many people after them).

Let us believe for a moment that 3D NSE in $S^2 \times (0, \varepsilon)$, perturbed by a random kick-force, describes the meteorology of Earth. By Thm 6 statistics of the horisontal component (u_1, u_2) of solution may be well approximated by that of a solution v for 2D NSE in S^2 with a random force, uniformly in time. By Thm 4 statistical characteristics of the 2D solution v continuously and uniformly in time depend on characteristics of the random force.

So, in this setting it is possible to calculate numerically statistical characteristics of the meteorology. I recall that its deterministic characteristics CANNOT be calculated due to the exponential instability.

I believe that this conclusion remains true if we replace the 3D NSE in the layer $S^2 \times (0, \varepsilon)$ by the primitive equations of the meteorology.

REFERENCES:

[1] S. Kuksin, A. Shirikyan "Mathematics of Two-Dimensional Turbulence"', CUP 2012.

[2] S. Kuksin, "Randomly Forced Nonlinear PDEs and Statistical Hydrodynamics in 2 Space Dimensions", Europ. Math. Soc. Publ. House, 2006