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QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS FOR MAPPINGS BETWEEN MANIFOLDS, I

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Abstract. A class of quasilinear elliptic differential operators defined on mappings between manifolds is introduced and studied. In arbitrary local coordinates, such an operator is a quasilinear elliptic differential operator that sends a mapping between manifolds to a vector field along this mapping.

Contents

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§0. Introduction

The present paper is devoted to the theory of quasilinear elliptic differential operators acting on the mappings $u : M \to M'$, where M and M' are smooth manifolds without boundary. By a differential operator A we mean a correspondence that sends every smooth mapping u to a vector field $M \ni x \mapsto Au(x) \in T_{u(x)}M'$ along u such that, in arbitrary local coordinates, this correspondence has the form of a differential operator. Such an operator is called *quasilinear elliptic* if it is a quasilinear elliptic differential operator in any local coordinates (it can easily be shown that this property is invariant under the changes of coordinates in M and M'). For brevity, in this Introduction, quasilinear elliptic differential operators will be called elliptic operators. An equation of the form $Au(x) = 0$, where A is an elliptic operator, is called an elliptic equation.[1]

Presently, there is no general theory for elliptic operators on mappings of manifolds. Despite this, the solutions of certain elliptic equations have been studied extensively during the last fifty years, mainly, for the needs of geometry. First of all, this concerns the harmonic map and pseudoholomorphic curve equations. The solutions of these equations (harmonic mappings and pseudoholomorphic curves) are effective tools in Riemannian and symplectic geometries (see the surveys [13, 14] and the fundamental paper [15]). As a result pertaining to geometry and partial differential equations simultaneously, we mention a Lefschetz type formula obtained in [20] for the algebraic number of contractible solutions of the Cauchy–Riemann type equations

$$\frac{\partial u}{\partial \bar z} = \mathsf{f}(z, u(z)), \quad u : \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to M', \tag{0.1}$$

where M' is a Kähler manifold.

§1 of the present paper is preliminary; here we introduce all necessary differential-geometric notation. In §2, we introduce the notions of a differential and a quasilinear differential operator and also define their symbols. After that, we define a (quasilinear differential) elliptic operator A as an operator the symbol of which is nondegenerate. The linearization $A_*(u)$ of A at a mapping u is a linear differential operator acting on the sections of the bundle u^*TM'. This operator is elliptic if so is A. Respectively, the index of an elliptic operator A can be defined as the index $\operatorname{ind} A_*(u)$ of its linearization. Since the index is homotopy invariant, the index of an elliptic operator only depends on the homotopy class $[u]$ of u, $\operatorname{ind}_{[u]} A = \operatorname{ind} A_*(u)$. Examples of several families of elliptic operators are given in §4 (by a *family* of operators we mean an operator depending on an additional functional parameter running over a certain infinite-dimensional space $\mathcal{F}$). In particular, in §4 we discuss variational elliptic operators A that are the Euler–Lagrange operators of the corresponding variational problems. The indices of such operators are equal to zero.

Next, we introduce the moduli space $\mathfrak{M}_{A,[u]}$ for solutions of an elliptic equation (more precisely, of a family of equations). This space is defined as the set of all pairs (u, f), where f is a functional parameter of the equation and u is a solution of a fixed homotopy type $[u]$. By definition, the natural projection of the moduli space is the mapping

$$\pi_A : \mathfrak{M}_{A,[u]} \to \mathcal{F}, \quad (u, \mathsf{f}) \mapsto \mathsf{f}.$$

[1]We note that, though the class of equations in question is sufficiently large (see §4), it does not exhaust all known elliptic equations. Thus, pseudoholomorphic curves [15, 21] are solutions of a homogeneous equation for the Cauchy–Riemann operator $\bar\partial u(x) = 0$, where $\bar\partial u(x)$ is not a vector in the tangent space $T_{u(x)}M'$ but a linear operator $T_xM \to T_{u(x)}M'$(see Subsection 4.2 below). Our definitions and methods can be generalized to a wider class of elliptic equations including the equations of pseudoholomorphic curves, but this would complicate the exposition.

Choosing the dependence of the elliptic equation on the parameter f so as to ensure a certain transversality condition, in Theorem 5.1 we prove that $\mathfrak{M}_{A,[u]}$ is a smooth manifold and π_A is a Fredholm mapping. Moreover, in Theorem 5.5 it is proved that the manifold $\mathfrak{M}_{A,[u]}$ is quasi-finite-dimensional, namely, $\mathfrak{M}_{A,[u]}$ admits an atlas such that the transition functions have the quasi-finite-dimensional form $\mathsf{f} \mapsto \mathsf{f} + K(\mathsf{f})$, where K is a finite-dimensional mapping.

By the Sard–Smale theorem [24], for a typical f the fiber $\pi_A^{-1}(\mathsf{f})$ (i.e., the set of all solutions of the equation for a fixed value of the functional parameter f) is a smooth manifold of dimension $\operatorname{ind}_{[u]} A$. We say that a family of elliptic problems has the *compactness property* if the mapping π_A is proper. From the results of [24] it follows that, in this case, the typical fibers of the projection π_A are cobordant. The corresponding cobordism class is an invariant; this is discussed in Theorem 5.6. In particular, if an elliptic equation has the compactness property and $\operatorname{ind}_{[u]} A = 0$ (e.g., if A is a variational operator), then a typical fiber $\pi_A^{-1}(\mathsf{f})$ consists of a finite number of points, and the parity of this number does not depend on f.

Thus, the classes of elliptic equations with compactness property are particularly interesting and promising objects of study. Unfortunately, no specific properties of an elliptic equation (and of the manifolds M and M') responsible for the compactness property are known at present. The corresponding families of elliptic equations can be presented on a short list. Three of these families are discussed in the last section of this paper.

The present paper is a survey of the basics of the theory of elliptic equations for mappings between manifolds. In addition to new results, it contains well-known statements and also generalizations of some facts known for specific operators to the case of general elliptic operators. The corresponding references and explanations are given in the text.

In a subsequent paper, the authors plan to discuss the orientability of moduli spaces and related problems and results. Cf. [20], where this was done for the moduli space of the contractible solutions of equations (0.1).

Agreements. Throughout the paper, unless otherwise stated, smoothness means C^∞-smoothness; all manifolds, bundles, and mappings are smooth, and the operators under consideration have smooth coefficients. By M and M' we denote connected smooth paracompact Hausdorff manifolds without boundary and having dimensions n and n', respectively. The Einstein summation rule with respect to repeating indices is adopted throughout.

§1. Basic notation

1.1. Covariant differentials of mappings. Let ∇ and ∇' be torsion-free connections on M and M', respectively, and let u be a smooth mapping from M to M'. For each $k \in \mathbb{N}$, we define a k-linear form

$$D^k u(x) : \times^k T_x M \to T_{u(x)} M' \tag{1.1}$$

as follows. If $k = 1$, then $Du = du$ is the differential of u. If the multilinear form $D^{k-1}u$ has already been defined, then we view it as a section of the bundle

$$\otimes^{k-1} T^* M \otimes u^* T M' \tag{1.2}$$

and define $D^k u$ as

$$D^k u(X_0, \dots, X_{k-1}) = \big(\widetilde{\nabla}_{X_0} D^{k-1} u\big)(X_1, \dots, X_{k-1}),$$

where $X_0, \dots, X_{k-1}$ are smooth vector fields on M, and $\widetilde{\nabla}$ is the connection on the bundle (1.2), i.e., the tensor product of the connections ∇ and ∇'. The multilinear form defined above is a mapping of the form (1.1), i.e., it only depends on the values of the vector fields at points. For any $X_0, \dots, X_{k-1} \in C^\infty(TM)$, we have

$$\begin{aligned} &D^k u(X_0, X_1, \dots, X_{k-1}) \\ &\quad = \nabla'_{du(X_0)} D^{k-1}u(X_1, \dots, X_{k-1}) - \sum_{i=1}^{k-1} D^{k-1}u(X_1, \dots, \nabla_{X_0} X_i, \dots, X_{k-1}) \end{aligned}$$

(for the case of classical tensor fields, see [4, Vol. 1, Chapter 3]).

Since the connections are torsion-free, the form D^2u is symmetric. The classical name of this form is the *second fundamental form* of u [13].

Example. Let M and M' be Riemannian manifolds, let ∇ and ∇' be their Levi-Civita connections, and let $u : M \to M'$ be a Riemannian embedding. Identifying the vector fields X and Y on M with the fields $du(X)$ and $du(Y)$, we see that

$$\nabla'_X Y = \nabla_X Y + D^2u(X, Y).$$

It follows [4, Vol. 2, Chapter 7] that the form D^2u takes values in the normal bundle on M and is the second fundamental form in the classical case.

If $k \geq 3$, then, in general, the form $D^k u$ is not symmetric. We denote by $\mathcal{D}^k u$ its symmetric part,

$$\mathcal{D}^k u(X_1, \dots, X_k) = \frac{1}{k!} \sum_{\sigma \in \mathrm{S_k}} D^k u(X_{\sigma(1)}, \dots, X_{\sigma(k)}),$$

where $X_1, \dots, X_k \in C^\infty(TM)$ and $\mathrm{S_k}$ is the symmetry group of degree k.

Definition 1.1. The form $\mathcal{D}^k u$ is called the *total covariant differential* of the mapping u of order k with respect to the connections ∇ and ∇'.

If $(x^1, \dots, x^n)$ and $(u^1, \dots, u^{n'})$ are local coordinates on M and M', respectively, then the components $(\mathcal{D}^k u)^\alpha_{i_1 \cdots i_k}$ of the n'-dimensional vector

$$(\mathcal{D}^k u)_{i_1 \cdots i_k} = \left(\mathcal{D}^k u\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)\right)^\alpha \frac{\partial}{\partial u^\alpha}$$

have the form

$$(\mathcal{D}^k u)^\alpha_{i_1 \cdots i_k}(x) = \frac{\partial^k u^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}(x) + \cdots, \tag{1.3}$$

where the dots hide the "covariant terms" containing only the derivatives of u of order less than k. If two multiindices $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ coincide up to permutation, then they determine the same vector. In particular,

$$\begin{aligned} (\mathcal{D}u)^\alpha_i &= \frac{\partial u^\alpha}{\partial x^i}, \\ (\mathcal{D}^2 u)^\alpha_{ij} &= \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \Gamma^p_{ij} \frac{\partial u^\alpha}{\partial x^p} + \Gamma'^\alpha_{\beta\gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}, \end{aligned} \tag{1.4}$$

where the Γ^p_{ij} and Γ'^p_{ij} are the Christoffel symbols of the connections ∇ and ∇', respectively. If the connections are canonically flat, then

$$(\mathcal{D}^k u)^\alpha_{i_1 \cdots i_k}(x) = \frac{\partial^k u^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}(x).$$

1.2. Jet manifolds. For smooth manifolds M and M', we denote by $J^k = J^k(M, M')$ the manifold of k-jets $j^k u(x)$, $x \in M$, of smooth mappings $u : M \to M'$. The mapping

$$j^k u : M \to J^k,$$

defined by the rule $x \mapsto j^k u(x)$ is called the *k-jet extension* of u. In particular, $J^0 = M \times M'$, and the 0-jet extension of u is simply its graph $j^0 u(x) = (x, u(x))$. The manifold J^k fibers over $M \times M'$ via the mapping

$$p_k : j^k u(x) \mapsto (x, u(x)).$$

For an integer s, $0 \le s < k$, we have the natural projection

$$J^k \to J^s, \quad j^k u(x) \mapsto j^s u(x),$$

$x \in M$, which determines a fibering of J^k over J^s. The mappings

$$\Pi_k : J^k \to M, \quad j^k u(x) \mapsto x,$$
$$\Pi'_k : J^k \to M', \quad j^k u(x) \mapsto u(x)$$

are called the *source* mapping and the *target* mappings (of jets), respectively. In particular, $\Pi = \Pi_0$ and $\Pi' = \Pi'_0$ are the usual projections to the factors.

We put

$$J^k(n, n') = \prod_{p=1}^{k} \mathcal{L}^p_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'}),$$

where $\mathcal{L}^p_{\mathrm{sym}}(V, W)$ is the space of p-linear symmetric mappings of a vector space V to a space W. If $M = \mathbb{R}^n$ and $M' = \mathbb{R}^{n'}$, then the manifold $J^k(M, M')$ is identified with the space $\mathbb{R}^n \times \mathbb{R}^{n'} \times J^k(n, n')$ via the mapping

$$j^k u(x) \mapsto (x, u(x), \partial u(x), \ldots, \partial^k u(x)).$$

Here, $\partial^p u(x)$ denotes the row of all partial derivatives of order p of u at x. This row determines an element of $\mathcal{L}^p_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'})$. If x and u belong to local charts $\mathcal{O} \subset M$ and $\mathcal{O}' \subset M'$ of the manifolds M and M', respectively, then the set $p_k^{-1}(\mathcal{O} \times \mathcal{O}')$ is identified with $\mathcal{O} \times \mathcal{O}' \times J^k(n, n')$ and serves as a local chart on $J^k(M, M')$. We call such coordinates on J^k *natural* and denote them by $(x, u, \partial u, \ldots, \partial^k u)$, where x and u are points belonging to the local charts $\mathcal{O}$ and $\mathcal{O}'$, and the $\partial^p u$ are elements of $\mathcal{L}^p_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'})$. If we pass from one natural local coordinate system to another, the coordinates $(\partial^p u)^\alpha_{i_1 \cdots i_p} = \partial^p u^\alpha / \partial x^{i_1} \cdots x^{i_p}$ of the symmetric p-linear mapping $\partial^p u$ change as follows:

$$(\partial^p u)^{\alpha'}_{i'_1 \cdots i'_p} = (\partial^p u)^\alpha_{i_1 \cdots i_p} \frac{\partial u^{\alpha'}}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial x^{i'_1}} \cdots \frac{\partial x^{i_p}}{\partial x^{i'_p}} + \cdots, \tag{1.5}$$

where the dots stand for the summands that only involve the derivatives of u of order less than p.

We note that, in general, J^k is not a vector bundle over $M \times M'$, because the isomorphism between the fiber over $(x, u) \in M \times M'$ and the space $J^k(n, n')$ may fail to be canonical, i.e., it depends on the choice of a local system of coordinates.

We denote by $L^k_{\mathrm{sym}} = L^k_{\mathrm{sym}}(M, M')$ the total space of the vector bundle over $M \times M'$ whose fiber over $(x, u) \in M \times M'$ coincides with

$$\mathcal{L}^k_{\mathrm{sym}}(T_x M, T_u M'). \tag{1.6}$$

If $k = 0$, then we put L^0_{sym} equal to the trivial bundle over $M \times M'$ with zero-dimensional fiber. We note that the mapping

$$J^1 \ni j^1 u(x) \mapsto (x, u(x), du(x)) \in L^1_{\mathrm{sym}}$$

yields a canonical isomorphism between the bundles L^1_{sym} and J^1. In particular, J^1 is always a vector bundle over $M \times M'$. The total space of the direct sum $\bigoplus_{p=0}^k L^p_{\mathrm{sym}}$ fibers naturally over the space $\bigoplus_{p=0}^s L^p_{\mathrm{sym}}$, $s < k$. We introduce the projections

$$\pi_k : \bigoplus_{p=0}^{k} L^p_{\mathrm{sym}} \to M \times M' \xrightarrow{\Pi} M,$$

$$\pi'_k : \bigoplus_{p=0}^{k} L^p_{\mathrm{sym}} \to M \times M' \xrightarrow{\Pi'} M'.$$

Observe that if M and M' are Riemannian manifolds, then the metrics g and g' give rise to the following natural metric $\langle \cdot, \cdot \rangle$ on the bundle L^k_{sym}:

$$\langle \mathcal{D}^k u_1, \mathcal{D}^k u_2 \rangle = g^{i_1 j_1} \cdots g^{i_k j_k} g'_{\alpha\beta} (\mathcal{D}^k u_1)^{\alpha}_{i_1 \cdots i_k} (\mathcal{D}^k u_2)^{\beta}_{j_1 \cdots j_k}, \tag{1.7}$$

where $\mathcal{D}^k u_1$ and $\mathcal{D}^k u_2$ are elements of (1.6). Moreover, the Levi-Civita connections of the metrics g and g' determine a connection in L^k_{sym} compatible with the metric (1.7). If $k = 1$, then the fiber (1.6) is a space of operators, and the metric (1.7) determines the Hilbert–Schmidt norm in this space.

We denote by $J^{k-1} \oplus L^k_{\mathrm{sym}}$ the total space of the Whitney sum of the bundles J^{k-1} and L^k_{sym} over $M \times M'$. This manifold is a vector bundle over J^{k-1} with the fiber $\mathcal{L}^k_{\mathrm{sym}}(T_x M, T_u M')$ over $j^{k-1}u$. Given torsion-free connections on M and M', we define the mapping

$$I_k^{-1} : J^k \to J^{k-1} \oplus L^k_{\mathrm{sym}}, \quad j^k u(x) \mapsto (j^{k-1} u(x), \mathcal{D}^k u(x)).$$

Lemma 1.1. *The mapping I_k^{-1} is a diffeomorphism yielding an isomorphism of bundles,*

$$\begin{array}{ccc} J^k & \xrightarrow{I_k^{-1}} & J^{k-1} \oplus L^k_{\mathrm{sym}} \\ \downarrow & & \downarrow \\ J^{k-1} & = & J^{k-1}. \end{array}$$

Moreover, in the natural local coordinates this mapping has the form

$$(j^{k-1}u, \partial^k u) \mapsto (j^{k-1}u, \partial^k u + T(j^{k-1}u)),$$

where $j^{k-1}u = (x, u, \partial^1 u, \ldots, \partial^{k-1} u)$ is an element of J^{k-1} and T is a smooth mapping.

The proof follows immediately from the local representation (1.3) of the forms $\mathcal{D}^k u$, and from the structure of I_k^{-1} in the natural local coordinates on J^k. Below, we denote by I_k the inverse mapping $J^{k-1} \oplus L^k_{\mathrm{sym}} \to J^k$.

Corollary 1.2. *For $k = 0, 1, \ldots$, the mappings*

$$J^k \ni j^k u(x) \longmapsto (x, u(x), \mathcal{D}u(x), \ldots, \mathcal{D}^k u(x)) \in \bigoplus_{p=0}^{k} L^p_{\mathrm{sym}}$$

are diffeomorphisms, and for $0 \le s \le k$ they yield isomorphisms of fiber bundles,

$$\begin{array}{ccc} J^k & \longrightarrow & \bigoplus_{p=0}^k L^p_{\mathrm{sym}} \\ \downarrow & & \downarrow \\ J^s & \longrightarrow & \bigoplus_{p=0}^s L^p_{\mathrm{sym}}. \end{array}$$

It is natural to call the row $(x, u(x), \mathcal{D}u(x), \dots, \mathcal{D}^k u(x))$ the *covariant k-jet* of the smooth mapping $u : M \to M'$, and the space of covariant k-jets (coinciding with the total space $\bigoplus_{p=0}^k L^p_{\mathrm{sym}}$) will be called the *manifold of covariant k-jets*.

§2. Differential operators

2.1. General definitions. Let A be a smooth section of the bundle $(\Pi'_k)^* TM'$, where $\Pi'_k : J^k \to M'$ is the target map of jets. Starting with this section, we define a differential operator A of order k on smooth mappings $u : M \to M'$ as follows:

$$Au(x) = \mathsf{A}(j^k u(x)) \in T_{u(x)}M', \quad x \in M. \tag{2.1}$$

The section A is called the *system of coefficients* of the differential operator A, and the bundle $(\Pi'_k)^* TM'$ is called the *bundle of coefficients*.

Let M'' be another smooth manifold, and let $\phi : M' \to M''$ be a smooth mapping. We denote by Π''_k the natural projection $J^k(M, M'') \to M''$. A section A of $(\Pi'_k)^* TM'$ and a section A' of $(\Pi''_k)^* TM''$ are said to be *ϕ-connected* if

$$\mathsf{A}'(j^k(\phi \circ u)) = d\phi \circ \mathsf{A}(j^k u)$$

for every smooth mapping $u : M \to M'$. In particular, if ϕ is a diffeomorphism, then the relation of ϕ-connectedness is bijective. If $\phi : M' \to M''$ is a smooth mapping, then two operators A and A' defined on the mappings from M to M' and to M'', respectively, are said to be ϕ-connected if

$$A'(\phi \circ u) = d\phi \circ A(u).$$

Obviously, ϕ-connected sections of the bundles $(\Pi'_k)^* TM'$ and $(\Pi''_k)^* TM''$ give rise to ϕ-connected differential operators.

Definition 2.1. A section A is said to be *quasilinear* if in arbitrary natural local coordinates it has the form

$$\mathsf{A}(j^{k-1}u, \partial^k u) = A_k(j^{k-1}u) \cdot \partial^k u + \mathsf{g}(j^{k-1}u), \tag{2.2}$$

where $j^{k-1}u = (x, u, \partial u, \dots, \partial^{k-1} u)$, the pair $(j^{k-1}u, \partial^k u)$ is an element of J^k, and $\mathsf{A}_k(j^{k-1}u)$ is a linear operator from $L^k_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'})$ to $\mathbb{R}^{n'}$.

By (1.5), the notion of quasilinearity is well defined relative to the change of coordinates. Moreover, from (1.5) we obtain the following statement.

Lemma 2.1. *Under the changes of local coordinates, the linear part $\mathsf{A}_k(j^{k-1}u)$ of a quasilinear section A transforms as a tensor and, for each $j^{k-1}u \in J^{k-1}$, it determines an element of*

$$\mathcal{L}\big(\mathcal{L}^k_{\mathrm{sym}}(T_x M, T_u M'), T_u M'\big). \tag{2.3}$$

Consider the bundle $\mathrm{Symb}_k(M, M')$ with the base J^{k-1} and with the fiber (2.3) over $j^{k-1}u$. By Lemma 2.1, every quasilinear section A gives rise to a section A_k of $\mathrm{Symb}_k(M, M')$; this A_k is called the *principal part* of A.

Example. Every section A of $(\Pi'_1)^* TM'$ can be regarded as a morphism of bundles,

$$\begin{array}{ccc} J^1 & \xrightarrow{\ \mathsf{A}\ } & TM', \\ \Big\downarrow & & \Big\downarrow \\ M \times M' & \xrightarrow{\ \Pi'\ } & M'. \end{array} \tag{2.4}$$

Since J^1 is a vector bundle, a section A is quasilinear if and only if the morphism (2.4) is fiberwise affine. Such a section A has the form

$$\mathsf{A}(x, u(x), du(x)) = \mathsf{A}_1(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \quad x \in M,$$

where A_1 is a morphism of the vector bundles J^1 and $(\Pi')^*TM'$ over $M \times M'$ (the principal part of A) and g is a section of $(\Pi')^*TM'$.

Definition 2.2. A differential operator of the form (2.1) is said to be *quasilinear* of order k if its system of coefficients is a quasilinear section of the bundle $(\Pi'_k)^*TM'$.

The set of quasilinear differential operators of order k is a vector space, which we denote by $DO_k(M, M')$. Now, we show that the notions of a symbol and ellipticity are well defined for such operators.

Let A be a quasilinear differential operator of order k, and let A_k be its principal part. The isomorphism

$$\mathcal{L}\big(\mathcal{L}^k_{\mathrm{sym}}(T_xM, T_uM'), T_uM'\big) \simeq \mathcal{L}\big(\vee^k T^*_xM, \mathcal{L}(T_uM', T_uM')\big), \tag{2.5}$$

where $\vee^k$ is the kth symmetric power, allows us to identify the fiber of the bundle $\mathrm{Symb}_k(M, M')$ over $j^{k-1}u \in J^{k-1}$ with the space

$$\mathcal{L}\big(\vee^k T^*_xM, \mathcal{L}(T_uM', T_uM')\big)$$

(cf. [9, Chapter IV]). This makes the following definition consistent.

Definition 2.3. By the *symbol* of a quasilinear differential operator A we mean the mapping $\sigma_A = \sigma_A(j^{k-1}u)$,

$$\sigma_A : T^*_xM \to \mathcal{L}(T_uM', T_uM'),$$

defined for all $j^{k-1}u \in J^{k-1}$ and $w \in T^*_xM$, $x \in M$, by the rule

$$\sigma_A(w) = \sigma_A(j^{k-1}u, w) = \mathsf{A}_\mathsf{k}(j^{k-1}u) \cdot \vee^k w.$$

Definition 2.4. A quasilinear operator A is said to be *elliptic* if for every $j^{k-1}u \in J^{k-1}$ and every nonzero $w \in T^*_xM$, $x \in M$, the linear operator

$$\sigma_A(w) : T_uM' \to T_uM'$$

is an isomorphism.

Example. If $M = \mathbb{R}^n$ and $M' = \mathbb{R}^{n'}$, the manifold $J^k(M, M')$ is the product $\mathbb{R}^n \times \mathbb{R}^{n'} \times J^k(n, n')$, and the k-jet extension of $u : \mathbb{R}^n \to \mathbb{R}^{n'}$ has the form

$$x \mapsto \big(x, u(x), \partial u(x), \ldots, \partial^k u(x)\big),$$

where, as before, $\partial^p u(x)$ is the row of all partial derivatives of u of order p at x. By the isomorphism (2.5), every linear operator

$$\mathsf{A}_\mathsf{k} : \mathcal{L}^k_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'}) \to \mathbb{R}^{n'}$$

is given by a system of $(n' \times n')$-matrices $(\mathsf{A}_\mathsf{k})^{i_1\cdots i_k}$, $i_p = 1, \ldots, n$, $p = 1, \ldots, k$, such that the collections $(i_1, \ldots, i_k)$ that coincide up to permutation correspond to the same matrix. Let $(x_1, \ldots, x_n)$ and $(u_1, \ldots, u_{n'})$ be bases of the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n'}$, respectively. The same letters with superscript indices will denote the dual bases, which determine coordinate systems in $\mathbb{R}^n$ and $\mathbb{R}^{n'}$. We have

$$\mathsf{A}_\mathsf{k} = (\mathsf{A}_\mathsf{k})^{\beta i_1\cdots i_k}_{\alpha} x_{i_1} \otimes \ldots \otimes x_{i_k} \otimes u^\alpha \otimes u_\beta.$$

Denoting by $(\mathsf{A}_\mathsf{k})^{i_1\cdots i_k}$ an $(n' \times n')$-matrix with entries $(\mathsf{A}_\mathsf{k})^{\beta i_1\cdots i_k}_{\alpha}$, where $\alpha, \beta = 1, \ldots, n'$, and writing the quasilinear mapping (2.2) explicitly, we obtain

$$\begin{aligned} Au = {} & (\mathsf{A}_\mathsf{k})^{i_1\cdots i_k}\big(x, u, \partial u, \ldots, \partial^{k-1}u\big) \cdot \big(\partial^k u / \partial x^{i_1} \cdots \partial x^{i_k}\big) \\ & + \mathsf{g}\big(x, u, \partial u, \ldots, \partial^{k-1}u\big). \end{aligned} \tag{2.6}$$

Thus, in the case where $M = \mathbb{R}^n$ and $M' = \mathbb{R}^{n'}$, the above definition of a quasilinear differential operator coincides with the classical one. In particular, the "characteristic polynomial" $\sigma_A(w)$ of A is the following $(n' \times n')$-matrix:

$$\sigma_A(j^{k-1}u, w) = (\mathsf{A}_\mathsf{k})^{i_1 \cdots i_k}(j^{k-1}u) \cdot w_{i_1} \cdots w_{i_k}, \tag{2.7}$$

where $w \in \mathbb{R}^{n*}$ and $w_1, \dots, w_n$ are the coordinates of w in the basis $x^1, \dots, x^n$. Ellipticity means that for all $j^{k-1}u \in J^{k-1}$ and $w \in \mathbb{R}^{n*}$, $w \neq 0$, the matrix (2.7) is nondegenerate.

The above example shows that, in local coordinates, the elements $DO_k(M, M')$ are usual quasilinear differential operators, and formula (2.7) gives the form of the symbols of such operators in the natural local coordinates.

In view of Definition 2.1, the next statement is clear. For the convenience of future references, we formulate it as a lemma.

Lemma 2.2. *The mapping*

$$C^\infty(M, M') \ni u \longmapsto Au \in C^\infty(u^*TM')$$

is a quasilinear differential operator of order k if and only if it is a quasilinear differential operator of order k in arbitrary local coordinates on M and M'. In particular, this mapping is an elliptic quasilinear differential operator if and only if it is an elliptic quasilinear differential operator in arbitrary local coordinates on M and M'.

Example (first order differential operators)**.** A first order differential operator is determined by a morphism A of bundles,

$$\begin{array}{ccc} J^1 & \xrightarrow{\mathsf{A}} & TM' \\ \big\downarrow & & \big\downarrow \\ M \times M' & \xrightarrow{\Pi'} & M'. \end{array}$$

A quasilinear first order differential operator A has the form

$$Au(x) = \mathsf{A}_1(x, u(x)) \cdot du + \mathsf{g}(x, u(x)), \quad x \in M,$$

where A_1 is a morphism of the vector bundles J^1 and $(\Pi')^*TM'$ over $M \times M'$ (the principal part), and g is a section of the bundle $(\Pi')^*TM'$. In local coordinates $x = (x^1, \dots, x^n)$ on M, we have

$$Au(x) = {\mathsf{A}_1}^i(x, u(x)) \cdot \frac{\partial u}{\partial x^i}(x) + \mathsf{g}(x, u(x)), \quad x \in M,$$

where, as before, the ${\mathsf{A}_1}^i$ are matrix-valued functions. The symbol σ_A of the operator A looks like this:

$$\sigma_A(w) = {\mathsf{A}_1}^i(x, u)w_i,$$

where $x \in M$, $u \in M'$, and $w = w_i dx^i \in T_x^*M$.

If we pass from one system of local coordinates on M to another, the matrices ${\mathsf{A}_1}^i$ change in accordance with the rule

$${\mathsf{A}_1}^{i'} = \frac{\partial x^{i'}}{\partial x^i} {\mathsf{A}_1}^i.$$

In particular, for an arbitrary pair of local coordinates on M and M', if the matrices ${\mathsf{A}_1}^i(x, u)$ in these coordinates are scalar, ${\mathsf{A}_1}^i(x, u) = a^i(x, u)I$ (in this case we say that the operator has a scalar symbol), then the collections $\{a^i(x, u)\}$ determine a nonautonomous smooth vector field on M depending on the additional parameter $u \in M'$. In other words, such an operator is given by the rule

$$u(x) \mapsto du(x) \cdot a(x, u(x)) + \mathsf{g}(x, u(x)), \quad x \in M, \tag{2.8}$$

and its symbol

$$\sigma_A(w) = (w \cdot a(x,u))I \tag{2.9}$$

is the operator of multiplication by $w \cdot a(x,u)$ on T_uM', where $\cdot$ denotes the pairing of a covector $w \in T_x^*M$ and the vector $a(x,u)$. In particular, A is elliptic only if $\dim M = 1$ and $a(x,u) \neq 0$.

2.2. Using connection. Quasilinearity. Suppose that manifolds M and M' are equipped with torsion-free connections ∇ and ∇', respectively. In this case, the class of quasilinear differential operators has an equivalent definition, which is more convenient for the treatment of specific operators. The corresponding definitions are given in the present and the next subsections.

Every section of the bundle $(\Pi'_k)^*TM'$, where $\Pi'_k : J^k \to M'$ is the target mapping, can be regarded as a morphism of bundles,

$$\begin{array}{ccc} J^k & \xrightarrow{\mathsf{A}} & TM' \\ \downarrow & & \downarrow \\ J^{k-1} & \xrightarrow{\Pi'_{k-1}} & M'. \end{array}$$

In Subsection 1.1, we used connections on M and M' to construct the isomorphism I_k of the bundles $J^{k-1} \oplus L^k_{\mathrm{sym}}$ and J^k over the base space J^{k-1}. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} J^{k-1} \oplus L^k_{\mathrm{sym}} & \xrightarrow{I_k} & J^k & \xrightarrow{\mathsf{A}} & TM' \\ \downarrow & & \downarrow & & \downarrow \\ J^{k-1} & = & J^{k-1} & \xrightarrow{\Pi'_{k-1}} & M'. \end{array} \tag{2.10}$$

Observe that the bundles $J^{k-1} \oplus L^k_{\mathrm{sym}}$ over J^{k-1} and TM' over M' are vector. For a morphism of such bundles, we have a well-defined notion of affinity as the fiber affinity of the mappings of the corresponding vector spaces.

Definition 2.5. A section A of the bundle $(\Pi'_k)^*TM'$ is said to be *quasilinear* if the morphism $\mathsf{A} \circ I_k$ of the bundles $J^{k-1} \oplus L^k_{\mathrm{sym}}$ and TM' (2.10) is affine.

For every $j^{k-1}u$ in J^{k-1}, the linear part $\mathsf{A}_k(j^{k-1}u)$ of the affine mapping $\mathsf{A} \circ I_k(j^{k-1}u)$ determines an element of the space $\mathcal{L}\big(\mathcal{L}^k_{\mathrm{sym}}(T_xM, T_uM'), T_uM'\big)$, i.e., a section of the bundle $\mathrm{Symb}_k(M, M')$.

Lemma 2.3. *Definitions* 2.1 *and* 2.5 *of the quasilinearity property of a section* A *of the bundle* $(\Pi'_k)^*TM'$ *are equivalent. The linear part* A_k *of the fiberwise affine mapping* $\mathsf{A} \circ I_k$ *coincides with the principal part of the section* A.

The proof follows immediately from Lemma 1.1.

In particular, Lemma 2.3 implies that the notions of quasilinearity and the principal part of A in the sense of Definition 2.5 do not depend on the choice of the connections on M and M' that determine the mapping I_k.

2.3. Using connection. The definition of a differential operator. Understanding the differentiation operation in the sense of connection, we can define the notion of a differential operator covariantly. Namely, let A be a morphism of bundles,

$$
\begin{array}{ccc}
\bigoplus_{p=0}^{k} L^p_{\mathrm{sym}} & \xrightarrow{\mathsf{A}} & TM', \\
\downarrow & & \downarrow \\
\bigoplus_{p=0}^{k-1} L^p_{\mathrm{sym}} & \xrightarrow{\pi'_{k-1}} & M'.
\end{array}
\tag{2.11}
$$

Definition 2.6. A morphism (2.11) is said to be *quasilinear* if it is fiberwise affine as a mapping of vector spaces.

The linear part A_k of a quasilinear mapping A determines a morphism of vector bundles in (2.11), called the principal part of A. As before, on the mappings $u : M \to M'$ each morphism A determines a differential operator A of order k:

$$Au(x) = \mathsf{A}\big(x, u(x), \mathcal{D}u(x), \dots, \mathcal{D}^k u(x)\big), \quad x \in M. \tag{2.12}$$

The morphism A is called the *system of coefficients* of the differential operator (2.12).

Definition 2.7. The differential operator A given by (2.12) is said to be *quasilinear of order k* if the morphism A of bundles (2.11) is quasilinear.

A quasilinear differential operator A (in the sense of Definition 2.7) with the principal part A_k has the form

$$Au = \mathsf{A}_\mathsf{k}\big(x, u, \mathcal{D}u, \dots, \mathcal{D}^{k-1}u\big) \cdot \mathcal{D}^k u + \mathsf{g}\big(x, u, \mathcal{D}u, \dots, \mathcal{D}^{k-1}u\big), \tag{2.13}$$

where g is a smooth section of the bundle $(\pi'_{k-1})^* TM'$. In particular, from (2.13) it follows that for every morphism A_k of vector bundles (2.11), there exists a differential operator with the given principal part A_k. For such operators, the notions of the symbol and ellipticity are defined in the same way as in the preceding subsection, with the replacement of the jet variety J^{k-1} by its covariant analog, namely, by the total space of the bundle $\bigoplus_{p=0}^{k-1} L^p_{\mathrm{sym}}$.

Lemma 2.4. *The definitions* (2.1) *and* (2.12) *of a differential operator are equivalent. Moreover, the classes of quasilinear differential operators and elliptic quasilinear differential operators given by Definitions* 2.2 *and* 2.7 *coincide.*

Proof. Let $\mathcal{J}^k$ be the mapping occurring in Corollary 1.2. Let A be a differential operator in the sense of (2.12), and let A be its system of coefficients. Then, by Corollary 1.2, the operator

$$Au(x) = (\mathsf{A} \circ \mathcal{J}^k)(j^k u(x))$$

is the same differential operator, but now in the sense of (2.1). The form of the mapping $\mathcal{J}^k$ in the natural local coordinates on J^k shows (see Lemma 1.1) that, under this bijection, the quasilinear differential operators correspond to quasilinear differential operators. Moreover, the elliptic quasilinear differential operators correspond to elliptic operators. □

Corollary 2.5. *The class of quasilinear differential operators of order k in the sense of Definition* 2.7 *is independent of the choice of connections on M and M'.*

Corollary 2.6. *For every section A_k of the bundle* $\mathrm{Symb}_k(M, M')$ *of symbols, there exists a differential operator with symbol A_k.*

§3. Linearized differential operators

3.1. Basic properties. In this section, we assume that M is a compact manifold, that M' is equipped with a Riemann structure g' and some connection ∇'.

Let u be a smooth mapping of M to M', and let ν be a smooth vector field on M' along u. There always exists $\varepsilon > 0$ and a smooth mapping U of the product $M \times (-\varepsilon, \varepsilon)$ to M' such that the family $u_t(\cdot) = U(\cdot, t)$, $t \in (-\varepsilon, \varepsilon)$ has the following properties:

$$u_0 = u, \quad \frac{\partial u_t}{\partial t}\Big|_{t=0} = \nu. \tag{3.1}$$

Indeed, it suffices to put $u_t(x) = \exp_{u(x)}(t\nu(x))$. Then the existence of the required positive ε follows from the compactness of M and the lower semicontinuity of the injectivity radius on M' (see [2]).

Now, let A be a differential operator of order k. We denote by ∇^* the connection on the bundle U^*TM' (the covariant differentiation along the mapping U) induced by the connection ∇' on TM'.

Definition 3.1. The *linearization* of an operator A at a point $u \in C^\infty(M, M')$ is the operator $A_*(u) : C^\infty(u^*TM') \to C^\infty(u^*TM')$ that acts on the sections ν of the bundle u^*TM' by the rule

$$A_*(u)\nu = \nabla^*_{\frac{\partial}{\partial t}}\Big|_{t=0} Au_t. \tag{3.2}$$

As is shown below (see Lemma 3.1), $A_*(u)$ is a linear differential operator whose value $A_*(u)\nu$ is determined only by the smooth mapping u and a vector field ν along u. It follows that Definition 3.1 is independent of the choice of a family u_t satisfying (3.1).

Relation (3.2) can be understood as follows. For each $x \in M$, the vector $A_*(u)\nu(x)$ is the value at $t = 0$ of the covariant derivative of the vector field $Au_t(x)$ along the curve $u_t(x)$, $t \in (-\varepsilon, \varepsilon)$, on M'.

Lemma 3.1. *Let A be a differential operator of order k, and let u be a smooth mapping from M to M'. Then $A_*(u)$ is a linear differential operator of the same order on the bundle u^*TM'. In particular, if A is a quasilinear differential operator, then the symbol $A_*(u)$ is given by the formula*

$$\sigma_{A_*(u)}(x) = \sigma_A(j^{k-1}u(x)), \quad x \in M. \tag{3.3}$$

Corollary 3.2. *A quasilinear differential operator A is elliptic if and only if for every u in $C^\infty(M, M')$ the linear differential operator $A_*(u)$ is elliptic.*

As another consequence of Lemma 3.1, we mention that the symbol of the linearized operator $A_*(u)$ is independent of the choice of a connection on M'.

Let $(x^1, \dots, x^n)$ and $(u^1, \dots, u^{n'})$ be local coordinates on M and M', respectively. Let $(x^1, \dots, x^n, t)$ be local coordinates on $M \times (-\varepsilon, \varepsilon)$. Then, in these coordinates, the derivative of a section of U^*TM' is given by the rule

$$(\nabla^*_{\frac{\partial}{\partial t}} Au_t)^\alpha = \frac{\partial}{\partial t}(Au_t)^\alpha + \Gamma'^\alpha_{\beta\gamma} \frac{\partial u_t^\beta}{\partial t}(Au_t)^\gamma, \tag{3.4}$$

where the $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of the connection on TM'. In particular, if u is a solution of the equation $Au = 0$, then

$$(A_*(u)\nu)^\alpha = \frac{\partial}{\partial t}\Big|_{t=0}(Au_t)^\alpha. \tag{3.5}$$

Thus, we arrive at the following statement.

Lemma 3.3. *Let u be a solution of the equation $Au = 0$. Then, in arbitrary local coordinates, the linearization of the operator A at a point u can be defined by relation* (3.5). *This definition survives if we change the coordinates and is independent of a Riemann structure on M'.*

Example. Let $\mathfrak{g}(x,u)$ be a nonautonomous vector field on M', $u \in M'$, $x \in M$. The differential operator G of order zero acts by the rule

$$u(x) \mapsto \mathfrak{g}(x, u(x)), \quad x \in M.$$

Its linearization $G_*(u)$ is determined by the operator field that has the following form in local coordinates:

$$G_*(u)^\alpha_\beta = \nabla'_\beta \mathfrak{g}^\alpha = \frac{\partial \mathfrak{g}^\alpha}{\partial u^\beta} + \Gamma'^\alpha_{\gamma\beta}\mathfrak{g}^\gamma.$$

For each $x \in M$, let $u_0(x)$ be a singular point of the vector field $\mathfrak{g}(x,\cdot)$. Then the mapping u_0 is a solution of the equation $Gu = 0$, and the linearization

$$G_*(u_0)^\alpha_\beta(x) = \frac{\partial \mathfrak{g}^\alpha}{\partial u^\beta}(x, u_0(x))$$

coincides with the linearization of the vector field $\mathfrak{g}(x,\cdot)$ at the singular point $u_0(x)$.

Remark. The assumption that M is compact (see the beginning of this section) was used only for proving the existence of a family u_t satisfying conditions (3.1). If M is not compact, then the existence of the required family u_t will be ensured if we assume, e.g., that M' is complete.

Let A be a quasilinear elliptic differential operator of order k, and let u be a smooth mapping from M to M'. Then the linearized operator $A_*(u)$ is elliptic and determines continuous mappings

$$[A_*(u)]_s : H^s(u^*TM') \to H^{s-k}(u^*TM'), \quad s \in \mathbb{Z},$$

of Sobolev spaces. (The same is true if u is a mapping of smoothness class C^s, $s \geq k$.) The index of the Fredholm operator $[A_*(u)]_s$, defined as

$$\operatorname{ind}[A_*(u)]_s = \dim \operatorname{Ker}[A_*(u)]_s - \operatorname{codim} \operatorname{Im}[A_*(u)]_s,$$

does not depend on s (see, e.g., [9, 11]) and is denoted by $\operatorname{ind} A_*(u)$. Since the index of a linear operator is homotopy invariant, we see that $\operatorname{ind} A_*(u)$ only depends on the homotopy class of the mapping u. Moreover, it is well known that the index $\operatorname{ind} A_*(u)$ is determined by the symbol of $A_*(u)$; therefore, by Lemma 3.1, it depends only on the symbol σ_A of A.

Definition 3.2. For a quasilinear elliptic differential operator A and a homotopy class $[u]$ of mappings from M to M', the integer

$$\operatorname{ind} A_*(u) = \operatorname{ind}(\sigma_A, [u])$$

is called the index of A and is denoted by $\operatorname{ind}_{[u]A}$.

If M is an oriented manifold, then a Riemann structure on M' determines an inner product in the space of smooth sections of the bundle u^*TM':

$$(\mathrm{v}, \mathrm{w}) = \int_M \langle \mathrm{v}, \mathrm{w}\rangle_{g'} \, dVol_g, \quad \mathrm{v}, \mathrm{w} \in C^\infty(u^*TM'). \tag{3.6}$$

Let $(A_*(u))^*$ denote the differential operator formally adjoint to $A_*(u)$ with respect to the inner product (3.6). As is well known (see [9, 11]), the operators $A_*(u)$ and $(A_*(u))^*$ have finite-dimensional kernels and

$$\operatorname{ind} A_*(u) = \dim \operatorname{Ker} A_*(u) - \dim \operatorname{Ker}(A_*(u))^*. \tag{3.7}$$

3.2. Proof of Lemma 3.1. Let $(x^1, \dots, x^n)$ and $(u^1, \dots, u^{n'})$ be local coordinates on M and M', respectively, and let $(x^1, \dots, x^n, t)$ be local coordinates on the product $M \times (-\varepsilon, \varepsilon)$. Let A denote the coefficients of the differential operator A. Considering the section A in the natural local coordinates $(x, u, \partial u, \dots, \partial^k u)$ on J^k, we can understand $\mathsf{A}_{\partial^p u}$ as a linear operator from the space $\mathcal{L}^p_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^{n'})$ to $\mathbb{R}^{n'}$. Then, for $p = 0, \dots, k$, we have

$$\mathsf{A}_{\partial^p u} = (\mathsf{A}_{\partial^p u})^{\beta i_1 \cdots i_p}_{\alpha} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes du^\alpha \otimes \frac{\partial}{\partial u^\beta}.$$

We denote by $\mathsf{A}^{i_1 \cdots i_p}_{\partial^p u}$ the $(n' \times n')$-matrices with the entries $(\mathsf{A}_{\partial^p u})^{\beta i_1 \cdots i_p}_{\alpha}$. Differentiating $\mathsf{A}(j^k u_t)$ with respect to t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0} A u_t &= \sum_{p=0}^{k} (\mathsf{A}_{\partial^p u})^{i_1 \cdots i_p}(j^k u_t)\big|_{t=0} \cdot \frac{\partial}{\partial t}\Big|_{t=0} \big(\partial^p u_t / \partial x^{i_1} \cdots \partial x^{i_p}\big) \\ &= \sum_{p=0}^{k} (\mathsf{A}_{\partial^p u})^{i_1 \cdots i_p}(j^k u) \cdot \big(\partial^p \mathsf{v} / \partial x^{i_1} \cdots \partial x^{i_p}\big).\end{aligned}$$

Thus, relation (3.4) implies that, in every system of local coordinates, the mapping defined by the relation

$$\mathsf{v}(x) \longmapsto \nabla^*_{\frac{\partial}{\partial t}}\Big|_{t=0} \mathsf{A}(j^k u_t(x)), \quad x \in M,$$

determines a differential operator of order k and, therefore, is a linear differential operator of the same order. Moreover, the symbol of this operator has the form

$$\sigma_{A_*(u)}(x) w = (\mathsf{A}_{\partial^k u})^{i_1 \cdots i_k}(j^k u(x)) w_{i_1} \cdots w_{i_k}, \quad x \in M,$$

where $w \in T^*_x M$ and $w = w_i dx^i$. In particular, if a section A is quasilinear and represents a system of coefficients of A with principal part A_k, then, by Definition 2.1, we have

$$\mathsf{A}_{\partial^k u}(j^k u) = \mathsf{A}_\mathsf{k}(j^{k-1} u).$$

It follows that the matrices $(\mathsf{A}_\mathsf{k})^{i_1 \cdots i_k}$ constructed as in Subsection 2.1 coincide with the matrices $(\mathsf{A}_{\partial^k u})^{i_1 \cdots i_k}$. Using the form (2.7) for the symbol of a quasilinear differential operator in local coordinates, we obtain relation (3.3). $\square$

§4. Examples

4.1. An ordinary differential operator. Let M be the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and let $s \in [0, 1]$ be a parameter on M. We assume that M' is a Riemannian manifold with metric g'.

The first order elliptic differential operator d/ds acts by the rule

$$C^\infty(S^1, M') \ni u \longmapsto \frac{du}{ds} \in C^\infty(u^* TM').$$

Its linearization $(d/ds)_*(u)$ maps each smooth vector field v along u to the vector field $\nabla^*_{\frac{\partial}{\partial s}} \mathsf{v}$, where ∇^* is the covariant differentiation along the curve u. Indeed, let u_t be a family of mappings satisfying (3.1). Then, since the connection is torsion-free, we obtain

$$\left(\frac{du}{ds}\right)_* \mathsf{v} = \nabla^*_{\frac{\partial}{\partial t}} \frac{du_t}{ds}\Big|_{t=0} = \nabla^*_{\frac{\partial}{\partial s}} \frac{du_t}{dt}\Big|_{t=0} = \nabla^*_{\frac{\partial}{\partial s}} \mathsf{v}. \tag{4.1}$$

The operator $(d/ds)_*$ is formally antiselfadjoint in the sense of the inner product (3.6), i.e.,

$$\int_{S^1} \langle \nabla^*_{\frac{\partial}{\partial s}} \mathsf{v}, \mathsf{w} \rangle_{g'}\, ds = -\int_{S^1} \langle \mathsf{v}, \nabla^*_{\frac{\partial}{\partial s}} \mathsf{w} \rangle_{g'}\, ds$$

for all $\mathsf{v}, \mathsf{w} \in C^\infty(u^*TM')$. This relation can be obtained by integrating the identity

$$\langle \nabla^*_{\frac{\partial}{\partial s}} \mathsf{v}, \mathsf{w}\rangle_{g'} = -\langle \mathsf{v}, \nabla^*_{\frac{\partial}{\partial s}} \mathsf{w}\rangle_{g'} + \frac{d}{ds}\langle \mathsf{v}, \mathsf{w}\rangle_{g'}.$$

In particular, relation (3.7) and formal antiselfadjointness imply that the index of $(d/ds)_*$ is zero for each homotopy class of mappings from M to M'. For a nonautonomous vector field $\mathsf{f}(s,u)$ on M', the solutions of the equation

$$\frac{du}{ds}(s) = \mathsf{f}(s, u(s)), \quad s \in S^1,$$

coincide with the periodic trajectories of the vector field $\mathsf{f}(s,u)$.

More generally, we can consider differential operators of an arbitrary order defined on the space of smooth loops in M'. It is natural to call such operators ordinary differential operators. The ordinary elliptic quasilinear differential operators have the following property: their index is zero at each homotopy class of loops in M'. Indeed, a linearized operator acts on the sections of a vector bundle over the circle, and the index of such an elliptic differential operator is equal to zero.

4.2. The Cauchy–Riemann operator. Let $M = \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the complex torus and M' an arbitrary complex manifold. The following first order elliptic operators are defined on smooth mappings $u : \mathbb{T}^2 \to M'$:

$$\frac{\partial u}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right), \quad \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right),$$

where $z = x + iy \in \mathbb{T}^2$. Their linearizations have the form

$$\left(\frac{\partial u}{\partial \bar z}\right)_* \mathsf{v} = \nabla^*_{\frac{\partial}{\partial \bar z}} \mathsf{v}, \quad \left(\frac{\partial u}{\partial z}\right)_* \mathsf{v} = \nabla^*_{\frac{\partial}{\partial z}} \mathsf{v},$$

where $\mathsf{v} \in C^\infty(u^*TM')$. The index of the operator $(\partial/\partial \bar z)_*$ is given by the formula (see, e.g., [21])

$$\operatorname{ind}\left(\frac{\partial u}{\partial \bar z}\right)_* = 2c_1(u^*TM'),$$

where c_1 is the first Chern number of the bundle u^*TM'. Since the operators $(\partial/\partial \bar z)_*$ and $(\partial/\partial z)_*$ are formally antiselfadjoint, they have equal indices by (3.7).

We note that an operator "of type $\partial/\partial \bar z$" cannot be defined for mappings from an arbitrary compact complex curve M to M'. More precisely, an elliptic differential operator of the form (in local coordinates)

$$a(z, u(z))\frac{\partial u}{\partial \bar z}(z) \in T_{u(z)}M', \quad z \in M,$$

where $a(z,u)$ is some complex-valued function (defined only in the given local coordinate $z \in M$, $u \in M'$) exists only if $M = \mathbb{T}^2$. Indeed, as in the example in Subsection 2.1, for a fixed mapping u the functions $\{a(z, u(z))\}$ corresponding to different local coordinates must change in accordance with the following rule under a holomorphic change of coordinates:

$$a(w, u(w)) = \frac{\partial \bar w}{\partial \bar z} a(z, u(z)).$$

Consequently, the set of such functions determines a section of the antiholomorphic tangent bundle of M, which cannot have zeros since the operator is elliptic. By the Poincaré–Hopf theorem, a vector field without zeros exists only if the Euler characteristic of M is zero. It follows that M is a torus.

We recall that an even-dimensional manifold N is said to be *almost complex* if it is equipped with a smooth operator field J that determines a complex structure on any tangent space T_yN, i.e., $J_y^2 = -I$, $y \in N$. A smooth mapping of a complex curve M to an almost complex manifold N is *pseudoholomorphic* if

$$du \circ i = J \circ du,$$

where i and J denote the complex structure and the almost complex structure on M and N, respectively. A smooth section f of $(\Pi')^*TM'$ gives rise to the following almost complex structure J^{f} on the product $\mathbb{T}^2 \times M'$ (see [15]):

$$J^{\mathsf{f}}_{(z,u)}(\zeta,\eta) = \big(i\zeta, i\eta + 2i\mathsf{f}(z,u)\bar{\zeta}\big), \quad (\zeta,\eta) \in T_{(z,u)} \simeq \mathbb{C} \times T_uM', \;\; z \in \mathbb{T}^2, \;\; u \in M'.$$

Let u_0 be a smooth mapping from $\mathbb{T}^2$ to M', and let $\tilde{u}_0$ be the graph of u_0,

$$\tilde{u}_0 : \mathbb{T}^2 \to \mathbb{T}^2 \times M', \quad z \mapsto (z, u_0(z)).$$

Proposition 4.1 [15, 20]**.** *A mapping $\mathbb{T}^2 \to \mathbb{T}^2 \times M'$ homotopic to $\tilde{u}_0$ is pseudoholomorphic with respect to the almost complex structure J^{f} if and only if it is the graph of a solution of the equation*

$$\frac{\partial u}{\partial \bar{z}}(z) + \mathsf{f}(z, u(z)) = 0, \quad z \in \mathbb{T}^2, \tag{4.2}$$

such that this solution is homotopic to u_0.

We note that if $M' = \mathbb{C}P^{n'}$, then solutions of equation (4.2) arise naturally when we projectivize nonlinear homogeneous eigenvalue problems

$$\frac{\partial U}{\partial \bar{z}} + \mathsf{F}(z,U) = \lambda U, \quad U : \mathbb{T}^2 \to \mathbb{C}^{n'+1} \backslash \{0\}, \;\; \lambda \in \mathbb{C}. \tag{4.3}$$

Here $\mathsf{F} : \mathbb{T}^2 \times \mathbb{C}^{n'+1} \to \mathbb{C}^{n'+1}$ is a mapping 1-homogeneous with respect to U (i.e., $\mathsf{F}(z,\mu U) = \mu \mathsf{F}(z,U)$ for all $\mu \in \mathbb{C}$) and smooth if $U \neq 0$. A solution of this equation is a pair $(U(z), \lambda)$, where the mapping U is defined up to multiplication by a nonzero constant. Let p denote the canonical projection $\mathbb{C}^{n'+1} \backslash \{0\} \to \mathbb{C}P^{n'}$. The relation

$$\mathsf{f}(z, p(U)) = dp(\mathsf{F}(z,U)), \quad U \in \mathbb{C}^{n'+1}, \tag{4.4}$$

determines a nonautonomous smooth vector field on $\mathbb{C}P^{n'}$. Moreover, if $\mathsf{f}(z,u)$ is such a vector field, then there exists a mapping $\mathsf{F}(z,U)$ satisfying (4.4) and 1-homogeneous with respect to the second argument. Observe that the group $\mathbb{Z} \oplus \mathbb{Z}$ acts naturally on the solutions of (4.3),

$$(m,n) : (U(x+iy), \lambda) \longmapsto \big(e^{2\pi i(mx+ny)} U(x+iy), \lambda + \pi(im - n)\big). \tag{4.5}$$

The following statement can be checked by direct calculation (see [20]).

Proposition 4.2. *A pair $(U(z), \lambda)$ satisfies* (4.3) *if and only if $u = p \circ U : \mathbb{T}^2 \to \mathbb{C}P^{n'}$ is a contractible solution of* (4.2)*. Distinct solutions of* (4.3) *correspond to the same solution of* (4.2) *if and only if they belong to the same orbit of the action* (4.5)*.*

4.3. Lagrangian problems. Here, we describe a class of differential operators arising from the Euler–Lagrange equations for a certain class of variational problems. We assume that manifolds M and M' have Riemannian metrics g and g' with Levi-Civita connections ∇ and ∇', respectively. For simplicity, we consider only the Lagrangians of the first order and assume that M is compact and orientable.

Let $\mathfrak{L}$ be a smooth real-valued function on $J^1(M,M')$. On the set of smooth mappings from M to M', we consider the functional

$$\mathfrak{J}(u) = \int_M \mathfrak{L}(x, u(x), du(x))\, dVol_g(x). \tag{4.6}$$

By definition, the *first variation* of $\mathcal{J}(u)$ is the linear mapping $\mathcal{J}'(u) : C^\infty(u^*TM') \to \mathbb{R}$ defined by the rule

$$\mathcal{J}'(u)\nu = \frac{d}{dt}\Big|_{t=0}\mathcal{J}(u_t), \quad \nu \in C^\infty(u^*TM'),$$

where u_t is a family of mappings satisfying (3.1). For the functional (4.6), the first variation has the form

$$\mathcal{J}'(u)\nu = \int\limits_M \langle L(u), \nu\rangle_{g'}\, d\,Vol_g, \tag{4.7}$$

where $L(u)$ is a second order differential operator called the *Euler–Lagrange operator*. In local coordinates on M and M', this operator can be represented in the form

$$L(u)^\alpha = g'^{\alpha\beta}\bigg(\frac{\partial\mathcal{L}}{\partial u^\beta} - \nabla_{\frac{\partial}{\partial x^i}}\bigg(\frac{\partial \mathcal{L}}{\partial u^\beta_{x^i}}\bigg)\bigg), \tag{4.8}$$

where

$$\nabla_{\frac{\partial}{\partial x^k}}\bigg(\frac{\partial\mathcal{L}}{\partial u^\beta_{x^i}}\bigg) = \frac{\partial}{\partial x^k}\bigg(\frac{\partial\mathcal{L}}{\partial u^\beta_{x^i}}\bigg) - \Gamma^m_{ik}\frac{\partial\mathcal{L}}{\partial u^\beta_{x^m}}$$

is the covariant derivative of the covector $(\partial\mathcal{L}\ /\partial u^\beta_{x^i})_i$, and $g'^{\alpha\beta}$ is the tensor inverse to the metric tensor on M' (for the case where M is a domain in $\mathbb{R}^n$, see [3, Chapter 6]). Thus, relation (4.8) and Lemma 2.2 imply that the Euler–Lagrange operator $L(u)$ is a quasilinear differential operator. Its symbol $\sigma_L = \sigma_L(x, u, du)$ has the form

$$\sigma_L(w) = \mathcal{L}^{ij}w_iw_j,$$

where $w \in T^*_xM$, $w = w_kdx^k$, and $\mathcal{L}^{ij}$ denotes the following $(n' \times n')$-matrix:

$$(\mathcal{L}^{ij})^\alpha_\beta = -g'^{\alpha\gamma}\frac{\partial^2\mathcal{L}}{\partial u^\gamma_{x^i}\partial u^\beta_{x^j}}.$$

We recall that there is a natural inner product (3.6) on the space of smooth sections of the bundle u^*TM'.

Lemma 4.1. *The linear differential operator $L_*(u)$ is formally selfadjoint*

Proof. For all $\nu,\ w \in C^\infty(u^*TM')$, there exists $\varepsilon > 0$ and a mapping

$$U : M \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M'$$

such that for the family $u_{t,s} = U(\cdot, t, s)$, $t,\ s \in (-\varepsilon, \varepsilon)$ we have

$$u_{t,s}\Big|_{t,s=0} = u, \quad \frac{\partial u_{t,s}}{\partial t}\Big|_{t,s=0} = \nu, \quad \frac{\partial u_{t,s}}{\partial s}\Big|_{t,s=0} = w.$$

Indeed, we can put $u_{t,s}(x) = \exp_{u(x)}(t\nu(x) + sw(x))$. Calculating the mixed derivatives of $\mathcal{J}(u_{t,s})$ with respect to t and s, we obtain the relations

$$\frac{\partial^2}{\partial s\partial t}\Big|_{t,s=0}\mathcal{J}(u_{t,s}) = (L_*(u)\nu, w) + \bigg(L(u), \nabla^*_{\frac{\partial}{\partial t}}\frac{\partial u_{t,s}}{\partial s}\Big|_{t,s=0}\bigg),$$

$$\frac{\partial^2}{\partial t\partial s}\Big|_{t,s=0}\mathcal{J}(u_{t,s}) = (L_*(u)w, \nu) + \bigg(L(u), \nabla^*_{\frac{\partial}{\partial s}}\frac{\partial u_{t,s}}{\partial t}\Big|_{t,s=0}\bigg).$$

Since the connection is torsion-free, we have

$$\nabla^*_{\frac{\partial}{\partial t}}\frac{\partial u_{t,s}}{\partial s} = \nabla^*_{\frac{\partial}{\partial s}}\frac{\partial u_{t,s}}{\partial t}.$$

Combining the above relations, we obtain

$$(L_*(u)\nu, w) = (\nu, L_*(u)w).$$

The lemma is proved. □

In particular, if $L_*(u)$ is elliptic, relation (3.7) implies that $\operatorname{ind} L_*(u) = 0$. By Lemma 3.3, the linearization $L_*(u)$ of the Euler–Lagrange operator at a critical point u (i.e., at a solution of the equation $L(u) = 0$) is independent of the choice of a connection on M'. The quadratic form determined by this operator is called the *Hessian* of the functional $\mathcal{J}$ at u.

Example (the harmonic map operator)**.** Riemannian metrics on M and M' give rise to a metric on the bundle $J^1(M, M')$ over the space $M \times M'$ (see (1.7)). We consider the functional (4.6) with the Lagrangian

$$\mathcal{L}(x, u(x), du(x)) = \frac{1}{2}\|du(x)\|^2.$$

As in [13, 14], we use the notation $e(u)(x) = \mathcal{L}(x, u(x), du(x))$ and call $e(u)$ the *energy density* of a map u. The corresponding functional (4.6) is denoted by E, and $E(u)$ is called the *energy* of u. The Euler–Lagrange operator of the energy functional taken with the minus sign is denoted by $\tau(u)$ and is called the *harmonic map operator*. The solutions of the equation $\tau(u) = 0$ are called the *harmonic mappings* from M to M'. Let $\{x^i\}$, $\{u^\alpha\}$ be local coordinates on M and M', respectively, and let $\{x^i, u^\alpha, u^\beta_{x^j}\}$ be the natural local coordinates on $J^1(M, M')$. In these coordinates, the energy density function e has the form

$$e(x, u, \partial u) = \frac{1}{2} g^{ij}(x) {g'}_{\beta\gamma}(u) u^\beta_{x^i} u^\gamma_{x^j}.$$

Now, we find the form of the corresponding Euler–Lagrange operator in local coordinates. (If $M = S^1$, the equation $\tau(u) = 0$ coincides with the equation for the closed geodesics in M', and such calculation can be found in [3, Chapter 6].) We have

$$\frac{\partial e}{\partial u^\beta_{x^i}} = g^{ij} {g'}_{\beta\gamma} u^\gamma_{x^j}, \quad \frac{\partial e}{\partial u^\beta} = \frac{1}{2} g^{ij} \frac{\partial g'_{\delta\gamma}}{\partial u^\beta} u^\delta_{x^i} u^\gamma_{x^j},$$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial e}{\partial u^\beta_{x^i}} = g^{ij} {g'}_{\beta\gamma} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial u^\gamma}{\partial x^j} + g^{ij} \frac{\partial u^\gamma}{\partial x^j} \frac{\partial g'_{\beta\gamma}}{\partial y^\delta} \frac{\partial u^\delta}{\partial x^i}.$$

Observe that the following relation is fulfilled:

$$g'^{\alpha\beta} g^{ij} \frac{\partial u^\gamma}{\partial x^j} \frac{\partial u^\delta}{\partial x^i} \frac{\partial g'_{\beta\gamma}}{\partial y^\delta} = \frac{1}{2} g^{ij} g'^{\alpha\beta} \left(\frac{\partial g'_{\beta\gamma}}{\partial y^\delta} + \frac{\partial g'_{\beta\delta}}{\partial y^\gamma} \right) \frac{\partial u^\gamma}{\partial x^j} \frac{\partial u^\delta}{\partial x^i}.$$

Recalling (4.8), we finally obtain

$$\tau(u)^\alpha = g^{ij} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial u^\alpha}{\partial x^j} + g^{ij} \Gamma'^\alpha_{\delta\gamma} \frac{\partial u^\delta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}. \tag{4.9}$$

In particular, the symbol $\sigma_\tau = \sigma_\tau(x, u, \partial u)$ is scalar and has the form

$$\sigma_\tau(w) = g^{ij}(x) w_i w_j I,$$

where $(x, u, \partial u) \in J^1(M, M')$, $w \in T^*_x M$, and $w = w_k dx^k$. The local form of $\tau(u)$ obtained above, Lemma 2.2, and the form (2.7) of the symbol in local coordinates show that the operator of harmonic mappings is a quasilinear elliptic differential operator. If $M' = \mathbb{R}^{n'}$ is the Euclidean space, then $\tau(u)$ is also called the *Laplace–Beltrami operator* on M.

The paper [14] presents a classical treatment of harmonic mappings. For the progress in this field, see [13].

Below, trace_g means the trace (of a quadratic form) with respect to the metric g on M, and R' denotes the connection curvature tensor on M'.

Proposition 4.2. *The linearization $\tau_*(u)$ of the harmonic map operator with respect to the Levi-Civita connection ∇' on M' is given by the formula*

$$\tau_*(u)\nu = \operatorname{trace}_g \nabla^2\nu + \operatorname{trace}_g R'(du,\nu)du. \tag{4.10}$$

If u is a harmonic mapping, then the Hessian of the energy functional is given by the operator $J(u) = -\tau_*(u)$, which is called the *Jacobi operator* for the mapping u.

We prove (4.10) for the case of the operator of closed geodesics. In the general case, the proof is only more technical.

Proof (the case of $\dim M = 1$). In the notation of Subsection 4.1, we must linearize the operator

$$u \longmapsto \nabla^*_{\frac{\partial}{\partial s}}\left(\frac{du}{ds}\right), \quad s \in S^1.$$

Let u_t be a family of mappings satisfying (3.1). Then from the definition of the curvature tensor it follows that

$$\nabla^*_{\frac{\partial}{\partial t}}\nabla^*_{\frac{\partial}{\partial s}}\left(\frac{du_t}{ds}\right) = \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial t}}\left(\frac{du_t}{ds}\right) + R'\left(\frac{du_t}{ds},\frac{du_t}{dt}\right)\frac{du_t}{ds}.$$

Moreover, by (4.1),

$$\nabla^*_{\frac{\partial}{\partial t}}\frac{du_t}{ds}\bigg|_{t=0} = \nabla^*_{\frac{\partial}{\partial s}}\nu.$$

Thus, the linearized operator has the form

$$\tau_*(u)\nu = \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}}\nu + R'\left(\frac{du}{ds},\nu\right)\frac{du}{ds}, \quad \nu \in C^\infty(u^*TM'),$$

which proves the proposition. □

4.4. A remark on the construction of second order operators. In this section, we present a method of constructing second order elliptic differential operators with scalar symbol.

Let ∇ and ∇' be arbitrary connections on M and M', respectively, and let $\mathcal{D}^2u$ be the second fundamental form of a mapping u, constructed with respect to these connections. By Definition 2.7, the trace of $\mathcal{D}^2u$ with respect to an arbitrary Riemannian metric g on M yields the quasilinear elliptic differential operator

$$u \longmapsto \operatorname{trace}_g \mathcal{D}^2u. \tag{4.11}$$

Example. Let M and M' be manifolds with Riemannian metrics g and g', respectively, and let ∇ and ∇' be their Levi-Civita connections. Comparison of the local form (1.4) of $\mathcal{D}^2u$ with the local form of $\tau(u)$ (see (4.9)) shows that, in this case, the operator (4.11) is the harmonic map operator.

Example. Let M be a complex manifold with a connection ∇ in TM, with a complex structure J, and with an almost Hermitian metric g (i.e., with a metric g such that $g(JX,JY) = g(X,Y)$ for all vector fields X and Y). The metric g extends to a complex-valued bilinear form g^c on the complexified real tangent bundle T^cM and determines a Hermitian form $\bar g$ on the holomorphic tangent bundle T_JM by the formula $\bar g(X,Y) = g^c(X,\bar Y)$. Let M' be an arbitrary Riemannian manifold with Levi-Civita connection ∇'. Using connections on M and M', we construct the second fundamental form $\mathcal{D}^2u$. We extend this form to a complex-bilinear mapping $T^c_xM \times T^c_xM \to T^c_{u(x)}M'$ and denote by $(\mathcal{D}^2u)^{1,1}$ the $(1,1)$-component of the extension. By direct calculation, we obtain the relation

$$\frac{1}{2}\operatorname{trace}_g \mathcal{D}^2u = \operatorname{trace}_{\bar g}(\mathcal{D}^2u)^{1,1}. \tag{4.12}$$

In local complex coordinates $(z^1,\dots,z^n)$ on M and in coordinates $(u^1,\dots,u^{n'})$ on M', the right-hand side of (4.12) has the form

$$\bar g^{i\bar j}\left(\frac{\partial^2 u^\alpha}{\partial z^i\partial z^{\bar j}}-\Gamma^l_{i\bar j}\frac{\partial u^\alpha}{\partial z^l}-\Gamma^{\bar l}_{i\bar j}\frac{\partial u^\alpha}{\partial z^{\bar l}}+\Gamma'^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial z^i}\frac{\partial u^\gamma}{\partial z^{\bar j}}\right),\quad \alpha=1,\dots,n'. \tag{4.13}$$

Assume that the connection ∇ on M is complex, i.e., ∇ is torsion-free and $\nabla J=0$. In this case, its Christoffel symbols involved in (4.13) vanish. The operator (4.12) takes the form

$$\sigma(u)^\alpha=\bar g^{i\bar j}\left(\frac{\partial^2 u^\alpha}{\partial z^i\partial z^{\bar j}}+\Gamma'^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial z^i}\frac{\partial u^\gamma}{\partial z^{\bar j}}\right)$$

and is called the *Hermitian-harmonic map operator*. In particular, if M is Kähler, then this operator coincides with the harmonic map operator. An important property of $\sigma(u)$ is that the holomorphic and antiholomorphic mappings are solutions of the equation $\sigma(u)=0$ even if the manifold M is not Kähler. This fact was used in [17] for the study of Hermitian manifolds.

§5. Moduli spaces for the solutions of quasilinear elliptic differential equations

In what follows, we assume that M is a compact manifold.

5.1. Manifolds of mappings. Let $H^m(M,\mathbb{R}^N)$, where $m\in\mathbb{Z}$, $m\ge 0$, be the Sobolev space of mappings from M to $\mathbb{R}^N$ (see [11, 21]). For a smooth manifold M' embedded in $\mathbb{R}^N$, we define the space $H^m(M,M')$ as follows:

$$H^m(M,M')=\{u\in H^m(M,\mathbb{R}^N): u(x)\in M' \text{ for almost all } x\in M\}.$$

By a Hilbert (Banach) manifold we mean a Hilbert (Banach) manifold with separable model space (see [6, 8, 21]).

Proposition 5.1 [22]**.** *If $2m>n=\dim M$, then $H^m(M,M')$ is a C^∞-smooth submanifold of the Hilbert space $H^m(M,\mathbb{R}^N)$. Moreover, $H^m(M,M')$ has the same homotopy type as $C^0(M,M')$.*

In a natural way, the Hilbert structure of the space $H^m(M,\mathbb{R}^N)$ yields a Riemannian metric on $H^m(M,M')$. Normalizing it (if necessary) and referring to the Sobolev embedding theorem, we may assume that the following inequality is fulfilled for all u, $v\in H^m(M,M')$:

$$\operatorname{dist}(u,v)\ge\max_{x\in M}\operatorname{dist}_{M'}(u(x),v(x)), \tag{5.1}$$

where dist and $\operatorname{dist}_{M'}$ are the distance functions related to the metrics on $H^m(M,M')$ and M', respectively. The tangent space $T_uH^m(M,M')$ can naturally be identified with the space $H^m(u^*TM')$ formed by the H^m-smooth vector fields along u. The system of spaces $H^m(u^*TM')$, where $u\in H^m(M,M')$, can be taken as a system of model spaces of the manifold $H^m(M,M')$. Indeed, a neighborhood of zero in $H^m(u^*TM')$ is identified with a neighborhood of u in $H^m(M,M')$ under the exponential mapping

$$H^m(u^*TM')\ni\mathsf{v}(x)\mapsto\exp_{u(x)}\mathsf{v}(x)\in H^m(M,M').$$

Arguing as in [6, 12], we can prove that the transition functions

$$H^m(u_1^*TM')\ni\mathsf{v}(x)\mapsto\exp^{-1}_{u_2(x)}\circ\exp_{u_1(x)}\mathsf{v}(x)\in H^m(u_2^*TM')$$

are C^∞-smooth.

Let $H^m_{[v]}(M,M')$ denote the Hilbert manifold of H^m-smooth mappings from M to M' that are C^0-homotopic to a fixed $v\in C^\infty(M,M')$.

Let M'' be another smooth manifold.

Proposition 5.2. *Suppose $\phi \in C^{m+r}(M', M'')$, where $r \geq 0$. If $2m > n = \dim M$, then the mapping*

$$H^m(M, M') \ni u \mapsto \phi \circ u \in H^m(M, M'') \tag{5.2}$$

is a well-defined C^r-smooth mapping from $H^m(M, M')$ to $H^m(M, M'')$. In particular, if ϕ is a diffeomorphism, then (5.2) *yields a diffeomorphism from $H^m(M, M')$ onto $H^m(M, M'')$.*

The proof follows from the preceding proposition and the fact that if ϕ belongs to the class $C^{m+r}(\mathbb{R}^{N_1}, \mathbb{R}^{N_2})$ with $r \geq 0$ and $2m > n = \dim M$, then

$$H^m(M, \mathbb{R}^{N_1}) \ni u \mapsto \phi \circ u \in H^m(M, \mathbb{R}^{N_2})$$

is a well-defined C^r-smooth mapping of Sobolev spaces.

5.2. Spaces of functional parameters. Let B be a smooth manifold, and let E be the total space of a finite-dimensional real vector bundle over B with fiber F_b over $b \in B$. We need Banach spaces of weighted C^s-smooth sections of E.

Suppose E is equipped with a Riemannian structure, i.e., with a metric and a connection that are compatible. Then, for a Riemannian metric on the base and a collection of continuous real-valued positive functions $\rho_l(b)$, $l = 0, \dots, s$, we introduce the weighted norm by

$$\|w\| = \sup_{b \in B} \sum_{l=0}^{s} \rho_l(b) \|\nabla^l w(b)\|. \tag{5.3}$$

Here $\nabla^l w(b)$ is an element of the space of l-linear mappings from $T_b B$ to F_b, and the metric on this space is induced by the metrics on $T_b B$ and F_b. We consider the space formed by the sections w such that $\|w\| < +\infty$; obviously, this is a Banach space with respect to the norm (5.3). This space depends on the choice of the functions ρ_l, on the Riemannian structure on the bundle E, and on the Riemannian metric on B. If the manifold B is compact, then all spaces defined in this way are isomorphic.

We note that each of the Banach spaces defined above contains all sections with compact support. Convergence in these spaces implies C^s-convergence on the compact subsets of B. In the sequel, by the space of C^s-smooth sections of a vector bundle we shall mean the Banach space formed by the sections with finite norm (5.3) for a fixed collection of positive functions ρ_l.

The Banach spaces of sections of vector bundles with the norm (5.3) that we consider below are spaces of functional parameters of perturbations of quasilinear elliptic differential equations. An important property of such spaces is that they satisfy a certain transversality condition. We consider the Banach space $\mathcal{V}$ formed by the C^s-smooth nonautonomous vector fields (the sections of the bundle $(\Pi')^* TM'$). We shall show (see Lemma 5.9) that the transversality property is fulfilled for this space. Moreover, this property is shared also by all spaces "containing" $\mathcal{V}$.

More precisely, let $\mathcal{F}$ be the Banach space of C^s-smooth systems of coefficients of differential operators of order k (i.e., $\mathcal{F}$ is formed by the C^s-smooth sections of the bundle $(\Pi'_k)^* TM'$). We denote by i the embedding

$$i : C^s((\Pi')^* TM') \to C^s((\Pi'_k)^* TM')$$

induced by the projection $J^k \to M \times M'$. We have

$$i(\mathsf{g})(j^k u) = \mathsf{g}(x, u) \quad \text{for every jet } j^k u \quad \text{in } J^k(M, M'),\ x \in M,\ u \in M'. \tag{5.4}$$

Definition 5.1. We say that a space $\mathcal{F}$ *is not less than* the space $\mathcal{V}$ if the mapping i given by (5.4) is a continuous embedding of the Banach space $\mathcal{V}$ into $\mathcal{F}$.

Example. Obviously, the space $\mathcal{V}$ is not less than itself.

Example (quasilinear first order perturbations)**.** Consider C^s-smooth quasilinear sections f of the bundle $(\Pi'_1)^*TM'$ (see Subsection 2.1). They have the form

$$\mathsf{f}(x, u(x), du(x)) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \quad x \in M,$$

where G is a morphism between the vector bundles $J^1(M, M')$ and TM' (the principal homogeneous part of f) and g is a nonautonomous vector field on M'. If M and M' are Riemannian manifolds, then the corresponding metrics give rise to Banach spaces $\mathcal{G}$ and $\mathcal{V}$ formed by the C^s-smooth morphisms G and the nonautonomous vector fields g, respectively. The elements of the product $\mathcal{F} = \mathcal{G} \times \mathcal{V}$ parametrize the set of first order quasilinear differential operators. Obviously, the Banach space $\mathcal{F}$ obtained in this way is not less than the space $\mathcal{V}$ of vector fields.

Example (construction of kth order perturbations)**.** Let M and M' be Riemannian manifolds with metrics g and g', respectively. The Levi-Civita connection of g' yields a Riemannian structure on the bundles $(\Pi')^*TM'$ and $(\Pi'_k)^*TM'$. Moreover, these metrics give rise to a natural Riemannian metric on the total space of the bundle $\bigoplus_{p=0}^{s} L^p_{\mathrm{sym}}$ over $M \times M'$ (see Subsection 1.2). (Indeed, the Riemannian structures on M and M' induce a Riemannian structure on the bundle $\bigoplus_{p=0}^{s} L^p_{\mathrm{sym}}$. The connection of this structure decomposes of the tangent space into the horizontal and vertical subspaces, which are canonically isomorphic to the fiber and to the tangent space of the base, respectively. This decomposition and the canonical isomorphisms allow us to define the required metric. For the details, see [2].)

The diffeomorphism from Corollary 1.2 pulls this metric back to $J^k(M, M')$. Using the collection of positive functions ρ_l and the Riemannian metric on $J^k(M, M')$ constructed above, we define the Banach space $\mathcal{F}$ with the norm (5.3) formed by the sections of the bundle $(\Pi'_k)^*TM'$. Similarly, the functions $\rho_l\big|_{M\times M'}$ and the metric $g \times g'$ on $M \times M'$ give rise to the Banach space $\mathcal{V}$ of sections of the bundle $(\Pi')^*TM'$. The resulting spaces are such that $\mathcal{F}$ is not less than $\mathcal{V}$.

5.3. Quasilinear elliptic problem. Let A be a quasilinear elliptic differential operator of order $k \geq 1$ defined on the mappings from M to M'. We choose integers m, r, and s such that

$$m > \frac{n}{2} + k - 1, \quad r \geq 0, \quad \text{and} \quad s = r + m - k + 1. \tag{5.5}$$

Let $\mathcal{F}$ be the Banach space of C^s-smooth systems of coefficients of $(k-1)$st order differential operators (see Subsection 5.2). For $\mathsf{f} \in \mathcal{F}$, we consider the differential equation

$$Au(x) + \mathsf{f}(j^{k-1}u(x)) = 0, \quad x \in M, \tag{5.6}$$

for mappings from M to M' of a fixed homotopy class,

$$u \in [v], \quad v \in C^\infty(M, M'). \tag{5.7}$$

A mapping u from M to M' is called an H^m-*solution* of the quasilinear elliptic problem (5.6), (5.7) (in the sequel, we call an H^m-solution simply a solution) if it belongs to the space $H^m(M, M')$ and satisfies (5.6), (5.7). The Banach space $\mathcal{F}$ is called the *space of functional parameters* of problem (5.6), (5.7).

We want to study the space of solutions of a quasilinear elliptic equation for different values of the functional parameter $\mathsf{f} \in \mathcal{F}$. The set

$$\mathfrak{M}_{A,[v]} = \{(u, \mathsf{f}) : u \in H^m(M, M'),\ \mathsf{f} \in \mathcal{F},\ \text{and } (u, \mathsf{f}) \text{ satisfies (5.6)–(5.7)}\}$$

is called the *moduli space* of solutions of the quasilinear elliptic problem. The mapping $\pi_A : \mathfrak{M}_{A,[v]} \to \mathcal{F}$ given by the rule

$$\mathfrak{M}_{A,[v]} \ni (u, \mathsf{f}) \mapsto \mathsf{f} \in \mathcal{F}$$

is called the *natural projection of the moduli space* onto the space $\mathcal{F}$ of functional parameters. The fibers $\pi^{-1}(\mathsf{f})$ are the sets of solutions of equation (5.6) for a fixed value of the functional parameter f.

We note that, by relations (5.5) and the elliptic regularity property, the solutions of (5.6) are sufficiently smooth in the classical sense. More precisely, we have

$$\mathfrak{M}_{A,[v]} \subset C^{m+r}(M, M') \times C^s((\Pi'_{k-1})^* TM'). \tag{5.8}$$

Indeed, this follows from the fact that, in local coordinates, equation (5.6) is a quasilinear elliptic system of differential equations (see Lemma 2.2). In the notation of the corresponding example in Subsection 2.1, we have

$$\begin{aligned} &\mathsf{A}^{i_1\dots i_k}\big(x, u, \partial u, \dots, \partial^{k-1} u\big) \cdot (\partial^k u / \partial x^{i_1} \cdots \partial x^{i^k}) + \mathsf{g}\big(x, u, \partial u, \dots, \partial^{k-1} u\big) \\ &\qquad = \mathsf{f}\big(x, u, \partial u, \dots, \partial^{k-1} u\big). \end{aligned} \tag{5.9}$$

Let $\mathsf{h} = \mathsf{f} - \mathsf{g}$. From the assumption that u is an H^m-solution and the corresponding embedding theorem it follows that the matrices

$$\mathsf{A}^{i_1\dots i_k}(x) = \mathsf{A}^{i_1\dots i_k}\big(x, u(x), \partial u(x), \dots, \partial^{k-1} u(x)\big) \tag{5.10}$$

and the vectors

$$\mathsf{h}(x) = \mathsf{h}\big(x, u(x), \partial u(x), \dots, \partial^{k-1} u(x)\big) \tag{5.11}$$

are $C^{0,\alpha}$-smooth for all $0 < \alpha < 1$. Using the Schauder estimate (see [7]) for the solutions of the linear elliptic system

$$\mathsf{A}^{i_1\dots i_k}(x) \cdot (\partial^k u / \partial x^{i_1} \cdots \partial x^{i^k}) = \mathsf{h}(x), \tag{5.12}$$

we show that u belongs to $C^{k,\alpha}$. Consequently, $\mathsf{A}^{i_1\dots i_k}(x)$ and $\mathsf{h}(x)$ are $C^{1,\alpha}$-smooth, and the Schauder estimate implies that $u \in C^{k+1,\alpha}$. Repeating this procedure several times, we see that the mapping u is $C^{m+r,\alpha}$-smooth.

As in Subsection 5.2, we denote by $\mathcal{V}$ the Banach space of C^s-smooth nonautonomous vector fields on M'. The facts stated below concern the moduli space $\mathfrak{M}_{A,[v]}$ and the projection π_A; the proofs are presented in the next subsection.

Theorem 5.1. *Let A be a quasilinear elliptic differential operator. We assume that $r \geq 1$ and that the space $\mathcal{F}$ of functional parameters is not less than the Banach space $\mathcal{V}$. Then the moduli space $\mathfrak{M}_{A,[v]}$ of solutions of problem* (5.6), (5.7) *is a C^r-smooth submanifold of $H^m_{[v]}(M, M') \times \mathcal{F}$. The projection π_A is a C^r-smooth Fredholm mapping, and the index* $\operatorname{ind}(\pi_A)_*(u, \mathsf{f})$ *does not depend on* $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$ *and is equal to* $\operatorname{ind}_{[v]} A$.

We recall that a point $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$ is called a *regular point* of π_A if the differential $(\pi_A)_*(u, \mathsf{f})$ is surjective. In this case, the mapping u is called a *regular solution* of problem (5.6), (5.7). A point f is called a *regular point* of the space of functional parameters $\mathcal{F}$ if $\pi_A^{-1}(\mathsf{f})$ consists only of regular points of π_A (in particular, if $\pi_A^{-1}(\mathsf{f}) = \varnothing$). In other words, an element f of the space formed by the systems of coefficients of differential operators is regular if problem (5.6), (5.7) has only regular solutions or does not have any solutions. By the Sard–Smale theorem [24], if $r > \max(\operatorname{ind}_{[v]} A, 0)$, then all points in $\mathcal{F}$, except, possibly, for a set of the first Baire category, are regular. In particular, the set of regular points is everywhere dense in $\mathcal{F}$. As in [24], we obtain the following statements.

Corollary 5.2. *If A is a quasilinear differential operator with negative index at a homotopy class $[v]$, then the set of all $\mathsf{f} \in \mathcal{F}$ such that problem* (5.6), (5.7) *is solvable is of the first Baire category. In particular, the set of functional parameters for which the problem has only regular solutions is empty.*

Corollary 5.3. *If $r > \max(\operatorname{ind}_{[v]} A, 0)$, then, for each regular point $\mathsf{f} \in \mathcal{F}$, the set $\pi_A^{-1}(\mathsf{f})$ of solutions of* (5.6), (5.7) *is either empty, or is a C^r-smooth submanifold in $\mathfrak{M}_{A,[v]}$ of dimension* $\operatorname{ind}_{[v]} A$.

For an arbitrary section $\mathsf{f} \in \mathcal{F}$, the coefficients of the differential operator $\mathsf{f}(j^{k-1}u)_*$ linearized at a solution u of equation (5.6) are C^{s-1}-smooth. The following lemma gives a simple criterion for u to be regular.

Lemma 5.4. *A solution u of problem* (5.6), (5.7) *is regular if and only if the equation*

$$A_*(u)\mathsf{v} + \mathsf{f}(j^{k-1}u)_*\mathsf{v} = 0, \ \mathsf{v} \in H^m(u^*TM'),$$

has exactly $\operatorname{ind}_{[v]} A$ *linearly independent solutions.*

In particular, we see that the fact that a section f is regular does not depend on the choice of a space $\mathcal{F}$ containing f and satisfying the conditions of Theorem 5.1.

Let $\mathcal{O}$ be a domain in a Banach space $\mathcal{F}$. A mapping $K : \mathcal{O} \to \mathcal{F}$ is said to be *finite-dimensional* if there exists a subspace in $\mathcal{F}$ that has finite dimension and contains $\operatorname{Im} K$. A mapping from $\mathcal{O}$ to $\mathcal{F}$ is *quasifinite-dimensional* if it has the form

$$\mathcal{O} \ni \mathsf{f} \mapsto \mathsf{f} + K(\mathsf{f}) \in \mathcal{F},$$

where K is a finite-dimensional mapping. It is obvious that a smooth finite-dimensional mapping is Fredholm and its index is zero.

Definition 5.2 (see [1, 20])**.** A C^r-smooth manifold with model Banach space $\mathcal{F}$ is said to be *quasifinite-dimensional* if it admits an atlas with C^r-smooth quasifinite-dimensional transition functions.

Theorem 5.5 (see [1, 20])**.** *Under the assumptions of Theorem* 5.1, *$\mathfrak{M}_{A,[v]}$ is a C^r-smooth quasifinite-dimensional Banach manifold with the model space $\mathcal{F}$, or $\mathcal{F} \oplus \mathbb{R}^{\operatorname{ind}_{[v]} A}$, or $\mathcal{F} \ominus \mathbb{R}^{-\operatorname{ind}_{[v]} A}$ depending on whether* $\operatorname{ind}_{[v]} A = 0$, $\operatorname{ind}_{[v]} A > 0$, *or* $\operatorname{ind}_{[v]} A < 0$, *respectively.*

Below, we follow the lines of [24] and make use of a quasilinear elliptic differential equation to define an invariant of the family of equations (5.6), (5.7).

We recall that a mapping between topological spaces is said to be *proper* if the preimage of every compact set is compact.

Definition 5.3. We say that the quasilinear elliptic problem (5.6), (5.7) is *compact* with respect to a domain $\mathcal{U} \subset \mathcal{F}$ if the projection mapping $\pi_A\big|_{\pi_A^{-1}(\mathcal{U})}$ restricted to the Banach manifold $\pi_A^{-1}(\mathcal{U})$ is proper.

Observe that if $\pi_A\big|_{\pi_A^{-1}(\mathcal{U})}$ is proper, then the set of regular $\mathsf{f} \in \mathcal{U}$ is open.

We say that two compact manifolds are *nonorientably cobordant* if the disjoint union of them is the boundary of a compact manifold. The cobordism relation between l-dimensional manifolds is an equivalence relation, and the operation of disjoint union turns the set of equivalence classes into a group called the group of l-dimensional nonoriented cobordisms.

Let $r > \operatorname{ind}_{[v]} A + 1$, where $\operatorname{ind}_{[v]} A \geq 0$. Suppose that, for problem (5.6)–(5.7), the compactness condition with respect to a region $\mathcal{U} \subset \mathcal{F}$ is fulfilled. Then, for all regular f_1 and f_2 in $\mathcal{U}$, the manifolds $\pi_A^{-1}(\mathsf{f}_1)$ and $\pi_A^{-1}(\mathsf{f}_2)$ are cobordant (see [24]). In particular, if $\operatorname{ind}_{[v]} A = 0$, then for each regular $\mathsf{f} \in \mathcal{U}$ problem (5.6), (5.7) has only finitely many solutions, and the cobordism class of $\pi_A^{-1}(\mathsf{f})$ is identified with the parity of the number of these solutions. The above-mentioned element of the group of $\operatorname{ind}_{[v]} A$-dimensional nonoriented cobordisms is an invariant of the family of equations (5.6), (5.7). Summarizing the above, we obtain the following theorem.

Theorem 5.6. *Let A be a quasilinear elliptic differential operator with $\operatorname{ind}_{[v]} A \geq 0$, and let $r > \operatorname{ind}_{[v]} A + 1$. Suppose that problem* (5.6), (5.7) *has the compactness property with respect to a connected region $\mathcal{U}$ in some space $\mathcal{F}$ of functional parameters. Then, under the conditions of Theorem* 5.1, *there exists an invariant of the family of equations* (5.6), (5.7) *defined as the nonoriented cobordism class of the set of solutions of problem* (5.6), (5.7) *with a regular value $\mathsf{f} \in \mathcal{U}$ of the functional parameter.*

Since the range of a proper mapping is closed, we obtain the following statement.

Corollary 5.7. *If the invariant described in Theorem* 5.6 *is nonzero, then problem* (5.6), (5.7) *has at least one solution for each $\mathsf{f} \in \mathcal{U}$.*

The invariant constructed in Theorem 5.6 is computable practically only if $\operatorname{ind}_{[v]} A = 0$. Actually, in part 2 of this paper (to be published elsewhere) we shall follow the lines of [20] to show that in this case the moduli space $\mathfrak{M}_{A,[v]}$ is an oriented manifold. Respectively, if problem (5.6), (5.7) has the compactness property, then there is a well-defined degree of the mapping π_A, which is equal to the algebraic number of solutions of the problem for a typical value of the functional parameter f. This degree is a finer invariant than the parity of the number of solutions.

We note that if a manifold M' is replaced with a diffeomorphic one and the operator A and the homotopy class $[v]$ are transformed accordingly (see Subsection 2.1), then the regular values f remain regular, and the sets of solutions of problem (5.6), (5.7) are transformed diffeomorphically. In particular, acting on the pairs $(A, [v])$, the diffeomorphisms of the manifold M' do not change the invariant in question.

5.4. Proofs of the facts stated in Subsection 5.3. Inequality (5.1) and the fact that M is compact imply that for every mapping $u_0 \in H^m_{[v]}(M, M')$ there exists $\delta > 0$ such that, for every $u \in H^m_{[v]}(M, M')$ with $\operatorname{dist}(u, u_0) < \delta$ and every $x \in M$, there exists a unique shortest geodesic connecting $u(x)$ with $u_0(x)$. The parallel transport $\Phi(u(x), u_0(x))$ along such a geodesic yields an isomorphism $\Phi(u, u_0)$ of the spaces of H^l-smooth vector fields $H^l(u^*TM')$ and $H^l = H^l(u_0^*TM')$ for every integer l with $0 \leq l \leq m$.

Let $u_0 \in H^m_{[v]}(M, M')$. We denote by $W(u_0)$ the δ-neighborhood of u_0 with $\delta > 0$ chosen as above, and consider the mapping $\Psi : W(u_0) \times \mathcal{F} \to H^{m-k}$ defined by the rule

$$(u, \mathsf{f}) \xmapsto{\Psi} \Phi(u, u_0) \circ (A(u) + \mathsf{f}(j^{k-1}u)). \tag{5.13}$$

A pair $(u, \mathsf{f}) \in W(u_0) \times \mathcal{F}$ satisfies (5.6), (5.7) if and only if $\Psi(u, \mathsf{f}) = 0$. Under the assumptions of Theorem 5.1, we have the following statement.

Lemma 5.7. *The mapping Ψ defined by* (5.13) *is C^r-smooth.*

The proof of Lemma 5.7 follows from Proposition 5.2 and is similar to the proof of the corresponding statement in [20, Appendix A.4].

The differential $d\Psi$ at $(u, \mathsf{f}) \in W(u_0) \times \mathcal{F}$ is a continuous linear mapping

$$d\Psi(u, \mathsf{f}) : H^m(u^*TM') \times \mathcal{F} \to H^{m-k}$$

of the form

$$d\Psi(u, \mathsf{f}) = \Psi'_u(u, \mathsf{f})du + \Psi'_{\mathsf{f}}(u, \mathsf{f})d\mathsf{f}$$

with

$$\Psi'_u(u,\mathsf{f}) : H^m(u^*TM') \to H^{m-k}, \quad \Psi'_{\mathsf{f}}(u,\mathsf{f}) : \mathcal{F} \to H^{m-k}.$$

Lemma 5.8. *For each* $\mathsf{f} \in \mathcal{F}$, *we have*

$$\Psi'_u(u_0,\mathsf{f}) = A_*(u_0) + \mathsf{f}(j^{k-1}u_0)_*.$$

In particular, the operator $\Psi'_u(u_0,\mathsf{f})$ *is Fredholm, and its index is equal to* $\operatorname{ind}_{[v]} A$.

Proof. First, we note that if $\Phi_\gamma(t,s) : T_{\gamma(t)}M' \to T_{\gamma(s)}M'$ is the parallel transport along a curve γ, then for every vector field $w(t) \in T_{\gamma(t)}M'$ along this curve we have

$$\frac{d}{dt}\Big|_{t=0} \Phi_\gamma(t,0)w(t) = \nabla^*_{\frac{d}{dt}}\Big|_{t=0} w(t), \tag{5.14}$$

where $\nabla^*_{\frac{d}{dt}}$ is the covariant derivative along γ. This fact becomes obvious if we represent relation (5.14) in the coordinates in some base of parallel vector fields along γ.

As in Subsection 3.1, for any $\mathsf{v} \in H^m(u^*TM')$ we can find a smooth path u_t in $W(u_0)$ such that (3.1) is satisfied with $u = u_0$ and for each $x \in M$ the curve $u_t(x)$, $t \in (-\varepsilon,\varepsilon)$, is a geodesic in M'. Then

$$\Psi'_u(u_0,\mathsf{f})\mathsf{v} = \frac{d}{dt}\Big|_{t=0} \Psi(u_t,\mathsf{f}).$$

By (5.14), the right-hand side of the above equation has the form

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} &\big[\Phi(u_t(x),u_0(x)) \circ \big(A(u_t)(x) + \mathsf{f}(j^{k-1}u_t(x))\big)\big] \\ &= \nabla^*_{\frac{d}{dt}}\Big|_{t=0} A(u_t)(x) + \nabla^*_{\frac{d}{dt}}\Big|_{t=0} \mathsf{f}(j^{k-1}u_t(x))\end{aligned}$$

for all $x \in M$, where $\Phi(u_t(x),u_0(x))$ is the parallel transport from $u_t(x)$ to $u_0(x)$ along the curve $u_s(x)$, and s takes values between 0 and t. By Definition 3.1, this fact implies the required statement. $\square$

Now, we consider the operator $\Psi'_{\mathsf{f}} = \Psi'_{\mathsf{f}}(u_0,\mathsf{f})\big|_{i(\mathcal{V})}$ restricted to the subspace $i(\mathcal{V}) \subset \mathcal{F}$, where $i : \mathcal{V} \to \mathcal{F}$ is the embedding (5.4). We have

$$i(\mathcal{V}) \ni i(\mathsf{g})(x) \mapsto \mathsf{g}(x,u_0(x)) \in C^s(u_0^*TM') \subset H^{m-k}, \quad x \in M. \tag{5.15}$$

Under the assumptions of Theorem 5.1, the following statement is true.

Lemma 5.9 (the transversality property)**.** *Let* $(u_0,\mathsf{f}) \in \mathfrak{M}_{A,[v]}$. *Then the operator* Ψ'_{f} *has a right inverse* $(\Psi'_{\mathsf{f}})^{-1}$, *which is a linear mapping from* $C^s(u_0^*TM')$ *to* $i(\mathcal{V})$.

Proof. With each tangent vector $\mathsf{v} \in T_yM'$, $y \in M'$, we associate a smooth vector field $\Theta(y,\mathsf{v})$ having a compact support on M' and constructed as follows. First, we choose a smooth function $\lambda : M' \times M' \to \mathbb{R}$ such that

$$\operatorname{supp}\lambda \subset \operatorname{Im}\exp \subset M' \times M',$$

$$\lambda|_{\mathcal{O}} = 1 \text{ in a neighborhood of } \mathcal{O} \text{ in the set } \operatorname{diag}(M' \times M'),$$

$$\lambda(y,\cdot) : M' \to \mathbb{R} \text{ has a compact support for every } y \in M'.$$

Here $\operatorname{Im}\exp$ is an open subset of $M' \times M'$ that is the diffeomorphic image of a neighborhood of the zero section in TM' under the mapping

$$TM' \ni (y,\mathsf{v}) \longmapsto (y, \exp_y \mathsf{v}) \in M' \times M'.$$

We spread a given vector $\mathsf{v} \in T_yM'$ over the neighborhood $\operatorname{Im}\exp_y$ of $y \in M'$ by using the parallel transport along the uniquely determined shortest geodesics. Multiplying the resulting vector field by the function $\lambda(y,\cdot)$, we obtain the required vector field $\Theta(y,\nu)$. Obviously, $\Theta(y,\nu)$ depends smoothly on (y,ν). By construction, we obtain

$$\Theta(y,\nu_1+\nu_2)=\Theta(y,\nu_1)+\Theta(y,\nu_2) \quad \text{and} \quad \Theta(y,\nu)\Big|_y=\nu. \tag{5.16}$$

For every vector field ν along the mapping u_0, we consider the section g of $(\Pi')^*TM'$ defined as follows:

$$\mathsf{g}(x,y)=\Theta(u_0(x),\nu(x))\Big|_y, \quad \nu\in C^{m-k}(u_0^*TM').$$

Since $(u_0,\mathsf{f})\in\mathfrak{M}_{A,[v]}$, the elliptic regularity property (5.8) implies that the mapping u_0 is of class $C^{m+r}(M,M')\subset C^s(M,M')$, and the section g is C^s-smooth. Moreover, this section has a compact support and, therefore, belongs to the space $\mathcal{V}$. Now, we define the right inverse operator $(\Psi'_{\mathsf{f}})^{-1}$ by putting $(\Psi'_{\mathsf{f}})^{-1}\nu=i(\mathsf{g})$. By (5.16), the operator $(\Psi'_{\mathsf{f}})^{-1}$ is linear. Moreover, from (5.15) and (5.16) it follows that

$$(\Psi'_{\mathsf{f}})i(\mathsf{g}(x))=\mathsf{g}(x,u_0(x))=\Theta(u_0(x),\nu(x))\Big|_{u_0(x)}=\nu(x),$$

which proves the lemma. □

Lemma 5.10. *Let $(u_0,\mathsf{f})\in\mathfrak{M}_{A,[v]}$. There exist decompositions of the spaces H^m and $\mathcal{F}$,*

$$H^m=H_1\oplus H_2,\quad \mathcal{F}=\mathcal{F}_1\oplus\mathcal{F}_2,$$

such that the subspaces H_1 and $\mathcal{F}_1$ are finite-dimensional, $\dim H_1-\dim\mathcal{F}_1=\operatorname{ind}_{[v]}A$, *and the linear mapping*

$$d\Psi(u_0,\mathsf{f})\Big|_{H_2\times\mathcal{F}_1}: H_2\times\mathcal{F}_1\to H^{m-k} \tag{5.17}$$

is an isomorphism.

Proof. First, we show that it suffices to prove the lemma for $\mathcal{F}=i(\mathcal{V})$ Indeed, assume that the lemma is proved for the space of functional parameters $i(\mathcal{V})$, and let $i(\mathcal{V})=\mathcal{V}_1\oplus\mathcal{V}_2$ be the corresponding decomposition. Then, for every space $\mathcal{F}$ satisfying the assumptions of Theorem 5.1, we put $\mathcal{F}_1=\mathcal{V}_1$. Since (by the Hahn–Banach theorem) every finite-dimensional subspace of a Banach space is complemented (see [10, Chapter 4]), we denote by $\mathcal{F}_2$ an arbitrary complement of $\mathcal{F}_1$ in $\mathcal{F}$ and obtain the required statement.

Now, we put $\mathcal{F}=i(\mathcal{V})$. Let $\Psi'_u=\Psi'_u(u_0,\mathsf{f})$, $\Psi'_{\mathsf{f}}=\Psi'_{\mathsf{f}}(u_0,\mathsf{f})$, and let $(\Psi'_u)^*$ be the differential operator formally adjoint to Ψ'_u. We put $Z=\operatorname{Ker}(\Psi'_u)^*$. The well-known properties of elliptic operators imply that the space Z is finite-dimensional. Moreover, since the operator $(\Psi'_u)^*$ is elliptic, we obtain the inclusion $Z\subset C^s(u_0^*TM')$ and the equation

$$Z\oplus\operatorname{Im}\Psi'_u=H^{m-k}.\text{[2]} \tag{5.18}$$

Putting $H_1=\operatorname{Ker}\Psi'_u$ and denoting by H_2 the orthogonal complement to H_1 in H^m, we see that $H^m=H_1\oplus H_2$, $\dim H_1-\dim Z=\operatorname{ind}\Psi'_u$, and the mapping

$$\Psi'_u: H_2\to\operatorname{Im}\Psi'_u$$

is an isomorphism. Putting $\mathcal{F}_1=(\Psi'_{\mathsf{f}})^{-1}Z$, we represent (5.18) in the form

$$(\Psi'_{\mathsf{f}})\mathcal{F}_1\oplus(\Psi'_u)H_2=H^{m-k},$$

[2] *Added in proofs of the English translation.* There is a flaw in the second paragraph of the proof of Lemma 5.10. For the vector space Z, the inclusion $Z\subset C^s(u_0^*TM')$ may fail. But this finite-dimensional subspace can always be approximated by a space contained in $C^s(u_0^*TM')$ and satisfying (5.18). Denoting this new space also by Z, we can finish the proof without changes.

which implies (5.17). Denoting by $\mathcal{F}_2$ an arbitrary complement to the finite-dimensional subspace $\mathcal{F}_1$ in $\mathcal{F}$, we obtain the required decomposition. The relation $\operatorname{ind}\Psi'_u = \operatorname{ind}_{[v]} A$ (see Lemma 5.8) completes the proof of Lemma 5.10. $\square$

Proof of Theorem 5.1. Let $\mathcal{O}$ be a sufficiently small neighborhood in $H^m_{[v]} \times \mathcal{F}$ of a point $(u_0, \mathsf{f}_0) \in \mathfrak{M}_{A,[v]}$; such that it can be identified with a ball in the space $H^m \times \mathcal{F}$ (see Subsection 5.1). By Lemma 5.10, the space $H^m \times \mathcal{F}$ can be decomposed as

$$H^m \times \mathcal{F} = (H_2 \times \mathcal{F}_1) \oplus (H_1 \times \mathcal{F}_2).$$

We denote by u_1 and u_2 the projections of an element $u \in H^m$ to the subspaces H_1 and H_2, respectively. Similarly, we represent the elements of $\mathcal{F}$ in the form $\mathsf{f}_1 + \mathsf{f}_2$, where $\mathsf{f}_1 \in \mathcal{F}_1$ and $\mathsf{f}_2 \in \mathcal{F}_2$. By Lemmas 5.7 and 5.10, we can apply the implicit function theorem to the mapping Ψ in a neighborhood of (u_0, f_0) in order to express (u_2, f_1) (possibly, in a smaller neighborhood) as a C^r-smooth function of (u_1, f_2). Thus, we obtain a C^r-smooth mapping of an open subset of $H_1 \times \mathcal{F}_2$ to an open subset of $\mathcal{O} \cap \mathfrak{M}_{A,[v]}$ containing (u_0, f_0),

$$(u_1, \mathsf{f}_2) \mapsto \big((u_1, u_2 = u_2(u_1, \mathsf{f}_2)), (\mathsf{f}_1 = \mathsf{f}_1(u_1, \mathsf{f}_2), \mathsf{f}_2)\big).$$

We view this map as a chart on $\mathfrak{M}_{A,[v]}$. Constructed for every $(u_0, \mathsf{f}_0) \in \mathfrak{M}_{A,[v]}$, such charts form a C^r-atlas of the moduli space.

We prove that the projection $\pi_* = (\pi_A)_*(u_0, \mathsf{f}_0) : \operatorname{Ker} d\Psi(u_0, \mathsf{f}_0) \to \mathcal{F}$ is a Fredholm mapping and that $\operatorname{ind}\pi_* = \operatorname{ind}\Psi'_u(u_0, \mathsf{f}_0)$. Indeed, it is easily seen that

$$\operatorname{Ker}\pi_* = \operatorname{Ker}\Psi'_u \times \{0\}, \quad \operatorname{Im}\pi_* = (\Psi'_{\mathsf{f}})^{-1}(\operatorname{Im}\Psi'_u), \tag{5.19}$$

where $(\Psi'_{\mathsf{f}})^{-1}$ stands for the preimage under the mapping Ψ'_{f}. We have the following (algebraic) isomorphisms of vector spaces:

$$\mathcal{F}\big/\operatorname{Im}\pi_* \simeq \mathcal{F}\big/(\Psi'_{\mathsf{f}})^{-1}(\operatorname{Im}\Psi'_u) \simeq \operatorname{Im}\Psi'_{\mathsf{f}}\big/\big(\operatorname{Im}\Psi'_u \cap \operatorname{Im}\Psi'_{\mathsf{f}}\big) \simeq H^{m-k}\big/\operatorname{Im}\Psi'_u.$$

Here, the second isomorphism is induced by the mapping Ψ'_{f}, and the third follows from the surjectivity of the operator

$$d\Psi(u_0, \mathsf{f}_0) = \Psi'_u \times \Psi'_{\mathsf{f}} : H^m \times \mathcal{F} \to H^{m-k}.$$

Thus, $\operatorname{codim}\operatorname{Im}\pi_* = \operatorname{codim}\operatorname{Im}\Psi'_u$, which proves the required statement if we take the first relation in (5.19) into account.

By Lemma 5.8, we have $\operatorname{ind}\Psi'_u(u_0, \mathsf{f}_0) = \operatorname{ind}_{[v]} A$, and this completes the proof of the theorem. $\square$

Proof of Lemma 5.4. A solution u_0 of problem (5.6), (5.7) is regular if and only if $\dim\operatorname{Ker}(\pi_A)_*(u_0, \mathsf{f}) = \operatorname{ind}(\pi_A)_*(u_0, \mathsf{f})$. However, from (5.19) it follows that

$$\dim\operatorname{Ker}(\pi_A)_*(u_0, \mathsf{f}) = \dim\operatorname{Ker}\Psi'_u(u_0, \mathsf{f}).$$

Since $\operatorname{ind}(\pi_A)_*(u_0, \mathsf{f}) = \operatorname{ind}_{[v]} A$ (see Theorem 5.1), Lemma 5.8 proves the required statement. $\square$

Proof of Theorem 5.5. We prove the statement in the case where $\operatorname{ind}_{[v]} A \geq 0$. (The case where $\operatorname{ind}_{[v]} A < 0$ is analyzed similarly.) By Lemma 5.10, the dimensions of the finite-dimensional spaces H_1 and $\mathcal{F}_1 \oplus \mathbb{R}^{\operatorname{ind}_{[v]} A}$ are equal. Fixing an arbitrary linear isomorphism $H_1 \simeq \mathcal{F}_1 \oplus \mathbb{R}^{\operatorname{ind}_{[v]} A}$, we identify the space $H_1 \times \mathcal{F}_2$ with $\mathcal{F} \oplus \mathbb{R}^{\operatorname{ind}_{[v]} A}$. Using the implicit function theorem as in the proof of Theorem 5.1, we construct a C^r-atlas of the space $\mathfrak{M}_{A,[v]}$, and the charts of this atlas are now homeomorphic to open subsets of $\mathcal{F} \oplus \mathbb{R}^{\operatorname{ind}_{[v]} A}$. We show that this atlas gives the structure of a finite-dimensional Banach manifold.

Let $\mathcal{O}$ and $\mathcal{O}'$ be charts with nonempty intersection. Then, in the notation of the proof of Theorem 5.1, the transition functions from $\mathcal{O}$ to $\mathcal{O}'$ can be represented in the form

$$\begin{aligned}&\mathcal{F}\oplus\mathbb{R}^{\operatorname{ind}_{[v]} A}\simeq H_1\oplus\mathcal{F}_2\supset\mathcal{O}\ni(u_1,\mathsf{f}_2)\\&\qquad\mapsto\big(u_1'(u_1,\mathsf{f}_2),\mathsf{f}_2'(u_1,\mathsf{f}_2)\big)\in\mathcal{O}'\subset H_1'\oplus\mathcal{F}_2'\simeq\mathcal{F}\oplus\mathbb{R}^{\operatorname{ind}_{[v]} A},\end{aligned}$$

where H_1 and H_1' are finite-dimensional spaces. The pairs (u_1,f_2) and (u_1',f_2') parametrize the same point (u,f) in $\mathfrak{M}_{A,[v]}$, so that f_2 and f_2' are the components of the vector $\mathsf{f}\in\mathcal{F}$ that correspond to the subspaces $\mathcal{F}_2$ and $\mathcal{F}_2'$, respectively. Consequently, f_2' takes the form

$$\mathsf{f}_2'(u_1,\mathsf{f}_2)=\mathsf{f}_2+\mathsf{f}_0(u_1,\mathsf{f}_2),\quad \mathsf{f}_0\in\mathcal{F}_1+\mathcal{F}_2,$$

which proves that the transition functions

$$(u_1,\mathsf{f}_2)\mapsto(u_1,\mathsf{f}_2)+(u_1',0)-(u_1,0)+\mathsf{f}_0$$

are finite-dimensional. Here, the sign $+$ means operation in $\mathcal{F}\oplus\mathbb{R}^{\operatorname{ind}_{[v]} A}$, and the summands are elements of the same space understood in the sense of the isomorphisms $H_1\simeq\mathcal{F}_1\oplus\mathbb{R}^{\operatorname{ind}_{[v]} A}$ and $H_1'\simeq\mathcal{F}_1'\oplus\mathbb{R}^{\operatorname{ind}_{[v]} A}$. □

§6. The compactness property for quasilinear elliptic differential equations

In this section, we describe conditions that ensure the compactness property for some classes of quasilinear elliptic differential equations.

6.1. Notation and remarks. As always, let A denote a quasilinear elliptic differential operator of order k that acts on the mappings from a compact manifold M to a manifold M'. We denote by m, r, and s some integers satisfying (5.5). We assume that the Banach space $\mathcal{F}$ of functional parameters satisfies the assumptions of Theorem 5.1. Let π_A be the natural projection of the moduli space $\mathfrak{M}_{A,[v]}$ to $\mathcal{F}$.

We recall that a family $\{u_\alpha\}_{\alpha\in A}$ of C^p-smooth mappings from M to M' is said to be *C^p-bounded* if there exists a compact subset N of $J^p(M,M')$ such that $j^pu_\alpha(M)\subset N$ for all $\alpha\in A$. This means that all sets $u_\alpha(M)$, $\alpha\in A$, lie in a compact subset of $K'\subset M'$, and, for each pair of charts (φ,U) and (ψ,V) on M and M', (respectively) and each compact subset K of U, there exists a constant $C>0$ such that for every mapping u_α, $\alpha\in A$, with $u_\alpha(K)\subset V$ we have

$$\max_{x\in\varphi(K)}\|\partial^l(\psi u_\alpha\varphi^{-1})(x)\|\le C\quad\text{for all } l,\ 1\le l\le p. \tag{6.1}$$

We denote by ∂^l the l-differential of a mapping from $\varphi(U)\subset\mathbb{R}^n$ to $\psi(V)\subset\mathbb{R}^{n'}$.

If M and M' are Riemannian manifolds, then the C^p-boundedness condition takes the following form. A family $\{u_\alpha\}$, $\alpha\in A$, is C^p-bounded if there exists a compact subset $K'\subset M'$ and a constant $C>0$ such that, for all $\alpha\in A$,

$$u_\alpha(M)\subset K'\quad\text{and}\quad\max_{x\in M}\|\mathcal{D}^lu_\alpha(x)\|\le C\text{ for all } l,\ 1\le l\le p.$$

Here, by the norm of the covariant differential $\mathcal{D}^lu(x)$ we mean the natural norm of an l-linear form with respect to the metrics g and g' (see Subsection 1.2).

Obviously, every sequence that converges in the C^p-topology is C^p-bounded. Conversely, the Arzelà–Ascoli theorem implies that if a sequence of C^p-smooth ($p\ge1$) mappings $u_i: M\to M'$ is C^p-bounded, then there exists a subsequence of it that converges in the C^{p-1}-topology.

In essence, the following lemma is a consequence of the elliptic regularity property.

Lemma 6.1. *Let A be a quasilinear elliptic differential operator of order $k\ge1$. The following conditions are equivalent.*

(i) *Problem* (5.6), (5.7) *has the compactness property with respect to a region* $\mathfrak{U}$ *of the space* $\mathfrak{F}$.
(ii) *For every sequence* $(u_i, \mathsf{f}_i) \in \mathfrak{M}_{A,[v]}$, $\mathsf{f}_i \in \mathfrak{U}$, *such that* $\mathsf{f}_i \to \mathsf{f} \in \mathfrak{U}$, *there exists a subsequence* u_{i_l} *that converges in the* C^{m+r}*-topology to a mapping* u *such that* $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$.
(iii) *For every sequence* $(u_i, \mathsf{f}_i) \in \mathfrak{M}_{A,[v]}$, $\mathsf{f}_i \in \mathfrak{U}$, *such that* $\mathsf{f}_i \to \mathsf{f} \in \mathfrak{U}$, *there exists a* C^k*-bounded subsequence* u_{i_l}.

Proof. (i) $\implies$ (ii). Suppose a sequence (u_i, f_i) satisfies the assumptions of item (ii). We denote by F the set $\{\mathsf{f}_i\}_{i\in\mathbb{N}} \cup \{\mathsf{f}\}$. Using the embedding (5.8) and the fact that π is proper, we see that the set $\pi^{-1}(F)$ is compact in $C^{m+r}(M, M') \times \mathfrak{F}$. Therefore, there exists a subsequence $(u_{i_l}, \mathsf{f}_{i_l})$ converging to $(u, \mathsf{f}) \in \pi^{-1}(F) \subset \mathfrak{M}_{A,[v]}$.

(ii) $\implies$ (i). Let F be a compact subset of $\mathfrak{U}$. Then, for every sequence $(u_i, \mathsf{f}_i) \in \pi^{-1}(F)$, we may assume without loss of generality (since F is compact) that $\mathsf{f}_i \to \mathsf{f} \in F$. The condition in (ii) means that (u_i, f_i) has a subsequence converging to $(u, \mathsf{f}) \in \pi^{-1}(F)$. This proves that $\pi^{-1}(F)$ is compact.

(ii) $\implies$ (iii). This is obvious.

(iii) $\implies$ (ii). Let u_{i_l} be a C^k-bounded subsequence of a sequence u_i as in (iii). We prove that this sequence has a subsequence that C^{m+r}-converges to a mapping u such that $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$. Since the sequence u_{i_l} is at least C^1-bounded, then, the Arzelá–Ascoli theorem implies the existence of a subsequence (also denoted by u_{i_l}) that converges in the C^0-topology to some mapping u of class $C^0(M, M')$. Consequently, there exists a finite cover of the manifold M by charts $\{Q_j\}$ and a system of charts $\{Q'_j\}$ on M' such that for almost all indices i_l we have the inclusion $u_{i_l}(Q_j) \subset Q'_j$ for each j. (Without loss of generality, we may assume that the closures $\overline{Q}_j$ are compact and lie in some charts of M.) Let $\{U_j\}$ be a cover of M such that $\overline{U}_j \subset Q_j$. By Lemma 2.2, the restrictions of u_{i_l} to Q_j, regarded as mappings $Q_j \to \mathbb{R}^{n'}$, satisfy an elliptic system of differential equations

$$\mathsf{A}_{i_l}^{i_1\cdots i_k}(x)(\partial^k u/\partial x^{i_1} \cdots \partial x^{i_k}) = \mathsf{h}_{i_l}(x) \tag{6.2}$$

(we use the notation of (5.9)–(5.12)), where

$$\mathsf{h}_{i_l}(x) = \mathsf{f}_{i_l}\big(x, u_{i_l}(x), \partial u_{i_l}(x), \dots, \partial^{k-1} u_{i_l}(x)\big) - \mathsf{g}\big(x, u_{i_l}(x), \partial u_{i_l}(x), \dots, \partial^{k-1} u_{i_l}(x)\big).$$

From our assumptions it follows that the differential operators in (6.2) are elliptic uniformly with respect to i_l, and that their coefficients are uniformly bounded and equicontinuous. Next, since the u_{i_l} are C^k-bounded, we see that the $\mathsf{h}_{i_l}(x)$ are bounded in the space $C^1(Q_j, \mathbb{R}^{n'})$. Let $\{V_j^1\}$ be a cover of M such that

$$\overline{U}_j \subset V_j^1 \subset \overline{V}_j^1 \subset Q_j \quad \text{for all } j.$$

Application of the Schauder estimate to (6.2) shows that the mappings u_{i_l} are uniformly bounded in the space $C^{k,\alpha}(V_j^1, \mathbb{R}^{n'})$ for all $0 < \alpha < 1$ (see [7]). The coefficients $\mathsf{A}_{i_l}^{i_1\cdots i_k}(x)$ and the right-hand sides $\mathsf{h}_{i_l}(x)$ are uniformly bounded in the space $C^{1,\alpha}(V_j^1, \mathbb{R}^{n'})$. Now, we cover M by sets $\{V_j^2\}$ such that

$$\overline{U}_j \subset V_j^2 \subset \overline{V}_j^2 \subset V_j^1,$$

and apply the Schauder estimate on the regions V_j^1. We obtain that the sequence u_{i_l} is uniformly bounded in the space $C^{k+1,\alpha}(V_j^2, \mathbb{R}^{n'})$. Repeating this procedure sufficiently many times, we see that the sequence u_{i_l} is bounded in the space $C^{m+r,\alpha}(U_j, \mathbb{R}^{n'})$. Since the family $\{U_j\}$ covers M, we can use the fact that the embedding $C^{m+r,\alpha}(U_j, \mathbb{R}^{n'}) \subset$

$C^{m+r}(U_j, \mathbb{R}^{n'})$ is compact to choose a subsequence that C^{m+r}-converges to u. Passing to the limit in (6.2), we obtain $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$. $\square$

Condition (iii) of Lemma 6.1 gives a general compactness criterion for a quasilinear elliptic problem. For specific classes of differential operators (depending on the order of the derivatives occurring in the coefficients), this condition can be refined. Below, we shall need the following lemma.

Lemma 6.2. *Let A be a quasilinear elliptic differential operator of order $k \geq 2$ whose symbol does not depend on the derivatives of order $k-1$. Then problem* (5.6), (5.7) *is compact with respect to a region $\mathfrak{U}$ if and only if for each sequence $(u_i, \mathsf{f}_i) \in \mathfrak{M}_{A,[v]}$, $\mathsf{f}_i \in \mathfrak{U}$, such that $\mathsf{f}_i \to \mathsf{f} \in U$ there exists a C^{k-1}-bounded subsequence u_{i_l}.*

Proof. The necessity of the existence of a C^{k-1}-bounded subsequence follows from Lemma 6.1. Now, we prove sufficiency. Let u_{i_l} be a C^{k-1}-bounded subsequence of u_i satisfying the conditions of the lemma. It suffices to prove that there is a subsequence that C^{m+r}-converges to an element u such that $(u, \mathsf{f}) \in \mathfrak{M}_{A,[v]}$. The Arzelá–Ascoli theorem shows that there exists a subsequence of u_{i_l} (also denoted by u_{i_l}) that converges to a $C^0(M, M')$-mapping in the C^0-topology. Therefore, there exists a finite cover of M by charts $\{Q_j\}$ and a system of charts $\{Q'_j\}$ on M' such that $u_{i_l}(Q_j) \subset Q'_j$ for almost all mappings u_{i_l} (it is assumed that the closures $\overline{Q}_j$ are compact and lie in some charts on M). Let $\{U_j\}$ be a finite cover of M such that $\overline{U}_j \subset Q_j$. Since the sequence u_{i_l} is C^{k-1}-bounded, the assumptions of the lemma imply that the differential operators in (6.2) are uniformly elliptic, and their coefficients are uniformly bounded in the space $W^{1,p}(Q_j, \mathbb{R}^{n'})$ for all $1 < p < \infty$. Next, the right-hand sides $\mathsf{h}_{i_l}(x)$ are uniformly bounded in $W^{0,p}(Q_j, \mathbb{R}^{n'})$. Therefore, using the Schauder estimate as in the proof of Lemma 6.1, we can show that the sequence u_{i_l} is bounded in $W^{m+r+1,p}(U_j, \mathbb{R}^{n'})$. Since, for $p > n$, this space compactly embeds in $C^{m+r}(U_j, \mathbb{R}^{n'})$ and the sets $\{U_j\}$ cover M, we can find a subsequence that converges to the map u. Passing to the limit in (6.2), we show that (u, f) belongs to $\mathfrak{M}_{A,[v]}$, which completes the proof. $\square$

6.2. The equation for closed trajectories of a nonautonomous vector field. Let $\mathsf{f}(t, u)$ be a C^3-smooth nonautonomous vector field on a manifold M', where $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Consider the equation of closed trajectories of this vector field

$$\frac{du}{dt}(t) + \mathsf{f}(t, u(t)) = 0, \quad t \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{6.3}$$

This is a quasilinear elliptic differential equation on the H^1-smooth loops in M'. For simplicity, we assume that the manifold M' is compact and fix a Riemannian metric g' on it. Denoting by $\|du(t)\|$ the norm of the differential du at a point t, we have

$$\|du(t)\| = \left\|\frac{du}{dt}(t)\right\|_{g'} = \|\mathsf{f}(t, u(t))\|_{g'}.$$

This means that, under the conditions of item (iii) in Lemma 6.1, the sequence u_i is always C^1-bounded. Consequently, problem (6.3) is compact by Lemma 6.1 (independently of a homotopy class of loops in M').

From the results of Subsection 5.3 it follows that, for a typical nonautonomous C^3-smooth vector field $\mathsf{f}(t, u)$, the number of H^1-smooth (and, therefore, also C^3-smooth) trajectories in a fixed homotopy class is finite. Moreover, the parity of this number is an invariant of M'.

In the case of a noncompact manifold M', Lemma 6.1 gives the following criterion. Equation (6.3) is compact with respect to a domain $\mathfrak{U}$ in the space of C^3-smooth fields if for every sequence (u_i, f_i) in the moduli space $\mathfrak{M}_{d/dt,[v]}$ such that $\mathcal{U} \ni \mathsf{f}_i \to \mathsf{f} \in \mathcal{U}$, there exists a subsequence u_{i_l} such that all loops $u_{i_l}(S^1)$ lie in a compact subset of M'.

6.3. Compactness for semilinear Cauchy–Riemann equations. Let $\mathbb{T}^2$ be a complex torus, and let M' be a compact complex Kähler manifold. Then the equation

$$\frac{\partial u}{\partial \bar{z}}(z) + \mathsf{f}(z, u(z)) = 0, \quad z \in \mathbb{T}^2, \tag{6.4}$$

is defined on the mappings $u : \mathbb{T}^2 \to M'$ (see Subsection 4.2). Here $\mathsf{f}(z, u)$ is a nonautonomous C^4-smooth vector field on M' and u belongs to $H^3(\mathbb{T}^2, M')$.

We recall that a complex manifold M' is called *Kähler* (see [11]) if it admits a Hermitian structure $\langle \cdot, \cdot \rangle$ such that the 2-form $\omega(\cdot, \cdot) = -\operatorname{Im}\langle \cdot, \cdot \rangle$ is closed. We denote by g' the Riemannian metric $\operatorname{Re}\langle \cdot, \cdot \rangle$. With each homotopy class $[v]$ of mappings from $\mathbb{T}^2$ to M' we associate the number

$$\mu_{[v]} = \langle \omega, v(\mathbb{T}^2) \rangle = \int_{\mathbb{T}^2} v^* \omega,$$

which is called the *symplectic area* of the curve v. In particular, for the class of contractible mappings we have $\mu_{[\mathrm{pt}]} = 0$. Let V denote the minimal area of a nontrivial holomorphic spheroid in M',

$$V = \inf\{\langle \omega, u(S^2) \rangle, \text{ where } u : S^2 \to M' \text{ is holomorphic and nontrivial}\}.$$

As usual, we assume that the infimum over the empty set is equal to infinity. In particular, if $\pi_2(M') = 0$, then $V = \infty$. It can be proved (see [20]) that the constant V is always positive.

Proposition 6.1. *Suppose that a compact Kähler manifold M' and a homotopy class $[v]$ of mappings from $\mathbb{T}^2$ to M' are such that $V > \mu_{[v]}$. Then problem* (6.4) *possesses the compactness property with respect to the set*

$$\mathcal{U} = \left\{ \mathsf{f} \in \mathcal{V} : \|\mathsf{f}(z, u)\|_{g'} < \sqrt{V - \mu_{[v]}} \text{ for all } z \in \mathbb{T}^2,\ u \in M' \right\}, \tag{6.5}$$

where $\mathsf{f}(z, u)$ denotes a nonautonomous C^4-smooth vector field on M'.

Proposition 6.1 follows from the Gromov compactness theorem [15], because, by Proposition 4.1, the graph of an arbitrary solution u of equation (6.4) is a pseudoholomorphic torus in $\mathbb{T}^2 \times M'$. A complete proof can be found in the paper [20], where the case of a trivial homotopy class was considered. The proof in [20] can be extended to the case of an arbitrary homotopy class almost without changes.

As was shown in [20], the compactness property for problem (6.4) may fail if we replace the set (6.5) by a larger set.

Thus, if $V > \mu_{[v]}$, then the invariant of a Kählerian manifold M' discussed in Theorem 5.6 is a class of nonoriented cobordisms of the set of J^{f}-holomorphic tori in $\mathbb{T}^2 \times M'$ belonging to a fixed homotopy class for a typical vector field $\mathsf{f}(z, u)$ in the set $\mathcal{U}$. In particular, in the case of the trivial homotopy class, the index of the Cauchy–Riemann operator is zero, and this invariant coincides with the parity of the number of J^{f}-holomorphic tori homotopic to $\mathbb{T}^2 \times \{\mathrm{pt}\}$.

6.4. Perturbations of the harmonic map equation. Let M and M' be closed Riemannian manifolds with metrics g and g', respectively. In what follows, we use the notation introduced in the example in Subsection 4.3. On mappings from M to M', we consider the following differential equation:

$$\tau(u)(x) + \mathsf{f}(x, u(x), du(x)) = 0, \quad x \in M. \tag{6.6}$$

We assume that f is quasilinear with respect to du, i.e.,

$$\mathsf{f}(x, u(x), du(x)) = \mathsf{G}(x, u(x)) \cdot du(x) + \mathsf{g}(x, u(x)), \quad x \in M. \tag{6.7}$$

Here G is a morphism of the vector bundles $J^1(M, M')$ and TM' (the principal part of f), and g is a nonautonomous vector field on M'. Let $\mathcal{F}$ be a Banach space of C^s-smooth coefficients of first order quasilinear differential operators (see the corresponding example in Subsection 5.2).

Below, we always denote by G the principal part of a quasilinear section f belonging to the space $\mathcal{F}$, and $\|\mathsf{G}(x, u)\|$ denotes the natural norm of a linear mapping $\mathsf{G}(x, u) : L(T_xM, T_uM') \to T_uM'$ with respect to the Riemannian metrics on M and M'.

Proposition 6.2. *Suppose that a Riemannian manifold M' has a nonpositive sectional curvature. Then, for each homotopy class $[v]$ of mappings from M to M', there exists a constant $C_{[v]} > 0$ such that, for problem* (6.6) *with a set $\mathcal{F}$ of functional parameters, the compactness condition is fulfilled with respect to the set*

$$\mathcal{U} = \big\{\mathsf{f} \in \mathcal{F} : \|\mathsf{G}(x, u)\| < C_{[v]} \text{ for all } x \in M,\ u \in M'\big\}. \tag{6.8}$$

In the case where $\dim M \le 3$ or $G = 0$, Proposition 6.2 was proved in the paper [18]. This restriction was lifted in [5]. At the end of the present section, we give a proof of Proposition 6.2 for two-dimensional manifolds M.

Examples given below show that compactness fails if we drop one of the following assumptions:

(i) quasilinearity of the functional parameter f;
(ii) smallness of the principal part of f;
(iii) nonpositivity of the sectional curvature of M'.

Example 6.1 (the necessity of (i))**.** Let M be the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and M' the two-dimensional torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Then $\tau(u) = -u_{tt}$. We consider the following equation for mappings u from S^1 to $\mathbb{T}^2$:

$$u_{tt} + i\varepsilon u_t |u_t|^2 = 0, \tag{6.9}$$

where ε is a positive real number. The contractible solutions of (6.9) can be regarded as mappings $S^1 \to \mathbb{C}$. Consider the equation

$$w_t + i\varepsilon w|w|^2 = 0,$$

where w is a mapping from S^1 to $\mathbb{C}$. Obviously, for each $l \in \mathbb{N}$ the function

$$w_l = \varepsilon^{-1}\sqrt{2\pi l}\exp(-2\pi l i t)$$

is a 1-periodic solution of this equation with zero mean. Therefore, the curves

$$u_l = \int_0^t w_l(s)\, ds, \quad l \in \mathbb{N},$$

form a family of solutions of (6.9). This family is not C^1-bounded and, therefore, is not compact. By Lemma 6.2, problem (6.9) is not compact.

Example 6.2 (the necessity of (ii))**.** Let M and M' be as in Example 6.1. We consider the equation

$$u_{tt} - 2\pi i u_t = 0. \tag{6.10}$$

As above, the contractible solutions are viewed as mappings from S^1 to $\mathbb{C}$. The sequence

$$u_l(t) = l\exp(2\pi i t), \quad l \in \mathbb{N},$$

is a family of solutions of (6.10), and this family is not C^1-bounded and, therefore, is not compact. We note that if we replace the factor $2\pi i$ in (6.10) by an arbitrary positive number $\lambda < 2\pi$, then the resulting equation will have only one solution, $u = 0$, and the compactness property will be fulfilled.

Example 6.3 (the necessity of (iii))**.** Again, let M be the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and let M' be the two-dimensional sphere S^2 regarded as the set of unit vectors in $\mathbb{R}^3$ with the metric induced from $\mathbb{R}^3$. As f in (6.6), we take the zero section. Then the solutions of (6.6) are periodic geodesics. We consider the family of geodesics $u_l(t)$ that go around the equator of the sphere l times. Obviously, this is a noncompact family of contractible solutions of (6.6).

We say that a C^1-smooth homotopy $H(s,x)$, $s \in [0,1]$, $x \in M$, between C^1-smooth mappings u and v is *geodesic* if for each $x \in M$ the curve

$$[0,1] \ni s \mapsto H_s(x) = H(s,x) \in M'$$

is a geodesic in M'. If M' has a nonpositive sectional curvature, then any two homotopic mappings are geodesically homotopic. Indeed, a geodesic homotopy is constructed by replacing the curve $s \mapsto H_s(x)$ with a unique geodesic that is homotopic to this curve.

For any two homotopic C^1-mappings u and v, we have the distance

$$N_p(u,v) = \inf \big\{ N_p(H) : H \ \text{ is a } \ C^1\text{-homotopy between } u \text{ and } v \big\},$$

where $1 \le p < \infty$ and

$$N_p(H) = \bigg(\int\limits_M \Big(\int\limits_0^1 \Big\| \frac{d}{ds} H_s(x) \Big\|_{g'} \, ds \Big)^p \, d\,Vol_g(x) \bigg)^{1/p}.$$

By the Hölder inequality, we have

$$N_{p_1}(u,v) \le N_{p_2}(u,v) (\,Vol\, M)^{(p_2 - p_1)/p_2 p_1}$$

for all $1 \le p_1 \le p_2 < \infty$. For a nonautonomous vector field g on M' and a morphism G of the vector bundles $J^1(M,M')$ and TM', we introduce the norms

$$\|\mathsf{g}\|_{C^0} = \max_{\substack{x \in M \\ u \in M'}} \|\mathsf{g}(x,u)\|_{g'}, \quad \|\mathsf{G}\|_{C^0} = \max_{\substack{x \in M \\ u \in M'}} \|\mathsf{G}(x,u)\|.$$

Proposition 6.3. *Let M' be a manifold of nonpositive sectional curvature. Then, for any* (*homotopic to v*) *solution u of equation* (6.6) *with a quasilinear* f, *we have the estimate*

$$E(u) \le \|\mathsf{g}\|_{C^0} N_1(u,v) + \sqrt{2} \|\mathsf{G}\|_{C^0} E(u)^{1/2} N_2(u,v) + E(v).$$

Proof. Let $H(s,x)$ be a geodesic homotopy between u and v. Since the sectional curvature is nonpositive, the formula for the second variation of the energy functional (see, e.g., [13, p. 28]) implies that

$$\frac{\partial^2}{\partial s^2} E(H_s) \ge 0 \quad \text{for all } s \in [0,1].$$

Therefore, the function $s \mapsto E(H_s)$ is convex, and we obtain the inequality

$$E(u) \le E(v) + (\partial/\partial s)\big|_{s=1} E(H_s). \tag{6.11}$$

By relation (4.7) and equation (6.6), we have

$$\frac{\partial}{\partial s}\Big|_{s=1} E(H_s) = - \int_M \Big\langle \mathsf{f}, \frac{\partial H}{\partial s} \Big\rangle \Big|_{s=1} \, d\,Vol.$$

Since f has the form (6.7), we obtain

$$\left|\int_M \left\langle \mathsf{f}, \frac{\partial H}{\partial s}\right\rangle d\,Vol\right| \le \|\mathsf{g}\|_{C^0} N_1(H) + \|\mathsf{G}\|_{C^0}\int_M \sqrt{2e(u)}\left\|\frac{\partial H}{\partial s}\right\| d\,Vol$$
$$\le \|\mathsf{g}\|_{C^0} N_1(H) + \sqrt{2}\|\mathsf{G}\|_{C^0} E(u)^{1/2} N_2(H).$$

This inequality and (6.11) imply the required estimate. □

A proof of the following geometric inequality can be found in [18].

Proposition 6.4 [18]**.** *Let M' be a manifold of nonpositive sectional curvature, and let $[v]$ be an arbitrary homotopy class, $v \in C^1(M, M')$. Then there exists a constant $C > 0$ (depending on $[v]$) such that*

$$N_2(u, v) \le C\big(E(u)^{1/2} + E(v)^{1/2} + 1\big) \tag{6.12}$$

for every C^1-smooth $u \in [v]$.

Remark. For manifolds of negative sectional curvature, the constant C in (6.12) can be chosen independent of a homotopy class (see [19]). Accordingly, in that case the constant $C_{[v]}$ in Proposition 6.2 is also independent of $[v]$.

Propositions 6.3 and 6.4 imply the following statement.

Corollary 6.3. *Let M' be a manifold of nonpositive sectional curvature, and let $[v]$ be a homotopy class of mappings from M to M'. Then, for any positive C_{g} and $C_{\mathsf{G}} < C_{[v]} = (\sqrt{2}C)^{-1}$ (where C is as in (6.12)), there is a positive constant C_* such that for every solution $u \in [v]$ of equation (6.6) with a quasilinear f for which*

$$\|\mathsf{G}\|_{C^0} \le C_{\mathsf{G}}, \quad \|\mathsf{g}\|_{C^0} \le C_{\mathsf{g}} \tag{6.13}$$

we have $E(u) \le C_*$.

Proof of Proposition 6.2 (for the case where $\dim M = 2$). For $i = 1, 2, \ldots$, let u_i be a solution of the equation

$$\tau(u)(x) + \mathsf{g}_i(x, u_i(x)) + \mathsf{G}_i(x, u_i(x)) \cdot du_i(x) = 0, \quad x \in M,$$

where g_i and G_i satisfy (6.13) and are C^s-bounded. Since M' is compact, Lemma 6.2 shows that for the proof of the theorem it suffices to check that the sequence $\max_{x\in M}\|du_i(x)\|$ is bounded.

Suppose the contrary. Then there exists a subsequence u_{i_l} such that

$$\max_{x\in M}\|du_{i_l}(x)\| = B_{i_l} \to \infty \quad \text{as } l \to \infty.$$

We choose points $x_l \in M$ such that

$$\|du_{i_l}(x_l)\| = B_{i_l}. \tag{6.14}$$

Without loss of generality, we may assume that the sequence x_l converges to a point $x_0 \in M$. We choose a chart $\mathcal{O}$ on M such that $x_0 \in \mathcal{O}$ and $g_{ij}(x_0) = \delta_{ij}$. For sufficiently large $l \in \mathbb{N}$, the mappings

$$\varphi_l : D_l = \big\{x \in R^2 : |x| < \sqrt{B_{i_l}}\big\} \to \mathcal{O}, \quad x \mapsto x/B_{i_l} + x_l,$$

are well defined. On the disks D_l we define a Riemannian metric g_l by putting $(g_l)_{ij} = g_{ij} \circ \varphi_l$. Then the mappings $w_l = u_{i_l} \circ \varphi_l : D_l \to M'$ satisfy the equations

$$\tau_l(w_l)(x) = -B_{i_l}^{-2}\mathsf{g}_{i_l}(\bar{x}, w_l) - B_{i_l}^{-1}\mathsf{G}_{i_l}(\bar{x}, w_{i_l}) \cdot dw_l, \quad x \in D_l, \tag{6.15}$$

where $\bar{x} = x/B_{i_l} + x_{i_l}$, and $\tau_l(w)$ is the operator of harmonic mappings from D_l to M' that corresponds to the metrics g_l and g'. By (6.14), we have

$$\|dw_l(x)\| \le 1, \quad \|dw_l(0)\| = 1. \tag{6.16}$$

We denote by $\tau_\infty(w)$ the operator of harmonic mappings from $\mathbb{R}^2$ to M' that corresponds to the Euclidean metric on $\mathbb{R}^2$ and the metric g' on M'. Observe that the coefficients of the operator $\tau_l(w)$ with all their derivatives converge to the corresponding coefficients of the operator $\tau_\infty(w)$, and the right-hand sides of equations (6.15) with all their derivatives of order less than s converge to zero uniformly on the compact subsets of $\mathbb{R}^2$. Therefore, using the estimate for dw_l in (6.16), applying the Schauder estimate for subsequent control of the higher order derivatives of w_l, and arguing as in the proof of Lemma 6.2, we obtain a subsequence of w_l that C^{m+r}-converges to a harmonic mapping $w : \mathbb{R}^2 \to M'$ on each compact subset of $\mathbb{R}^2$. From (6.16) it follows that

$$\|dw(0)\| = 1. \tag{6.17}$$

By Corollary 6.3, condition (6.13) (which is fulfilled for every u_i) implies the inequality $E(u_i) \leq C_*$. Since the mappings $\varphi_l : D_l \to M'$ are conformal and the energy functional is conformally invariant (see [14]), we have $E(w_l) \leq C_*$. Passing to the limit, we obtain $E(w) \leq C_*$. Identifying $\mathbb{R}^2$ with $S^2 \backslash \infty$ via the stereographic projection, we can view w as a harmonic mapping $S^2 \backslash \infty \to M'$ with finite energy. (Here S^2 is the unit sphere in the Eucledean $\mathbb{R}^3$ with the induced Riemannian metric.) From [23, Theorem 3.6] it follows that the singularity of w at ∞ is removable, and w extends to a smooth harmonic mapping $S^2 \to M'$. Since M' has a nonpositive sectional curvature, the Hadamard–Cartan theorem (see, e.g., [2]) yields $\pi_2(M') = 0$. Thus, w is a contractible harmonic mapping. Since harmonic mappings into a nonpositively curved manifold minimize the energy in their homotopy class (see [16]) and since, for contractible mappings, the minimum value of the energy is zero, we conclude that $E(w) = 0$. Consequently, $dw = 0$, which contradicts (6.17). □

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