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PERTURBATIONS OF THE HARMONIC MAP EQUATION

T. KAPPELER* and V. SCHROEDER†

Institut für Mathematik, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
**tk@math.unizh.ch*
†*vschroed@math.unizh.ch*

S. KUKSIN

Department of Mathematics, Heriot–Watt University, Edinburgh EH14 4AS, Scotland, UK
Steklov Institute of Mathematics, 8 Gubkina St., 117966 Moscow, Russia
kuksin@ma.hw.ac.uk

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We consider perturbations of the harmonic map equation in the case where the target manifold is a closed Riemannian manifold of nonpositive sectional curvature. For any semilinear and, under some extra conditions, quasilinear perturbation, the space of classical solutions within a homotopy class is proved to be compact. An important ingredient for our analysis is a new inequality for maps in a given homotopy class which can be viewed as a version of the Poincaré inequality for such maps.

Keywords:

0. Introduction

In this paper we study semilinear and quasilinear perturbations of the harmonic map equation $\tau(u) = 0$. This is an equation for maps $u : M \to M'$ between closed Riemannian manifolds, defined as the Euler–Lagrange equation for the energy functional

$$E(u) := \int_M e(u)(x)\, d\,\mathrm{vol}(x)\,,$$

where $e(u)(x)$ denotes the energy density,

$$e(u)(x) := \frac{1}{2} g^{ij}(x) \left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right\rangle .$$

Here $g^{ij}(x)$ is the inverse of the (smooth) metric tensor on M, $d\,\mathrm{vol}(x)$ the corresponding volume element, and $\langle \cdot\,, \cdot \rangle$ the scalar product in TM'. In local coordinates for M and M', the operator τ can be written as the Laplace–Beltrami operator, perturbed by terms which are quadratic in the derivatives of u (cf. Sec. 1).

Our aim is to consider perturbations of the equation $\tau(u) = 0$ by a semilinear term $F(x, u(x))$ with F being a x-dependent vector field on M', and by terms linear in the derivatives of u (and of sufficiently small size) of the form

$$L(x, u(x))\ u_*(G(x, u(x)))\,,$$

where $L(x, u(x))$ is a linear operator on $T_{u(x)}M'$, u_* the differential of u, and G a vector field on M, $G(x, u(x)) \in T_xM$. More precisely we want to study the set of solutions $u : M \to M'$ of

$$\tau(u)(x) + F(x, u(x)) + L(x, u(x))u_*(G(x, u(x))) = 0 \tag{0.1}$$

in a given homotopy class ζ of C^1-maps $u : M \to M'$,

$$u \in \zeta\,. \tag{0.2}$$

Note that (0.1) is not necessarily of variational form. The main assumption we impose is that

$$M' \text{ has nonpositive sectional curvature}\,. \tag{0.3}$$

Unless otherwise stated this assumption will be made throughout the paper. Given F, G and L of class C^k with $k \geq 2$, denote by $S_{F,G,L} \equiv S^{(k)}_{F,G,L}$ the set of all C^{k+1} solutions of (0.1). In the case $L = 0$ and $G = 0$ we simply write S_F instead of $S_{F,0,0}$. We note that by the regularity theory of elliptic equations (cf. Proposition 8.1), given F of class C^k, any C^3-solution $v \in S^{(2)}_F$ is C^{k+1}-smooth, i.e.

$$S^{(2)}_F = S^{(k)}_F\,.$$

Similarly, if $\dim M \leq 3$ and F, G and L are of class C^k, one has $S^{(2)}_{F,G,L} = S^{(k)}_{F,G,L}$ (cf. Proof of Proposition 8.2). Our aim is to prove that for any F, G, L of class C^k with $k \geq 2$, $S_F \cap \zeta$ is compact in the C^{k+1} topology and, if $\dim M \leq 3$, $S_{F,G,L} \cap \zeta$ is compact in the C^{k+1} topology for G and L of sufficiently small size,

$$\max_{\substack{x \in M \\ y \in M'}} (\|L(x, y)\| \cdot \|G(x, y)\|) \leq c_*$$

where $c_* > 0$ is a constant which only depends on ζ and the manifolds M and M'. Here $\|L(x, y)\|$ denotes the operator norm of $L(x, y) : T_yM' \to T_yM'$ and $\|G(x, y)\|$ is the norm of $G(x, y)$ in T_xM.

In fact we prove slightly stronger results. To state them we first need to introduce some more notation. Let us denote by $\mathcal{C}^k$ the space of C^k-maps from M to M', by $\mathcal{F}^{(k)}$ the vector space of x-dependent vector fields $F(x, y)$ on M' of class C^k in x and y, by $\mathcal{G}^{(k)}$ the vector space of y-dependent vector fields $G(x, y)$ on M of class C^k in x and y and by $\mathcal{L}^{(k)}$ the vector space of linear operators $L(x, y)$ on T_yM' of class C^k in x and y. Let

$$\mathcal{M}^{(k)}_\zeta := \{(u, F) \mid F \in \mathcal{F}^{(k)};\ u \in S_F \cap \zeta\}$$

considered as a subset of $\mathcal{C}^{k+1} \times \mathcal{F}^{(k)}$ and, for any $c_* > 0$

$$\mathcal{N}^{(k)}_{\zeta,c_*} := \Big\{(u, F, G, L) \mid (F, G, L) \in \mathcal{F}^{(k)} \times \mathcal{G}^{(k)} \times \mathcal{L}^{(k)};\ u \in S_{F,G,L} \cap \zeta\,;$$

$$\max_{x,y} \|L(x, y)\| \cdot \|G(x, y)\| < c_*\Big\}$$

considered as a subset of $\mathcal{C}^{k+1} \times \mathcal{F}^{(k)} \times \mathcal{G}^{(k)} \times \mathcal{L}^{(k)}$. By π we denote the natural projections

$$\pi : \mathcal{M}^{(k)}_\zeta \to \mathcal{F}^{(k)} \quad \text{or} \quad \pi : \mathcal{N}^{(k)}_\zeta \to \mathcal{F}^{(k)} \times \mathcal{G}^{(k)} \times \mathcal{L}^{(k)}\,.$$

Recall that a continuous map between topological spaces is called *proper* if the preimage of any compact set is compact.

The main results of this paper are the following ones:

Theorem 0.1. *Let M and M' be closed Riemannian manifolds with M' having nonpositive sectional curvature and ζ be a homotopy class of C^1-maps from M to M'. Then for any $k \geq 2$, the projection $\pi : \mathcal{M}^{(k)}_\zeta \to \mathcal{F}^{(k)}$ is proper.*

Theorem 0.2. *Let M and M' be closed Riemannian manifolds with M' having nonpositive sectional curvature and ζ be a homotopy class of C^1-maps from M to M'. Assume that $k \geq 2$ and $\dim M \leq 3$. Then there exists $c_* > 0$ such that $\pi : \mathcal{N}^{(k)}_{\zeta,c_*} \to \mathcal{F}^{(k)} \times \mathcal{G}^{(k)} \times \mathcal{L}^{(k)}$ is proper.*

In particular, Theorem 0.1 contains the following generalization to solutions of perturbations of the harmonic map equation of a result due to Schoen–Yau for harmonic maps [15] (cf. also Hartmann [9]) concerning the compactness of the space of harmonic maps within a homotopy class.

Corollary 0.1. *Let $F \in \mathcal{F}^{(k)}$ with $k \geq 2$. Then $S_F \cap \zeta$ is compact in the C^{k+1}-topology.*

We note that a corollary of Theorem 0.2 similar to Corollary 0.1 holds.

Simple examples show that the stated result of Corollary 0.1 no longer holds if M' is not of nonpositive sectional curvature and the statement of Theorem 0.2 is no longer true if the perturbation is not affine in the differential u_* or the part which is linear in u_* is not sufficiently small.

We remark that no efforts have been made to see if Theorems 0.1 and 0.2 hold for k smaller than two. Moreover, most likely Theorem 0.2 holds for manifolds M of arbitrary dimension.

Our results are similar in flavour to the compacity results due to Kuksin [11] for double periodic solutions of quasilinear Cauchy–Riemann equations which originated in a compacity result of Gromov [8] for J-holomorphic curves. In future work we plan to establish similar results for other important nonlinear elliptic equations.

Theorems 0.1 and 0.2 are proven below in Secs. 1–8. To simplify our exposition we have assumed that in (0.1), $L(x, y)$ is the identity map on T_yM' for any $x \in M, y \in M'$. The main ingredient of the proof is an *a priori* estimate for the energy

$E(u)$ for a solution u of (0.1) in a given homotopy class ζ: As a first step (cf. Sec. 1) we introduce canonical distance functions $N_p(u,v)(p \geq 1)$ between two C^3-maps $u, v : M \to M'$ in ζ and prove that the energy $E(u)$ can be bounded by

$$E(u) \leq \|F\|_{C^0} N_1(u,v) + \sqrt{2}\|G\|_{C^0} E(u)^{1/2} N_2(u,v) + E(v)\,.$$

Here $N_p(u,v)$ is defined by

$$N_p(u,v) := \inf\{N_p(H) \mid H \text{is a } C^1\text{-homotopy between } u \text{and } v\}$$

with

$$N_p(H) := \left(\int_M \left(\int_0^1 \left\|\frac{d}{ds}H_s(x)\right\| ds\right)^p d\,\mathrm{vol}(x)\right)^{1/p},$$

where we consider throughout this paper only continuous homotopies $H : M \times [0,1] \to M'$ so that for any $x \in M$, the path $s \mapsto H(x,s)$ is C^1-smooth.

In a second step (cf. Secs. 3–6) we show that $N_2(v,u)$ can be bounded in terms of $E(u)$ and $E(v)$:

Theorem 0.3. *Let M and M' be closed Riemannian manifolds with M' having nonpositive sectional curvature and ζ be a homotopy class of C^1-maps from M to M'. Then there exists a constant $C > 0$ such that for any $u, v \in \zeta$*

$$N_2(u,v) \leq C(E(u)^{1/2} + E(v)^{1/2} + 1)\,. \tag{0.4}$$

Our proof of Theorem 0.3 uses in an essential way that M' has nonpositive sectional curvature.

Estimate (0.4) is a new inequality which can be viewed as a version of the Poincaré inequality for maps between manifolds and is of independent interest. It has also the flavour of a quadratic isoperimetric inequality. We illustrate this by considering the case when $M = S^1$. Viewing $E(u)^{1/2}$ as a measure for the length of u, inequality (0.4) says that there exists a homotopy such that the area of the cylinder induced by the homotopy can be bounded in terms of the square of the length of its boundary. Here the area of the cylinder is measured in terms of its L_2-averaged "length" $N_2(u,v)$ and the length of its boundary by $E(u)^{1/2}+E(v)^{1/2}$. We recall that for a Hadamard space X the following isoperimetric inequality holds: given any simple, closed curve γ in X of length L, there exists a *disc* D with $\partial D = \gamma$ so that area $(D) \leq \pi L^2$.

Theorems 0.1 and 0.2 form the basis for a more detailed study of the set of solutions of (0.1) in a given homotopy class ζ which will be presented in a subsequent paper using arguments similar to the ones in [11, 12]. In the remainder of this introduction we state conjectural results of this study in the case $G = 0$, i.e. for

$$\Phi(u) := \tau(u) + F(x, u(x)) = 0 \tag{0.5}$$

$$u \in \zeta \tag{0.6}$$

and relate them to the corresponding results for the harmonic map equation.

One verifies in a straight forward way that $\mathcal{M}_\zeta^{(k)}$ is a C^1-manifold modeled by $\mathcal{F}^{(k)}$, hence the projection $\pi : \mathcal{M}_\zeta^{(k)} \mapsto \mathcal{F}^{(k)}$ is C^1-smooth. Let us denote by $\mathcal{F}_{\mathrm{reg}}^{(k)}$ the set of *regular values* of π. As π is proper, $\mathcal{F}_{\mathrm{reg}}^{(k)}$ is an open subset of $\mathcal{F}^{(k)}$. By the Sard–Smale theorem [14], it is dense in $\mathcal{F}^{(k)}$ and, for any $F \in \mathcal{F}_{\mathrm{reg}}^{(k)}$, the inverse image $\pi^{-1}(F)$ is a submanifold of $\mathcal{M}_\zeta^{(k)}$. One can show that this submanifold is of dimension 0, i.e. that $\pi^{-1}(F)$ is a discrete set. By associating appropriate signs to each element of $\pi^{-1}(F)$, one can define an "algebraic" number D_ζ,

$$D_\zeta := \sharp_{\mathrm{algebraic}}(S_F \cap \zeta)\,,$$

which is constant on $\mathcal{F}_{\mathrm{reg}}^{(k)}$.

If M' has negative sectional curvature we can compute D_ζ and obtain

$$D_\zeta = \pm\chi(S_0 \cap \zeta)$$

where S_0 is the set of harmonic maps $u : M \to M'$, $S_0 \cap \zeta$ turns out to be a manifold, and $\chi(S_0 \cap \zeta)$ denotes its Euler characteristic. Note that in the case where M' has negative sectional curvature, the energy functional $E(u)$ is Morse–Bott and , according to [9], the set of harmonic maps $S_0 \cap [u]$ in the homotopy class $[u]$ of a harmonic map u has the property that either $S_0 \cap [u] = \{u\}$ or $S_0 \cap [u]$ consists of all constant maps, or $u(M)$ is a closed geodesic γ and any other element in $S_0 \cap [u]$ is obtained by composing u with a translation along γ. The integer D_ζ is then computed in each of the three cases by considering special regular vector fields, leading to the claimed identity $D_\zeta = \pm\chi(S_0 \cap \zeta)$. In particular, it follows from this identity that for any $F \in \mathcal{F}_{\mathrm{reg}}^{(k)}$,

$$\sharp(S_F \cap \zeta) \geq |\chi(S_0 \cap \zeta)|\,.$$

Counter examples show that this inequality is sharp.

Throughout this paper, $M \equiv M^n$ and $M' \equiv {M'}^{n'}$ denote closed manifolds with fixed smooth Riemannian metrics g respectively g'. Moreover, (M', g') is supposed to have nonpositive sectional curvature.

Points in M will (often) be denoted by $x, z, \ldots$ whereas points in M' will (often) be denoted by y. For the inner product $g'(y)$ of T_yM', we will use the notation $\langle\cdot\,,\cdot\rangle_{g'(y)}$ or simply $\langle\cdot\,,\cdot\rangle$.

We follow mostly the notation established in [7]. For ease of notation we write $\nabla \equiv \nabla^{M'}$ for the Levi–Cività connection on M' and ∇^M for the one on M.

1. *A priori* Estimate for the Energy

Let $u : M \to M'$ be a given C^1-map.[a] Recall that the energy $E(u)$ of u is defined by

$$E(u) = \int_M e(u)(x)\, d\,\mathrm{vol}(x)\,,$$

[a]No efforts have been made to obtain minimal regularity assumptions for our results.

where $e(u)(x)$ is the energy density,

$$e(u)(x) = \frac{1}{2} g^{ij}(x) \left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right\rangle .$$

Let ζ be a homotopy class of C^1-maps from M to M'. By embedding M' isometrically into some Euclidean space $\mathbb{R}^N$ and using a mollifier argument one verifies that for C^3-smooth maps $u, v \in \zeta$, there exists a C^3-homotopy $H : M \times [0,1] \to M', (x,s) \mapsto H_s(x)$ between $v = H_0(\cdot)$ and $u = H_1(\cdot)$.

A homotopy H is said to be *geodesic* if

(GH1) $s \mapsto H_s(x)$ is a geodesic in M' $\forall\, x \in M$;
(GH2) the parameter $s \in [0,1]$ is proportional to arc length.

As M' is assumed to have nonpositive sectional curvature, a C^1-homotopy determines a geodesic homotopy, obtained by replacing, for any $x \in M$, the curve $s \mapsto H_s(x)$ by the unique geodesic in its homotopy class. If the homotopy is chosen to be C^3-smooth, then the corresponding geodesic homotopy H is again C^3.

For arbitrary maps $u, v \in \zeta$ and any $1 \le p < \infty$, introduce the distance function

$$N_p(v,u) := \inf\{N_p(H) \mid H \text{ is a homotopy between } v \text{ and } u\}$$

where

$$N_p(H) := \left(\int_M \left(\int_0^1 \left\| \frac{d}{ds} H_s(x) \right\| ds \right)^p d\,\mathrm{vol}(x) \right)^{1/p} .$$

By Hölder's inequality we have for any $1 \le p_1 \le p_2 < \infty$

$$N_{p_1}(H) \le N_{p_2}(H) (\mathrm{vol}\ M)^{(p_2 - p_1)/p_1 p_2} . \tag{1.1}$$

For an x-dependent C^2-vector field F on M', $F(x,y) \in T_y M'$, denote by $\|F\|_{C^0}$ the sup-norm,

$$\|F\|_{C^0} := \sup_{\substack{x \in M \\ y \in M'}} \|F(x,y)\| .$$

Below we study (0.1). To simplify our exposition we restrict ourselves to the case when the operator L is the identity,

$$\tau(u) + F(x, u(x)) + u_*(G(x, u(x)) = 0 .$$

Denote by ζ an arbitrary homotopy class of C^1-maps from M to M' and by $S_{F,G}$ the set of all C^3-solutions of the above equation. In case $G \equiv 0$, we write S_F instead of $S_{F,0}$.

Proposition 1.1. *For any* $u \in S_F \cap \zeta$,

$$E(u) \le \|F\|_{C^0}\ N_1(v,u) + E(v) .$$

Note that by the same proof, one can obtain similar estimates for the energy density $e(u)(x)$ of $u \in S_F \cap \zeta$.

To prove Proposition 1.1 we need to establish several auxilary results and some more notation.

Given a C^1-map $u : M \to M'$, a C^0-map $Y : M \to TM'$ with $Y(x) \in T_{u(x)}M'$ is said to be a vector field along u. The covariant derivative defined by the Levi–Civitá connection $\nabla \equiv \nabla^{M'}$ on M' can be extended to such vectorfields: $\nabla_X Y$ denotes a vector field along u defined for a C^0-vector field X on M and a C^1-vector field Y along u (cf. [7, Sec, 2.5]). Let $x^1, \dots, x^n$ be coordinates on an open set U of M, $H : M \times [0,1] \to M'$ a C^3-homotopy and $X_i := \frac{\partial}{\partial x^i}$.

Lemma 1.1. (i) *For* $x \in U$ *and* $0 \le s \le 1$,

$$\frac{\partial}{\partial s} e(H_s)(x) = g^{ij}(x) \left\langle \nabla_{X_i} \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle .$$

(ii) *If the homotopy* H *is geodesic, then*

$$\frac{\partial^2}{\partial s^2} e(H_s)(x) = g^{ij}(x) \left\langle \nabla_{X_i} \frac{\partial H}{\partial s}, \nabla_{X_j} \frac{\partial H}{\partial s} \right\rangle - g^{ij}(x) \left\langle R'\left(\frac{\partial H}{\partial x^i}, \frac{\partial H}{\partial s}\right) \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle ,$$

where R' *denotes the Riemannian curvature tensor on* M'.

Proof. (i) Since the Levi–Cività connection ∇ is Riemannian, one has

$$\frac{\partial}{\partial s} e(H_s)(x) = g^{ij}(x) \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial x^i}, \frac{\partial H}{\partial x^j} \right\rangle , \tag{1.2}$$

where $\nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial x^i}$ is a vectorfield along H. As the torsion of the Levi–Cività connection vanishes and

$$\left[\frac{\partial}{\partial s}, X_i\right] = 0 , \tag{1.3}$$

we conclude that

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial x^i} = \nabla_{X_i} \frac{\partial H}{\partial s} , \tag{1.4}$$

and hence (1.2) leads to (i). To show (ii), use (i) to obtain

$$\frac{\partial^2}{\partial s^2} e(H_s)(x) = g^{ij}(x) \left(\left\langle \nabla_{\frac{\partial}{\partial s}} \nabla_{X_i} \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle + \left\langle \nabla_{X_i} \frac{\partial H}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial x^j} \right\rangle \right) . \tag{1.5}$$

Applying again (1.4) one sees that

$$\left\langle \nabla_{X_i} \frac{\partial H}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial x^j} \right\rangle = \left\langle \nabla_{X_i} \frac{\partial H}{\partial s}, \nabla_{X_j} \frac{\partial H}{\partial s} \right\rangle , \tag{1.6}$$

and, by the definition of the Riemannian curvature and (1.3), one has

$$\begin{aligned} \left\langle \nabla_{\frac{\partial}{\partial s}} \nabla_{X_i} \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle &= \left\langle \nabla_{X_i} \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle \\ &\quad - \left\langle R'\left(\frac{\partial H}{\partial x^i}, \frac{\partial H}{\partial s}\right) \frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j} \right\rangle . \end{aligned} \tag{1.7}$$

Further, as $s \mapsto H_s(x)$ is a geodesic in M', one has $\nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial s}(x) = 0$ and the first term on the right side of (1.7) vanishes. Substituting (1.6)–(1.7) into (1.5) leads to (ii). □

Given any map $u : M \to M'$, $-\tau(u)$ denotes the variational derivative of the energy functional. It is a vector field along u and for any C^3-homotopy $H : M \times [0,1] \to M$ we have for any $0 \leq s \leq 1$,

$$\frac{\partial}{\partial s} E(H_s) = -\int_M \left\langle \tau(H_s)(x), \frac{\partial H}{\partial s}(x) \right\rangle d\,\mathrm{vol}(x)\,. \tag{1.8}$$

In local coordinates, $\tau(u)$ is given by

$$\tau(u)^\alpha(x) = \Delta_M u^\alpha(x) + g^{ij}(x)\Gamma'^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}\,,$$

where Δ_M denotes the Laplace–Beltrami operator on M.

Corollary 1.1. *For any $x \in M$, the function $s \mapsto e(H_s)(x)$ is convex*

$$\frac{\partial^2}{\partial s^2} e(H_s)(x) \geq 0 \quad \forall\, 0 \leq s \leq 1\,. \tag{1.9}$$

Proof. M' having nonpositive sectional curvature, one has

$$g^{ij}(x)\left\langle R'\left(\frac{\partial H}{\partial x^i}, \frac{\partial H}{\partial s}\right)\frac{\partial H}{\partial s}, \frac{\partial H}{\partial x^j}\right\rangle \leq 0$$

and (1.9) follows from Lemma 1.1(ii). □

Proof of Proposition 1.1. Let H be a geodesic C^3-homotopy between $H_0 = v$ and $H_1 = u$. By Corollary 1.1, the function $E(H_s)$ is convex

$$\frac{\partial^2 E}{\partial s^2}(H_s) = \int_M \frac{\partial^2 e}{\partial s^2}(H_s)(x)\,d\,\mathrm{vol}(x) \geq 0\,,$$

hence

$$E(u) = E(v) + \int_0^1 \frac{\partial E}{\partial s}(H_s)ds \leq E(v) + \frac{\partial E}{\partial s}(H_s)|_{s=1}\,. \tag{1.10}$$

As $u \in S_F \cap \zeta$, one has $\tau(u)(x) = -F(x, u(x))$ and, in view of (1.8)

$$\frac{\partial E}{\partial s}(H_s)|_{s=1} = \int_M \left\langle F, \left.\frac{\partial H}{\partial s}\right|_{s=1}\right\rangle d\,\mathrm{vol}(x)\,. \tag{1.11}$$

Recall that, for any $x \in M$, the geodesic $s \mapsto H_s(x)$ is parametrized proportional to arclength, and thus, for any $x \in M$,

$$\left\|\frac{\partial H(x)}{\partial s}\right\| = \ell_H(x)\,,$$

where $\ell_H(x)$ is the length of the curve $s \mapsto H_s(x)$ with $0 \le s \le 1$. It follows that

$$\left| \int_M \left\langle F, \left. \frac{\partial H}{\partial s} \right|_{s=1} \right\rangle d\,\mathrm{vol}(x) \right| \le \|F\|_{C^0} \; N_1(H) \,. \tag{1.12}$$

Combining (1.10)–(1.12) leads to

$$E(u) \le E(v) + \|F\|_{C^0} N_1(H) \,.$$

As H is an arbitrary geodesic C^3-homotopy with $H_0 = v$ and $H_1 = u$, the claimed statement follows. □

Proposition 1.1 can be generalized to hold for solutions $u \in S_{F,G} \cap \zeta$ where G is a y-dependent C^2-vector field on M, $G(x,y) \in T_x M$. Denote by $\|G\|_{C^0}$ the sup-norm

$$\|G\|_{C^0} := \sup_{\substack{x \in M \\ y \in M'}} \|G(x,y)\| \,.$$

Proposition 1.2. *For any* $u \in S_{F,G} \cap \zeta$,

$$E(u) \le \|F\|_{C^0} N_1(v,u) + \sqrt{2}\|G\|_{C^0} E(u)^{1/2} N_2(v,u) + E(v) \,.$$

Proof. For $u \in S_{F,G} \cap \zeta$ we have $\tau(u) = -F - u_* G$. Therefore

$$\begin{aligned} &\left| \int_M \left\langle F + u_* G, \left. \frac{\partial H}{\partial s} \right|_{s=1} \right\rangle d\,\mathrm{vol}(x) \right| \\ &\quad \le \|F\|_{C^0} \; N_1(H) + \int_M \sqrt{2e(u)} \|G\|_{C^0} \; \ell_H(x)\, d\,\mathrm{vol}(x) \\ &\quad \le \|F\|_{C^0} \; N_1(H) + \sqrt{2}\|G\|_{C^0} \, E(u)^{1/2} N_2(H) \,, \end{aligned}$$

hence the assertion follows in view of (1.8) and (1.10). □

2. *A priori* Estimate for the Energy Density

In this section we obtain a C^0-bound of the energy density in terms of the energy of a solution u in S_F. In the case $F = 0$, these estimates are due to Eells–Sampson [6, p. 142] and it turns out that they can be extended to the case $F \notin 0$. At the end of this section we prove an *a priori* estimate for $\|\nabla du\|_{L^2}$ for $u \in S_{F,G}$ in the case where $\dim M \le 3$.

First we recall a Bochner type formula due to [6] (cf. [4]). Throughout this section it is convenient to work with Riemannian normal coordinates in M. Choose $x_0 \in M$. Then, at x_0, Riemannian normal coordinates $x^1, \ldots, x^n$, defined in a chart $U \subseteq M$ containing x_0, have the following properties

$$g_{ij}(x_0) = \delta_{ij} \,; \quad \frac{\partial g_{ij}}{\partial x_k}(x_0) = 0 \quad (\forall\, i,j,k) \,. \tag{2.1}$$

In particular, the Christoffel symbols Γ_{ij}^k of the Levi–Cività connection vanish at x_0. The following result is well known (cf. [10]).

Lemma 2.1. *Let $u : M \to M'$ be a C^3-map. Then, $\Delta_M e(u)(x)$ at $x = x_0$ takes the form*

$$\begin{aligned}\Delta_M e(u)(x_0) &= \sum_i \left\langle \nabla_{X_i}\tau(u),\, \frac{\partial u}{\partial x^i}\right\rangle + \sum_{i,k}\left\langle \nabla_{X_k}\frac{\partial u}{\partial x^i},\, \nabla_{X_k}\frac{\partial u}{\partial x^i}\right\rangle \\ &\quad + \sum_{i,j}\mathrm{Ric}_M(X_i, X_j)\left\langle \frac{\partial u}{\partial x^i},\, \frac{\partial u}{\partial x^j}\right\rangle \\ &\quad - \sum_{i,k}\left\langle R'\left(\frac{\partial u}{\partial x^i},\, \frac{\partial u}{\partial x^k}\right)\frac{\partial u}{\partial x^k},\, \frac{\partial u}{\partial x^i}\right\rangle, \end{aligned} \tag{2.2}$$

where Ric_M denotes the Ricci curvature on M, $R' \equiv R_{M'}$ the Riemannian curvature on M', and $X_i = \frac{\partial}{\partial x^i} (1 \le i \le n)$.

Given two Hilbert spaces V, V' and a linear map $S : V \to V'$, denote by $\|S\|_{HS}$ the Hilbert–Schmidt norm of S. With respect to an orthonormal basis (f_i) of V, the norm $\|S\|_{HS}$ can be computed as

$$\|S\|_{HS} = \left(\sum_i \langle Sf_i,\, Sf_i\rangle_{V'}\right)^{1/2}.$$

Given a C^3-map u, let $\nabla.\tau(u)(x)$ be the linear map obtained from the vector field $\tau(u)$ along u

$$\nabla.\tau(u)(x) : T_xM \to T_{u(x)}M', \quad X \mapsto \nabla_X\tau(u)$$

and denote its Hilbert–Schmidt norm by $\|\nabla.\tau(u)(x)\|_{HS}$. If $(e_i)_{1\le i\le n}$ is any orthonormal basis of T_xM, then

$$\|\nabla.\tau(u)(x)\|_{HS} = \left(\sum_i \|\nabla_{e_i}\tau(u)(x)\|^2\right)^{1/2}. \tag{2.3}$$

Identifying the space of linear operators from T_xM to $T_{u(x)}M'$ with the Hilbert space $T_x^*M \otimes T_{u(x)}M'$ (with norm given by the Hilbert–Schmidt norm), the differential $u_*(x) \equiv d_xu : T_xM \to T_{u(x)}M'$ can be viewed as a map

$$du : x \mapsto d_xu \in T_x^*M \otimes T_{u(x)}M'.$$

The covariant derivative of du then defines a map

$$\nabla.d_xu : T_xM \to T_x^*M \otimes T_{u(x)}M', \quad X \mapsto \nabla_X d_xu.$$

Given any orthonormal basis $(e_i)_{1\le i\le n}$ of T_xM, the Hilbert–Schmidt norm $\|\nabla.d_xu\|_{HS}$ can be computed as

$$\|\nabla.d_xu\|_{HS} = \left(\sum_{i,j} \|\nabla_{e_i} d_xuX_j\|^2\right)^{1/2} . \tag{2.4}$$

Further, $\|\mathrm{Ric}_M(x)\|_{HS}$ denotes the Hilbert–Schmidt norm of the Ricci curvature $\mathrm{Ric}_M(x)$ viewed as a linear map $\mathrm{Ric}_M(x) : T_xM \to T_x^*M$.

Corollary 2.1. *For any C^3-map $u : M \to M'$ and any $x \in M$,*

$$\begin{aligned} &-\Delta_M e(u)(x) + \|\nabla.d_xu\|_{HS}^2 \\ &\quad\le \sqrt{2}\|\nabla.\tau(u)(x)\|_{HS}\sqrt{e(u)(x)} + 2\|\mathrm{Ric}_M(x)\|_{HS} \cdot e(u)(x) . \end{aligned} \tag{2.5}$$

Proof. To verify this inequality at an arbitrary point $x_0 \in M$, choose Riemannian normal coordinates at x_0. Using that M' has nonpositive sectional curvature, one obtains from Lemma 2.1 and (2.4)

$$\begin{aligned} &-\Delta_M e(u)(x_0) + \|\nabla.d_{x_0}u\|_{HS}^2 \\ &\quad\le -\sum_i \left\langle \nabla_{X_i}\tau(u), \frac{\partial u}{\partial x^i}\right\rangle - \sum_{i,j} \mathrm{Ric}_M(X_i, X_j)\left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\right\rangle . \end{aligned} \tag{2.6}$$

Further, as $X_i = \frac{\partial}{\partial x^i} (1 \le i \le n)$ is an orthonormal base of $T_{x_o}M$, one has

$$2e(u)(x_0) = \sum_i \left\|\frac{\partial u}{\partial x^i}\right\|^2 ,$$

hence in view of (2.3)

$$\begin{aligned} \left|\sum_i \left\langle \nabla_{X_i}\tau(u), \frac{\partial u}{\partial x^i}\right\rangle\right| &\le \left(\sum_i \|\nabla_{X_i}\tau(u)\|^2\right)^{1/2} \left(\sum_i \left\|\frac{\partial u}{\partial x^i}\right\|^2\right)^{1/2} \\ &\le \|\nabla.\tau(u)(x_0)\|_{HS}\sqrt{2e(u)(x_0)} , \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\left|\sum_{i,j} \mathrm{Ric}_M(X_i, X_j)\left\langle \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\right\rangle\right| \\ &\quad\le \left(\sum_{i,j} |\mathrm{Ric}_M(X_i, X_j)|^2\right)^{1/2} \left(\sum_{i,j} \left\|\frac{\partial u}{\partial x^i}\right\|^2 \left\|\frac{\partial u}{\partial x^j}\right\|^2\right)^{1/2} \\ &\quad\le \|\mathrm{Ric}_M(x_0)\|_{HS} \cdot 2e(u)(x_0) . \end{aligned} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6) leads to the claimed estimate at $x = x_0$. □

Given finite atlases $(U_j)_{j\in I}$ of M and $(U'_i)_{i\in I'}$ of M', define the norm $\|F\|_{C^1}$ of the x-dependent vectorfield F on M' as follows:

$$\|F\|_{C^1} := \sup_{\substack{j\in I\\ i\in I'}} \sup_{\substack{x\in U_j\\ y\in U'_i}} \sup_{\substack{1\le k\le n\\ 1\le\alpha\le n'}} \left(\|F\|, \left\|\frac{\partial F}{\partial x^k}\right\|, \left\|\frac{\partial F}{\partial y^\alpha}\right\|\right).$$

Proposition 2.1. *There exists a constant $C_1 \ge 1$ so that for any $u \in S_F$ and any $x \in M$,*

$$-\Delta_M e(u)(x) \le C_1(1 + \|F\|_{C^1})(1 + e(u)(x)).$$

Proof. We apply the inequality (2.5). To estimate $\|\nabla.\tau(u)(x)\|_{HS}$ in (2.5) notice that for $u \in S_F$, we have $\tau(u)(x) = -F(x, u(x))$. In local coordinates of M and M' introduced above,

$$\nabla_{X_i} F^\alpha = \frac{\partial F^\alpha}{\partial x^i} + \frac{\partial F^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i} + \Gamma'^\alpha_{\beta\gamma} F^\beta \frac{\partial u^\gamma}{\partial x^i}$$

and thus

$$\|\nabla_{X_i} F(x, u(x))\|_{u(x)} \le C\|F\|_{C^1}\left(1 + \left\|\frac{\partial u}{\partial x^i}(x)\right\|\right),$$

where $C > 0$ depends on M and M' and the atlases $(U_j)_{j\in I}, (U'_i)_{i\in I'}$. Hence we obtain for $C > 0$ sufficiently large

$$\|\nabla.F(x, u(x))\|_{HS} \le C\|F\|_{C^1}(1 + \sqrt{e(u)(x)}),$$

which in turn leads to

$$\|\nabla.F(x, u(x))\|_{HS}\sqrt{e(u)(x)} \le C\|F\|_{C^1}(1 + e(u)(x)). \tag{2.9}$$

As M is compact, $\|\mathrm{Ric}_M(x)\|_{HS}$ is bounded and the claimed estimate follows. □

Theorem 2.1. *There exists a constant $C_2 > 0$ so that for any $u \in S_F$,*

$$e(u)(x) \le C_2(1 + \|F\|_{C^1})^n(E(u) + 1) \quad \forall\, x \in M.$$

In the case $F = 0$ (i.e. in the case of harmonic maps), this theorem is due to Eells–Sampson ([6, p. 142]). Their proof can be generalized to the situation at hand.

Let us first make a few preliminary considerations. Following Eells–Sampson ([6, p. 141]) denote by $P(x, z)$ the kernel of a parametrix of the Laplacian $-\Delta_M$, defined as

$$P(x,z) := \begin{cases} \kappa_n(\varphi(d^2(x,z)))^{-n/2+1} & n \ge 3, \\ -\dfrac{1}{2\pi}\log\sqrt{\varphi(d^2(x,z))} & n = 2, \\ \dfrac{1}{2}(\varphi(d^2(x,z))^{1/2} + 1 & n = 1, \end{cases}$$

where $1/\kappa_n = (n-2)\mathrm{vol}(S^{n-1}), d(x,z)$ denotes the distance between x and z and for $\lambda \geq 0, \varphi(\lambda)$ is a non decreasing C^∞ function satisfying

$$\varphi(\lambda) = \begin{cases} \lambda & 0 \leq \lambda \leq a\,, \\ 2a_0 & \lambda \geq 2a\,, \end{cases}$$

with $0 < a < a_0 < 1/2$ chosen in such a way that $d^2(x,z)$ is C^∞-smooth for $d^2(x,z) < 3a$.

Then, for any $x \in M, P(x,\cdot)$ is C^∞ on $M\backslash\{x\}$ and

$$P(x,z) \geq C_0 \quad \forall\, z \in M\backslash\{x\}\,, \tag{2.10}$$

where $C_0 > 0$ can be chosen independently of $x \in M$. Denoting by B the parametrix of $-\Delta_M$ given by the kernel $P(x,z)$, one has

$$B(-\Delta_M) = \mathrm{Id} + S\,,$$

where S is a smoothing operator with kernel $Q(x,y)$. Therefore

$$\mathrm{Id} = B(-\Delta_M) - S\,.$$

Applied to a function f in $L^2(M)$, this identity reads

$$f(x) = \int_M (P(x,z)(-\Delta_M)f(z) - Q(x,z)f(z))\, d\,\mathrm{vol}(z)\,. \tag{2.11}$$

As the kernel $Q(x,z)$ is smooth and thus bounded and since $P(x,z) \geq C_0 > 0$ (cf. (2.10)), there exists a constant $C_3 > 0$ such that for any $x \in M$,

$$|Q(x,z)| \leq C_3 P(x,z) \quad \forall\, z \in M\backslash\{x\}\,.$$

Thus (2.11) leads to the inequality

$$f(x) \leq \int_M (P(x,z)(-\Delta_M)f(z) + C_3 P(x,z)|f(z)|)\, d\,\mathrm{vol}(z)\,. \tag{2.12}$$

Proof of Theorem 2.1. The inequality (2.12) is applied to $f(x) = e(u)(x) + 1$ with $u \in S_F$. By Proposition 2.1, as $P(x,z) \geq 0$,

$$P(x,z)(-\Delta_M)f(z) \leq P(x,z)C_1(1 + \|F\|_{C^1})f(z)\,. \tag{2.13}$$

As $f(x) \geq 1$ we then obtain from (2.12) the inequality

$$f(x) \leq C \int_M P(x,z)f(z)\, d\,\mathrm{vol}(z) \tag{2.14}$$

with $C := C_1(1 + \|F\|_{C^1}) + C_3$.

Estimate (2.14) can be iterated to get

$$f(x) \leq C^k \int_M P_k(x,z)f(z)\, d\,\mathrm{vol}(z)\,, \tag{2.15}$$

where P_k is defined inductively by $P_1 := P$ and, for $k \geq 2$,

$$P_k(x,z) = \int_M P_{k-1}(x,z')P(z',z)\, d\,\mathrm{vol}(z')\,.$$

$P_k(x,z)$ is the kernel of a parametrix for $(-\Delta_M)^k$ and thus continuous for $k > n/2$. Hence, as M is compact, $P_k(x,z)$ is bounded on M for $k = n$,

$$P_n(x,z) \le C'$$

and (2.15) leads to

$$\begin{aligned} e(u)(x) + 1 &\le C'C^n \int_M (e(u)(z) + 1)\, d\,\mathrm{vol}(z) \\ &\le C'C^n (E(u) + \mathrm{vol}(M))\,. \end{aligned}$$

This establishes the claimed estimate. □

The Bochner type formula stated in Lemma 2.1 can also be used to obtain an *a priori* estimate for a solution u in $S_{F,G}$ when $G \ne 0$. For this purpose define the norm $\|G\|_{C^1}$, similarly as $\|F\|_{C^1}$, as follows: Given finite atlases $(U_j)_{j\in I}$ (of M) and $(U_i')_{i\in I'}$ (of M'), define the norm $\|G\|_{C^1}$ as follows

$$\|G\|_{C^1} := \sup_{\substack{k,\alpha \\ x,y}} \left\{ \|G\|, \left\| \frac{\partial G}{\partial x^k} \right\|, \left\| \frac{\partial G}{\partial y^\alpha} \right\| \right\},$$

where the supremum is taken over any k, α and any $(x,y) \in U_j \times U_i'$ with $(j,i) \in I \times I'$.

Proposition 2.2. *There exists a constant $C_1' \ge 1$ so that for any $u \in S_{F,G}$ and any $x \in M$,*

$$\begin{aligned} &-\Delta_M e(u)(x) + \frac{1}{2}\|\nabla . d_x u\|_{HS}^2 \\ &\qquad \le C_1'(1 + \|F\|_{C^1} + \|G\|_{C^1}^2)(1 + (e(u)(x))^{3/2})\,. \end{aligned} \tag{2.16}$$

Proof. By (2.5),

$$\begin{aligned} &-\Delta_M e(u)(x) + \|\nabla . d_x u\|_{HS}^2 \\ &\qquad \le \|\nabla . \tau(u)(x)\|_{HS} \sqrt{e(u)(x)} + \|\mathrm{Ric}_M(x)\|_{HS} e(u)(x)\,. \end{aligned} \tag{2.17}$$

For $u \in S_{F,G}$,

$$\|\nabla . \tau(u)(x)\|_{HS} \le \|\nabla . F(x, u(x))\|_{HS} + \|\nabla . u_* G(x, u(x))\|_{HS}\,. \tag{2.18}$$

By (2.9)

$$\|\nabla . F(x, u(x))\|_{HS} \sqrt{e(u)(x)} \le C\|F\|_{C^1}(1 + e(u)(x))\,. \tag{2.19}$$

To estimate $\|\nabla . u_* G(x, u(x))\|_{HS}$, write in local coordinates of a chart U_j of the atlas chosen above

$$\begin{aligned} \nabla_{X_i} u_* G(x, u(x)) &= \nabla_{X_i} \left(\frac{\partial u}{\partial x^\ell} G^\ell(x, u(x)) \right) \\ &= G^\ell(x, u(x)) \left(\nabla_{X_i} \frac{\partial u}{\partial x^\ell} \right) + \frac{\partial u}{\partial x^\ell} \frac{\partial G^\ell}{\partial x^i} + \frac{\partial u}{\partial x^\ell} \frac{\partial G^\ell}{\partial u^\beta} \frac{\partial u^\beta}{\partial x^i}\,. \end{aligned}$$

Thus

$$\begin{aligned}\|\nabla . u_* G(x, u(x))\|_{HS} \leq C\|G\|_{C^1} &\left(\sum_\ell \left\|\nabla . \frac{\partial u(x)}{\partial x^\ell}\right\|_{HS}^2\right)^{1/2} \\ &+ C\|G\|_{C^1}\left(\sum_\ell \left\|\frac{\partial u(x)}{\partial x^\ell}\right\|^2\right)^{1/2} \\ &+ C\|G\|_{C^1}\left(\sum_\ell \left\|\frac{\partial u(x)}{\partial x^\ell}\right\|^2\right).\end{aligned}$$

Hence, using $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, one gets

$$\begin{aligned}&\|\nabla . u_* G(x, u(x))\|_{HS}\sqrt{e(u)(x)} \\ &\quad \leq \frac{1}{2}\|\nabla . d_x u\|_{HS}^2 + \frac{1}{2}C^2\|G\|_{C^1}^2 e(u)(x) \\ &\qquad + C\|G\|_{C^1} e(u)(x) + C\|G\|_{C^1}(e(u)(x))^{3/2}\,. \end{aligned} \tag{2.20}$$

Combining (2.17)–(2.20) leads to the claimed estimate. □

If $\dim M \leq 3$, estimate (2.16) can be used as follows: integrate (2.16) to get

$$\begin{aligned}&\frac{1}{2}\int_M \|\nabla . du\|_{HS}^2\, d\,\mathrm{vol}(x) \\ &\quad \leq C_1'(1 + \|F\|_{C^1} + \|G\|_{C^1}^2)\int_M (1 + e(u)(x)^{3/2})\, d\,\mathrm{vol}(x)\,. \end{aligned} \tag{2.21}$$

By the Gagliardo–Nirenberg inequality and $\dim M \leq 3$,

$$\|du\|_{L^3} \leq C\|du\|_{L^2}^{1/2}\left(\int_M \|\nabla . du\|_{HS}^2\, d\,\mathrm{vol}(x)\right)^{1/4} + C\|du\|_{L^2}\,.$$

Thus, using $(a+b)^3 \leq 2^3(a^3 + b^3)$, one gets for $C > 0$ sufficiently large

$$\int_M e(u)(x)^{3/2}\, d\,\mathrm{vol}(x) \leq CE(u)^{3/4}\left(\int_M \|\nabla . du\|_{HS}^2\, d\,\mathrm{vol}(x)\right)^{3/4} + CE(u)^{3/2}\,.$$

Hence, there exists $C > 0$ so that for any $\varepsilon > 0$

$$\int_M e(u)(x)^{3/2} d\,\mathrm{vol}(x) \leq \frac{CE(u)^{3/2}}{\varepsilon^2} + \varepsilon^2\left(\int_M \|\nabla . du\|_{HS}^2\, d\,\mathrm{vol}(x)\right)^{3/2}$$

and we deduce from (2.21) the following

Theorem 2.2. *Assume that* $\dim M \leq 3$. *Then there exists a constant* $C_4 > 0$ *so that for any* $u \in S_{F,G}$

$$\int_M \|\nabla . du\|_{HS}^2\, d\,\mathrm{vol}(x) \leq C_4(1 + \|F\|_{C^1} + \|G\|_{C^1}^2)(E(u)^{3/2} + 1)\,. \tag{2.22}$$

3. Estimates for the Diameter of a Homotopy Class

In this section we start with the proof of the estimate of the distance $N_2(v,u)$ between maps $v,u : M \to M'$ in a given homotopy class of C^1-maps from M to M' stated in Theorem 0.3. In this section, we do not assume that M' has nonpositive sectional curvature but only require that M' has no conjugate points. In that case, there is a unique geodesic in any homotopy class of curves on M' connecting two given points. Note that a geodesic homotopy H from v to u is completely determined by the geodesic $c : [0,1] \to M', s \mapsto H_s(z)$ from $v(z)$ to $u(z)$ for an arbitrary given point $z \in M$. For any other point $x \in M$, the curve $s \mapsto H_s(x)$ is the unique geodesic in the homotopy class of the curve obtained by composing the three curves $v\gamma_{z,x}^{-1}, c$, and $u\gamma_{z,x}$ where $\gamma_{z,x}$ is some C^1-curve in M from z to x and $\gamma_{z,x}^{-1}$ denotes its inverse. In order to estimate the length of the geodesic curve $s \mapsto H_s(x)$ it is therefore sufficient to estimate $L(u\gamma_{z,x}), L(v\gamma_{z,x})$, and $L(c)$ where L denotes the length functional.

In this section we make a choice of C^1-paths $\gamma_{z,x}$ for an open, nonempty set of pairs of points (z,x) of the form $B_r(x_0) \times M, B_r(x_0)$ being the open ball with center x_0 and radius $r = \bar{r}/2$ where $\bar{r}$ is the convexity radius of M (cf. Appendix A) and give for any z in an open set of $B_r(x_0)$ and any $u \in \zeta$ an L^2-estimate for $L(u\gamma_{z,x})$ in terms of the energy of u. This leads naturally to a hypothesis (Z) defined below which implies an estimate of $N_2(v,u)$ in terms of the energies of v and u as claimed in Theorem 0.3. In Sec. 6 we show that (Z) holds in our situation.

Choose an arbitrary base point $x_0 \in M$. Given any unit vector $v \in T_{x_0}M$, let $t \mapsto c_v(t)$ be the geodesic with $c_v(0) = x_0$ and $\dot{c}_v(0) := \frac{d}{dt}|_{t=0} c_v(t) = v$, defined for $0 \le t \le m_v$ where $[0, m_v]$ is the interval of maximal length so that c_v is a minimal geodesic. Denote by $\varphi_{c_v} : (-r, m_v + r) \times B_r(0) \to M$ the corresponding Fermi coordinates (cf. Appendix A) and by $V_v \subseteq M$ the image of φ_{c_v}. Here $B_r(0)$ denotes the open ball in $\mathbb{R}^{n-1}$ with center 0 and radius r. As any two points in M can be joined by a geodesic of minimal length and M is compact, there are finitely many vectors $v_1, \ldots, v_k$ so that $V_j := V_{v_j} (1 \le j \le k)$ is an open cover of M. Note that $B_r(x_0) \subseteq V_j$ for any j. For $z \in B_r(x_0)$ and $x \in V_j$, let $\gamma_{z,x}^j : [0,1] \to M$ be the path, parametrized proportionally to arclength, such that $\gamma_{z,x}^j$ corresponds to a straight line in the Fermi coordinates defined by $\varphi_j := \varphi_{v_j}$ and for any $z \in B_r(x_0)$ and $x \in M$ define $\gamma_{z,x} := \gamma_{z,x}^j$ where $j := \min\{i \mid x \in V_i\}$. Note that $\gamma_{z,x}$ depends continuously on (z,x) in $B_r(x_0) \times (V_j \setminus \bigcup_{i=1}^{j-1} V_i)$.

Proposition 3.1. *Let* $0 < \lambda < 1$ *and* $x_0 \in M$ *be given. Then there exists a constant* $C_5 > 0$ *with the following property*: *for any* C^1*-map* $u : M \to M'$ *into an arbitrary Riemannian manifold* M', *there exists an open subset* $A_u \subseteq B_r(x_0)$ *with* $\mathrm{vol}(A_u) \ge \lambda \,\mathrm{vol}(B_r(x_0))$ *such that for any* $z \in A_u$,

$$\left(\int_M L^2(u\gamma_{z,x}) d\,\mathrm{vol}(x) \right)^{1/2} \le C_5 E(u)^{1/2}.$$

Note that Proposition 3.1 is valid for any target manifold M'.

Before proving the proposition above we introduce hypothesis (Z) and show (cf. Proposition 3.2) that our desired estimate of $N_2(v,u)$ is an immediate consequence of (Z) and Proposition 3.1.

Let ζ be a homotopy class of C^1-maps $v : M \to M'$ where M' is an arbitrary closed Riemannian manifold and $x_0 \in M$ a given base point.

We say that (ζ, x_0) satisfies hypothesis (Z) if the following holds:

(Z) There exist constants $C_6 > 0$ and $0 < \mu < 1$, so that for any C^1-maps $v, u \in \zeta$, there exists an open subset $A_{vu} \subseteq B_r(x_0)$ with $\mathrm{vol}(A_{vu}) > \mu\,\mathrm{vol}(B_r(x_0))$ so that for any $z \in A_{vu}$ there exists a geodesic homotopy $H \equiv H^z$ from v to u satisfying

$$\ell_H(z) \le C_6(E(u)^{1/2} + E(v)^{1/2} + 1)\,.$$

We will show that hypothesis (Z) holds for any (ζ, x_0) if M' has nonpositive sectional curvature (cf. Sec. 6). First we want to prove the following application of Proposition 3.1:

Proposition 3.2. *Assume (ζ, x_0) satisfies (Z). Then there exists $C_7 > 0$, so that for any C^1-maps $u, v \in \zeta$,*

$$N_2(v,u) \le C_7(E(u)^{1/2} + E(v)^{1/2} + 1)\,.$$

Proof. Let $0 < \mu < 1$ and $A_{vu} \subseteq B_r(x_0)$ be the open set with $\mathrm{vol}(A_{vu}) > \mu \cdot \mathrm{vol}\,B_r(x_0)$ given by (Z). Choose λ with

$$1 > \lambda > 1 - \frac{\mu}{2} \ge \frac{1}{2}\,. \tag{3.1}$$

By Proposition 3.1 there exists $C_5 > 0$ so that there are open subsets A_v, A_u of $B_r(x_0)$ with

$$\mathrm{vol}(A_v),\ \mathrm{vol}(A_u) > \lambda\,\mathrm{vol}(B_r(x_0))$$

with the property that for any $z \in A_v$,

$$\left(\int_M L^2(v\gamma_{z,x})d\,\mathrm{vol}(x)\right)^{1/2} \le C_5 E(v)^{1/2} \tag{3.2}$$

and for any $z \in A_u$,

$$\left(\int_M L^2(u\gamma_{z,x})d\,\mathrm{vol}(x)\right)^{1/2} \le C_5 E(u)^{1/2}\,. \tag{3.3}$$

By a simple volume computation one has

$$\begin{aligned}\mathrm{vol}(A_u \cap A_v) &\ge \mathrm{vol}(A_u) - \mathrm{vol}(B_r(x_0)\backslash A_v)\\ &\ge (2\lambda - 1)\mathrm{vol}(B_r(x_0))\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{vol}(A_u \cap A_v \cap A_{vu}) &\geq \mathrm{vol}(A_u \cap A_v) - \mathrm{vol}(B_r(x_0)\backslash A_{vu})\\ &\geq (2\lambda + \mu - 2)\mathrm{vol}(B_r(x_0)) > 0\end{aligned}$$

by the choice of (3.1) of λ. Therefore $A_{vu} \cap A_u \cap A_v$ is an open nonempty set. In view of hypothesis (Z) there exists for any $z \in A_{vu} \cap A_u \cap A_v$ a geodesic homotopy $H \equiv H^z$ from v to u so that

$$\ell_H(z) \leq C_6(E(u)^{1/2} + E(v)^{1/2} + 1)\,.$$

As H is a geodesic homotopy one has for any $x \in M$,

$$L(s \mapsto H_s(x)) \leq L(v\gamma_{z,v}) + \ell_H(z) + L(u\gamma_{z,x})\,,$$

we then conclude from (3.2)–(3.3) that

$$\begin{aligned}&\int_M L^2(s \mapsto H_s(x))d\,\mathrm{vol}(x)\\ &\quad\leq \int_M (3L^2(v\gamma_{z,x}) + 3\ell_H^2(z) + 3L^2(u\gamma_{z,x}))d\,\mathrm{vol}(x)\\ &\quad\leq 3C_5^2(E(v) + E(u)) + 3C_6^2(E(u)^{1/2} + E(v)^{1/2} + 1)^2 \cdot \mathrm{vol}(M)\\ &\quad\leq C_7^2(E(u)^{1/2} + E(v)^{1/2} + 1)^2\,,\end{aligned}$$

where

$$C_7 := (3(C_5^2 + C_6^2 \cdot \mathrm{vol}(M)))^{1/2}\,. \qquad \square$$

The remainder of this section is devoted to the proof of Proposition 3.1. First we need to establish an auxilary result. Let $U \subseteq \mathbb{R}^n$ be open and convex and $\varphi : U \to M$ a coordinate chart with $\|\varphi_*\|_{C^0} < \infty$ where $\|\varphi_*\|_{C^0}$ denotes the C^0-norm of the differential φ_* of φ for φ in the space of C^1-maps from U into M. For $a, b \in U$, the straight line $t \mapsto (1-t)a + tb (0 \leq t \leq 1)$ between a and b is in U and hence the following curve $\gamma_{a,b}$ well defined,

$$[0,1] \to M, t \mapsto \gamma_{a,b}(t) := \varphi((1-t)a + tb)\,.$$

Lemma 3.1. *For any C^1-map $u : M \to M'$,*

$$\int_{U\times U} L^2(u\gamma_{a,b})d\,\mathrm{vol}(a)d\,\mathrm{vol}(b) \leq C_8 E(u)\,,$$

where $C_8 := (2C)^n\|\varphi_*\|_{C^0}^2(\mathrm{diam}(U))^2\mathrm{vol}(M)$, $d\,\mathrm{vol}(a)$ and $d\,\mathrm{vol}(b)$ denote the volume elements of U with respect to the pull back g_φ of the metric g on M by φ and $C \geq 1$ is a bound for the metric tensor g_φ and its inverse on U.

Proof. For any C^1-map $u : M \to M'$ one has by Cauchy's inequality

$$\begin{aligned}
&\int_{U\times U} L^2(u\gamma_{a,b})d\,\mathrm{vol}(a)d\,\mathrm{vol}(b)\\
&\quad= \int_{U\times U}\left(\int_0^1 \|u_*(\gamma_{a,b}(t))\cdot\varphi_*((1-t)a+tb)\cdot(b-a)\|dt\right)^2 d\,\mathrm{vol}(a)d\,\mathrm{vol}(b)\\
&\quad\le (\mathrm{diam}(U))^2\|\varphi_*\|^2_{C^0}\int_{U\times U}\int_0^1 2e(u(\gamma_{a,b}(t)))dt\,d\,\mathrm{vol}(a)d\,\mathrm{vol}(b)\,.
\end{aligned}$$

Since $\gamma_{a,b}(t) = \gamma_{b,a}(1-t)$, we have

$$\begin{aligned}
&\int_{U\times U}\int_0^1 e(u(\gamma_{a,b}(t)))dt\,d\,\mathrm{vol}(a)d\,\mathrm{vol}(b)\\
&\quad= 2\int_{U\times U}\left(\int_{1/2}^1 e(u(\gamma_{a,b}(t)))dt\right)d\,\mathrm{vol}(a)d\,\mathrm{vol}(b)\,.
\end{aligned}$$

For a and $\frac{1}{2}\le t\le 1$ fixed, consider the transformation

$$\psi_{a,t} : U \to U_{a,t}\,,\quad b\mapsto \bar{b} := (1-t)a+tb$$

where $U_{a,t} := \{(1-t)a+tb \mid b\in U\}$. As U is convex, $U_{a,t}\subseteq U$ and

$$\begin{aligned}
\int_U e(u(\varphi((1-t)a+tb)))d\,\mathrm{vol}(b) &= \int_{U_{a,t}} e(u(\varphi(\bar{b})))(\psi_{a,t}^{-1})^*(d\,\mathrm{vol})(\bar{b})\\
&\le (C/t)^n E(u) \le (2C)^n E(u)\,,
\end{aligned}$$

where $(\psi_{a,t}^{-1})^*(d\,\mathrm{vol})(\bar{b})$ denotes the pull back of the volume element $d\,\mathrm{vol}(b)$ by $\psi_{a,t}^{-1}$ and $C\ge 1$ is a bound for the metric tensor g_φ (the pullback of g by φ) and its inverse on the coordinate chart U,

$$\|g_\varphi\|_{C^0}\,,\quad \|g_\varphi^{-1}\|_{C^0}\le C\,.$$

Thus

$$\begin{aligned}
\int_{U\times U}\int_0^1 e(u(\gamma_{a,b}(t)))dtd\,\mathrm{vol}(a)d\,\mathrm{vol}(b) &\le 2\cdot(2C)^n E(u)\int_U\int_{1/2}^1 dtd\,\mathrm{vol}(a)\\
&\le (2C)^n E(u)\mathrm{vol}(M)\,,
\end{aligned}$$

and the claimed inequality follows. □

Proof of Proposition 3.1. For any $1\le j\le k$, let

$$\varphi_j : (-r, m_j+r)\times B_r(0)\to V_j\subseteq M$$

be the Fermi coordinate chart $\varphi_j\equiv\varphi_{c_{v_j}}$ with $m_j\equiv m_{v_j}$ as introduced above.

Since $B_r(x_0)\subseteq V_j\ \forall\, 1\le j\le k$, Lemma 3.1 implies that there exists a constant $C>0$ independent of u such that $\forall\, 1\le j\le k$

$$\int_{B_r(x_0)\times V_j} L^2(u\gamma^j_{z,x})d\,\mathrm{vol}(z)d\,\mathrm{vol}(x)\le CE(u)\,.$$

For any $1 \leq j \leq k$ and $n \in \mathbb{N}$, let

$$A_n^j := \left\{ z \in B_r(x_0) \middle| \int_{V_j \setminus \bigcup_1^{j-1} V_i} L^2(u\gamma_{z,x}^j) d\,\mathrm{vol}(x) \geq nE(u) \right\}.$$

Notice that A_n^j is closed and

$$CE(u) \geq \int_{A_n^j \times V_j} L^2(u\gamma_{z,x}^j) d\,\mathrm{vol}(z) d\,\mathrm{vol}(x) \geq n\,\mathrm{vol}(A_n^j) E(u)\,.$$

Hence for any given $0 < \lambda < 1$, there exists $m \in \mathbb{N}$, so that

$$\mathrm{vol}(A_m^j) < \frac{1-\lambda}{k} \mathrm{vol}(B_r(x_0))\,, \quad \forall\, 1 \leq j \leq k\,.$$

Then $A := B_r(x_0) \backslash (\bigcup_{j=1}^k A_m^j)$ is open and satisfies

$$\mathrm{vol}(A) \geq \mathrm{vol}(B_r(x_0)) - (1-\lambda)\mathrm{vol}(B_r(x_0)) \geq \lambda \mathrm{vol}(B_r(x_0))$$

and

$$\int_{V_j \setminus \bigcup_1^{j-1} V_i} L^2(u\gamma_{z,x}^j) d\,\mathrm{vol}(x) \leq mE(u) \quad \forall\, z \in A\,.$$

As a consequence one gets for any $z \in A$,

$$\int_M L^2(u\gamma_{z,x}) d\,\mathrm{vol}(x) \leq \sum_{j=1}^k \int_{V_j \setminus \bigcup_1^{j-1} V_i} L^2(u\gamma_{z,x}^j) d\,\mathrm{vol}(x) \leq kmE(u)\,,$$

which implies Proposition 3.1 with $A_u := A$ and $C_5 := \sqrt{km}$. □

4. Closed Manifolds of Nonpositive Curvature

In this section we collect some material about closed Riemannian manifolds M' of nonpositive sectional curvature (cf. [2]) which we need to prove property (Z).

The manifold M' can be represented as X'/Γ, where $X' \to M'$ is the universal covering of M' and $\Gamma \cong \pi_1(M')$ is the discrete and cocompact group of isometries on X'. The universal covering is a Hadamard manifold, i.e. a complete and contractible Riemannian manifold of nonpositive sectional curvature. In particular, $\pi_k(M') = \{1\}$ for any $k \geq 2$. Any isometry $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ acts freely on X', i.e. has no fixed points.

To an isometry $\gamma \in \Gamma$ we associate the displacement function $d_\gamma : X' \to [0,\infty), d_\gamma(x) := d(x, \gamma x)$ where here d is the distance function on X'. The function d_γ is convex, i.e. $d_\gamma(x(t))$ is convex in t for any geodesic, parametrized proportional to arclength, and 2-Lipschitz continuous, i.e.

$$|d_\gamma(x) - d_\gamma(z)| \leq 2d(x,z)\ (x, z \in X')\,.$$

Thus the set

$$\mathrm{MIN}(\gamma) := \{x \in X' \mid d_\gamma(x) = \inf\, d_\gamma\}$$

is a closed, convex subset of X'. If $\alpha \in \Gamma$ commutes with γ, then d_γ is α invariant since

$$d_\gamma(\alpha x) = d(\alpha x, \gamma\alpha x) = d(\alpha x, \alpha\gamma x) = d(x, \gamma x) = d_\gamma(x)\,.$$

Hence the centralizer $Z(\gamma)$ leaves $\mathrm{MIN}(\gamma)$ invariant. Corollary 4.1 below shows that $\mathrm{MIN}(\gamma)$ is not empty and $Z(\gamma)$ operates with compact quotient on $\mathrm{MIN}(\gamma)$.

More generally, consider finitely many elements $\gamma_1, \ldots, \gamma_m$ in Γ and introduce the function $f : X' \to [0,\infty), f := \sum_{i=1}^m d_{\gamma_i}$. As each d_{γ_i} is convex as well and 2-Lipschitz continuous it follows that f is convex and $2m$-Lipschitz continuous. Let

$$Z(\gamma_1, \ldots, \gamma_m) := \{\alpha \in \Gamma \mid \alpha\gamma_i = \gamma_i\alpha \text{ for all } i = 1, \ldots, m\}$$

be the centralizer of $\gamma_1, \ldots, \gamma_m$. Then f is $Z(\gamma_1, \ldots, \gamma_m)$ invariant and hence induces a convex $2m$-Lipschitz continuous function $\bar{f}$ on $X'/_{Z(\gamma_1,\ldots,\gamma_m)}$.

Lemma 4.1. *$\bar{f}$ is a proper function on $X'/_{Z(\gamma_1,\ldots,\gamma_m)}$.*

Proof. (cf. [5]) Let $a > \inf(f)$ be given. We have to show that the sublevel $\{\bar{f} \le a\}$ is compact. Let $(\overline{x_i})_{i\ge1}$ be a sequence of points in $\{\bar{f} \le a\}$ and let $x_i \in X'$ be such that $\pi(x_i) = \overline{x_i}$ where $\pi : X' \to X'/_{Z(\gamma_1,\ldots,\gamma_m)}$ is the canonical projection. Since Γ operates cocompactly on X', there are elements $\alpha_i \in \Gamma$ such that $\alpha_i \cdot x_i \in D$ where $D \subset X'$ is a fixed compact fundamental domain for Γ. Hence there is a subsequence of $\alpha_i \cdot x_i$ converging to $x_0 \in D$. For any $j \in \{1, \ldots, m\}$ and any $i \in \mathbb{N}$ we have

$$d_{\alpha_i\gamma_j\alpha_i^{-1}}(\alpha_i \cdot x_i) = d_{\gamma_j}(x_i) \le f(x_i) = \bar{f}(\overline{x_i}) \le a$$

and thus, using the 2-Lipschitz continuity of d_γ, one obtains that for any $1 \le j \le m$

$$\begin{aligned} d_{\alpha_i\gamma_j\alpha_i^{-1}}(x_0) &\le |d_{\alpha_i\gamma_j\alpha_i^{-1}}(x_0) - d_{\alpha_i\gamma_j\alpha_i^{-1}}(\alpha_i \cdot x_i)| + d_{\alpha_i\gamma_j\alpha_i^{-1}}(\alpha_i \cdot x_i) \\ &\le 2d(x_0, \alpha_i \cdot x_i) + a \le 1 + a \end{aligned}$$

for i large enough. It then follows from the discreteness of the group Γ, that there are only finitely many elements in the set $\{\alpha_i\gamma_j\alpha_i^{-1}x_0 \mid i \in \mathbb{N}\}$ and thus, as Γ acts freely on X', the set $\{\alpha_i\gamma_j\alpha_i^{-1} \mid i \in \mathbb{N}\}$ is finite. Hence passing to a subsequence of $(x_i)_{i\ge1}$ if necessary, we can assume that there are elements $\delta_1, \ldots, \delta_m \in \Gamma$ such that

$$\alpha_i\gamma_j\alpha_i^{-1} = \delta_j \quad \text{for all } i \in \mathbb{N}$$

and hence $\alpha_i\gamma_j\alpha_i^{-1} = \alpha_k\gamma_j\alpha_k^{-1}$ for all $i, k \in \mathbb{N}, j \in \{1, \ldots, m\}$ which implies $\alpha_k^{-1}\alpha_i \in Z(\gamma_1, \ldots, \gamma_k)$. Thus, with $\bar{d}$ denoting the distance function on $X'/Z(\gamma_1, \ldots, \gamma_m)$ we have

$$\bar{d}(\overline{x_i}, \overline{x_k}) \le d(\alpha_k^{-1}\alpha_i \cdot x_i, x_k) = d(\alpha_i \cdot x_i, \alpha_k \cdot x_k) \le \mathrm{diam}(D)$$

and hence there exists a subsequence of $(\bar{x}_i)_{i\ge1}$ which converges. $\square$

As $\bar{f}$ is continuous, the above lemma implies that $\bar{f}$ and hence f assumes its infimum. Therefore $\mathrm{MIN}(f) := \{x \in X' \mid f(x) = \inf(f)\}$ has the following properties:

Corollary 4.1. MIN(f) *is nonempty and closed. Further it is convex and* MIN$(f)/Z(\gamma_1,\dots,\gamma_m)$ *is compact.*

The fact that MIN(f) is closed and convex allows to introduce the metric projection $\pi_{\mathrm{MIN}(f)} : X' \to \mathrm{MIN}(f)$ defined by

$$d(x,\pi_{\mathrm{MIN}(f)}(x)) = \min\{d(x,y) \mid y \in \mathrm{MIN}(f)\}\,.$$

Lemma 4.2. *There is a constant $\varrho > 0$ such that for any $x \in X'$ with $d(x, \mathrm{MIN}(f)) \geq 1$*

$$f(x) \geq \varrho\; d(x,\pi_{\mathrm{MIN}(f)}(x))\,.$$

Hence for any $x \in X'$,

$$d(x,\pi_{\mathrm{MIN}(f)}(x)) \leq \frac{1}{\varrho} f(x) + 1\,.$$

Proof. Let $a_0 := \inf(f)$. If $a_0 = 0$, then every $x \in \mathrm{MIN}(f)$ is a common fixed point of the isometries γ_j $(1 \leq j \leq m)$. As $\gamma = \mathrm{id}$ is the only element in Γ with a fixed point $m = 1$ and $\gamma_1 = \mathrm{id}$, hence $\mathrm{MIN}(f) = X'$ and the claimed statement clearly holds. Now consider the case $a_0 > 0$. Given any $x \in X' \backslash \mathrm{MIN}(f)$ denote by $c : [0,\infty) \to X'$ the geodesic ray starting from $c(0) = \pi_{\mathrm{MIN}(f)}(x)$ and passing through x, parametrized by arclength. As c is a geodesic we then have $d(c(t), \mathrm{MIN}(f)) = t$. Modulo the operation of the group $Z(\gamma_1,\dots,\gamma_m)$ the set of these rays is compact by Lemma 4.1. Thus there exists a constant $a_1 > a_0$ with $f(c(1)) \geq a_1$ for all such rays. As f is convex and $c(t)$ is a geodesic, $f \circ c$ is convex and hence for any $t \geq 1$

$$f(c(t)) \geq a_0 + t(a_1 - a_0) = a_0 + (a_1 - a_0)d(c(t), \mathrm{MIN}(f))\,. \qquad \square$$

5. Short Homotopies Between Graphs

Let $M' = X'/\Gamma$ be a compact Riemannian manifold with nonpositive sectional curvature. In this section we show that two homotopic maps from a graph into M' can be joined by a short homotopy.

Let G be a finite metric graph, i.e. a finite graph, where every edge has some positive length (cf. [3, I.1.9]). We also assume for simplicity that G has no terminals, i.e. that every edge is incident to at least two edges. A map $u : G \to M'$ is called differentiable, if the restriction of u to every edge is differentiable. In an obvious way one defines the length $L(u)$ of a differentiable map $u : G \to M'$ by summing up the lengths of the restriction of u to any of the edges of G.

Theorem 5.1. *Let ζ be a homotopy class of C^1-maps from G to M'. Then there is a constant $C_9 > 0$ such that for any $u, v \in \zeta$ there exists a homotopy $H : G \times [0,1] \to M'$ from v to u, such that for any point $z \in G$,*

$$\ell_H(z) \leq C_9(L(u) + L(v) + 1)\,.$$

The constant C_9 does not depend on the choice of the metric on G.

For the convenience of the reader we first outline the proof in the special case where G is a circle with a given metric, the homotopy class ζ is nontrivial, and the sectional curvature of M' is *strictly* negative. Denote by $T \subset \mathbb{R}$ an interval of the same length as G and by p^+, p^- its two endpoints. Let $\varphi : T \to G$ be the canonical map identifying p^+ and p^-, choose $t_0 = p^+ \in T$ as a basepoint and let $H^G : G \times [0,1] \to M'$ be a given homotopy from v to u. Consider the map $H^T : T \times [0,1] \to M'$ defined by $H^T(t,s) = H^G(\varphi(t), s)$ which can be lifted to a map $\bar{H}^T : T \times [0,1] \to X'$. Since $H^T(p^+, s) = H^T(p^-, s)$ for any $0 \le s \le 1$ and Γ acts discretely, there is a deck transformation $\gamma \in \Gamma$ so that $\gamma \bar{H}^T(p^+, s) = \bar{H}^T(p^-, s)$ for any $0 \le s \le 1$. Furthermore γ is not the trivial element, since ζ is nontrivial. As, by assumption, the curvature is strictly negative, γ translates a unique geodesic which coincides as a set with $\mathrm{MIN}(\gamma)$ (cf. [2, Sec. 6]). Note that by Lemma 4.2

$$d(\bar{H}^T(t_0, 1), \mathrm{MIN}(\gamma)) \le \frac{1}{\varrho} d_\gamma(\bar{H}^T(t_0, 1)) + 1 \le \frac{1}{\varrho} L(u) + 1\,, \tag{5.1}$$

and, similarly,

$$d(\bar{H}^T(t_0, 0), \mathrm{MIN}(\gamma)) \le \frac{1}{\varrho} L(v) + 1\,. \tag{5.2}$$

As above let $\pi_{\mathrm{MIN}(\gamma)} : X' \to \mathrm{MIN}(\gamma)$ be the metric projection onto $\mathrm{MIN}(\gamma)$. The cyclic group $\langle \gamma \rangle$ operates with compact quotient on the geodesic $\mathrm{MIN}(\gamma)$ (cf. Corollary 4.1). Thus there exists $m \in \mathbb{Z}$ such that for $\alpha = \gamma^m$

$$d(\alpha \pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0, 1), \pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0, 0)) \le C\,, \tag{5.3}$$

where C is the minimum of d_γ which depends only on the homotopy class of ζ. With the help of α we define a new homotopy $\hat{H}^T : T \times [0,1] \to X'$ by $\hat{H}^T(t,s) = c_t(s)$ where for any $t \in T$, $c_t : [0,1] \to X'$ is the geodesic from $\bar{H}^T(t,0)$ to $\alpha \bar{H}^T(t,1)$. Since α commutes with γ it is easily checked that for any $0 \le s \le 1$, $\gamma \hat{H}^T(p^+, s) = \hat{H}^T(p^-, s)$. Hence $\hat{H}^T$ induces a homotopy H from v to u. By (5.1), (5.2) and (5.3), we estimate

$$\begin{aligned} d(\hat{H}^T(t_0,1), \hat{H}^T(t_0,0)) &\le d(\hat{H}^T(t_0,1), \alpha \pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0,1)) \\ &\quad + d(\alpha \pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0,1), \pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0,0)) \\ &\quad + d(\pi_{\mathrm{MIN}(\gamma)} \bar{H}^T(t_0,0), \bar{H}^T(t_0,0)) \\ &\le \left(\frac{1}{\varrho} L(v) + 1\right) + C + \left(\frac{1}{\varrho} L(u) + 1\right). \end{aligned} \tag{5.4}$$

Let $H : G \times [0,1] \to M'$ be defined by composing $\hat{H}^T$ with the projection $X' \to M'$. Then the above inequality implies

$$\ell_H(z_0) \le C'(L(u) + L(v) + 1)$$

for $z_0 = \varphi(t_0)$ and C' is given by $C' := C + 2 + 1/\varrho$ which is independent of the metric on G. Using the triangle inequality we obtain

$$\ell_H(z) \le (C' + 1)(L(u) + L(v) + 1)$$

for an arbitrary point $z \in G$. The argument in the general case is essentially the same. The interval T has to be replaced by a suitable metric tree, $\mathrm{MIN}(\gamma)$ by $\mathrm{MIN}(f)$ for a suitable function $f = \sum_{i=1}^{m} d_{\gamma_i}$ and the group $\langle\gamma\rangle$ by the centralizer $Z(\gamma_1, \ldots \gamma_m)$.

To be precise, let G be an arbitrary finite metric graph assumed to be connected. Recall that the Euler characteristic $\chi(G)$ of a graph G is defined by

$$\chi(G) := \sharp \text{ vertices} - \sharp \text{ edges}\,.$$

By a straight forward inductive argument one sees that $\chi(G) \leq 1$ as G is connected. Further we recall that a connected graph is said to be a tree if it does not contain any loop. Again by a straight forward inductive argument one verifies that a connected graph G is a tree if and only if $\chi(G) = 1$. Let $T_1 \subset G$ be a maximal connected subgraph of G such that T_1 is in addition a tree. T_1 is obtained from G by removing m edges, denoted by $e_1, \ldots, e_m$. It then follows from the above characterization of trees that $m = 1 - \chi(G)$. Let $p_1, \ldots, p_m$ be the midpoints of $e_1, \ldots, e_m$ and consider the abstract metric tree T which is obtained from G by removing the points p_j and then completing the metric tree. A point p_i then gives rise to two points, p_i^+ and p_i^-, in T. Thus T is a metric tree whose terminals are the vertices $p_i^+, p_i^-, i = 1, \ldots, m$, and G is obtained from T by identifying p_i^+ with p_i^- for any $1 \leq i \leq m$. Let us denote by $\varphi : T \to G$ this identification map. We choose a base point t_0 in the interior of the tree T. For every terminal p_i^+, p_i^- of T there is a unique path $\sigma_i^+, \sigma_i^- : [0, 1] \to T$ parametrized proportionally to arclength from t_0 to p_i^+, p_i^-. By our assumption there exists a homotopy $H^G : G \times [0, 1] \to M'$ with $H_0^G = v$ and $H_1^G = u$. Let $H^T : T \times [0, 1] \to M'$ be the map

$$H^T(t, s) = H^G(\varphi(t), s)\,.$$

Since T is contractible, we can lift H^T to a map

$$\bar{H}^T : T \times [0, 1] \to X'$$

where $\pi : X' \to M'$ is the universal covering of M'. Since $H^T(p_i^+, s) = H^T(p_i^-, s)$ for any $i = 1, \ldots, m$ and $s \in [0, 1]$, the points $\bar{H}^T(p_i^+, s)$ and $\bar{H}^T(p_i^-, s)$ are identified by deck transformations. Hence there are isometries $\gamma_1, \ldots, \gamma_m$ in the deck transformation group Γ so that for any $0 \leq s \leq 1$,

$$\gamma_i(\bar{H}^T(p_i^+, s)) = \bar{H}^T(p_i^-, s)\,.$$

Introduce

$$L(\sigma_i^\pm, s) := \mathrm{length}(\tau \mapsto \bar{H}^T(\sigma_i^\pm(\tau), s))\,,$$

and note that

$$L(\sigma_i^\pm, 0) \leq L(v)\,,$$

and

$$L(\sigma_i^\pm, 1) \leq L(u)\,.$$

It then follows by the triangle inequality that

$$\begin{aligned} d_{\gamma_i}(\bar{H}^T(p_i^+,0)) &= d(\bar{H}^T(p_i^+,0),\gamma_i\bar{H}^T(p_i^+,0)) \\ &\le d(\bar{H}^T(p_i^+,0),\bar{H}^T(t_0,0)) + d(\bar{H}^T(t_0,0),\bar{H}^T(p_i^-,0)) \\ &\le L(\sigma_i^+,0) + L(\sigma_i^-,0) \le 2L(v)\,, \end{aligned}$$

and hence, for any $1 \le j \le m$,

$$\begin{aligned} d_{\gamma_j}(\bar{H}^T(p_i^+,0)) &\le d(\bar{H}^T(p_i^+,0),\bar{H}^T(p_j^+,0)) + d_{\gamma_j}(\bar{H}^T(p_j^+,0)) \\ &\quad + d(\gamma_j\bar{H}^T(p_j^+,0),\gamma_j\bar{H}^T(p_i^+,0)) \\ &\le 2\cdot d(\bar{H}^T(p_i^+,0),\bar{H}^T(p_j^+,0)) + 2L(v) \le 4L(v)\,. \end{aligned}$$

Summing up these inequalities then leads to

$$f(\bar{H}^T(p_i^+,0)) \le 4mL(v)\,,$$

where

$$f = \sum_{j=1}^{m} d_{\gamma_j}\,.$$

Since f is $2m$-Lipschitz continuous we have

$$|f(\bar{H}^T(t_0,0)) - f(\bar{H}^T(p_i^+,0))| \le 2m\cdot d(\bar{H}^T(t_0),\bar{H}^T(p_i^+,0)) \le 2m\cdot L(\sigma_i^+,0)\,,$$

and hence

$$f(\bar{H}^T(t_0,0)) \le 2m\cdot L(\sigma_i^+,0) + f(\bar{H}^T(p_i^+,0)) \le 6m\cdot L(v)\,.$$

Similarly one obtains

$$f(\bar{H}^T(t_0,1)) \le 6m\cdot L(u)\,.$$

Together with Lemma 4.2 one then gets

$$\begin{aligned} d(\bar{H}^T(t_0,0),\pi_{\mathrm{MIN}(f)}\bar{H}^T(t_0,0)) &\le \frac{1}{\varrho}f(\bar{H}^T(t_0,0)) + 1 \\ &\le C\cdot(L(v)+1)\,, \end{aligned} \tag{5.5}$$

where $C := 1 + 6m/\varrho$. Similarly,

$$d(\bar{H}^T(t_0,1),\pi_{\mathrm{MIN}(f)}\bar{H}^T(t_0,1)) \le C\cdot(L(u)+1)\,. \tag{5.6}$$

By Corollary 4.1, $Z(\gamma_1,\ldots,\gamma_m)$ operates with compact quotient on $\mathrm{MIN}(f)$ and thus for some constant $C > 0$

$$\mathrm{diam}(\mathrm{MIN}(f)/Z(\gamma_1,\ldots,\gamma_m)) \le C\,.$$

Hence there is an element $\alpha \in Z(\gamma_1, \dots, \gamma_m)$ with

$$d(\alpha\pi_{\mathrm{MIN}(f)}\bar{H}^T(t_0, 1), \pi_{\mathrm{MIN}(f)}\bar{H}^T(t_0, 0)) \le C\,. \tag{5.7}$$

Combining the inequalities (5.5)–(5.7), we obtain for some constant $C > 0$, independent of the metric on G,

$$d(\bar{H}^T(t_0, 0), \alpha\bar{H}^T(t_0, 1)) \le C \cdot (L(u) + L(v) + 1)\,. \tag{5.8}$$

We now define a new map

$$\hat{H}^T : T \times [0, 1] \to X'\,, \quad (t, s) \mapsto \hat{H}^T\ (t, s) := c_t(s)\,,$$

where $c_t : [0, 1] \to X'$ is the geodesic from $\bar{H}^T(t, 0)$ to $\alpha\bar{H}^T(t, 1)$ with $\alpha \in Z(\gamma_1, \dots, \gamma_m)$ given as above. We claim that

$$\gamma_i\hat{H}^T(p_i^+, s) = \hat{H}^T(p_i^-, s) \quad (i \in \{1, \dots, m\},\ s \in [0, 1])\,. \tag{5.9}$$

To see it, note that as $c_{p_i^+}$ is the geodesic from $\bar{H}^T(p_i^+, 0)$ to $\alpha\bar{H}^T(p_i^+, 1)$, it follows that $\gamma_i c_{p_i^+}$ is the geodesic from $\gamma_i\bar{H}^T(p_i^+, 0) = \bar{H}^T(p_i^-, 0)$ to

$$\gamma_i\alpha\bar{H}^T(p_i^+, 1) = \alpha\gamma_i\bar{H}^T(p_i^+, 1) = \alpha\bar{H}^T(p_i^-, 1)$$

as α is an element in the centralizer $Z(\gamma_1, \dots, \gamma_m)$. Thus $\gamma_i c_{p_i^+}$ is the geodesic $c_{p_i^-}$ and hence (5.9) established. By (5.9), $\hat{H}^T$ induces a homotopy $H : G \times [0, 1] \to M'$. For $z_0 := \varphi(t_0)$ we have by (5.8)

$$\ell_H(z_0) = d(\bar{H}^T(t_0, 0), \alpha\bar{H}^T(t_0, 1)) \le C(L(u) + L(v) + 1)\,.$$

By the triangle inequality we then obtain again that

$$\ell_H(z) \le (C + 1)(L(u) + L(v) + 1)$$

for any $z \in G$ with $C_9 := C + 1$ independent of the metric on G.

6. Validity of (Z)

In this section we prove that condition (Z), introduced in Sec. 3, always holds in the case where M' has nonpositive sectional curvature. In fact, we prove that a stronger version of (Z) holds.

Denote by $\bar{r} \equiv \bar{r}(M)$ the convexity radius of M, let $r := \bar{r}/2$ and denote by $B_r(x_0) \subseteq M$ the ball of radius r centered at a point x_0 in M.

Theorem 6.1. *Let $0 < \lambda < 1$, a homotopy class ζ of C^1-maps $v : M \to M'$, and $x_0 \in M$ be given and assume that M' has nonpositive sectional curvature. Then there exists a constant $C_{10} > 0$ so that for any C^1-maps $u, v : M \to M'$ in ζ there is an open subset $A_{uv} \subseteq B_r(x_0)$ with $\mathrm{vol}(A_{uv}) \ge \lambda B_r(x_0)$ so that for any $z \in A_{uv}$ one can find a homotopy H from v to u satisfying*

$$\ell_H(z) \le C_{10}(E(u)^{1/2} + E(v)^{1/2} + 1)\,.$$

The idea of the proof is to construct homotopies between maps v and u by using short homotopies between graphs (cf. Sec. 5). Throughout this section we assume that M' has nonpositive sectional curvature.

First we need to establish an auxilary result. Let G be a finite graph and $\psi : G \to M$ a contiuous map such that ψ_* is surjective where here $\psi_* : \pi_1(G, g) \to \pi_1(M, \psi(g))$ is the induced map on the fundamental groups with $g \in G$ an arbitrary base point. As above let $v, u : M \to M'$ be homotopic C^1-maps.

Lemma 6.1. *Let $H : \psi(G) \times [0,1] \to M'$ be a continuous map such that for every $p \in \psi(G), s \mapsto H(p,s)$ is a geodesic from $v(p)$ to $u(p)$. Then H can be extended in a unique way to a geodesic homotopy, again denoted by H, on all of M,*

$$H : M \times [0,1] \to M' .$$

Proof. Choose $z := \psi(g)$ as base point of M and let c be the geodesic $s \mapsto H(z,s)$ from $v(z)$ to $u(z)$. Introduce a map $\tilde{H} : M \times [0,1] \to M'$ as follows: choose for any point $x \in M$ a continuous path $\gamma_{z,x}$ in M from z to x. The path $\gamma_{z,x}$ is arbitrary except in the case where $x \in \psi(G)$. For such a point, $\gamma_{z,x}$ is chosen so that it lies entirely in $\psi(G)$. Let $s \mapsto \tilde{H}(x,s) \equiv \tilde{H}_s(x)$ be the unique geodesic, parametrized proportional to arclength, from $v(x)$ to $u(x)$ in the homotopy class of the curve obtained by composing $v(\gamma_{z,x}^{-1}), c$, and then $u(\gamma_{z,x})$. Clearly $\tilde{H}$ is an extension of H since for any point $x \in \psi(G)$, $s \mapsto H_s(x)$ is a geodesic from $v(x)$ to $u(x)$ which is homotopic to the composition $v(\gamma_{z,x}^{-1}), c$, and $u(\gamma_{z,x})$. For simplicity, denote the extension $\tilde{H}$ again by H. To show that H is a C^1-map we first prove that H does not depend on the choice of the family of pathes $(\gamma_{z,x})_{x \in M}$. Indeed, assume that $(\breve{\gamma}_{z,x})_{x \in M}$ is any other choice of such a family and denote by $\breve{H} : M \times [0,1] \to M'$ the corresponding extension of $H|_{\psi(G)\times[0,1]}$. It is to show that for any point $x \in M, s \mapsto H_s(x)$ and $s \mapsto \breve{H}_s(x)$ are homotopic. As both are geodesics parametrized proportional to arclength it then folllows that $H_s(x) = \breve{H}_s(x)$ for any $0 \le s \le 1$. To see that $s \mapsto H_s(x)$ and $s \mapsto \breve{H}_s(x)$ are homotopic note that $\breve{\gamma}_{z,x}^{-1} \circ \gamma_{z,x}$ is a closed curve in M passing through z. By assumption $\psi_* : \pi_1(G, g) \to \pi_1(M, z)$ is onto, hence $\breve{\gamma}_{z,x}^{-1} \circ \gamma_{z,x}$ is homotopic to a curve which is entirely contained in $\psi(G)$. It follows that there are continuous maps $\gamma^{\pm} : [0,1] \times [0,1] \to M, (t,\tau) \mapsto \gamma_\tau^{\pm}(t) \equiv \gamma^{\pm}(t,\tau)$ with the following properties

(i) $\gamma_1^+ = \gamma_{z,x};\ \gamma_1^- = \breve{\gamma}_{z,x}$,
(ii) $\gamma_\tau^+(0) = \gamma_\tau^-(0) = z,\ \forall\, 0 \le \tau \le 1$,
(iii) $x(\tau) := \gamma_\tau^+(1) = \gamma_\tau^-(1),\ \forall\, 0 \le \tau \le 1$,
(iv) γ_0^+ and γ_0^- are entirely contained in $\psi(G)$.

Denote by $s \mapsto H_s^{\pm}(x(\tau))$ the geodesic from $v(x(\tau))$ to $u(x(\tau))$ in the homotopy class of the composition $v(\gamma_\tau^{\pm})^{-1} \circ c \circ u\gamma_\tau^{\pm}$. Hence for $\tau = 0, s \mapsto H_s^+(x(\tau))$ and $s \mapsto H_s^-(x(\tau))$ are in the same homotopy class, thus by continuity, they are homotopic for any $0 \le \tau \le 1$. In particular ($\tau = 1$), the pathes $s \mapsto H_s^+(x(1)) = H_s(x)$ and $s \mapsto H_s^-(x(1)) = H_s(x)$ are homotopic as claimed. The regularity of $H : M \times [0,1] \to$

M' is now easily established: Given any $x_0 \in M$, choose in a (sufficiently small) neighborhood U of x_0 a family of pathes $(\gamma_{zx})_{x\in U}$ depending smoothly on x. By construction, H is a C^1-map on $U \times [0,1]$ as both u and v are C^1-maps. The uniqueness of the extension follows from the observation made at the beginning of Sec. 3 that the homotopy H is completely determined by the geodesic c. □

Proof of Theorem 6.1. We want to apply Lemma 6.1. For the graph G and the continuous map $\psi : G \to M$ we choose a parametrization of a graph in M as follows: Let $x_0 \in M$ be our chosen base point and $\{x_0, x_1, \dots, x_m\}$ be a maximal system of points in M such that $d(x_i, x_j) \geq \bar{r}$ for all $i \neq j$ where $\bar{r}$ is the convexity radius of M. The graph $G' \subseteq M$ is then defined as follows: The vertices of G' are the points $x_0, \dots, x_m$. We join two different vertices x_i and x_j by an edge iff $d(x_i, x_j) < 2\bar{r}$ where the edge, denoted by c_{ij}, is given by the unique minimal geodesic between x_i and x_j. We first claim that the canonical map $\pi_1(G', x_0) \to \pi_1(M, x_0)$, induced by the inclusion $G' \hookrightarrow M$, is onto. In order to prove this we have to show that a continuous curve $\eta : [0,1] \to M$ with $\eta(0) = \eta(1) = x_0$ is homotopic to a curve in G'. As the set $\{x_0, \dots, x_m\}$ is maximal with the property that $d(x_i, x_j) \geq \bar{r}\ \forall\, i \neq j$ and M is compact there exists $\varepsilon > 0$ so that

$$\sup_{z\in M} d(z, \{x_0, \dots, x_m\}) \leq \bar{r} - 2\varepsilon\,.$$

Subdivide the interval $[0,1]$ by $0 = t_0 \leq \cdots \leq t_k = 1$ such that for any $i \in \{0, \dots, k\}, \eta([t_i, t_{i+1}]) \subset B_\varepsilon(\eta(t_i))$. Choose for any $i \in \{0, \dots, k-1\}$ a point $x_{\nu(i)}$ with

$$d(\eta(t_i), x_{\nu(i)}) = d(\eta(t_i), \{x_0, \dots, x_m\})\,.$$

As $\eta(0) = \eta(1) = x_0$ one has $x_{\nu(k)} = x_{\nu(0)} = x_0$. Then $d(\eta(t_i), x_{\nu(i)}) \leq \bar{r} - \varepsilon$ by the maximality of the set $\{x_0, \dots, x_m\}$. Let β_i be the unique (in M) minimal geodesic from $\eta(t_i)$ to $x_{\nu(i)}$ and define $\eta_i := \eta|_{[t_i, t_{i+1}]}$ for $i = 0, \dots, k-1$. Then η can be considered as the composition of curves, $\eta = \eta_0, \dots, \eta_{k-1}$, the latter being clearly homotopic to

$$\eta_0 \beta_1 \beta_1^{-1} \eta_1 \beta_2 \cdots \beta_{k-1}^{-1} \eta_{k-1}\,.$$

Note that the segments $\beta_i^{-1} \eta_i \beta_{i+1}$ are curves connecting $x_{\nu(i)}$ and $x_{\nu(i+1)}$ which are contained in $B_{\bar{r}}(\eta(t_i))$. Thus this segment is either homotopic (with fixed boundary points) to the constant map $x_{\nu(i)}$ in the case $x_{\nu(i)} = x_{\nu(i+1)}$ or to the geodesic from $x_{\nu(i)}$ to $x_{\nu(i+1)}$ and the claim follows.

We use (i,j) as a mark for $c_{i,j}$ and introduce

$$\mathcal{J} := \{(i,j) \mid (i,j) \text{ is a mark for an edge of } G'\}$$

which is a subset of $\{(i,j) \mid 0 \leq i, j \leq m\}$. The graph G is now defined as a graph on the standard simplex in $\mathbb{R}^m$ whose vertices $v_0, \dots, v_m$ are the vertices of the simplex and whose edges are the edges e_{ij} of the simplex with $(i,j) \in \mathcal{J}$. Clearly e_{ij} denotes the straight line joining v_i with v_j. The metric is chosen to be the uniform metric normalized in such a way that each edge has length 1. The map

$\psi : G \to M$ is now defined by setting $\psi(v_j) := x_j (0 \le j \le m)$ and by defining $\psi|_{e_{ij}}$ as the parametrization of the minimal geodesic c_{ij} from x_i to x_j which is proportional to arclength. We need to introduce variations of the map ψ of the following type: For a point $z := (z_0, \ldots, z_m) \in B_r(x_0) \times \cdots \times B_r(x_m)$ define a map

$$\psi_Z : G \to M$$

by setting $\psi_Z(v_j) := z_j$ for any $0 \le j \le m$ and defining $\psi_Z|_{e_{ij}}$ as being the curve $\gamma_{z_i z_j}$. Let us recall from Appendix A how the curves $\gamma_{ij} \equiv \gamma_{z_i z_j}$ are defined. Denote by $\varphi_{ij} \equiv \varphi_{c_{ij}}$ the Fermi coordinates associated to the minimal geodesic c_{ij} (cf. Appendix A), $\varphi_{ij} : (-r, m_{ij} + r) \times B_r(0) \to M$ where m_{ij} is the length of c_{ij} and $B_r(0)$ is the ball in R^{n-1} with center 0 and radius r. Let a_i and a_j be the points

$$a_i := \varphi_{ij}^{-1}(z_i)\,, \quad a_j := \varphi_{ij}^{-1}(z_j)\,.$$

Then γ_{ij} is the image of the straight line $(1-t)a_i + t a_j (0 \le t \le 1)$ by the Fermi coordinates φ_{ij}.

We can identify G via the map ψ_Z with a metric graph $G_Z \subseteq M$, with metric induced from M, for any z in $B(x_0, \ldots, x_m) = B_r(x_0) \times \cdots \times B_r(x_m)$. For any $z \in B(x_0, \ldots, x_m)$, the restrictions v_Z and u_Z of v and u to G_Z are homotopic maps from G_Z into M'. By Theorem 5.1, there is a constant $C_9 > 0$, independent of z, such that there is a geodesic homotopy

$$H : G_Z \times [0,1] \to M'$$

from v_Z to u_Z with the property that

$$\ell_H(z_0) \le C_9(L(u_Z) + L(v_Z) + 1)\,. \tag{6.1}$$

By Lemma 6.1, the homotopy H can be extended to a geodesic homotopy $H : M \times [0,1] \to M'$. The claimed statement then follows immediately from (6.1) and the proposition below. ☐

The following result is a version of Proposition 3.1 for graphs. For its statement and proof we use the notation introduced in the proof of Theorem 6.1.

Proposition 6.1. *Let $0 < \lambda < 1$ and $x_0 \in M$ be given. Then there exists a constant $C_{11} \ge 1$, with the following property: for any homotopic C^1-maps $u, v : M \to M'$ into an arbitrary Riemannian manifold M', there exists an open subset $A_{uv} \subseteq B_r(x_0)$ with $\mathrm{vol}(A_{uv}) \ge \lambda\,\mathrm{vol}(B_r(x_0))$ so that for any $z_0 \in A_{uv}$ one can find $(z_1, \ldots, z_m) \in B_r(x_1) \times \cdots \times B_r(x_m)$ with*

$$L(u_Z) \le C_{11} E(u)^{1/2}\,; \quad L(v_Z) \le C_{11} E(v)^{1/2}$$

where $z = (z_0, \ldots, z_m)$, u_z denotes the restriction of u to G_z, and $L(u_Z)$ denotes the length of the graph u_Z,

$$L(u_Z) := \sum_{(i,j) \in \mathcal{J}} L(u \gamma_{z_i z_j})\,.$$

Proof. Let $B_j := B_r(x_j)$ for convenience. As B_i and B_j are in the image of the Fermi coordinate map φ_{ij} one concludes from Lemma 3.1, that for any $0 \le i, j \le m$

$$\int_{B_i \times B_j} L^2(u\gamma_{z_i z_j}) d\,\mathrm{vol}(z_i) d\,\mathrm{vol}(z_j) \le CE(u)$$

and

$$\int_{B_i \times B_j} L^2(v\gamma_{z_i z_j}) d\,\mathrm{vol}(z_i) d\,\mathrm{vol}(z_j) \le CE(v)$$

where $C := \max_{i,j} C_{ij}$ with C_{ij} denoting the constant $C_8 = C_8(\varphi_{ij})$ given by Lemma 3.1 for the chart given by φ_{ij}. Introduce the following open subset $W_{ij}^{(n)} \subseteq B_i \times B_j$

$$W_{ij}^{(n)} := B_i \times B_j \backslash U_{ij}^{(n)} \cup V_{ij}^{(n)}$$

where $U_{ij}^{(n)}$ and $V_{ij}^{(n)}$ are closed subsets of $B_i \times B_j$ given by

$$U_{ij}^{(n)} := \{(z_i, z_j) \in B_i \times B_j \mid L(u\gamma_{z_i z_j}) \ge nE(u)^{1/2}\}$$

and

$$V_{ij}^{(n)} := \{(z_i, z_j) \in B_i \times B_j \mid L(v\gamma_{z_i z_j}) \ge nE(v)^{1/2}\}\,.$$

Then

$$n^2 E(u)\mathrm{vol}(U_{ij}^{(n)}) \le \int_{B_i \times B_j} L^2(u\gamma_{z_i z_j}) d\,\mathrm{vol}(z_i) d\,\mathrm{vol}(z_j) \le CE(u)\,,$$

and it follows that

$$\mathrm{vol}(U_{ij}^{(n)}) \le \frac{C}{n^2}\,.$$

Similarly one has that

$$\mathrm{vol}(V_{ij}^{(n)}) \le \frac{C}{n^2}\,.$$

Hence there exists a sequence $\nu^{(n)}$ with $0 \le \nu^{(n)} < 1$, only depending on M and the combinatorial structure of G so that for any $0 \le i, j \le m$

$$\nu^{(n)} \le \mathrm{vol}(W_{ij}^{(n)})/\mathrm{vol}(B_i \times B_j)\,; \quad \lim_{n\to\infty} \nu^{(n)} = 1\,.$$

As a consequence,

$$W^{(n)} := \{z = (z_0, \ldots, z_m) \in B(x_0, \ldots, x_m) \mid (z_i, z_j) \in W_{ij}^{(n)}\ \forall\,(i,j) \in \mathcal{J}\}$$

is an open subset of $B_0 \times \cdots \times B_m$ with the property that there exists $0 \le \lambda^{(n)} < 1$, so that

$$\lambda^{(n)} \le \mathrm{vol}(W^{(n)})/\mathrm{vol}(B_0 \times \cdots \times B_m)\,; \quad \lim_{n\to\infty} \lambda^{(n)} = 1\,.$$

Finally let $A^{(n)} := \Pi_0(W^{(n)})$ where $\Pi_0 : B_0 \times \cdots \times B_m \to B_0$ denotes the canonical projection on the first factor. Then $A^{(n)}$ is an open subset of B_0 and there exists $0 \le \lambda_0^{(n)} < 1$ so that

$$\lambda_0^{(n)} \le \mathrm{vol}(A^{(n)})/\mathrm{vol}(B_0)\,; \quad \lim_{n\to\infty} \lambda_0^{(n)} = 1\,.$$

For any given $0 < \lambda < 1$ choose $n_0 \ge 1$ so large that $\lambda_0^{(n_0)} > \lambda$ and set $A_{uv} := A_{n_0}$. Then $A_{uv} \subseteq B_r(x_0)$ is open with

$$\lambda < \mathrm{vol}(A_{uv})/\mathrm{vol}(B_r(x_0))\,,$$

and for any $z_0 \in A_{uv}$ there exists $(z_1, \ldots, z_m) \in B_1 \times \cdots \times B_m$ so that for $z := (z_0, z_1, \ldots, z_m)$ and $0 \le i, j \le m$,

$$L(u\gamma_{z_i z_j}) \le nE(u)^{1/2}\,; \quad L(v\gamma_{z_i z_j}) \le nE(v)^{1/2}\,.$$

Hence by choosing $C_{11} := n \cdot |\mathcal{J}|$, the claimed statement follows. □

The proof of Theorem 0.3 stated in the introduction is an immediate consequence of Proposition 3.2 and Theorem 6.1:

Proof of Theorem 0.3. By Theorem 6.1, for any $x_0 \in M$ and any homotopy class ζ of C^1-maps $v : M \to M'$, the pair (ζ, x_0) satisfies hypothesis (Z). Hence by Proposition 3.2, there exists C_7 so that for any C^1-maps $u, v \in \zeta$

$$N_2(v, u) \le C_7(E(u)^{1/2} + E(v)^{1/2} + 1)\,.$$ □

7. Summary of *a priori* Estimates

Let us summarize the results of Secs. 1–6 as follows:

Theorem 7.1. *Assume that ζ is a homotopy class of C^1-maps from M to M'. Then for any $K > 0$ there exists a constant $C > 0$ so that for any $u \in S_F \cap \zeta$ with F of class C^2 and satisfying $\|F\|_{C^1} \le K$,*

(i) *$E(u) \le C$.*
(ii) *$e(u)(x) \le C\ \forall\, x \in M$.*

Proof. (i) follows from Proposition 1.1, combined with Proposition 3.2 and Theorem 6.1 and (ii) follows from Theorem 2.1 and (i). □

Theorem 7.2. *Assume that ζ is a homotopy class of C^1-maps from M to M' and* $\dim M \le 3$. *Then there exist $c_* > 0$ and, given $K > 0$, a constant $C > 0$ so that for any $u \in S_{F,G} \cap \zeta$ with F and G C^2-smooth and satisfying*

$$\|G\|_{C^0} \le c_*\,, \quad \|F\|_{C^1} + \|G\|_{C^1} \le K$$

one has

(i) $E(u) \leq C$;
(ii) $\int_M \|\nabla . d_x u\|_{HS}^2 \, d\,\mathrm{vol}(x) \leq C$.

Remark 7.1. No effort was made to extend Theorem 7.2 to manifolds M of higher dimension. Most likely, Theorem 7.2 holds without any restriction on the dimension of M.

Proof of Theorem 7.2. (i) follows from Proposition 1.2, combined with Proposition 3.2 and Theorem 6.1 and (ii) follows from Theorem 2.2 and (i). □

8. On the Compactness of the Set of Solutions Within a Homotopy Class

In this section we prove the two theorems stated in the introduction. Concerning Theorem 0.1, it is an immediate consequence of the proposition below. To state it, let ζ be a homotopy class of C^1-maps from M to M', $k \geq 2$ and $(F_m)_{m \geq 1}$ an arbitrary sequence of x dependent C^k-vector fields on M' with

$$F := \lim_{m \to \infty} F_m \text{ in } C^k .$$

Introduce the corresponding sequence of equations $\Phi_{F_m}(u) = 0$ where

$$\Phi_{F_m}(u)(x) := \tau(u)(x) + F_m(x, u(x)) . \tag{8.1}$$

Proposition 8.1. *Let $k \geq 2$. Given any sequence $(u_m)_{m \geq 1}$ of C^{k+1}-maps with $u_m \in S_{F_m} \cap \zeta$, the solution u_m is C^3-smooth for any $m \geq 1$ and there exists a subsequence $(u_{m_j})_{j \geq 1}$ which converges to a C^{k+1}-map $u : M \to M'$ in C^{k+1}-topology so that $u \in \zeta$ is a solution of the limiting equation*

$$\Phi_F(u) = 0 .$$

Proof. By Theorem 7.1(i) and (ii) there exists $C > 0$ so that for any $m \geq 1$

$$E(u_m) \leq C , \tag{8.2}$$

$$e(u_m)(x) \leq C , \quad \forall \, x \in M . \tag{8.3}$$

By the Arzelà–Ascoli theorem and the compactness of M, (8.3) implies that there exists a subsequence $(m_j)_{j \geq 1}$ so that $\lim_{j \to \infty} u_{m_j} = u$ in $C^0(M, M')$. Hence there exist an atlas $Q_r (1 \leq r \leq R)$ of M and coordinate charts $Q'_r (1 \leq r \leq R)$ of M' so that

$$u_{m_j}(Q_r) \subseteq Q'_r \quad \forall \, 1 \leq r \leq R , \quad \forall \, j \geq 1 .$$

The restriction of u_{m_j} to Q_r can be viewed as a map with values in $\mathbb{R}^{n'}$. Choose an open, finite covering $(U_r)_{1 \leq r \leq R}$ of M with $\bar{U}_r \subset Q_r$ and smooth cutoff functions $(\varphi_r)_{1 \leq r \leq R}$ with $\varphi_r \in C_0^\infty(Q_r)$ and $0 \leq \varphi_r \leq 1$ so that $U_r \subseteq \{\varphi_r = 1\}$. Then, for any $1 \le r \le R, v_{r,j} := \varphi_r u_{m_j}$ vanishes on ∂Q_r and satisfies, on Q_r, an equation of the form

$$-\Delta_M v_{r,j} = \xi_{r,j} \tag{8.4}$$

with $\xi_{r,j} = (\xi^\alpha_{r,j})_{1\le\alpha\le n'}$ given by

$$\xi^\alpha_{r,j} := \varphi_r \left(g^{i\ell}\Gamma'^{\alpha}_{\beta\gamma} \frac{\partial u^\beta_{m_j}}{\partial x_i} \frac{\partial u^\gamma_{m_j}}{\partial x_\ell} + F^\alpha_{m_j} \right) + \eta^\alpha_{r,j}\,, \tag{8.5}$$

where $\eta_{r,j} := \varphi_r \Delta_M u_{m_j} - \Delta_M(\varphi_r u_{m_j})$ is linear in u_{m_j} and its first derivatives and does not contain higher order derivatives of u_{m_j}. Hence by (8.3), for any $1 < p < \infty$, there exists a constant $K_{0,p}$ with

$$\sup_{j,r} \|\xi_{r,j}\|_{L^p} \le K_{0,p}\,,$$

where $L^p \equiv L^p(Q_r; Q'_r)$. Hence we conclude from (8.4) that $K'_{2,p} > 0$,

$$\sup_{j,r} \|v_{r,j}\|_{W^{2,p}} \le K'_{2,p}\,, \tag{8.6}$$

where $W^{2,p} \equiv W^{2,p}(Q_r; Q'_r)$ denote the usual Sobolev spaces. By the Sovolev embedding theorem, $W^{\ell,p} = W^{\ell,p}(Q_r, Q'_r)$ continuously embeds into $C^{\ell-1} \equiv C^{\ell-1}(Q_r, Q'_r)$ for any $\ell \ge 1$ and $p > n$. Thus by (8.6) and the definition (8.5), it then follows that for any $p > n$,

$$\sup_{j,r} \|\xi_{r,j}\|_{W^{1,p}} \le K_{1,p}$$

and, by (8.4),

$$\sup_{j,r} \|v_{r,j}\|_{W^{3,p}} \le K'_{3,p}\,.$$

This procedure can be iterated to conclude that for any $p > n$,

$$\begin{aligned} \sup_{j,r} \|\xi_{r,j}\|_{W^{k,p}} &\le K_{k,p}\,, \\ \sup_{j,r} \|v_{r,j}\|_{W^{k+2,p}} &\le L_{k+2,p}\,. \end{aligned} \tag{8.7}$$

As the Sobolev embedding $W^{\ell,p} \hookrightarrow C^{\ell-1}$ is compact for any $\ell \ge 1$ and $p > n$ it follows from (8.7) that $(v_{r,j})_{j\ge1}$ is relatively compact in $C^{k+1}(Q_r; Q'_r)$ for any $1 \le r \le R$. It follows that for any $m \ge 1$ u_m is C^{k+1}-smooth and that there exists a subsequence, again denoted by $(u_{m_j})_{j\ge1}$, so that, for any $1 \le r \le R$, $(\varphi_r u_{m_j})_{j\ge1}$ converges in C^{k+1}-topology. As $(U_r)_{1\le r\le R}$ is a covering of M, we can define a limiting map $u \in C^{k+1}(M, M')$ by setting

$$u(x) := \lim_{j\to\infty} u_{m_j}(x) \quad \left(= \lim_{j\to\infty} v_{r,j}(x) \quad \forall\, x \in U_r \right).$$

Hence u is C^{k+1}-smooth and by the C^{k+1}-convergence of $(u_{m_j})_{j\geq 1}$ and $k \geq 2$, u satisfies the limiting equation $\Phi_F(u) = 0$. □

Similarly, Theorem 0.2 is an immediate consequence of the proposition stated below. Let ζ, k, and $(F_m)_{m\geq 1}$ be as above and assume that $(G_m)_{m\geq 1}$ is a sequence of y-dependent C^k-vector fields on M with

$$G := \lim_{m\to\infty} G_m \text{ in } C^k\,.$$

The corresponding sequence of equations $\Phi_{F_m,G_m}(u) = 0$ are now given by

$$\Phi_{F_m,G_m}(u) := \tau(u) + F_m(x, u(x)) + u_* G_m(x, u(x))\,.$$

Proposition 8.2. *Assume that* $\dim M \leq 3, k \geq 2,$ *and*

$$\sup_{m\geq 1} \|G_m\|_{C^0} \leq c_* \quad (c_* \text{ given in Theorem 7.2})\,.$$

Then, for any sequence $(u_m)_{m\geq 1}$ *of* C^3*-maps satisfying* $u_m \in S_{F_m,G_m} \cap \zeta$*, it follows that for any* $m \geq 1$*,* u_m *is* C^{k+1}*-smooth and that there exists a subsequence* $(u_{m_j})_{j\geq 1}$ *which converges to a* C^{k+1}*-map* $u : M \to M'$ *in* C^{k+1}*-topology so that* $u \in \zeta$ *is a solution of the limiting equation*

$$\Phi_{F,G}(u) = 0\,.$$

Remark 8.1. Most likely the hypothesis $\dim M \leq 3$ can be removed. However no efforts were made to extend Theorem 0.2 in this way.

Proof of Proposition 8.2. By Theorem 7.2, there exists $C > 0$, so that for any $m \geq 1$,

$$E(u_m) \leq C\,, \tag{8.8}$$

$$\int_M \|\nabla . d_x u_m\|^2_{HS}\, d\,\mathrm{vol}(x) \leq C\,. \tag{8.9}$$

As $\dim M \leq 3$ the Sobolev space $H^2 = H^2(M, M')$ is embedded compactly in $C^0(M, M')$. It then follows from the Azelà–Ascoli theorem that there exists a subsequence $(m_j)_{j\geq 1}$ so that $\lim_{j\to\infty} u_{m_j} = u$ in $C^0(M, M')$. Hence there exist an atlas $Q_r (1 \leq r \leq R)$ of M and coordinate charts $Q'_r (1 \leq r \leq R)$ of M' so that

$$u_{m_j}(Q_r) \subseteq Q'_r \quad \forall\, 1 \leq r \leq R\,, \quad \forall\, j \geq 1\,.$$

The restriction of u_{m_j} to Q_r can be viewed as a map with values in $\mathbb{R}^{n'}$. Choose an open covering $(U_r)_{1\leq r\leq R}$ of M with $\bar{U}_r \subset Q_r$ and smooth cut-off functions $(\varphi_r)_{1\leq r\leq R}$ with $\varphi_r \in C_0^\infty(Q_r)$ and $0 \leq \varphi_r \leq 1$ so that $U_r \subset \{\varphi_r = 1\}$. Then, for any $1 \leq r \leq R$, $v_{r,j} := \varphi_r u_{m_j}$ vanishes on ∂Q_r and satisfies, on Q_r, an equation of the form

$$-\Delta_M v_{r,j} = \xi_{r,j} \tag{8.10}$$

with $\xi_{r,j} = (\xi^\alpha_{r,j})_{1\le\alpha\le n'}$ given by

$$\xi^\alpha_{r,j} := \varphi_r \left(g^{i\ell}\Gamma'^{\alpha}_{\beta\gamma} \frac{\partial u^\beta_{m_j}}{\partial x_i} \frac{\partial u^\gamma_{m_j}}{\partial x_\ell} + F^\alpha_{m_j} + (u^*_{m_j} G_{m_j})^\alpha \right) + \eta^\alpha_{r,j}\,, \tag{8.11}$$

where $\eta_{r,j} := \varphi_r \Delta_M u_{m_j} - \Delta_M \varphi_r u_{m_j}$ is linear in u_{m_j} and its first derivatives and does not contain higher order derivatives of u_{m_j}. By (8.10) and the Sobolev embedding $H^2 \equiv W^{2,2} \hookrightarrow W^{1,5}$ we conclude from (8.11) that

$$\sup_{j,r} \|\xi_{r,j}\|_{L^{5/2}} \le K_0\,. \tag{8.12}$$

Hence by (8.10),

$$\sup_{j,r} \|v_{r,j}\|_{W^{2,5/2}} \le K'_2\,. \tag{8.13}$$

We need to improve estimate (8.12). To this aim, use (8.13) and the Sobolev embedding $W^{2,5/2} \hookrightarrow W^{1,8}$ to conclude from (8.11) that

$$\sup_{j,r} \|\xi_{r,j}\|_{L^4} \le \hat{K}_0 \tag{8.14}$$

hence by (8.10),

$$\sup_{j,r} \|v_{r,j}\|_{W^{2,4}} \le \hat{K}'_2\,. \tag{8.15}$$

For $\dim M \le 3$, the space $W^{\ell,4}(Q_r, Q'_r)$ is continuously embedded into $C^{\ell-1}(Q_r; Q'_r)$. This is used to iterate the procedure above and prove the claimed statement with arguments similar to the ones used in the proof of Theorem 8.1. □

Appendix A. Fermi Coordinates

In this appendix we recall the notion of Fermi coordinates along geodesics in a Riemannian manifold. For the convenience of the reader we provide proofs for those statements which are not completely standard. Let M be a closed Riemannian manifold of dimension n with distance function $d : M \times M \to [0,\infty)$ and denote by $\bar{r}$ its convexity radius (cf. [7, §5.2]), i.e. $\bar{r}$ is the maximal number so that for any $\varrho \le \bar{r}$ and $p \in M$ the ball $B_\varrho(p) := \{x \in M \mid d(p,x) < \varrho\}$ has the following convexity property: for any two points $x, z \in B_\varrho(p)$,

(i) there exists a unique geodesic $c \equiv c_{x,z} : [0,1] \to B_\varrho(p)$ from x to z, parametrized proportional to arclength;
(ii) the geodesic $c_{x,z}$ is the unique minimal geodesic in M from x to z.

In particular, such a geodesic does not intersect itself and its length $L(c)$ is given by $L(c) = d(x,z)$. Clearly $c_{x,z}$ depends smoothly on the endpoints x and z. In fact, the restriction of the exponential map $\exp_p$ at p to the ball $B_{\bar{r}}(0) \subseteq T_pM$ of radius $\bar{r}$ at 0 is a diffeomorphism onto the ball $B_{\bar{r}}(p)$ and satisfies

$$\exp_p(\dot{c}_{p,x}(0)) = c_{p,x}(1)\,,$$

where $\dot{c}_{p,x}(0) = \frac{d}{ds}|_{s=0} c_{p,x}(s)$.

First we need to establish some auxilary results. Given $p \in M$ and a nonconstant geodesic $c : [0,1] \to B_{\bar{r}}(p)$ parametrized proportional to arclength, introduce

$$f : [0,1] \to \mathbb{R}\,, \quad t \mapsto d(p, c(t))\,.$$

Lemma A.1. *There exists a point $t_0 \in [0,1]$ so that f is strictly decreasing on $[0,t_0]$ and strictly increasing on $[t_0,1]$.*

Proof. As f is continuous one can choose $t_0 \in [0,1]$ so that

$$f(t_0) = \inf\{f(t) \mid 0 \le t \le 1\}\,.$$

It remains to show that for any $s \in \mathbb{R}$,

(i) $\sharp(f^{-1}(s) \cap [t_0,1]) \le 1$,

and

(ii) $\sharp(f^{-1}(s) \cap [0,t_0]) \le 1$.

Statement (i) and (ii) are proved in the same way, so let us concentrate on (i). To prove (i), assume the contrary. Then there are two points $t_0 \le t_1 < t_2 \le 1$ so that $f(t_1) = f(t_2)$ and $f(t) \ge f(t_1)$ for any $t_1 \le t \le t_2$. Given any $0 \le t \le 1$,

$$c_t \equiv c_{c(t),p} : [0,1] \to B_{\bar{r}}(p)$$

denotes the unique geodesic from $c(t)$ to p parametrized proportional to arclength. Then $t \mapsto \dot{c}_t(1) := \frac{d}{ds}|_{s=1} c_t(s)$ is a smooth curve in T_pM with $\|\dot{c}_{t_3}(1)\| \ge a$ where $t_3 = \frac{t_1+t_2}{2}$ and

$$a := \|\dot{c}_{t_1}(1)\|\, (= \|\dot{c}_{t_2}(1)\|)\,.$$

As c is nonconstant (by assumption) and parametrized proportional to arclength, the map $t \mapsto c(t)$ is injective. In view of the identity $c(t) = \exp_p(-\dot{c}_t(1))$ this implies that $t \mapsto \dot{c}_t(1)$ is injective as well. In particular, $\dot{c}_{t_3}(1)$ is distinct from $\dot{c}_t(1)$ for any $t \in [0,1]\backslash\{t_3\}$. Define $q := \exp_p(\tau \dot{c}_{t_3}(1))$ with $0 < \tau \le 1$ chosen so small that $c(t)$ lies in $B_{\bar{r}}(q)$ for any $0 \le t \le 1$. Clearly,

$$d(q, c(t_3)) = d(q,p) + d(p, c(t_3)) \ge d(q,p) + a\,.$$

On the other hand, the curve obtained by composing the geodesic from q to p with the one from p to $c(t)$ is a broken geodesic for any $t \in [0,1]\backslash\{t_3\}$ as $\dot{c}_{t_3}(1) \ne \dot{c}_t(1)$ for such t's and hence for $j = 1, 2$

$$d(q, c(t_j)) < d(q,p) + d(p, c(t_j)) = d(q,p) + a\,.$$

As a consequence one can choose $0 < \delta < d(q, c(t_3))$ so that $c(t_1)$ and $c(t_2)$ are in $B_\delta(q)$ but $c(t_3) \notin B_\delta(q)$. But as $\delta < d(q, c(t_3)) < \bar{r}$ this contradicts the convexity of $B_\delta(q)$, hence statement (i) holds. $\square$

We will need a simple application of Lemma A.1 later. Let $c : [0,1] \to B_{\bar{r}}(p)$ be again a nonconstant geodesic parametrized proportional to arclength and set

$$\dot{c}(0) := \frac{d}{dt}\Big|_{t=0} c(t)\,; \quad \dot{c}_0(0) := \frac{d}{ds}\Big|_{s=0} c_{c(0),p}(s)\,.$$

Corollary A.1. *If $\dot{c}(0)$ and $\dot{c}_0(0)$ are orthogonal, then $t \mapsto f(t)$ is strictly increasing on $[0,1]$.*

Proof. Approximate $\dot{c}_0(0)$ by a smooth 1-parameter family of tangent vectors $w_\tau \in T_{c(0)}M, \tau \geq 0$, with $\|w_\tau\| = \|\dot{c}_0(0)\|$, $\lim_{\tau\to 0} w_\tau = \dot{c}_0(0)$, and

$$\langle w_\tau,\, \dot{c}(0)\rangle < 0 \quad \text{for } \tau > 0\,.$$

Denote by $c_\tau : [0,1] \to M$ the geodesic, parametrized proportional to arclength so that $c_\tau(0) = c(0)$ and $\dot{c}_\tau(0) = w_\tau$ and let $p_\tau := c_\tau(1)$. Then $\lim_{\tau\to 0} p_\tau = p$. Notice that for τ sufficiently small, c_τ lies entirely in the ball $B_{\bar{r}}(p)$. Moreover, for any $\tau > 0$, the function $f_\tau(t) = d(p_\tau, c(t))$ satisfies

$$\frac{d}{dt}\Big|_{t=0} f_\tau(t) = \left\langle \frac{\partial}{\partial x}\Big|_{x=c(0)} d(p_\tau, x),\ \dot{c}(0)\right\rangle = \left\langle -\frac{\dot{c}_\tau(0)}{\|\dot{c}_\tau(0)\|},\, \dot{c}(0)\right\rangle > 0\,.$$

As c is nonconstant and contained in the ball $B_{\bar{r}}(p_\tau)$ for τ sufficiently small it then follows from Lemma A.1 that f_τ is strictly increasing on $[0,1]$ for $\tau > 0$ sufficiently small. By continuity it then follows that f is nondecreasing on $[0,1]$. Invoking once more Lemma A.1 it then follows that f is strictly increasing on $[0,1]$ as well. $\square$

Given a minimal geodesic $c : [0,a] \to M$, parametrized *by arclength*, and an orthonormal basis $\bar{e}$ in $T_{c(0)}M$, given by $\bar{e}_1, \ldots, \bar{e}_{n-1}, \bar{e}_n := \frac{\partial}{\partial t}|_{t=0}c(t)$, we now construct Fermi coordinates as follows: Let $B_r(0) \equiv B_r^{n-1}(0)$ denote the open ball in $\mathbb{R}^{n-1}$, centered at the origin 0 with radius $r := \bar{r}/2$. Then we define

$$\varphi_c \equiv \varphi_{c,\bar{e}} : (-r, a+r) \times B_r(0) \to M$$

as follows:

(i) the curve $(-r, a+r) \to M$, $t \mapsto \varphi_c(t,0)$ is the arclength parametrization of the geodesic coinciding with c on $[0,a]$.

(ii) For any unit vector $v = \sum_1^{n-1} v_k e_k \in \mathbb{R}^{n-1}$, the curve $s \mapsto \varphi_c(0, sv)$ is the arclength parametrization of the geodesic with

$$\frac{\partial}{\partial s}\Big|_{s=0} \varphi_c(0, sv) = \sum_1^{n-1} v_k \bar{e}_k\,.$$

(iii) For any unit vector $v \in \mathbb{R}^{n-1}$ and any $t \in (-r, a+r)\backslash\{0\}$, the curve $s \mapsto \varphi_c(t, sv)$ is the arclength parametrization of the geodesic with $\frac{\partial}{\partial s}|_{s=0}\varphi_c(t,sv)$ obtained by parallel transport of $\frac{\partial}{\partial t}|_{t=0}\varphi_c(0,tv)$ along c. Here the parallel transport is the one induced by the Levi–Cività connection on M.

Clearly, φ_c is well defined and smooth. We claim that it is a coordinate map.

Proposition A.1. *The map φ_c is injective and of maximal rank.*

Remark A.1. The coordinates $(t, v) \in (-r, a+r) \times B_r(0)$ are referred to as Fermi coordinates.

Remark A.2. The construction above also works for the limit case where c shrinks to a point p. In this case the coordinate map φ is parametrized by a unit tangent vector w instead of the geodesic c. Given an orthonormal basis $\bar{e}$ of $T_pM, \bar{e}_1, \ldots, \bar{e}_{n-1}, \bar{e}_n := w$, the map

$$\varphi_p \equiv \varphi_{p,\bar{e}} : (-r, r) \times B_r(0) \to M$$

is defined similarly as above except that $t \mapsto \varphi_p(t, 0)$ is chosen to be the arclength parametrization of the minimal geodesic uniquely determined by $\varphi_p(0, 0) = p$ and $\frac{\partial}{\partial t}|_{t=0}\varphi_p(t, c) = w$.

Remark A.3. For any $0 < a \leq \mathrm{diam}(M)$, denote by $\mathcal{F}_a$ the set of all Fermi coordinate maps $\varphi_{c,\bar{e}} : (-r, a+r) \times B_r(0) \to M$ where c is the arclength parametrization of an arbitrary minimal geodesic of length a and $\bar{e}$ is an arbitrary orthonormal basis of $T_{c(0)}M$ of the type as above. Similarly, we define $\mathcal{F}_a$ for $a = 0$. As M is compact, $\mathcal{F}_a$ is, in particular, relatively compact in the space of C^1-maps $\mathcal{C}^1((-r, a+r) \times B_r(0); M)$ for any $0 \leq a \leq \mathrm{diam}(M)$. Hence it follows that $\|d\varphi_{c,\bar{e}}\|_{C^0}$ is universally bounded for any $\varphi_{c,\bar{e}}$ in $\bigcup_{0<a\leq \mathrm{diam}(M)} \mathcal{F}_a$.

Proof of Proposition A.1. Let us first show that φ_c is $1-1$. Assume that there exist $t_1, t_2 \in (-r, a+r)$, unit vectors $v_1, v_2 \in \mathbb{R}^{n-1}$ and $0 \leq s_1, s_2 < r$ so that

$$p := \varphi_c(t_1, s_1 v_1) = \varphi_c(t_2, s_2 v_2) .$$

Without loss of generality, we assume that $t_1 \leq t_2$. Notice that

$$d(p, \varphi_c(t_1, 0)) = d(\varphi_c(t_1, s_1 v_1), \varphi_c(t_1, 0)) < r ,$$

and

$$d(p, \varphi_c(t_2, 0)) = d(\varphi_c(t_2, s_2 v_2), \varphi_c(t_2, 0)) < r .$$

Thus $\varphi_c(t_1, 0)$ and $\varphi_c(t_2, 0)$ are elements in $B_r(p)$ and as $B_r(p)$ is convex it follows that the (in M) minimal geodesic between $\varphi_c(t_1, 0)$ and $\varphi_c(t_2, 0)$ is contained in $B_r(p)$. However, we do not know yet that this minimal geodesic coincides with $\varphi_c(t), t_1 \leq t \leq t_2$, as φ_c might not be contained in $B_{\bar{r}}(p)$. To see that this cannot happen, introduce

$$p_1 := \varphi_c(t_1^*, 0) ; \quad p_2 := \varphi_c(t_2^*, 0) ,$$

where

$$t_1^* := \max(0, t_1) ; \quad t_2^* := \min(a, t_2) .$$

In case $t_1 < 0$ we have

$$d(p, p_1) < d(p, \varphi_c(t_1, 0)) + d(\varphi_c(t_1, 0), \varphi_c(0, 0)) \leq 2r\,,$$

whereas for $t_1 \geq 0, p = p_1$ and thus $d(p, p_1) = 0$. Hence in both cases

$$d(p, p_1) < \bar{r}\,.$$

Similarly, one sees that

$$d(p, p_2) < \bar{r}\,.$$

By the convexity of $B_{\bar{r}}(p)$, it then follows that there is a unique geodesic in $B_{\bar{r}}(p)$ from p_1 to p_2 and that this geodesic is minimal in M. By definition, $\varphi_c(t)$ is the (in M) minimal geodesic from $\varphi_c(t_1^*)$ to $\varphi_c(t_2^*)$ and thus the two geodesics coincide. Hence $\varphi_c(t) \in B_{\bar{r}}(p)$ for $t_1^* \leq t \leq t_2^*$. As $\varphi_c(t) \in B_r(p) \subseteq B_{\bar{r}}(p)$ for $t_1 \leq t \leq t_1^*$ and for $t_2^* \leq t \leq t_2$ we conclude that $\varphi_c(t) \in B_{\bar{r}}(p)$ for any t with $t_1 \leq t \leq t_2$ and we have proved that $\varphi_c(t)(t_1 \leq t \leq t_2)$ is in M the minimal geodesic from $\varphi_c(t_1, 0)$ to $\varphi_c(t_2, 0)$. As $\varphi_c(t_1, 0)$ and $\varphi_c(t_2, 0)$ are in $B_r(p)$, and $B_r(p)$ is convex we have that $\varphi_c(t) \in B_r(p)$ for any $t_1 \leq t \leq t_2$. In particular, as $t \mapsto \varphi_c(t, 0)$ is parametrized proportional to arclength, it follows that $t \mapsto \varphi_c(t, 0)$ is injective for $t_1 \leq t \leq t_2$. Hence in case either of the two numbers s_1 or s_2 is 0, it follows that $t_1 = t_2$ and hence $s_1 = s_2 = 0$. Now let us consider the case where $s_1 \neq 0$ and $s_2 \neq 0$. By the definition of φ_c, $\frac{d}{dt}|_{t=t_j}\varphi_c(t, 0)$ and $\frac{d}{ds}|_{s=0}\varphi_c(t_j, sv_j)$ are orthogonal for $j = 1, 2$. Hence by Lemma A.1 and Corollary A.1, it follows from $p = \varphi_c(t_1, s_1v_1)$ that $d(p, \varphi_c(t, 0))$ attains its unique minimum at $t = t_1$ where as from $p = \varphi_c(t_2, s_2v_2)$, one concludes that $d(p, \varphi_c(t, 0))$ attains its unique minimum at $t = t_2$. Thus we conclude that $t_1 = t_2$. As the exponential map is a diffeomorphism it then follows from $\varphi_c(t_1, s_1v_1) = \varphi_c(t_1, s_2v_2)$ that $s_1 = s_2$ and $v_1 = v_2$. It remains to prove that φ_c has maximal rank at any point. Assume to the contrary that φ_c does not have maximal rank at a point $(t_0, s_0v_0) \in (-r, a + r) \times B_r(0)$ where v_0 is assumed to be of unit length and $0 \leq s_0 < r$. From the definition of φ_c it follows that $s_0 > 0$. Since φ_c is the normal exponential map of the geodesic $\bar{c} = \varphi_c((-r, a + r) \times \{0\})$ this implies that the geodesic $s \mapsto \varphi_c(t_0, sv_0)$ has a focal point at s_0 and it follows that beyond s_0, the geodesic $s \mapsto \varphi_c(t_0, sv_0)$ is no longer a minimizing geodesic to $\bar{c}$ (cf. [13]). This means that for a point $p_1 := \varphi_c(t_0, s_1v_0)$ with $s_0 < s_1 < r$, the distance function $t \mapsto d(p_1, \varphi_c(t, 0))$ does not assume the minimum in t_0. But this contradicts Lemma A.1 and Corollary A.1. $\square$

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