A coupling approach to randomly forced nonlinear PDE's. I

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Abstract

We develop a coupling approach to prove that a randomly forced dissipative PDE has a unique stationary measure and to study ergodic properties of this measure.

Contents

0 Introduction

Let H be a Hilbert space with a norm $\|\cdot\|$ and an orthonormal basis $\{e_j\}$ and let $S\colon H \to H$ be a locally Lipschitz operator such that $S(0)=0$. It is assumed that S satisfies some additional conditions which, roughly speaking, mean that S is compact and that $S^n(u)\to 0$ as $n\to\infty$ uniformly on bounded subsets of H. (For the exact statement, see conditions (A) – (C) in Section 2.) Let η_k, $k\in\mathbb{Z}$, be a sequence of i.i.d. random variables in H of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j. \tag{0.1}$$

Here $b_j \ge 0$ are some constants such that $\sum b_j^2 < \infty$ and $\{\xi_{jk}\}$ are independent random variables such that $\mathcal{D}(\xi_{jk}) = p_j(r)\,dr$, where p_j, $j=1,2,\dots$, are functions of bounded variation supported by the interval $[-1,1]$. (We denote by $\mathcal{D}(\xi)$ the distribution of a random variable ξ.)

Our goal is to study the random dynamical system (RDS)

$$u^k = S(u^{k-1}) + \eta_k. \tag{0.2}$$

For any $v\in H$, we denote by $u^k = u^k(v)$, $k\ge 0$, the solution of (0.2) such that $u^0 = v$. Let $C_b(H)$ be the space of bounded continuous functions on H and $\mathcal{P}(H)$ be the set of probability Borel measures on H. The RDS (0.2) defines a family of Markov chains in H. We shall denote by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the corresponding Markov semigroups acting in $C_b(H)$ and $\mathcal{P}(H)$, respectively:

$$\mathfrak{P}_k f(v) = \mathbb{E} f\big(u^k(v)\big), \quad f\in C_b(H),$$
$$\mathfrak{P}_k \mu(\Gamma) = \int_H \mathbb{P}\big\{u^k(v)\in\Gamma\big\}\mu(dv), \quad \mu\in\mathcal{P}(H).$$

Let us recall that $\mu\in\mathcal{P}(H)$ is called a *stationary measure* for (0.2) if $P_1^*\mu = \mu$.

The goal of this paper is to present a new, simple proof of the uniqueness and ergodicity of a stationary measure and to specify the rate of convergence to it. Namely, we prove the following result:

Theorem 0.1. *There is an integer $N\ge 1$ such that if*

$$b_j \ne 0 \quad \textit{for} \quad 1\le j\le N, \tag{0.3}$$

then (0.2) *has a unique stationary measure μ. Moreover, for any $R>0$ there is $C_R>0$ such that*

$$\big|\mathfrak{P}_k f(u) - (\mu, f)\big| \le C_R e^{-c\sqrt{k}}\big(\sup_H |f| + \mathrm{Lip}(f)\big) \quad \textit{for} \quad k\ge 0, \tag{0.4}$$

where $\|u\|\le R$, f is an arbitrary bounded Lipschitz function on H, and $c>0$ is a constant not depending on u, f, R, and k.

Example 0.2. Let us consider the 2D Navier–Stokes (NS) equations perturbed by a random kick-force:

$$\begin{aligned} &\dot u - \nu\Delta u + (u,\nabla)u + \nabla p = \eta(t,x) \equiv \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t-k), \\ &\operatorname{div} u = 0, \quad \langle u(t,\cdot)\rangle = 0, \end{aligned} \tag{0.5}$$

where $u=u(t,x)$, $x\in\mathbb{T}^2$, and $\langle u\rangle = \int_{\mathbb{T}^2} u(x)\,dx$. Let H be the space of divergence-free vector fields $u\in L^2(\mathbb{T}^2,\mathbb{R}^2)$ such that $\langle u\rangle = 0$ and let $\{e_j\}$ be the normalised trigonometric basis in H. Assuming that the kicks $\eta_k\in H$ have the form (0.1) and normalising solutions $u(t)$ for (0.5) to be continuous from the right, we observe that (0.5) can be written in the form (0.2), where $u^k = u(k,\cdot)\in H$ and $S\colon H\to H$ is the time-one shift along trajectories of the free NS system (i.e., of Equations (0.5) with $\eta\equiv 0$). As it is shown in [KS1], the operator S satisfies all the required assumptions, and therefore Theorem 0.1 applies to (0.5).

Theorem 0.1 can also be applied to many other dissipative nonlinear PDE's perturbed by a random kick-force, in particular, to the complex Ginzburg–Landau equation

$$\dot{u} - \nu(\Delta - 1)u + i|u|^2 u = \eta(t,x), \quad x \in \mathbb{T}^n,$$

where $u = u(t,x)$ and $\nu > 0$ (see [KS1, KS2]).

Uniqueness of a stationary measure for (0.2) was first established[1] in [KS1]. The proof in [KS1] is based on a Lyapunov–Schmidt type reduction of the system (0.2) to an N-dimensional RDS with delay (the integer N is the same as in Theorem 0.1). Due to this reduction, the problem of uniqueness of a stationary measure for (0.2) reduces to a similar question for an abstract 1D Gibbs system with an N-dimensional phase space. The uniqueness for the reduced Gibbs system is then established using a version of the Ruelle–Perron–Frobenius theorem.

E, Mattingly, Sinai [EMS] and Bricmont, Kupiainen, Lefevere [BKL] used later similar approaches to show that the NS system (0.5) perturbed by a white (in time) force of the form

$$\eta(t,x) = \sum_{j=1}^{N'} b_j \dot{\beta}_j(t) e_j(x), \quad N' < \infty,$$

also has a unique stationary measure $\mu \in \mathcal{P}(H)$, provided that $b_j \neq 0$ for $1 \le j \le N \le N'$ with some sufficiently large $N = N(\nu)$. Moreover, it is shown in [BKL] that for the case of white noise the convergence in (0.4) is exponentially fast for μ-almost all $u \in H$.

In [KS3] the NS equations (0.5) with an unbounded kick-force $\eta(t,x)$ is studied and the scheme of [KS1] is used to prove the uniqueness and ergodicity of a stationary measure.

The approach presented in this work does not use a Lyapunov–Schmidt type reduction and the Gibbs measure technique. Instead it exploits some ideas from [KS2], interpreting them in terms of the coupling. The new approach gives rise to a shorter proof and is more flexible.

The coupling is a well-known effective tool for studying finite-dimensional Markov chains (e.g., see [Lin] and the Appendix in [V]) and dynamical systems (e.g., see [Y, BL]). In [EMS] a coupling is used to study the auxiliary finite-dimensional RDS with delay which arises as a result of the Lyapunov–Schmidt reduction. Our work shows that a form of coupling applies directly to infinite-dimensional Markov chains and randomly forced PDE's.

When a preprint of this paper was sent around, we learned from L.-S. Young that a similar approach to prove Theorem 0.1 is developed by her and Nader Masmoudi in their work under preparation.

[1] It is shown in [KS1, KS2] that the left-hand side of (0.4) converges to zero as $k \to \infty$ for any $f \in C_b(H)$; however, the rate of convergence is not specified.

Notation

We abbreviate a pair of random variables ξ_1, ξ_2 or points u_1, u_2 to $\xi_{1,2}$ and $u_{1,2}$, respectively. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any integer $k \ge 1$ we denote by Ω^k the space $\Omega \times \cdots \times \Omega$ (k times) endowed with the σ-algebra $\mathcal{F} \times \cdots \times \mathcal{F}$ and the measure $\mathbb{P} \times \cdots \times \mathbb{P}$. For a random variable ξ, we denote by $\mathcal{D}(\xi)$ its distribution.

For a Banach space H, we shall use the following spaces and sets:

$C_b(H)$ is the space of bounded continuous functions on H with the supremum norm $\|\cdot\|_\infty$.

$L(H)$ is the space of bounded Lipschitz functions on H endowed with the natural norm $\|\cdot\|_L$ (see Section 1).

$\mathcal{M}(H)$ is the space of signed Borel measures on H with bounded variation.

$\mathcal{P}(H)$ is the set of probability measures $\mu \in \mathcal{M}(H)$; this space is endowed with two different metrics described in Section 1.

$\mathcal{P}(H, \mathcal{A})$ is the set of measures $\mu \in \mathcal{P}(H)$ with support in a closed set $\mathcal{A}$.

$\mu_v(k)$ is the measure $\mathfrak{P}(k, v, \cdot)$, where $\mathfrak{P}$ is the Markov transition function for (0.2).

$B_H(R)$ is the closed ball of radius $R > 0$ centred at zero.

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1 Measures on Hilbert spaces

Let H be a Hilbert space with the Borel σ-algebra $\mathcal{B}(H)$ and let $\mathcal{M}(H)$ be the space of signed Borel measures with bounded variation. We denote by $\mathcal{P}(H)$ the set of probability measures $\mu \in \mathcal{M}(H)$ and by $\mathcal{P}(H, \mathcal{A})$ the subset in $\mathcal{P}(H)$ consisting of measures supported by a closed set $\mathcal{A} \subset H$. For any measure $\mu \in \mathcal{M}(H)$ and any function $f \in C_b(H)$, we write

$$(\mu, f) = \int_H f(u)\, d\mu(u) = \int_H f(u) \mu(du).$$

We shall use two different topologies on $\mathcal{P}(H)$. The first of them is given by the variation norm on $\mathcal{M}(H)$:

$$\|\mu\|_{\text{var}} = \sup_{\Gamma \in \mathcal{B}(H)} |\mu(\Gamma)|.$$

The distance defined by this norm on $\mathcal{P}(H)$ can be characterised in terms of densities. Namely, let us assume that $\mu_1, \mu_2 \in \mathcal{P}(H)$ are absolutely continuous with respect to a fixed probability measure $m \in \mathcal{P}(H)$. (Such a measure always exists; for instance, one can take $m = (\mu_1 + \mu_2)/2$.) In this case, we have

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \frac{1}{2} \int_H |p_1(u) - p_2(u)| \, dm(u), \tag{1.1}$$

where $p_i(u)$, $i = 1, 2$, is the density of μ_i with respect to m. The space $\mathcal{P}(H)$ is complete with respect to $\|\cdot\|_{\mathrm{var}}$.

To define a second topology, we denote by $L(H)$ the space of real-valued bounded Lipschitz functions on H with the norm

$$\|f\|_L := \Bigl(\sup_{u \in H} |f(u)|\Bigr) \vee \Bigl(\sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|}\Bigr).$$

Let $\|\cdot\|_L^*$ be the dual norm on $\mathcal{M}(H)$:

$$\|\mu\|_L^* = \sup_{\|f\|_L \le 1} \bigl|(\mu, f)\bigr|.$$

It is clear that the norm $\|\cdot\|_L^*$ defines a metric on $\mathcal{P}(H)$.

Lemma 1.1. *The space $\mathcal{P}(H)$ is complete with respect to the metric $\|\cdot\|_L^*$.*

Proof. Suppose that $\{\mu_n\} \subset \mathcal{P}(H)$ is a sequence such that $\|\mu_n - \mu_m\|_L^* \to 0$ as $m, n \to \infty$. Let $L^*(H)$ be the space of continuous functionals on $L(H)$. Regarding μ_n as elements of $L^*(H)$, we conclude that the sequence $\{\mu_n\}$ converges (in the norm $\|\cdot\|_L^*$) to a limit $\ell \in L^*(H)$, and we have

$$\ell(f) = \lim_{n \to \infty} (\mu_n, f), \quad f \in L(H). \tag{1.2}$$

In view of the corollary[2] from Theorem 1 in [GS, Chapter VI, §1], there is a measure $\mu \in \mathcal{P}(H)$ such that $\ell(f) = (\mu, f)$. This completes the proof. $\square$

Note that, in the case when H is finite-dimensional, the fact that the functional ℓ in (1.2) is a measure is implied by the following well-known result (for instance, see [H, Theorem 2.1.7]): any nonnegative distribution is a measure; in particular, any positive functional $\ell \in L^*(H)$ is a measure as well.

Let $\mathfrak{P}(k, u, \Gamma)$, $k \ge 0$, $u \in H$, $\Gamma \in \mathcal{B}(H)$, be a Markov transition function. A set $\mathcal{A} \subset \mathcal{B}(H)$ is said be *invariant* for $\mathfrak{P}$ if

$$\mathfrak{P}(k, u, \mathcal{A}) = 1 \quad \text{for all} \quad k \ge 0, \quad u \in \mathcal{A}.$$

Lemma 1.2. *Let $\mathcal{A} \in \mathcal{B}(H)$ be an invariant set for $\mathfrak{P}(k, u, \Gamma)$. Suppose that there is $k_0 \ge 1$ and a sequence ζ_k, $k \ge k_0$, going to zero as $k \to \infty$ such that*

$$\|\mathfrak{P}(k, u, \cdot) - \mathfrak{P}(k, v, \cdot)\|_L^* \le \zeta_k \quad \text{for} \quad k \ge k_0, \quad u, v \in \mathcal{A}. \tag{1.3}$$

Then there is a unique measure $\mu \in \mathcal{P}(H, \mathcal{A})$ such that

$$\|\mathfrak{P}(k, u, \cdot) - \mu\|_L^* \le \zeta_k \quad \text{for} \quad k \ge k_0, \quad u \in \mathcal{A}. \tag{1.4}$$

[2] The corollary of Theorem 1 in [GS, Chapter VI, §1] claims, in fact, that if the limit in (1.2) exists for any $f \in C_b(H)$, then the functional ℓ can be represented in the form $\ell(f) = (\mu, f)$, where $\mu \in \mathcal{P}(H)$. However, the same proof works also in the case under study.

Proof. Let $f \in L(H)$, $\|f\|_L \le 1$. Then, by (1.3) and the Chapman–Kolmogorov relation, for $l \ge k \ge k_0$ and $u, v \in \mathcal{A}$ we have

$$\begin{aligned} \bigl|(\mathfrak{P}(l, v, \cdot) - \mathfrak{P}(k, u, \cdot), f)\bigr| &\le \\ \le \Bigl| \int_H \mathfrak{P}(l - k, v, dz) \int_H \bigl(\mathfrak{P}(k, z, dw) f(w) - \mathfrak{P}(k, u, dw) f(w)\bigr) \Bigr| &\le \\ \le \zeta_k \int_H \mathfrak{P}(l - k, v, dz) &= \zeta_k. \end{aligned} \tag{1.5}$$

By Lemma 1.1, the space $\mathcal{P}(H)$ is complete with respect to $\|\cdot\|_L^*$. Hence, there is a unique measure $\mu \in \mathcal{P}(H)$ such that $\|\mathfrak{P}(l, v, \cdot) - \mu\|_L^* \to 0$ as $l \to \infty$. It is clear that $\operatorname{supp} \mu \subset \mathcal{A}$ and therefore $\mu \in \mathcal{P}(H, \mathcal{A})$. Passing to the limit in (1.5) as $l \to \infty$, we obtain (1.4). $\square$

We now recall that a pair of random variables (ξ_1, ξ_2) defined on the same probability space is called a *coupling* for given measures $\mu_1, \mu_2 \in \mathcal{P}(H)$ if $\mathcal{D}(\xi_j) = \mu_j$, $j = 1, 2$. For some basic results on the coupling, see [Lin, V] and the Appendix (Section 4).

Lemma 1.3. *If measures $\mu_1, \mu_2 \in \mathcal{P}(H)$ admit a coupling (ξ_1, ξ_2) such that*

$$\mathbb{P}\bigl\{\|\xi_1 - \xi_2\| > \varepsilon\bigr\} \le \theta, \tag{1.6}$$

where $\varepsilon > 0$ and $\theta > 0$ are some constants, then

$$\|\mu_1 - \mu_2\|_L^* \le 2\theta + \varepsilon. \tag{1.7}$$

Proof. Let $f \in L(H)$, $\|f\|_L \le 1$. Then $(\mu_{1,2}, f) = \mathbb{E} f(\xi_{1,2})$ and, therefore,

$$|(\mu_1 - \mu_2, f)| \le \bigl|\mathbb{E} \chi_Q (f(\xi_1) - f(\xi_2))\bigr| + \bigl|\mathbb{E} \chi_{Q^c} (f(\xi_1) - f(\xi_2))\bigr|, \tag{1.8}$$

where χ_Q and χ_{Q^c} are characteristic functions of the event $\|\xi_1 - \xi_2\| > \varepsilon$ and of its complement, respectively. By (1.6), the first term in the right-hand side of (1.8) is bounded by 2θ, while the second does not exceed $\varepsilon \|f\|_L \le \varepsilon$. This completes the proof of (1.7). $\square$

2 A class of random dynamical systems

Let H be a Hilbert space with a norm $\|\cdot\|$ and an orthonormal basis $\{e_j\}$ and let $S \colon H \to H$ be an operator satisfying conditions (A) – (C) below:

(A) *For any $R > r > 0$ there exist positive constants $a = a(R, r) < 1$ and $C = C(R)$ and an integer $n_0 = n_0(R, r) \ge 1$ such that*

$$\|S(u_1) - S(u_2)\| \le C(R) \|u_1 - u_2\| \quad \text{for all} \quad u_1, u_2 \in B_H(R), \tag{2.1}$$

$$\|S^n(u)\| \le \max\{a\|u\|, r\} \quad \text{for} \quad u \in B_H(R), \quad n \ge n_0. \tag{2.2}$$

Let η_k, $k \geq 1$, be a sequence of i.i.d. H-valued random variables that are defined on a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and have the form (0.1), where $b_j \geq 0$ are some constants such that

$$\sum_{j=1}^{\infty} b_j^2 < \infty, \tag{2.3}$$

and $\{\xi_{jk}\}$ is a family of independent real-valued random variables such that $|\xi_{jk}| \leq 1$ for all j, k, and $\omega_1 \in \Omega_1$. We consider the following RDS in H:

$$u^k = S(u^{k-1}) + \eta_k =: F^{\omega_1}(u^{k-1}), \quad k \geq 1. \tag{2.4}$$

It follows from (0.1) and (2.3) that the distribution of η_k is supported by the Hilbert cube K,

$$K = \Big\{ u = \sum_{j=1}^{\infty} u_j e_j : |u_j| \leq b_j \text{ for all } j \geq 1 \Big\}.$$

Therefore, if the initial state u^0 of the RDS (2.4) belongs to a set $B \subset H$, then $u^k \in \mathcal{A}_k(B)$ for all $k \geq 1$ and $\omega_1 \in \Omega_1$, where $\mathcal{A}_0(B) = B$ and

$$\mathcal{A}_k(B) = S\big(\mathcal{A}_{k-1}(B)\big) + K \quad \text{for} \quad k \geq 1.$$

The next condition expresses the property of existence of a bounded absorbing set for the system in question.

(B) *There exists $\rho > 0$ such that for any bounded set $B \subset H$ there is an integer $k_0 \geq 1$ such that $\mathcal{A}_k(B) \subset B_H(\rho)$ for $k \geq k_0$.*

Clearly, inequality (2.2) and condition (B) are satisfied if $\|S(u)\| \leq \gamma \|u\|$ for all $u \in H$ and some positive constant $\gamma < 1$.

To formulate the last condition, we introduce some notations. For a subspace $E \subset H$, we denote by $E^\perp$ its orthogonal complement in H. For an integer $N \geq 1$, let H_N be the finite-dimensional subspace generated by the vectors $e_1, \dots, e_N$ and let P_N and Q_N be the orthogonal projections onto H_N and $H_N^\perp$, respectively.

(C) *For any $R > 0$ there is a decreasing sequence $\gamma_N(R) > 0$ tending to zero as $N \to \infty$ such that*

$$\big\|\mathsf{Q}_N\big(S(u_1) - S(u_2)\big)\big\| \leq \gamma_N(R)\|u_1 - u_2\| \quad \textit{for all} \quad u_1, u_2 \in B_H(R).$$

Finally, we specify the random variables $\{\xi_{jk}\}$:

(D) *For any j, the random variables ξ_{jk}, $k \geq 1$, have the same distribution $\pi_j(dr) = p_j(r)\,dr$, where the densities $p_j(r)$ are functions of bounded variation such that $\operatorname{supp} p_j \subset [-1, 1]$ and $\int_{|r| \leq \varepsilon} p_j(r)\,dr > 0$ for all $j \geq 1$ and $\varepsilon > 0$. We normalise the functions p_j to be continuous from the right.*

The RDS (2.4) defines a family of Markov chains in H with the transition function

$$P(k, v, \Gamma) = \mathbb{P}\{u^k \in \Gamma\},$$

where $(u^k, k \geq 0)$ is the solution of (2.4) such that $u^0 = v$. Let $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ be the corresponding semigroups (see the Introduction for their definition). Continuity of S (see condition (A)) and the Lebesgue theorem on dominated convergence imply that the transition function satisfies the Feller condition: if $f \in C_b(H)$, then $\mathfrak{P}_k f \in C_b(H)$ for all $k \geq 1$.

Let $\rho > 0$ be the constant in condition (B). We introduce the set

$$\mathcal{A} = \overline{\bigcup_{k \geq 1} \mathcal{A}_k\big(B_H(\rho)\big)}. \tag{2.5}$$

It is clear that $\mathcal{A}$ is an invariant set for the RDS (2.4): if $u^0 \in \mathcal{A}$, then $u^k \in \mathcal{A}$ for all $k \geq 1$ and $\omega_1 \in \Omega_1$. Moreover, it follows from condition (C) that the set $\mathcal{A}$ is compact in H. (Note that the union in (2.5) is taken over $k \geq 1$ and therefore $B_H(\rho)$ is not a subset of $\mathcal{A}$.)

Our goal is to prove the following result:

Theorem 2.1. *There is an integer $N \geq 1$ such that if* (0.3) *holds, then the RDS* (2.4) *has a unique stationary measure $\mu \in \mathcal{P}(H, \mathcal{A})$. Moreover, for any $R > 0$ there is $C_R > 0$ such that*

$$\big|\mathfrak{P}_k f(u) - (\mu, f)\big| \leq C_R e^{-c\sqrt{k}} \|f\|_L \quad \textit{for} \quad k \geq 0, \quad \|u\| \leq R,$$

where $f \in L(H)$ is an arbitrary function and $c > 0$ is a constant not depending on f, u, R, and k.

Condition (B) and the definition of $\mathcal{A}$ imply that for any $R > 0$ there is an integer $l \geq 1$ depending on R such that $P(l, u, \mathcal{A}) = 1$ for any $u \in B_H(R)$. Hence, we can restrict our consideration to the invariant set $\mathcal{A}$. In view of Lemma 1.2, Theorem 2.1 will be established if we show that there are positive constants C and c and an integer $k_0 \geq 1$ such that

$$\|P(k, u, \cdot) - P(k, v, \cdot)\|_L^* \leq C\, e^{-c\sqrt{k}} \quad \text{for} \quad k \geq k_0, \quad u, v \in \mathcal{A}. \tag{2.6}$$

3 Proof of the main result

We first establish some auxiliary assertions and then use them to prove inequality (2.6), which implies the required result.

3.1 Auxiliary assertions

We begin with a simple observation. Let $R > 0$ be so large that $B_H(R) \supset \mathcal{A}$. To simplify notation, we denote $B = B_H(R)$.

Lemma 3.1. *For any $d > 0$ there is an integer $l = l(d) \geq 0$ and a constant $\varkappa = \varkappa(d) > 0$ such that*

$$\mathbb{P}\big\{\|u^l(v)\| \leq d/2 \text{ for all } v \in B\big\} \geq \varkappa. \tag{3.1}$$

Proof. Let a and n_0 be the constants in condition (A) that correspond to the parameters R (the radius of B) and $r = d/4$ and let $l = n_0 m$, where m is the smallest integer such that $a^m R \leq d/4$. If $\eta_k = 0$ in (2.4) for $1 \leq k \leq l$, then, in view of (2.2), we have

$$\|u^l(v)\| \leq \max\{a^m R, d/4\} = d/4 \quad \text{for all} \quad v \in B.$$

By continuity, there is $\gamma > 0$ such that if

$$\|\eta_k\| \leq \gamma \quad \text{for} \quad 1 \leq k \leq l, \tag{3.2}$$

then

$$\|u^l(v)\| \leq d/2. \tag{3.3}$$

It follows from (2.3) and condition (D) that the event (3.2) has a positive probability $\varkappa$. Inequality (3.1) follows now from (3.3). ☐

To simplify notation, for any $v \in H$ we denote by $\mu_v(k)$ the measure $P(k, v, \cdot) \in \mathcal{P}(H)$. For any measurable space $(X, \mathcal{B}(X))$ and any integer $k \geq 1$, we denote by X^k the direct product $X \times \cdots \times X$ endowed with the product σ-algebra $\mathcal{B}^k(X) = \mathcal{B}(X) \times \cdots \times \mathcal{B}(X)$.

Lemma 3.2. *There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an integer $N \geq 1$, and a constant $C > 0$ such that if (0.3) holds, then for any $u_1, u_2 \in B$ the measures $\mu_{u_{1,2}}(1)$ admit a coupling $V_{1,2} = V_{1,2}(u_1, u_2; \omega)$ that possesses the following properties:*

(i) *The maps $V_{1,2}$ are measurable with respect to the σ-algebra $\mathcal{B}^2(H) \times \mathcal{F}$ as functions of $(u_1, u_2, \omega) \in B^2 \times \Omega$.*

(ii) *Let $d = \|u_1 - u_2\|$. Then*

$$\mathbb{P}\big\{\|V_1 - V_2\| \geq d/2\big\} \leq Cd. \tag{3.4}$$

Let us note that inequality (3.4) is nontrivial only in the case $Cd < 1$.

Proof. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be the probability space on which the random variables $\{\eta_k\}$ are defined and let $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be the probability space constructed in Theorem 4.2 for the measures $\nu_{1,2}$ specified below. We shall show that the set $\Omega = \Omega_1 \times \Omega_2$ endowed with the natural σ-algebra and probability of direct product is the required probability space.

The random variables $V_{1,2}$ are sought in the form

$$V_1 = S(u_1) + \xi_1, \quad V_2 = S(u_2) + \xi_2,$$

where $\xi_{1,2}$ are some random variables on Ω such that $\mathcal{D}(\xi_1) = \mathcal{D}(\xi_2) = \mathcal{D}(\eta_1)$. It is clear that $\mathcal{D}(V_{1,2}) = \mu_{u_{1,2}}(1)$ and that (i) holds. To define the random variables $\xi_{1,2}$, we specify their projections $\mathsf{P}_N\xi_{1,2}$ and $\mathsf{Q}_N\xi_{1,2}$, where $N \geq 1$ is a sufficiently large integer which is chosen below.

We set

$$\mathsf{Q}_N\xi_1 = \mathsf{Q}_N\xi_2 = \mathsf{Q}_N\tilde{\eta}_1,$$

where $\tilde{\eta}_1$ is the natural extension of η_1 to Ω, i.e., $\tilde{\eta}_1(\omega) = \eta_1(\omega_1)$ for $\omega = (\omega_1, \omega_2) \in \Omega$. To define $\mathsf{P}_N\xi_{1,2}$, let us write $\nu_{1,2} := \mathsf{P}_N\mu_{u_{1,2}}(1)$ and assume that we have proved the inequality

$$\|\nu_1 - \nu_2\|_{\text{var}} \leq Cd, \tag{3.5}$$

where $C > 0$ is a constant not depending on $u_{1,2} \in B$. In view of Theorem 4.2, there is a maximal coupling $\Xi_{1,2}(u_1, u_2; \omega_2)$ for the measures $\nu_{1,2}$ that is measurable with respect to $(u_1, u_2, \omega_2) \in B^2 \times \Omega_2$:

$$\mathbb{P}\{\Xi_1 \neq \Xi_2\} = \|\nu_1 - \nu_2\|_{\text{var}} \leq Cd. \tag{3.6}$$

Retaining the same notation for the natural extensions of Ξ_1 and Ξ_2 to Ω, we now set

$$\mathsf{P}_N\xi_{1,2} = \Xi_{1,2} - \mathsf{P}_N S(u_{1,2})$$

and note that $\mathsf{P}_N V_1 \neq \mathsf{P}_N V_2$ if and only if $\Xi_1 \neq \Xi_2$. Let $N \geq 1$ be so large that $\gamma_N(R) \leq 1/2$ (see condition (C)). In this case, if $\mathsf{P}_N V_1 = \mathsf{P}_N V_2$, then

$$\|V_1 - V_2\| = \big\|\mathsf{Q}_N(V_1 - V_2)\big\| = \big\|\mathsf{Q}_N(S(u_1) - S(u_2))\big\| \leq \|u_1 - u_2\|/2 \leq d/2.$$

Inequality (3.4) follows now from (3.6).

Thus, it remains to establish (3.5). To this end, we set $v_{1,2} = \mathsf{P}_N S(u_{1,2})$ and note that, in view of (2.1),

$$\|v_1 - v_2\| \leq C(R)d. \tag{3.7}$$

Since $b_j \neq 0$ for $1 \leq j \leq N$, condition (D) implies that $\mathcal{D}(\mathsf{P}_N\eta_1) = p(x)\,dx$, where dx is the Lebesgue measure on the finite-dimensional space H_N and

$$p(x) = \prod_{j=1}^{N} q_j(x_j), \quad q_j(x_j) = b_j^{-1} p_j(x_j/b_j), \quad x = (x_1, \ldots, x_N) \in H_N,$$

is a bounded function with support in the set $\mathsf{P}_N K$. It follows that

$$\nu_{1,2} = \mathcal{D}(v_{1,2} + \mathsf{P}_N\eta_1) = p(x - v_{1,2})\,dx.$$

Therefore, by (1.1),

$$\|v_1 - v_2\|_{\text{var}} = \frac{1}{2}\int_{H_N} |p(x - v_1) - p(x - v_2)|\,dx.$$

We claim that

$$\int_{H_N} |p(x-v_1)-p(x-v_2)|\,dx \le |v_1-v_2| \sum_{j=1}^N b_j^{-1}\operatorname{Var}(p_j), \tag{3.8}$$

where $\operatorname{Var}(p_j)$ stands for the total variation of p_j. The required inequality (3.5) follows immediately from (3.7) and (3.8).

To prove (3.8), we first assume that p_j are C^1-smooth functions. In this case, we have

$$\begin{aligned}
&\int_{H_N} |p(x-v_1)-p(x-v_2)|\,dx \\
&\le |v_1-v_2| \int_{H_N}\int_0^1 \big|(\nabla p)(x-\theta v_1-(1-\theta)v_2)\big|\,d\theta dx \\
&= |v_1-v_2| \int_{H_N} \big|(\nabla p)(x)\big|\,dx \le |v_1-v_2| \sum_{j=1}^N \int_{\mathbb{R}} \big|\partial_{x_j} q_j(x_j)\big|\,dx_j \\
&= |v_1-v_2| \sum_{j=1}^N \operatorname{Var}(q_j).
\end{aligned}$$

It remains to note that $\operatorname{Var}(q_j)=b_j^{-1}\operatorname{Var}(p_j)$.

Inequality (3.8) in the general case can be easily derived by a standard approximation procedure; we omit the corresponding arguments. $\square$

We now combine Lemmas 3.1 and 3.2 to obtain a coupling $U_{1,2}^k(u_1,u_2)$ for the measures $\mu_{u_{1,2}}(k)$, $k\ge 1$. Let $l=l(d)$ and $C>0$ be the constants in Lemmas 3.1 and 3.2 and let $d_0>0$ be so small that

$$Cd_0 \le 1/4.$$

We set $d_r = 2^{-r}d_0$, $r\ge 1$.

For a probability space $(\Omega,\mathcal{F},\mathbb{P})$, we shall denote by $(\Omega^k,\mathcal{F}^k,\mathbb{P}^k)$ the direct product of its k independent copies. Points of the latter will be denoted by $\boldsymbol{\omega}^k = (\omega_1,\dots,\omega_k)$.

Lemma 3.3. *Suppose that the conditions of Lemma 3.2 are satisfied. Let $u_1,u_2\in\mathcal{A}$ and $d=\|u_1-u_2\|$. Then for any $k\ge 1$ the measures $\mu_{u_{1,2}}(k)$ admit a coupling $U_{1,2}^k = U_{1,2}^k(u_1,u_2;\boldsymbol{\omega}^k)$, $\boldsymbol{\omega}^k\in\Omega^k$, such that the following assertions hold:*

(i) *The maps $U_{1,2}^k(u_1,u_2;\boldsymbol{\omega}^k)$ are measurable with respect to $(u_1,u_2,\boldsymbol{\omega}^k)\in\mathcal{A}^2\times\Omega^k$.*

(ii) *There is a constant $\theta>0$ not depending on u_1, u_2, and k such that*

$$\mathbb{P}^k\big\{\|U_1^k-U_2^k\|\le d_r\big\}\ge\theta \quad \text{for all} \quad k\ge r+l(d_0), \quad u_1,u_2\in\mathcal{A}. \tag{3.9}$$

(iii) *If $\|u_1-u_2\|\le d_r$, then*

$$\mathbb{P}^k\big\{\|U_1^k-U_2^k\|\le d_{k+r}\big\}\ge 1-2^{-r-1} \quad \text{for all} \quad k\ge 1, \quad r\ge 0. \tag{3.10}$$

Proof. Let us recall that for any $(u_1,u_2)\in B\times B$ a coupling $V_{1,2}(u_1,u_2;\omega)$ was constructed in Lemma 3.2. We set

$$U_j(u_1,u_2;\omega)=\begin{cases} V_j(u_1,u_2;\omega) & \text{if} \quad \|u_1-u_2\|\le d_0,\\ F^\omega(u_j) & \text{if} \quad \|u_1-u_2\|>d_0,\end{cases}$$

where $j=1,2$ and $F^\omega(u)$ is given by (2.4). We define random variables $U_{1,2}^k$ on $(\Omega^k,\mathcal{F}^k)$ by the following rule: if $\|u_1-u_2\|>d_0$, then

$$U_j^k(u_1,u_2;\boldsymbol{\omega}^k)=F^{\omega_k}\circ\dots\circ F^{\omega_1}(u_j)$$

for $k\le l(d_0)$ and

$$U_j^k(u_1,u_2;\boldsymbol{\omega}^k)=U_j\big(U_1^{k-1}(u_1,u_2;\boldsymbol{\omega}^{k-1}),U_2^{k-1}(u_1,u_2;\boldsymbol{\omega}^{k-1});\omega_k\big) \tag{3.11}$$

for $k>l(d_0)$, where $\boldsymbol{\omega}^k=(\boldsymbol{\omega}^{k-1},\omega_k)=(\omega_1,\dots\omega_k)$ and $U_j^0(u_1,u_2)=u_j$. If $\|u_1-u_2\|\le d_0$, then $U_{1,2}^0(u_1,u_2)=u_{1,2}$ and for $k\ge1$ the random variables $U_j^k(u_1,u_2;\boldsymbol{\omega}^k)$ are inductively defined by (3.11).

We claim that $U_{1,2}^k$ satisfy assertions (i) – (iii) of the lemma. Indeed, the measurability of the maps $U_{1,2}^k$ is obvious since they are compositions of measurable maps. To prove (3.9), we first note that it is sufficient to consider the case $k=l+r$, $l=l(d_0)$. We introduce the following events in Ω^{l+r}:

$$\begin{aligned}
Q^+ &= \big\{\|U_1^l-U_2^l\|\le d_0\big\},\\
Q^- &= \big\{\|U_1^l-U_2^l\|> d_0\big\},\\
Q &= \big\{\|U_1^{l+r}-U_2^{l+r}\|\le d_r\big\}.
\end{aligned}$$

By Lemma 3.1, we have

$$\mathbb{P}^k(Q)=\mathbb{P}^k(Q|Q^+)\mathbb{P}(Q^+)+\mathbb{P}^k(Q|Q^-)\mathbb{P}(Q^-)\ge\varkappa\,\mathbb{P}^k(Q|Q^+). \tag{3.12}$$

If we assume that (3.10) is proved for $r=0$, then (3.12) will imply the required estimate (3.9) with $\theta=\varkappa/2$. Thus, it remains to established (iii).

For a fixed $r\ge0$, we set

$$Q_k^+=\big\{\|U_1^k-U_2^k\|\le d_{k+r}\big\},\quad Q_k^-=\big\{\|U_1^k-U_2^k\|> d_{k+r}\big\}$$

and denote by p_k^+ and p_k^- the probabilities of Q_k^+ and Q_k^-, respectively. Using (3.4) with $d=d_{k+r-1}$, we derive

$$p_k^+=p_{k-1}^+\mathbb{P}^k(Q_k^+|Q_{k-1}^+)+p_{k-1}^-\mathbb{P}^k(Q_k^+|Q_{k-1}^-)\ge(1-Cd_{k+r-1})p_{k-1}^+.$$

Since $p_0^+=1$, iteration of this estimate results in

$$p_k^+\ge\lambda:=\prod_{j=0}^{k-1}(1-Cd_{j+r}). \tag{3.13}$$

Since $d_m = 2^{-m} d_0$ and $C d_0 \le 1/4$, we have

$$\log\lambda = \sum_{j=0}^{k-1} \log(1 - C d_{j+r}) \ge -C \sum_{j=0}^{k-1} d_{j+r}$$
$$\ge -C d_0 \sum_{j=0}^{\infty} 2^{-(j+r)} = -2^{1-r} C d_0 \ge -2^{-r-1}.$$

Therefore, $\lambda \ge 1 - 2^{-r-1}$. □

3.2 Proof of Theorem 2.1

As was mentioned at the end of Section 2, it is sufficient to establish inequality (2.6). In what follows, to simplify notation, we shall write $\mathbb{P}$ instead of $\mathbb{P}^k$.

1) Let us fix arbitrary $u_1, u_2 \in \mathcal{A}$ and set $T_0 = 0$ and $T_r = T_{r-1} + r + l$ for $r \ge 1$, i.e.,

$$T_r = r(r+1)/2 + rl.$$

We claim that for any integer $r \ge 0$ there is a coupling $y_{1,2}(T_r)$ on Ω^{T_r} for the measures $\mu_{u_{1,2}}(T_r)$ such that

$$\mathbb{P}\bigl\{\|y_1(T_r) - y_2(T_r)\| > d_r\bigr\} \le C_1 \gamma^r, \tag{3.14}$$

where C_1 and $\gamma < 1$ are some positive constants.

The construction of $y_{1,2}(T_r) = y_{1,2}(T_r, u_1, u_2; \boldsymbol{\omega}^{T_r})$ and the proof of (3.14) are by induction. For $r = 0$, we set $y_j(0) = u_j$, and inequality (3.14) with $C_1 \ge 1$ is trivial in this case. Assuming that $y_{1,2}(T_i)$ are constructed for $0 \le i \le r$, we set

$$y_j(T_{r+1}, u_1, u_2; \boldsymbol{\omega}^{T_{r+1}}) = U_j^{r+l+1}\bigl(y_1(T_r, u_{1,2}; \boldsymbol{\omega}^{T_r}), y_2(T_r, u_{1,2}; \boldsymbol{\omega}^{T_r}); \boldsymbol{\omega}^{r+l+1}\bigr), \tag{3.15}$$

where $U_{1,2}^k(u_1, u_2; \boldsymbol{\omega}^k)$ are defined in Lemma 3.3 and $\boldsymbol{\omega}^{T_{r+1}} = (\boldsymbol{\omega}^{T_r}, \boldsymbol{\omega}^{r+l+1})$. Let us introduce the events

$$Q_r^+ = \bigl\{\|y_1(T_r) - y_2(T_r)\| \le d_r\bigr\}, \quad Q_r^- = \bigl\{\|y_1(T_r) - y_2(T_r)\| > d_r\bigr\}$$

and denote by p_r^+ and p_r^- their probabilities. Then, in view of (3.9) and (3.10) with $k = r + l$, we have (cf. (3.12))

$$p_{r+1}^- = \mathbb{P}(Q_{r+1}^- | Q_r^+)\mathbb{P}(Q_r^+) + \mathbb{P}(Q_{r+1}^- | Q_r^-)\mathbb{P}(Q_r^-)$$
$$\le 2^{-r-1} p_r^+ + (1-\theta) p_r^- \le 2^{-r-1} + \gamma p_r^-, \tag{3.16}$$

where $\gamma = 1 - \theta$. Without loss of generality, we can assume that $0 < \theta < 1/2$, and therefore $1 < 2\gamma < 2$. Iterating (3.16), we obtain

$$p_{r+1}^- \le 2^{-r-1} \sum_{j=0}^{r} (2\gamma)^j + \gamma^{r+1} p_0^- \le 2^{-r-1} \frac{(2\gamma)^{r+1} - 1}{2\gamma - 1} + \gamma^{r+1} \le C_1 \gamma^{r+1}.$$

This completes the induction.

2) We can now prove (2.6). Let us fix arbitrary positive integers r and $m \le r + l$ and set $k = T_r + m$, so that $T_r + 1 \le k < T_{r+1}$. We define a coupling $y_{1,2}(k) = y_{1,2}(k, u_1, u_2)$ for the measures $\mu_{u_{1,2}}(k)$ by the formula (cf. (3.15))

$$y_j(k, u_1, u_2; \boldsymbol{\omega}^k) = U_j^m\bigl(y_1(T_r, u_1, u_2; \boldsymbol{\omega}^{T_r}), y_2(T_r, u_1, u_2; \boldsymbol{\omega}^{T_r}); \boldsymbol{\omega}^m\bigr).$$

In view of (3.10) and (3.14), we have (cf. (3.16))

$$\mathbb{P}\bigl\{\|y_1(k) - y_2(k)\| > d_{r+1}\bigr\} \le \mathbb{P}(Q_r^-) + 2^{-r-1}\mathbb{P}(Q_r^+) \le C_2 \gamma^r, \tag{3.17}$$

where $C_2 > 0$ is a constant. Now note that $r^2/2 \le T_r \le (l+1) r^2$ for any $r \ge 0$ and therefore there are positive constants C and c such that

$$d_{r+1} \le C\, e^{-c\sqrt{k}}, \quad C_2 \gamma^r \le C\, e^{-c\sqrt{k}} \quad \text{for} \quad T_r \le k < T_{r+1}.$$

Combining this with (3.17), we derive

$$\mathbb{P}\bigl\{\|y_1(k, u_1, u_2) - y_2(k, u_1, u_2)\| \ge C\, e^{-c\sqrt{k}}\bigr\} \le C\, e^{-c\sqrt{k}}. \tag{3.18}$$

By Lemma 1.3, inequality (3.18) implies that

$$\bigl\|\mu_{u_1}(k) - \mu_{u_2}(k)\bigr\|_L^* \le 3C\, e^{-c\sqrt{k}}$$

which completes the proof of (2.6) with $k_0 = T_1$. Theorem 2.1 is proved.

4 Appendix: coupling

In this appendix, we present some results on the coupling in finite-dimensional spaces in the form which we learned from S. Foss. These results are well known (e.g., see [Lin, V] for Lemma 4.1 and [BF] for Lemma 4.3).

Let $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^N)$ be two measures absolutely continuous with respect to the Lebesgue measure dx:

$$\nu_{1,2}(dx) = p_{1,2}(x)\, dx.$$

We set

$$\rho := \|\nu_1 - \nu_2\|_{\mathrm{var}} = \frac{1}{2} \int_{\mathbb{R}^N} |p_1(x) - p_2(x)|\, dx \tag{4.1}$$

and assume first that $0 < \rho < 1$. Let

$$p := (1-\rho)^{-1} p_1 \wedge p_2, \quad \hat{p}_{1,2} := \rho^{-1}(p_{1,2} - p). \tag{4.2}$$

For $\rho = 1$ or 0, we define $p(x)$ and $p_{1,2}(x)$ as follows:

$$p(x) \equiv 0, \qquad \hat{p}_{1,2}(x) \equiv p_{1,2}(x) \qquad \text{if} \qquad \rho = 1, \tag{4.3}$$
$$p(x) \equiv p_1(x), \qquad \hat{p}_{1,2}(x) \equiv 0 \qquad \text{if} \qquad \rho = 0. \tag{4.4}$$

It is clear that

$$p_{1,2}(x) = (1-\rho)p(x) + \rho\hat{p}_{1,2}(x) \quad \text{almost everywhere.}$$

If (ξ_1, ξ_2) is a coupling for the measures (ν_1, ν_2), then for any $\Gamma \in \mathcal{B}(\mathbb{R}^N)$ we have

$$\begin{aligned}\nu_1(\Gamma) - \nu_2(\Gamma) &= \mathbb{E}\{\chi_\Gamma(\xi_1) - \chi_\Gamma(\xi_2)\} \\ &= \mathbb{E}\{\chi_{\{\xi_1\neq\xi_2\}}\bigl(\chi_\Gamma(\xi_1) - \chi_\Gamma(\xi_2)\bigr)\} \\ &\leq \mathbb{P}\{\xi_1 \neq \xi_2\}.\end{aligned}$$

Therefore,

$$\mathbb{P}\{\xi_1 \neq \xi_2\} \geq \rho \equiv \|\nu_1 - \nu_2\|_{\mathrm{var}}.$$

A coupling (ξ_1, ξ_2) for (ν_1, ν_2) is said to be *maximal* if

$$\mathbb{P}\{\xi_1 \neq \xi_2\} = \rho \equiv \|\nu_1 - \nu_2\|_{\mathrm{var}}.$$

Lemma 4.1. *Let $\xi_{1,2}$, ξ, and α be independent random variables such that*

$$\mathbb{P}\{\alpha = 1\} = 1-\rho, \quad \mathbb{P}\{\alpha = 0\} = \rho, \quad \mathcal{D}(\xi) = p(x)\,dx, \quad \mathcal{D}(\xi_{1,2}) = \hat{p}_{1,2}(x)\,dx. \tag{4.5}$$

Then the random variables

$$\Xi_{1,2} = \alpha\xi + (1-\alpha)\xi_{1,2} \tag{4.6}$$

form a maximal coupling for $\nu_{1,2}$.

Proof. Since ξ_1 and ξ_2 are independent and their distributions possess densities with respect to the Lebesgue measure, we have $\mathbb{P}\{\xi_1 = \xi_2\} = 0$. Taking into account the relation $\alpha(1-\alpha) \equiv 0$, we get

$$\mathcal{D}(\Xi_{1,2}) = p_{1,2}(x)\,dx = \nu_{1,2}, \quad \mathbb{P}\{\Xi_1 \neq \Xi_2\} = \mathbb{P}\{\alpha = 0\} = \rho,$$

which completes the proof. □

Let us now assume that φ is a random variable in $\mathbb{R}^N$ with the distribution $\mathcal{D}(\varphi) = q(x)\,dx$, where $q \in L^1(\mathbb{R}^N)$. Consider the following family of measures depending on a parameter $v \in \mathbb{R}^N$:

$$\nu_v(dx) = \mathcal{D}(v + \varphi) = q(x-v)\,dx.$$

Let $\rho(v_1, v_2)$ be the variation distance between ν_{v_1} and ν_{v_2}. It is clear from (4.1) that $\rho(v_1, v_2)$ is measurable with respect to $v_1, v_2 \in \mathbb{R}^{2N}$. In the construction above, let us take $\nu_{1,2} = \nu_{v_{1,2}}$. Then

$$p(x) = p(x; v_1, v_2), \quad \hat{p}_{1,2}(x) = \hat{p}_{1,2}(x; v_1, v_2).$$

Clearly, the functions $p(x; v_1, v_2)$ and $\hat{p}_{1,2}(x; v_1, v_2)$ are measurable with respect to (x, v_1, v_2). Using the above observations, we construct a coupling for (ν_{v_1}, ν_{v_2}) that is measurable with respect to (v_1, v_2, ω). Namely, we have the following result:

Theorem 4.2. *There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any pair $(v_1, v_2) \in \mathbb{R}^{2N}$ there are random variables $\Xi_{1,2} = \Xi_{1,2}(v_1, v_2; \omega)$ satisfying the following properties:*

(i) *The pair (Ξ_1, Ξ_2) is a maximal coupling for (ν_{v_1}, ν_{v_2}).*

(ii) *The map $\Xi(v_1, v_2; \omega) : \mathbb{R}^{2N} \times \Omega \to \mathbb{R}^N$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R}^{2N}) \times \mathcal{F}$.*

To prove the theorem, we shall need the lemma below:

Lemma 4.3. *Let $\mu_z \in \mathcal{P}(\mathbb{R}^N)$, $z \in \mathbb{R}^d$, be a family of probability measures such that*

$$\mu_z(dx) = p_z(x)\,dx,$$

where $p_z \in L^1(\mathbb{R}^N_x)$ for each $z \in \mathbb{R}^d$ and $p_z(x)$ is measurable as a function of $(x, z) \in \mathbb{R}^N \times \mathbb{R}^d$. Then there is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a family of random variables $\zeta_z : \Omega \to \mathbb{R}^N$ such that $\mathcal{D}(\zeta_z) = \mu_z$ for all $z \in \mathbb{R}^d$ and $\zeta_z(x)$ is measurable with respect to (z, x).

Proof. If $N = 1$, then we take $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}, dt)$, where $\mathcal{B}$ is the Borel σ-algebra and dt is the Lebesgue measure. Denoting by $F_z(\lambda)$ the distribution function of the measure μ_z, $F_z(\lambda) = \mu_z((-\infty, \lambda])$, we set

$$\zeta_z(t) = \min\{\lambda : F_z(\lambda) \geq t\}.$$

The map $(t, z) \mapsto \zeta_z(t)$ from $[0,1] \times \mathbb{R}^d$ to $\mathbb{R}$ is measurable, and the distribution function of $\mathcal{D}(\zeta_z)$ is equal to F_z. Thus, for $N = 1$ the lemma is proved.

We now assume that the required assertion is established for $N = L$ and prove it for $N = L+1$. Let us write $x \in \mathbb{R}^{L+1}$ as $x = (x', y)$, where $x' \in \mathbb{R}^L$ and $y \in \mathbb{R}$. Decomposing μ_z in terms of the conditional density (see [GS]), we write

$$\mu_z(dx) = p_z(x)\,dx = p'_z(x'\,|\,y)\,dx'\,q_z(y)\,dy. \tag{4.7}$$

Here

$$q_z(y) = \int_{\mathbb{R}^L} p_z(x', y)\,dx', \quad p'_z(x'\,|\,y) = \frac{p_z(x', y)}{q_z(y)},$$

where we set $0/0 = \infty/\infty = 0$. Applying the induction hypothesis with z replaced by (z, y), we find a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and a measurable map

$$\zeta'_z(\omega', y) : \Omega' \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^L$$

such that $\mathcal{D}\bigl(\zeta'_z(\cdot, y)\bigr) = p'_z(x'\,|\,y)\,dx'$ for each $(z, y) \in \mathbb{R}^d \times \mathbb{R}$. Applying the first step of the proof, we construct a measurable map $\xi_z(t) : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ such that $\mathcal{D}(\xi_z) = q_z(\lambda)\,d\lambda$. We now set $\Omega = \Omega' \times [0,1]$ and

$$\zeta_z(\omega', t) = \bigl(\zeta'_z(\omega', \xi_z(t)), \xi_z(t)\bigr) \in \mathbb{R}^{L+1}.$$

We have constructed a measurable map $\Omega \times \mathbb{R}^d \to \mathbb{R}^{L+1}$ such that, for any fixed $z \in \mathbb{R}^d$, its distribution is given by the right-hand side of (4.7). □

Proof of Theorem 4.2. Applying Lemma 4.2 to measures in $\mathbb{R}^N$ given by the densities p and $\hat{p}_{1,2}$, we construct probability spaces $(\Sigma_j, \mathcal{S}_j, \mathbb{P}_j)$, $j = 0, 1, 2$, and random variables $\xi^j_{(v_1,v_2)}$ on Σ_j such that

$$\mathcal{D}(\xi^0_{(v_1,v_2)}) = p(x; v_1, v_2)\, dx, \quad \mathcal{D}(\xi^j_{(v_1,v_2)}) = \hat{p}_j(x; v_1, v_2)\, dx, \quad j = 1, 2. \tag{4.8}$$

We also define a random variable $\alpha_\rho : [0,1] \to \{0,1\}$, $\rho = \rho(v_1, v_2)$, by the formula

$$\alpha_\rho(t) = \chi_{[0,1-\rho]}(t),$$

where $[0,1]$ is endowed with the Borel σ-algebra and the Lebesgue measure, and $\chi_{[0,r]}$ is the characteristic function of the interval $[0,r]$.

We now define the required probability space as the set

$$\Omega = \Sigma_0 \times \Sigma_1 \times \Sigma_2 \times [0,1]$$

with the σ-algebra and the probability of direct product. The natural extensions[3] of α_ρ and $\xi^j_{(v_1,v_2)}$, $j = 0, 1, 2$, to Ω (for which we retain the same notations) form a quadruple of independent random variables satisfying (4.8) and also the relations

$$\mathbb{P}\{\alpha_\rho = 1\} = 1 - \rho(v_1, v_2), \quad \mathbb{P}\{\alpha_\rho = 0\} = \rho(v_1, v_2).$$

A maximal coupling (Ξ_1, Ξ_2) for the measures (ν_{v_1}, ν_{v_2}) that satisfies assertion (ii) of the theorem can now be defined by formula (4.6), in which $\alpha = \alpha_\rho$, $\xi = \xi^0_{(v_1,v_2)}$, and $\xi_j = \xi^j_{(v_1,v_2)}$, $j = 1, 2$. □

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[3]For instance, the extension of α_ρ is given by $\alpha_\rho(\omega) = \alpha_\rho(t)$, where $\omega = (\omega_0, \omega_1, \omega_2, t) \in \Omega$.

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