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Fifteen Years of KAM for PDE

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Dedicated to Sergei Petrovich Novikov on his 65-th birthday

1. Introduction

In this paper we discuss results, obtained in the KAM for PDEs theory since it was originated 15 years ago in [**Kuk87, Kuk88, Kuk89, Way90**]. We hope that our work covers all relevant topics except the theory of time-periodic solutions of Hamiltonian PDEs (since, first, this theory is an extensive subject and, second, many results there have been recently proved and re-proved without the KAM-machinery, see, e.g., [**Bam00**]). We avoid completely the classical finite-dimensional KAM-theory which deals with time-quasiperiodic solutions of finite-dimensional Hamiltonian systems and instead refer the reader to the recent survey [**Sev03**]. Our bibliography is by no means complete.

Notation. By $\mathbb{T}^n$ we denote the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and write $\mathbb{T}^1 = S^1$; by $\mathbb{R}^n_+$ – the open positive octant in $\mathbb{R}^n$; by $\mathbb{Z}_0$ – the set of nonzero integers. Abusing notation, we denote by x both the space-variable and an element of an abstract Banach space X. For an invertible linear operator J we set $\overline{J} = -J^{-1}$. The Lipschitz norm of a map f from a metric space M to a Banach space is defined as $\sup_{m\in M}\|f(m)\| + \sup_{m_1\neq m_2}\frac{\|f(m_1)-f(m_2)\|}{\mathrm{dist}(m_1,m_2)}$.

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2. Hamiltonian partial differential equations

2.1. Hilbert scales and their morphisms. Let X_0 be a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$ and a Hilbert basis $\{\varphi_k, k\in\mathcal{Z}\}$, where $\mathcal{Z}$ is a countable set. Let us take a positive sequence $\{\theta_k, k\in\mathcal{Z}\}$ which goes to infinity with k. For any s we define X_s as a Hilbert space with the Hilbert basis $\{\varphi_k\theta_k^{-s}, k\in\mathcal{Z}\}$. By $\|\cdot\|_s$ and $\langle\cdot,\cdot\rangle_s$ we denote the norm and the scalar product in X_s (in particular, $X_0 = X$). The totality $\{X_s\}$ is called a *Hilbert scale*, the basis $\{\varphi_k\}$ is called the *basis of the scale* and the scalar product $\langle\cdot,\cdot\rangle$ is called the *basic scalar product*. For any s, the basic scalar product extends to a paring $X_s\times X_{-s}\to\mathbb{R}$, still

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denoted as $\langle\,\cdot\,,\,\cdot\,\rangle_s$. This paring identifies the conjugated space $(X_s)^*$ with X_{-s}. See [**RS75, Kuk00**].

For a scale $\{X_s\}$ we denote by $X_{-\infty}$ and X_∞ the linear spaces $X_{-\infty} = \bigcup X_s$ and $X_\infty = \bigcap X_s$.

A Hilbert scale may be continuous or discrete, depending on whether $s \in \mathbb{R}$ or $s \in \mathbb{Z}$. The objects we define below are meaningful and the theorems we discuss are valid in both cases.

EXAMPLE 2.1. Basic for us is the Sobolev scale of functions on the d-dimensional torus $\{H^s(\mathbb{T}^d;\mathbb{R}) = H^s(\mathbb{T}^d)\}$. The space $H^s(\mathbb{T}^d)$ is formed by the functions $u\colon \mathbb{T}^d \to \mathbb{R}$ such that

$$u = \sum_{l\in\mathbb{Z}^d} u_l e^{il\cdot x}, \qquad \mathbb{C} \ni u_l = \overline{u}_{-l},\ \|u\|_s^2 = \sum_l (1+|l|)^{2s}|u_l|^2 < \infty.$$

The basis $\{\varphi_k\}$ is formed by properly normalised functions $\operatorname{Re} e^{il\cdot x}$ and $\operatorname{Im} e^{il\cdot x}$, $l \in \mathbb{Z}^d$.

We shall also use the sub-scale $\{H^s(\mathbb{T}^d)_0\}$, where the space $H^s(\mathbb{T}^d)_0$ consists of the functions from $H^s(\mathbb{T}^d)$ with zero mean-value. □

EXAMPLE 2.2. Consider the scale $\{H_0^s(0,\pi)\}$, where the space $H_0^s = H_0^s(0,\pi)$ is formed by the odd 2π-periodic functions $u = \sum_{k=1}^\infty u_k \sin kx$ such that $\|u\|_s^2 = \sum |k|^{2s}|u_k|^2 < \infty$. Since $\{\sin nx\}$ is a complete system of eigenfunctions of the operator $-\Delta$ in $L_2(0,\pi)$ with the domain of definition $\{u \in H^2(0,\pi) \mid u(0) = u(\pi) = 0\}$, then an equivalent definition of these spaces is that $H_0^s = \mathcal{D}(-\Delta)^{s/2}$ (see [**RS75**]). In particular,

$$\begin{aligned} &H_0^1 = \{u \in H^1(0,\pi) \mid u(0) = u(\pi) = 0\},\\ &H_0^2 = H^2(0,\pi) \cap H_0^1,\\ &H_0^3 = \{u \in H^3(0,\pi) \mid u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0\}. \quad \square \end{aligned} \tag{1}$$

Given a scale $\{X_s\}$ and a linear map $L\colon X_\infty \to X_{-\infty}$, we denote by $\|L\|_{s_1,s_2} \le \infty$ the norm of L as a map $X_{s_1} \to X_{s_2}$. We say that L defines a *morphism of the scale of order* d and write $\operatorname{ord} L = d$ if $\|L\|_{s,s-d} < \infty$ for every s. If in addition the inverse map L^{-1} exists and defines a morphism of order $-d$, we say that L defines an *automorphism of order* d. If L is a morphism of order d, then the adjoint maps L^* form a morphism of the same order, called the *adjoint morphism.* If $L = L^*$ ($L = -L^*$), then the morphism L is called self-adjoin (anti-selfadjoint). For example, the operator Δ defines a selfadjoint morphism of order 2 of the Sobolev scale $\{H^s(\mathbb{T}^n)\}$ and the operators $\partial/\partial x_j$, $1 \le j \le n$, define anti-selfadjoint morphisms of order one.

Let $O_s \subset X_s$ be an open domain and F be an analytic (C^k-smooth) map $F\colon O_s \to X_{s-d}$. Then F is called an analytic (C^k-smooth) map of order d on O_s and we write $\operatorname{ord} F = d$.

EXAMPLE 2.3. Let $\{X_s\}$ be the Sobolev scale $\{H^s(\mathbb{T}^d)\}$ and $f(u,x)$ a smooth function. Then the map $F : u(x) \mapsto f(u(x),x)$, $X_a \to X_a$, is smooth if $a > \frac{1}{2}$, so on these spaces $\operatorname{ord} F = 0$. If f is analytic, then so is F.

Now let us assume that $d = 1$, f is analytic, $f(0,x) \equiv 0$ and consider F as a map in the scale $\{H_0^s = H_0^s(0,\pi),\ s \in \mathbb{Z}\}$. For $s \ge 1$ the map $F : H_0^s \to H^s(0,\pi)$ is analytic. Since $Fu(0) = Fu(\pi) = 0$, then due to (1) for $s = 1$ and $s = 2$ $F(H_0^s) \subset H_0^s$. So on the spaces H_0^1 and H_0^2 we have $\operatorname{ord} F = 0$. Since in general for $u \in H_0^\infty$, $F(u) \in H_0^2$ but $\notin H_0^3$ (see (1)), then on the spaces $H_0^s, s \geq 3$ we have $\operatorname{ord} F > 0$.

If $f(u,x)$ is odd in u and even in x (e.g., is x-independent), or vice versa, then $F(H_0^s) \subset H_0^s$ for $s \geq 1$, so $\operatorname{ord} F = 0$ for any $s \geq 1$. □

Given a C^k-smooth function $H \colon X_d \supset O_d \to \mathbb{R}$, $k \geq 1$, we consider its *gradient map* with respect to the paring $\langle\,\cdot\,,\,\cdot\,\rangle$:

$$\nabla H \colon O_d \to X_{-d}, \qquad \langle \nabla H(u), v\rangle = dH(u)v \;\; \forall v \in X_d.$$

The map ∇H is C^{k-1}-smooth. If the gradient map ∇H defines a C^{k-1}-smooth map $O_d \to X_{d-d_H}$, we write that $\operatorname{ord} \nabla H = d_H$.

2.2. Symplectic structures and Hamiltonian equations. For simplicity we restrict ourselves to constant coefficient symplectic structures (for the general case see [**Kuk00**]). Let $\{X_s\}$ be a Hilbert scale and J be its anti-selfadjoint automorphism of order d_J. Then the operator $\overline{J} = -J^{-1}$ defines an anti-selfadjoint automorphism of order $-d_J$. We define a two-form α_2 as $\alpha_2 = \overline{J}\, dx \wedge dx$, where by definition $\overline{J}\, dx \wedge dx[\xi,\eta] = \langle \overline{J}\xi, \eta\rangle$. Clearly, $\overline{J}\, dx \wedge dx$ is a continuous skew-symmetric bilinear form on $X_r \times X_r$ if $r \geq -d_J/2$. Therefore any space X_r, $r \geq -d_J/2$, becomes a *symplectic* (*Hilbert*) *space* and we shall write it as a pair (X_r, α_2). The pair $(\{X_s\}, \alpha_2)$ is called a *symplectic* (*Hilbert*) *scale.*

EXAMPLE 2.4. Let us take the index set $\mathcal{Z}$ to be the union of disjoint subsets $\mathcal{Z}_+$ and $\mathcal{Z}_-$, provided with an involution $\mathcal{Z} \to \mathcal{Z}$ which will be denoted $j \mapsto -j$, such that $-\mathcal{Z}_\pm = \mathcal{Z}_\mp$. Let us consider a Hilbert scale $\{X_s\}$ with a basis $\{\phi_k, k \in \mathcal{Z}\}$ and a sequence $\{\theta_k, k \in \mathcal{Z}\}$, such that $\theta_{-j} \equiv \theta_j$. Take J to be the linear operator, defined by the relations

$$J\phi_k = \phi_{-k} \quad \forall\, k \in \mathcal{Z}_+\,, \quad J\phi_k = -\phi_{-k} \quad \forall\, k \in \mathcal{Z}_-.$$

It defines an anti-selfadjoint automorphism of the scale of zero order, and $\overline{J} = J$. The symplectic scale $(\{X_s\}, \alpha_2 = \overline{J} dx \wedge dx = J dx \wedge dx)$ will be called a *Darboux scale.* □

To a C^1-smooth function h on a domain $O_{d_0} \subset X_{d_0}$, the symplectic form α_2 as above corresponds the Hamiltonian vector field V_h, defined by the usual relation $\alpha_2[V_h,\xi] = -dh(\xi) \quad \forall \xi$ (cf. [**Arn89, HZ94**]). That is, $\langle \overline{J} V_h(x), \xi\rangle \equiv -\langle \nabla h(x), \xi\rangle$ and $V_h(x) = J\nabla h(x)$. So the corresponding Hamiltonian equation is

$$\dot{x} = J\nabla_x h(x) = V_h(x). \tag{2}$$

The vector field V_h defines a continuous map $O_{d_0} \to X_{-d_0-d_J}$. Usually we shall assume that V_h is smoother than that and $\operatorname{ord} V_h = d_1 \leq 2d_0 + d_J$.

A partial differential equation is called a *Hamiltonian partial differential equation* (HPDE) if under a suitable choice of a symplectic Hilbert scale $(\{X_s\}, \alpha_2)$, a domain $O_{d_0} \subset X_{d_0}$ and a Hamiltonian h, it can be written in the form (2). In this case the vector field V_h is unbounded, $\operatorname{ord} V_h = d_1 > 0$.

A continuous curve $x \colon [t_0,t_1] \to O_{d_0}$ is called a *solution of* (2) *(in the space* X_{d_0}*)* if it defines a C^1-smooth map $x \colon [t_0,t_1] \to X_{d_0-d_1}$ and both parts of (2) coincide as curves in $X_{d_0-d_1}$. A solution x is called *smooth* if it defines a smooth curve in each space X_s.

We shall call a Hamiltonian equation (2) *quasilinear* if $h(x) = \frac{1}{2}\langle Ax, x\rangle + h_0(x)$, where A is a linear operator which defines a selfadjoint morphism of the scale, and $\operatorname{ord}\nabla h_0 < \operatorname{ord} A$. Accordingly, (2) takes the form

$$\dot{x} = J(Ax + \nabla h_0(x)). \tag{3}$$

A quasilinear equation (3) will be called *semilinear* if $\operatorname{ord} J\nabla h_0 \le 0$.

Example 2.5 (Korteweg–de Vries equation). Let us take for $\{X_s\}$ the scale of zero mean-value Sobolev spaces $H^s(S^1)_0$ as in Example 2.1 and choose $J = \partial/\partial x$, so $d_J = 1$. For a Hamiltonian h we take $h(u) = \int_0^{2\pi}(-\frac{1}{8}u'(x)^2 + \frac{1}{4}u^3)\,dx$. Then $\nabla h(u) = \frac{1}{4}(u'' + 3u^2)$ and (2) becomes the KdV equation:

$$\dot{u}(t,x) = \frac{1}{4}\frac{\partial}{\partial x}(u'' + 3u^2). \tag{4}$$

It has the form (3). Taking $O_{d_0} = X_{d_0}$ with any $d_0 \ge 1$, we have $\operatorname{ord} JA = 3$ and $\operatorname{ord} J\nabla h_0 = 1$. So it is quasilinear, but not semilinear. □

Example 2.6 (NLS – nonlinear Schrödinger equation). Let $X_s = H^s(\mathbb{T}^n;\mathbb{C})$, where this Sobolev space is treated as a real Hilbert space, and the basic scalar product is $\langle u, v\rangle = \operatorname{Re}\int u\overline{v}\,dx$. For J we take the operator $Ju(x) = iu(x)$, so $\operatorname{ord} J = 0$ and $(\{X_s\}, \overline{J}du \wedge du)$ is a Darboux scale. We choose

$$h(u) = \frac{1}{2}\int_{\mathbb{T}^n}\big(|\nabla u|^2 + V(x)|u|^2 + g(x,u,\bar{u})\big)dx,$$

where V is a smooth real function and $g(x,u,v)$ is a smooth function taking real values if $v = \bar{u}$. Then $\nabla h(u) = -\Delta u + V(x)u + \frac{\partial}{\partial\bar{u}}g$ and (2) takes the form

$$\dot{u} = i\Big(-\Delta u + V(x)u + \frac{\partial}{\partial\bar{u}}\,g(x,u,\bar{u})\Big), \quad u = u(t,x),\ x \in \mathbb{T}^n. \tag{5}$$

This is a semilinear Hamiltonian equation in any space X_{d_0}, $d_0 > d/2$, with $\operatorname{ord} A = 2$ and $\operatorname{ord}\nabla h_0 = 0$. Equation (5) with $V = 0$ and $g = C|u|^4$, $C \ne 0$, is called the *Zakharov–Shabat equation*. The equation with $C > 0$ is called *defocusing* and with $C < 0$, *focusing*. □

Example 2.7 (1D NLS with Dirichlet boundary conditions). Let us choose for X_s the space $H_0^s(0,\pi;\mathbb{C})$ (see Example 2.2), $Ju(x) = iu(x)$ and

$$h(u) = \frac{1}{2}\int_0^\pi (|u_x|^2 + V(x)|u|^2 + g(x,|u|^2)\,dx,$$

where g is smooth and 2π-periodic in x. Now $\nabla h(u) = -u_{xx} + V(x)u + f(x,|u|^2)u$, where $f = \frac{\partial g}{\partial |u|^2}$, and (2) becomes

$$\dot{u} = i(-u_{xx} + V(x)u + f(x,|u|^2)u), \quad u(0) = u(\pi) = 0. \tag{6}$$

For $s = 1$ and 2 the nonlinear term defines a smooth map $X_s \to X_s$ (see Example 2.3), so in these spaces this is a semilinear equation with $\operatorname{ord} A = 2$ and $\operatorname{ord}\nabla h_0 = 0$. If in addition f is even in x, then the nonlinear term defines a smooth map for every $s \ge 1$. This map is analytic if f is. □

2.3. Integrable equations and their finite-gap solutions. Let us take a Hamiltonian PDE and write it as a Hamiltonian equation in a suitable symplectic Hilbert scale $(\{X_s\}, \alpha_2 = \overline{J}\, du \wedge du)$:

$$\dot{u} = J\nabla H(u). \tag{7}$$

We assume that for any $u_0 \in X_\infty$ the equation has, locally in time, a unique smooth solution, equal u_0 at $t = 0$. Equation (7) is called *Lax-integrable*, or just *integrable* if there exists an additional real or complex Hilbert scale $\{Z_s\}$ and finite order linear morphisms of this scale $\mathcal{L}_u$ and $\mathcal{A}_u$ depending on the parameter $u \in X_\infty$, such that a curve $u(t)$ is a smooth solution for (7) if and only if

$$\frac{d}{dt}\mathcal{L}_{u(t)} = [\mathcal{A}_{u(t)}, \mathcal{L}_{u(t)}]. \tag{8}$$

The operators $\mathcal{A}_u$ and $\mathcal{L}_u$, treated as morphisms of the scale $\{Z_s\}$, are assumed to depend smoothly on $u \in X_d$ with a sufficiently large d, so the left-hand side of (8) is well defined (for details see [**Kuk00**]). The pair of operators $\mathcal{L}$, $\mathcal{A}$ is called the *Lax pair.*

In most known examples of integrable equations relations between the scales $\{X_s\}$ and $\{Z_s\}$ are the following: spaces X_s are formed by T-periodic Sobolev vector-functions, while $\mathcal{A}$ and $\mathcal{L}$ are differential or integro-differential operators with u-dependent coefficients, acting in a scale $\{Z_s\}$ of TL-periodic Sobolev vector-functions. Here L is a fixed integer.

Let $u(t)$ be a smooth solution for (7). We set $\mathcal{L}_t = \mathcal{L}_{u(t)}$ and $\mathcal{A}_t = \mathcal{A}_{u(t)}$.

LEMMA 2.8. *Let $\chi_0 \in Z_\infty$ be a smooth eigenvector of $\mathcal{L}_0$, i.e., $\mathcal{L}_0\chi_0 = \lambda\chi_0$. Let us also assume that the initial-value problem*

$$\dot{\chi} = \mathcal{A}_t\chi, \qquad \chi(0) = \chi_0, \tag{9}$$

has a unique smooth solution $\chi(t)$. Then

$$\mathcal{L}_t\chi(t) = \lambda\chi(t) \quad \forall t. \tag{10}$$

PROOF. Let us denote the left-hand side of (10) by $\xi(t)$, the right-hand side by $\eta(t)$, and calculate their derivatives. We have:

$$\frac{d}{dt}\xi = \frac{d}{dt}\mathcal{L}\chi = [\mathcal{A}, \mathcal{L}]\chi + \mathcal{L}\mathcal{A}\chi = \mathcal{A}\mathcal{L}\chi = \mathcal{A}\xi$$

and $\frac{d}{dt}\eta = \frac{d}{dt}\lambda\chi = \lambda\mathcal{A}\chi = A\eta$. Thus, both $\xi(t)$ and $\eta(t)$ solve the problem (9) with χ_0 replaced by $\lambda\chi_0$ and coincide by the uniqueness assumption. □

Due to this lemma the discrete spectrum of the operator $\mathcal{L}_u$ is an integral of motion for equation (7). In particular, a set $\mathcal{T}$ formed by all smooth vectors $u \in X_\infty$ such that the eigenvalues of the operator $\mathcal{L}_u$ belong to a fixed subset of $\mathbb{C}\times\mathbb{C}\times\dots$, is invariant for the flow of equation (7). A remarkable discovery, made almost 30 years ago by Novikov [**Nov74**] and Lax [**Lax75**], is that for integrable Hamiltonian PDEs, considered on finite space-intervals with suitable boundary conditions, some sets $\mathcal{T}$ as above are finite dimensional symplectic submanifolds $\mathcal{T}^{2n}$ of X_∞, and restriction of equation (7) to any $\mathcal{T}^{2n}$ is an integrable Hamiltonian system. Moreover, for some integrable equations it is known that the union of all these manifolds $\mathcal{T}^{2n}$ is dense in every space X_s. Solutions that fill a manifold $\mathcal{T}^{2n}$ are called *finite-gap solutions*, and the manifold itself, a *finite-gap manifold.* See, e.g., [**DMN76, ZMNP84, BBE$^+$94, Kuk00**].

In particular, the KdV equation (4) has the form (7) in the Hilbert scale $\{X_s = H^s(S^1)_0\}$, is integrable and the corresponding Lax pair is

$$\mathcal{L}_u = -\frac{\partial^2}{\partial x^2} - u, \qquad \mathcal{A}_u = \frac{\partial^3}{\partial x^3} + \frac{3}{2}\,u\,\frac{\partial}{\partial x} + \frac{3}{4}\,u_x.$$

Taking for $\{Z_s\}$ the Sobolev scale of 4π-periodic functions and applying the Lemma above, we get that smooth 4π-periodic spectrum of the operator $\mathcal{L}_u$ is an integral of motion. It is well known that this spectrum has the form

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \cdots \nearrow \infty.$$

Let us take any n-vector $\mathbf{V} = (V_1, \dots, V_n) \in \mathbb{N}^n$ such that $V_1 < \cdots < V_n$. Denoting $\Delta_j = \lambda_{2j} - \lambda_{2j-1} \ge 0$, $j = 1, 2, \dots$, we define the set $\mathcal{T}^{2n}_{\mathbf{V}}$ as

$$\mathcal{T}^{2n}_{\mathbf{V}} = \{u(x) \mid \Delta_j \ne 0 \text{ iff } j \in \{V_1, \dots, V_n\}\}.$$

Clearly $\mathcal{T}^{2n}_{\mathbf{V}}$ equals to the union $\mathcal{T}^{2n}_{\mathbf{V}} = \bigcup_{r \in \mathbb{R}^n_+} T^n_{\mathbf{V}}(r)$, where

$$T^n_{\mathbf{V}}(r) = \{u(x) \mid \lambda_{2j} - \lambda_{2j-1} = r_j \;\forall\, j \in \{V_1, \dots, V_n\}\}.$$

The sets $T^n_{\mathbf{V}}(r)$ are invariant for the KdV-flow. By the classical theory of the Sturm–Liouville operator $\mathcal{L}_u$, the set $\mathcal{T}^{2n}_{\mathbf{V}}$ is a smooth submanifold of any space X_s, foliated to the smooth n-tori $T^n_{\mathbf{V}}(r)$ (see, e.g., [**KP03**]).

Due to Novikov and Lax, there exist an analytic map $\Phi = \Phi_{\mathbf{V}} \colon \{(r, \xi)\} = \mathbb{R}^n_+ \times \mathbb{T}^n \to X_s$ and an analytic function $h = h^n(r)$ such that $T^n_{\mathbf{V}}(r) = \Phi(\{r\} \times \mathbb{T}^n)$, and for any $\xi_0 \in \mathbb{T}^n$ the curve $u(t) = \Phi(r, \xi_0 + t\nabla h(r))$ is a smooth solution for (4). As a function of t, this solution is quasiperiodic. The celebrated Its–Matveev formula explicitly represents Φ in terms of theta-functions, see [**DMN76, Dub81, BBE$^+$94, Kuk00**].

Situations with other Lax–integrable equations is similar. For example, see [**BBE$^+$94**] for the Sine-Gordon (SG) and Zakharov-Shabat equations; for (SG), see also [**Kuk00, EKMY02**]. Integrable HPDEs with self-adjoint L-operator (e.g., the defocusing Zakharov–Shabat equation) are very similar to (KdV), while those with non-selfadjoint L-operator (e.g., (SG) and the focusing Zakharov–Shabat equation) differ from them in some respects and are more complicated. In particular, for those equations the manifolds $\mathcal{T}^{2n}$ are not smooth but have algebraic singularities.

3. Perturbations of parameter-depending linear problems

3.1. An abstract theorem. Let $(\{X_s\}, \alpha_2 = \bar{J} du \wedge du)$ be a symplectic Hilbert scale, $-d_J = \operatorname{ord} \bar{J} \le 0$; A an operator which defines a selfadjoint automorphism of the scale of order $d_A \ge -d_J$ and H a Frechet–analytic functional on X_{d_0}, $d_0 \ge 0$, such that $\operatorname{ord} \nabla H = d_H < d_A$:

$$\nabla H : X_{d_0} \to X_{d_0 - d_H}.$$

We assume that $d_A \le 2d_0$, so that the quadratic form $\frac{1}{2}\langle Au, u\rangle$ is well defined on the space X_{d_0}.

In this section we consider the quasilinear Hamiltonian equation with the Hamiltonian $H_\varepsilon(u) = \frac{1}{2}\langle Au, u\rangle + \varepsilon H(u)$:

$$\dot{u}(t) = J(Au(t) + \varepsilon \nabla H(u(t))). \tag{11}$$

We assume that the scale $\{X_s\}$ admits a basis $\{\varphi_k, k \in \mathbb{Z}_0 = \mathbb{Z}\backslash\{0\}\}$ such that

$$A\varphi_j^{\pm} = \lambda_j^A \varphi_j^{\pm}, \quad J\varphi_j^{\pm} = \mp \lambda_j^J \varphi_j^{\mp} \quad \forall j \ge 1, \tag{12}$$

with some real numbers λ_j^J, λ_j^A. In particular, the spectrum of the operator JA is $\{\pm i\lambda_j \mid \lambda_j = \lambda_j^J\lambda_j^A\}$. The numbers λ_j are called the *frequencies* of the linear system

$$\dot u = JAu. \tag{13}$$

Let us fix any $n \geq 1$. Then the $2n$-dimensional linear space

$$\operatorname{span}\{\varphi_j^\pm \mid 1 \leq j \leq n\} \tag{14}$$

is invariant under the equation (13) and is foliated to the invariant tori

$$T^n = T^n(I) = \left\{ \sum_{j=1}^n u_j^\pm \varphi_j^\pm \mid {u_j^+}^2 + {u_j^-}^2 = 2I_j \ \forall j \right\}. \tag{15}$$

If $I \in \mathbb{R}_+^n$, then $T^n(I)$ is an n-torus. Providing it with the coordinates $q = (q_1, \dots, q_n)$, where $q_j = \operatorname{Arg}(u_j^+ + iu_j^-)$, we see that equation (13) defines on $T^n(I)$ the motion

$$\dot q = (\lambda_1, \dots, \lambda_n) =: \omega. \tag{16}$$

So all solutions for the linear equation in $T^n(I)$ are quasiperiodic curves with the frequency-vector ω. Our goal in this section is to present and discuss a KAM-theorem which implies that under certain conditions 'most of' trajectories of the equation (16) on the torus $T^n(I)$ persist as time-quasiperiodic solutions of the perturbed equation (11), if $\varepsilon > 0$ is sufficiently small.

To formulate the result we assume that the operator A and the function H analytically depend on an additional n-dimensional parameter $a \in \mathcal{A}$, where $\mathcal{A}$ is a connected bounded open domain in $\mathbb{R}^n$. Then $\lambda_j = \lambda_j(a)$. We assume that the first n frequencies $\lambda_l = \omega_l$ depend on a in a nondegenerate way:

H1) $\det\{\partial\omega_l/\partial a_k \mid 1 \leq k,\, l \leq n\} \not\equiv 0$;

and that the following spectral asymptotic holds:

H2) $\left|\lambda_j(a) - K_1 j^{d_1} - K_1^1 j^{d_1^1} - K_1^2 j^{d_1^2} - \dots\right| \leq K j^{\tilde d}$, $\operatorname{Lip}\lambda_j \leq j^{\tilde d}$,

where $d_1 := d_A + d_J \geq 1$, $K_1 > 0$, $\tilde d < d_1 - 1$ and the dots stand for a finite sum with exponents $d_1 > d_1^1 > d_1^2 > \dots$.

Let us denote by X_s^c the complexification of a space X_s and assume that the equation (11) is quasilinear and analytic:

H3) the set $X_{d_0} \times \mathcal{A}$ admits a complex neighbourhood Q in $X_{d_0}^c \times \mathbb{C}^n$ such that the map $\nabla_x H : Q \to X_{d_0 - d_H}^c$ is complex-analytic and bounded uniformly on bounded subsets of Q. Moreover, $d_H + d_J \leq \tilde d$.

Finally, we shall need the following non-resonance condition:

H4) For all integer n-vectors s and $(M_2 - n)$-vectors l such that $|s| \leq M_1$, $1 \leq |l| \leq 2$ we have

$$s\cdot\omega(a) + l_{n+1}\lambda_{n+1}(a) + \dots + l_{M_2}\lambda_{M_2}(a) \not\equiv 0, \tag{17}$$

where the integers $M_1 > 0$ and $M_2 > n$ are to be specified.

Let us fix $I_0 \in \mathbb{R}_+^n$ and denote by Σ_0 the map $\mathbb{T}^n \times \mathcal{A} \to X_{d_0}$ which sends (q, a) to the point of the torus $T^n(I_0)$ with the coordinate q.

THEOREM 3.1. *Suppose the assumptions H1)–H3) hold. Then there exist integers $M_1 > 0$ and $M_2 > n$ such that if H4) is fulfilled, then for arbitrary $\gamma > 0$ and for sufficiently small $\varepsilon < \bar\varepsilon(\gamma)$, a Borel subset $\mathcal{A}_\varepsilon \subset \mathcal{A}$ and a Lipschitz map $\Sigma_\varepsilon : \mathbb{T}^n \times \mathcal{A}_\varepsilon \to X_{d_0}$, analytic in $q \in \mathbb{T}^n$, can be found with the following properties:*

a) meas $(\mathcal{A}\backslash\mathcal{A}_\varepsilon) \le \gamma$;

b) the map Σ_ε is $C\varepsilon$-close to $\Sigma_0|_{\mathbb{T}^n\times\mathcal{A}_\varepsilon}$ in the Lipschitz norm;

c) each torus $\Sigma_\varepsilon(\mathbb{T}^n \times \{a\})$, $a \in \mathcal{A}_\varepsilon$, is invariant for the flow of equation (11) *and is filled with its time-quasiperiodic solutions of the form $u_\varepsilon(t;q) = \Sigma_\varepsilon(q+\omega' t, a)$, $q \in \mathbb{T}^n$, where the frequency vector $\omega'(a)$ is $C\varepsilon$-close to $\omega(a)$ in the Lipschitz norm;*

d) the solutions u_ε are linearly stable.

We note that, strictly speaking, if equation (11) is not semilinear (i.e., if $d_J + d_H > 0$), then the last assertion of the theorem is proved provided that the equation satisfies some mild regularity condition, see Theorem 8.4 in [**Kuk00**].

Let us assume that ∇H defines an analytic map of order d_H on every space X_d, $d \ge d_0$. If $u_\varepsilon(t)$ is any solution of (11) as in the theorem, then due to the equation, $JAu(t)$ is a smooth curve in $X_{d_0-d_H-d_J}$. Since JA is an automorphism of the scale of order d_1, then $u_\varepsilon(t)$ is a smooth curve in $X_{d_0-d_H-d_J+d_1} \subset X_{d_0+1}$. Iterating this arguments we see that u_ε is a smooth curve in each space X_s. I.e., it is a smooth solution of the equation.

In the semilinear case (i.e., when $d_H + d_J \le \tilde d < d_1 - 1$ and $\tilde d \le 0$) the theorem is proved in [**Kuk87, Kuk88**] (see also [**Kuk93, Pös96a**]). The semilinearity restriction $\tilde d \le 0$ was removed in [**Kuk98**] (see also ([**Kuk00**] and [**KP03**]).

We note that for some specific HPDEs (11) the assertions of Theorem 3.1 can be proven for any $n \ge 1$ even if the parameter a is only one-dimesional. In particular, this can be done for the nonlinear wave equation as in Example 3.3 below, where $V(x) \equiv a$ and the constant a is the one-dimensional parameter. See [**Bou94**] and [**Bam99b**].

The proof of Theorem 3.1 is rather technical. For a well-written outline in the semilinear case see [**Cra00**]. Below we present the scheme of this proof in the form which suits our further purposes.

3.2. On the proof of Theorem 3.1. We start with the semilinear case and assume for simplicity that $\lambda_j^J \equiv 1$. Then $I = (I_1,\dots,I_n)$ and $q = (q_1,\dots,q_n)$ form a symplectic coordinate system in the space (2.3). We set $Y = \operatorname{span}\{\varphi_j^\pm,\ j > n\} \subset X$, and denote by $y_j^\pm$, $j > n$, the coordinates in Y with respect to the basis $\{\varphi_j^\pm\}$. To study the vicinity of a torus $T^n(I_0)$, we make the substitution $I = I_0 + p$. Then $\bar J du \wedge du = dp \wedge dq + dy^+ \wedge dy^-$, and $T^n(I_0) = \{p = 0,\ y = 0\}$. In the new variables equation (2.1) takes the form

$$\dot q = \nabla_p \mathcal{H}_\varepsilon, \quad \dot p = -\nabla_q \mathcal{H}_\varepsilon, \quad \dot y = J\nabla_y \mathcal{H}_\varepsilon,$$

with the hamiltonian

$$\mathcal{H}_\varepsilon = H_0(p,y) + \varepsilon H_1(p,q,y), \quad H_0 = \omega\cdot p + \tfrac12\langle Ay, y\rangle. \tag{18}$$

The vector ω and the operator A depend on the parameter a; the function H_1 depends on a and I_0. We call H_0 the integrable part of the hamiltonian $\mathcal{H}_\varepsilon$.

Retaining the terms of H_1 which are affine in p and quadratic in y, we write H_1 as

$$H_1 = H_1^1 + H_1^3, \quad H_1^1 = h(q) + h^p(q)\cdot p + \langle h^y(q), y\rangle + \langle h^{yy}(q)y, y\rangle,$$
$$H_1^3 = O(|p|^2 + \|y\|^3 + |p|\,\|y\|) =: \mathcal{O}(p,q,y).$$

Next in the vicinity of the torus $T^n = \{p = 0,\, y = 0\}$ we make a symplectic change of variable to kill the part εH_1^1 of the perturbation εH_1. This change of variable is a transformation S_1 which is the time-ε shift along trajectories of an additional hamiltonian F. Here the recipe is that to kill H_1^1, F should be of the same structure, so $F = f(q) + f^p(q)\cdot p + \langle f^y(q), y\rangle + \langle f^{yy}(q)y, y\rangle$. As in the finite-dimensional situation, the transformed hamiltonian $\mathcal{H}_\varepsilon \circ S_1$ can be written as the Lie series in ε. Retaining only terms linear in ε, we have

$$\mathcal{H}_\varepsilon \circ S_1 = H_0 + \varepsilon H_1 + \varepsilon\langle J\nabla_y F,\, \nabla_y H_0\rangle + \varepsilon\nabla_p F\cdot\nabla_q H_0 - \varepsilon\nabla_q F\cdot\nabla_p H_0 + O(\varepsilon^2) + \mathcal{O}.$$

Since $\nabla_p H_0 = \omega$, $\nabla_q H_0 = 0$ and $\nabla_y H_0 = Ay$, then the linear in ε term vanishes if the following relations hold:

$$(\omega\cdot\nabla)f = h, \quad (\omega\cdot\nabla)f^p = h^p,$$
$$(\omega\cdot\nabla)f^y - JAf^y = h^y, \quad (\omega\cdot\nabla)f^{yy} + [f^{yy},\, JA] = h^{yy}.$$

We take these relations as equations on f, f^p, f^y and f^{yy} (called *'the homological equations'*) and try to solve them.

Since the equations have constant coefficients, expanding $f, f^p, \dots$ in Fourier series in q, we find for their components (and for matrix components of the operator f^{yy}) explicit formulas. Certain terms in these formulas contain small divisors, which vanish for some values of the vector $\omega = \omega(a)$. Careful analysis of these divisors shows that all of them are bounded away from zero if $a \notin \mathcal{A}_1$, where $\mathcal{A}_1$ is a Borel subset of $\mathcal{A}$ of small measure. When the equations are solved, we get a transformation which in a sufficiently small neighbourhood of T^n transforms the hamiltonian $\mathcal{H}_\varepsilon$ to a hamiltonian which differs from its integrable part by $O(\varepsilon^2)$.

The explanation above has some flaws. The most important one is that the first and the second homological equations can be solved only if the mean values of h and h^p vanish. To fulfill the first condition we change the hamiltonian εH_1 by a constant (this change is irrelevant since it does not affect the equations of motion), while to fulfill the second we subtract from εH_1 the average $\varepsilon\langle h^p\rangle\cdot p$ and add it to the integrable part H_0, thus changing the term $\omega\cdot p$ to $\omega^2\cdot p$, where $\omega^2 = \omega + \varepsilon\langle h^p\rangle$. Similarly, to solve the last homological equation we subtract from the operator h^{yy} the average of its diagonal part and add the corresponding quadratic form to H_0. Thus, the transformed hamiltonian becomes

$$\mathcal{H}_2 := \mathcal{H}_\varepsilon \circ S_1 = \omega_2\cdot p + \tfrac12\langle A_2 y,\, y\rangle + \varepsilon^2 H_2(p,\, q,\, y) + \mathcal{O}(p,\, q,\, y).$$

This transformation is called *the KAM-step*.

Next we perform the second KAM-step. Under the condition that $a \notin \mathcal{A}_2$ we find a transformation S_2 which sends the hamiltonian $\mathcal{H}_2$ to $\mathcal{H}_3 = \mathcal{H}_2 \circ S_2 = \omega_3\cdot p + \frac12\langle A_3 y,\, y\rangle + (\varepsilon^2)^2 H_2 + \mathcal{O}(p,\, q,\, y)$, etc. After m steps we find transformations $S_1, \dots, S_m$ such that

$$\mathcal{H}_\varepsilon \circ S_1 \circ \cdots \circ S_m = \omega_m\cdot p + \tfrac12\langle A_m y,\, y\rangle + \varepsilon^{2^m} H_m + \mathcal{O}(p,\, q,\, y) =: \mathcal{H}_m.$$

The torus $T^n = \{p = 0, y = 0\}$ is 'almost invariant' for the equation with the hamiltonian $\mathcal{H}_m$. Hence, the torus $S_1 \circ \cdots \circ S_m(T^n)$ is 'almost invariant' for the original equation. Since the sequence ε^{2^m} converges to zero superexponentially fast, we can choose the sets $\mathcal{A}_1, \mathcal{A}_2, \dots$ in such a way that meas$(\mathcal{A}_\infty = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots) < \gamma$, for any $a \notin \mathcal{A}_\infty$ the vectors $\omega_m(a)$ converge to a limiting vector $\omega'(a)$, and the transformations $S_1 \circ \dots \circ S_m$ converge to a limiting map $\Sigma_\varepsilon(\cdot,\, a)$, defined on T^n. Then the torus $\Sigma_\varepsilon(T^n,\, a)$ is invariant for the equation (11) and is filled with its quasiperiodic solutions $t \to \Sigma_\varepsilon(q + \omega' t,\, a)$.

If the equation is not semilinear, then the situation becomes more complicated since to solve the forth homological equation we have to remove from the operator h^{yy} the entire diagonal part (and not only its average). Because of that the operator A in the integrable part of the hamiltonian gets terms which form a small q-dependent diagonal operator of a positive order. Accordingly, the forth homological equation becomes more difficult and cannot be solved by the direct Fourier method. Its solvability follows from a nontrivial lemma, based on properties of fast-oscillating Fourier integrals proved in [**Kuk98**] (see also [**Kuk00, KP03**]).

3.3. Applications to 1D HPDEs. Theorem 1 applies well to parameter-depending quasilinear HPDEs with one-dimensional space variable in a finite interval, supplemented by boundary conditions such that the spectrum of the linear operator JA is simple. Indeed, for such equations assumption H2) follows from usual spectral asymptotics, H3) is obvious if the nonlinearity is analytic, while H1) and H4) hold if the equation depends on the additional parameter in a non-degenerate way. More explicitly it means the following. In the examples which we consider below, the equations depend on a potential $V(x;\, a)$, which is analytic in a and smooth in x. The non-degeneracy means that in a functional space, formed by functions of x and a of the required smoothness, the potential V should not belong to some analytic subset of infinite codimension.

Below we just list the examples. In each case application of Theorem 3.1 is straightforward. The theorem applies if dimension of the parameter a is $\geq n$ and dependence of the potential V on a is nondegenerate as it was explained above. In the first three examples the potential $V(x;\, a)$ is real, smooth in x, and analytic in a. The function $f(x,\, v;\, a)$ is real, smooth in x and analytic in v and a. Details can be found in [**Kuk93, Kuk00, Kuk94, Kuk98**].

Example 3.2. Nonlinear Schrödinger equation (NLS), cf. Example 2.6:

$$\dot u = i(-u_{xx} + V(x;\, a)u + \varepsilon f(x,\, |u|^2;\, a)u), \tag{19}$$

$$u = u(t,\, x),\ x \in [0,\, \pi]; \quad u(t,\, 0) \equiv u(t,\, \pi) \equiv 0. \tag{20}$$

Now $d_J = 0$, $d_A = 2$, $\tilde d = d_H = 0$ and we view the Dirichlet boundary conditions as odd periodic ones (cf. Example 2.7). The theorem applies in the scale of odd periodic functions with $d_0 = 1$ or 2. If f is even and 2π-periodic in x, then the theorem applies with any $d_0 \geq 1$ and the constructed quasiperiodic solutions are smooth. □

Example 3.3. Nonlinear string equation: $w(t,x)$ satisfies (20) and

$$\ddot w = w_{xx} - V(x;\, a)w + \varepsilon f(x,\, w;\, a),$$

where now $V > 0$ and $f(x,w) = 0$ if $w = 0$ or $x = 0$. Let us denote $U = (u, (-\Delta)^{-1/2}\dot u)$. It is a matter of direct verification that U satisfies a semilinear Hamiltonian equation (11) in a suitable symplectic Hilbert scale, formed by odd periodic Sobolev functions. Now $d_A = 1$, $d_J = 0$, $\tilde d = d_H = -1$. Cf. [**Way90**] □

EXAMPLE 3.4. KdV-type equations:

$$\dot{u} = \frac{\partial}{\partial x}(-u_{xx} + V(x;\, a)u + \varepsilon f(x,\, u;\, a)); \quad x \in S^1,\ \int_{S^1} u\, dx \equiv 0, \tag{21}$$

cf. Example 2.5. Now $d_J = 1$, $d_A = 2$, $\tilde{d} = d_H = 0$. □

Theorem 3.1 also applies if $x \in \mathbb{R}^1$ and the potential $V(x;\, a)$ grows sufficiently fast as $x \to \infty$.

EXAMPLE 3.5. Nonlinear Schrödinger equation on the line:

$$\dot{u} = i(-u_{xx} + (x^2 + \mu x^4 + V(x;\, a))u + \varepsilon f(|u|^2;\, a)u), \quad \mu > 0,$$
$$u = u(t,\, x), \quad x \in \mathbb{R}, \quad u \to 0 \text{ as } |x| \to \infty.$$

Here the potential V is smooth, analytic in a and vanishes as $|x| \to \infty$. The real-valued function f is analytic. Now $d_J = 0$, $d_A = 4/3$, $d_H = 0$. Another example of this sort see in [**Kuk93**], Section 2.5. □

The time-quasiperiodic solutions, constructed in Examples 3.2–3.5, are linearly stable. Therefore they should be observable in numerical models for the corresponding equations. Indeed, quasiperiodic behaviour of solutions for 1D HPDEs with small nonlinearity was observed in many experiments, starting from the famous numerics of Fermi–Pasta–Ulam [**FPU65**]; e.g., see [**ZIS79**].

3.4. Multiple spectrum. In Examples 3.2, 3.3 the equations are considered under the Dirichlet boundary conditions. If we replace them by the periodic ones

$$u(t,\, x) \equiv u(t,\, x + 2\pi),$$

then Theorem 3.1 would not apply since now the frequencies of the corresponding linear equations are asymptotically double: they have the form $\{\lambda_j^{\pm}, j \geq 1\}$, where $|\lambda_j^+ - \lambda_j^-| \to 0$ as $j \to \infty$. It is clear that the numbers $\{\lambda_j^{\pm}\}$ cannot be reordered to meet the spectral asymptotic condition H2). Still, for some semilinear equations (11) assertions of the theorem remain true if the frequencies λ_j are not single, but asymptotically they have the same multiplicity $m \geq 2$ and behave regularly. Corresponding result is proved by Chierchia–You in [**CY00**], using the scheme, explained in Section 3.2. We do not give precise statement of their theorem, but note that it applies to the nonlinear string equation in Examples 3.3 under the periodic boundary conditions. The result is the same: if the non-degeneracy condition holds, then for ε small enough and for most (in the sense of measure) values of the n-dimensional parameter a, solutions of the linear equation (13) which fill in a torus $T^n(I)$, $I \in \mathbb{R}^n_+$, persist as linearly stable time-quasiperiodic solutions of the corresponding non-linear equation (11).

We note that earlier this persistence result was proven by Bourgain [**Bou94**], who used another KAM-scheme, discussed in the next section.

3.5. Space-multidimensional problems. The abstract Theorem 3.1 is a flexible tool to study 1D HPDEs, but it *never* applies to space-multidimensional equations since the spectral assumption H2) never holds in high dimensions. At the time of this writing, the only published KAM result, which applies to higher-dimensional HPDEs, is due to Bourgain [**Bou98**]. In that work the 2D NLS equation as in the Example 2.6 is considered. For technical reasons the potential term

Vu is replaced there by the convolution $V * u$:

$$\dot{u} = i\big(-\Delta u + V(x;\, a) * u + \varepsilon \frac{\partial}{\partial \bar{u}} g(u,\, \bar{u})\big), \quad u = u(t,\, x),\ x \in \mathbb{T}^2. \tag{22}$$

The potential $V(x;\, a)$ is real analytic and $g(u,\, \bar{u})$ is a real-valued polynomial of u and $\bar{u}$. This equation has the form (11), where $Au = -\Delta u + V * u$ and $Ju = iu$. The basis $\{\varphi_k\}$ as in (12) is formed by the exponents $\{e^{is\cdot x}$ and $ie^{is\cdot x}$, $s \in \mathbb{Z}^2\}$, renumerated properly, and

$$\lambda_s^J \equiv 1, \quad \lambda_s^A = |s|^2 + \hat{V}(s;\, a),$$

where $\{\hat{V}(s;\, a)\}$ are the Fourier coefficients of V. For any n, the linear equation $(22)|_{\varepsilon=0}$ has quasiperiodic solutions

$$u = \sum_{j=1}^{n} z_{s_j} e^{i\lambda_{s_j}^A t} \varphi_{s_j}(x) \tag{23}$$

(these are trajectories of the equation (16) on the n-torus (15), where $I_j = \frac{1}{2}|z_{s_j}|^2$ and $I_s = 0$ if s differs from all s_j). For simplicity let us assume that $a_j = \hat{V}(s_j;\, a)$, $j = 1, \dots, n$. Then the result of [**Bou98**] is that for most values of the parameter a (in the same sense as in Theorem 3.1), the solution (23) persists as a time-quasiperiodic solution of the equation (22). In difference with the 1D case, it is unknown if the persisted solutions are linearly stable.

Presumably, in the nearest future similar results will be obtained for a number of other semilinear equations. We do not expect that an abstract KAM-theorem, which applies to different classes of space-multidimensional HPDEs, will be proven. Certainly it will be much more difficult to handle quasi-linear equations that are not semi-linear.

The proof in [**Bou98**] is based on a KAM-scheme, different from that described in Section 3.2. Originally this scheme is due to Craig and Wayne [**CW93**] who used it to construct periodic solutions of nonlinear wave equations. Also see [**Bou94, Bou97**].

Now we briefly describe the scheme, using the notations from Section 3.2. When the perturbation εH_1 is decomposed as in (18), we extract the term $\varepsilon\langle h^{yy}(q)y,\, y\rangle$ from εH_1^1 and add it to the integrable part H_0. After this the hamiltonian to be killed is the sum of the three terms $h(q) + h^p(q) + \langle h^y(q),\, y\rangle$; accordingly the hamiltonian F is a sum of three terms as well. We have to find them from the first three homological equations. The first two are not difficult, but the third one is a real problem since the operator A is not any more an operator with constant coefficients but equals $A_0 + \hat{A}(q)$, where $\hat{A}$ is a bounded operator of order ε (it changes from one KAM-step to another). The resolution of this equation for high KAM steps is the most difficult part of implementation the Craig–Wayne–Bourgain KAM-scheme.

4. Perturbations of integrable equations

Let us consider a quasilinear HPDE on a finite space-interval, which is an integrable Hamiltonian equation (7) in some symplectic Hilbert scale ($\{X_s\}$, $\alpha_2 = \bar{J}dx \wedge dx$). As we explained in Section 2.3, this equation has invariant finite-gap symplectic manifolds $\mathcal{T}^{2n}$ such that restriction of (7) to any of them is integrable. In this section we discuss the results on persistence of quasiperiodic solutions that fill in these manifolds, provided by the KAM for PDEs theory. We shall see that they are very similar to the celebrated Kolmogorov theorem, which states that *most of quasiperiodic solutions of a nondegenerate analytic integrable (finite-dimensional) Hamiltonian system persist under small perturbations of the hamiltonian*; see [**Arn63, MS71**] and Addendum in [**Kuk00**]. We state the main result as a

THEOREM 4.1 (Metatheorem). *Most of quasiperiodic solutions that fill in any finite-gap manifold $\mathcal{T}^{2n}$ as above persist under small Hamiltonian quasilinear analytic perturbations of the integrable equation. If the finite-gap solutions in $\mathcal{T}^{2n}$ are linearly stable, then the persisted solutions are linearly stable as well.*

In the assertion above the statement 'most of quasiperiodic solutions persist' means the following. Due to the Liouville–Arnold theorem [**Arn89, HZ94**], the manifold $\mathcal{T}^{2n}$ can be covered by charts, diffeomorphic to $B \times \mathbb{T}^n = \{p, q\}$ (B is a ball in $\mathbb{R}^n$), with chart-maps $\Phi_0 : B \times \mathbb{T}^n \to \mathcal{T}^{2n}$ such that $\Phi_0^* \alpha_2 = dp \wedge dq$, and the curves $\Phi_0(p, q + t\nabla h(p))$ are solutions of the integrable equation, where $h(p) = H \circ \Phi_0(p, q)$. Let us denote by ε the small coefficient in front of the perturbation. Then for every chart there exists a Borel subset $B_\varepsilon \subset B$ and a map $\Phi_\varepsilon : B_\varepsilon \times \mathbb{T}^n \to X_d$ (d is fixed), with the following properties:

i) $\operatorname{mes}(B \backslash B_\varepsilon) \to 0$ as $\varepsilon \to 0$;

ii) the map $\Phi_\varepsilon : B_\varepsilon \times \mathbb{T}^n \to X_d$ is $C\sqrt{\varepsilon}$-close to Φ_0 in the Lipschitz norm and is analytic in $q \in \mathbb{T}^n$;

iii) there exists a map $\omega_\varepsilon : B_\varepsilon \to \mathbb{R}^n$, $C\varepsilon$-close to the gradient map ∇h in the Lipschitz norm, such that the curves $t \mapsto \Phi_\varepsilon(p, q + t\omega_\varepsilon(p))$, $p \in B_\varepsilon$, $q \in \mathbb{T}^n$, are solutions for the perturbed equation.

The statement of Theorem 4.1 is proven under a number of assumptions (see [**Kuk00**], [**EKMY02**]). These assumptions are checked for such basic integrable HPDEs as KdV, Sine– and Sinh–Gordon equations. There are no doubts that they also hold for the Zakharov–Shabat equations[1] (but the theorem in [**Kuk00, EKMY02**] does not apply to the Kadomtsev–Petviashvili equation). Below we present a scheme of the proof and discuss the restrictions on the integrable HPDE which allow to implement it.

We view (7) as an equation in the Hilbert space X_d, and denote the quasilinear hamiltonian of the perturbed equation as

$$H_\varepsilon = \tfrac{1}{2}\langle Ax, x\rangle + h_0(x) + \varepsilon h_1(x).$$

Accordingly, $H_0 = \frac{1}{2}\langle Ax, x\rangle + h_0$ is the hamiltonian H of the unperturbed equation (7).

Step 1. Let us consider any finite-gap solution $u_0(t) = \Phi_0(p_0, q_0 + t\nabla h(p_0))$ and linearize (7) about it:

$$\dot{v} = J(\nabla H(u_0(t)))_* v. \tag{24}$$

The theory of integrable equations provides tools to reduce this equation to constant coefficients by means of a time-quasiperiodic substitution $v(t) = G(p_0, q_0 + t\nabla h(p_0))\tilde{v}(t)$, where $G(p, q)$, $(p, q) \in B \times \mathbb{T}^n$, is a symplectic linear map $G(p, q) : Y_d \to Z_d$. Here Y_d is a fixed symplectic subspace of Z_d of codimension $2n$. The restriction, which we impose at this step, is that the operator $G(p, q)$ is a compact perturbation of the embedding $Y_d \to Z_d$, which analytically depends on (p, q).

[1] See [**GK03**] for an *ad hoc* KAM-theorem for the defocusing equation.

Step 2. The map G from the Step 1 defines an analytic map

$$B \times \mathbb{T}^n \times Y_d \to X_d,$$

linear and symplectic in $y \in Y_d$. This map defines a symplectomorphism

$$B \times \mathbb{T}^n \times B_\delta(Y_d) \to X_d, \quad B_\delta(Y_d) = \{\|y\|_d < \delta\}, \tag{25}$$

such that linearisation in y at $y = 0$ of the latter equals the former.

Step 3. We use the map (25) to pass to the variables (p, q, y) in the hamiltonian H_ε. We get

$$H_\varepsilon(p,\, q,\, y) = h(p) + \tfrac{1}{2}\langle \mathcal{A}(p)y,\, y\rangle + h_3(p,\, q,\, y) + \varepsilon h_1(p,\, q,\, y), \tag{26}$$

where $h_3 = O(\|y\|_d^3)$. Calculations show that $h_3(p, q, y)$ contains terms such that their gradient maps have the same order as the operator $\mathcal{A}(p)$. If this really were the case, then the Hamiltonian equation would not be quasilinear, which would complicate its study a lot. Fortunately, this does not happen due to a cancellation of a very general nature (see Lemma 7.5 in [**Kuk00**] and Lemma 12 in [**EKMY02**]), and we have

$$\operatorname{ord} \nabla h_3 < \operatorname{ord} \mathcal{A}(p) - 1. \tag{27}$$

Step 4. Invariant tori of the unperturbed system with the hamiltonian $H_0(p,\, q,\, y)$ have the form $\{p = \text{const}, y = 0\}$. Let us scale the variables near the torus $\{p = a, y = 0\}$ as follows: $p = a + \varepsilon^{2/3}\tilde{p}$, $q = \tilde{q}$, $y = \varepsilon^{1/3}\tilde{y}$. In the scaled variables the perturbed equation has the hamiltonian

$$\text{const} + \omega(a) \cdot \tilde{p} + \tfrac{1}{2}\langle \mathcal{A}(a)\tilde{y},\, \tilde{y}\rangle + O(\varepsilon^{1/3}), \quad \omega(a) = \nabla h(a). \tag{28}$$

So we obtain the system (11), written in the form (18), with ε replaced by $\varepsilon^{1/3}$. If Theorem 3.1 applies, then most of the finite-gap tori $\{p = \text{const}\}$ persist in the perturbed equation, as the Metatheorem claims. To be able to use the theorem we have to check the assumptions H1)–H4).

The condition H2) holds if the integrable equation is 1D (if the spectrum is asymptotically double, e.g., if the unperturbed equation is the Sine–Gordon equation under the periodic boundary conditions, then one should use a version of the Metatheorem, based on the Chierchia–You result). The quasilinearity condition H3) holds due to (27). The assumption H1) now takes the form

$$\operatorname{Hess} h(p) \not\equiv 0. \tag{29}$$

This is exactly Kolmogorov's nondegeneracy condition for the integrable system on $\mathcal{T}^{2n}$. The assumption H4) with $\omega = \nabla h(a)$ is the second nondegeneracy condition, which needs verification.

Summarizing what was said above, we see that Theorem 3.1 implies the Metatheorem if the unperturbed integrable equation is 1D quasilinear, the linear operator $G(p,\, q)$ from Step 1 possesses the required regularity properly and the nondegeneracy assumptions (29) and (17) hold true.

The scheme we have just explained was suggested in [**Kuk89**], where it was used to prove an abstract KAM-theorem, which next was applied to Birkhoff-integrable infinite dimensional systems and to perturbed KdV equations. See [**Kuk00**], [**EKMY02**] for a more general abstract theorem, based on the same scheme.

Steps 1–2 are not the only way to reduce an integrable equation to the normal form (26). Another approach to get it had been initiated by Kappeler [**Kap91**].

It was developed further in a number of publications and finally in [**KM01**] it was proved that the KdV equation is Birkhoff-integrable. It means the following. Let us take the Darboux scale $(\{X_s\}, \alpha_2)$ with the index set $\mathcal{Z} = \mathbb{Z}_0$, and $\theta_k = |k|$ (see Example 2.4). Then there exists a map $\Phi : X_\infty \to H_0^\infty(S^1)$ which extends to analytic maps $X_s \to H^s(S^1)_0$, $s \geq 0$, such that

$$h \circ \Phi(u) = \sum_{j=1}^{\infty} j^3(u_j^2 + u_{-j}^2) + \langle \text{a function of } \; u_l^2 + u_{-l}^2,\, l = 1, 2, \dots \rangle. \tag{30}$$

Here $\{u_k,\, k \in \mathbb{Z}_0\}$ are coefficients of decomposition of $u \in X_s$ in the basis $\{\varphi_k\}$ and h is the KdV-hamiltonian (see Example 2.5). Moreover, the hamiltonian (30) defines an analytic Hamiltonian vector field of order three in each space X_d, $d \geq 1$. In the transformed variables the N-gap tori of the KdV equation take the form (15), where $n \geq N$ and exactly N numbers I_j are non-zero. Now let us take a torus (15), where $I \in \mathbb{R}^n_+$. Making a change of variables as in Section 3.2, we arrive at the hamiltonian (28). For a detailed and readable derivation of the normal form (30), see [**KP03**].

Reduction to the Birkhoff normal form (30) essentially uses specifics of the KdV's L-operator. Still, similar arguments apply to the defocusing Zakharov–Shabat equation as well. Presumably, Birkhoff normal forms exist for many integrable equations with self-adjoint L-operators (see [**GK03**] for the defocusing Zakharov–Shabat), but not for equations with non-selfadjoint operators. In particular, the focusing Zakharov–Shabat equation cannot be reduced to the form (30) since for this equation some finite-gap tori are linearly unstable [**CMM02**], while all invariant tori of the form (15) for the hamiltonian (30) are linearly stable.

Example 4.2 (perturbed KdV equation). Consider the equation

$$\dot u(t,\, x) = \frac{1}{4}\frac{\partial}{\partial x}(u'' + 3u^2 + \varepsilon f(x,\, u)), \quad x \in S^1; \quad \int_{S^1} u\, dx \equiv 0, \tag{31}$$

where f is smooth in x, u and analytic in u. The Metatheorem applies and implies that most of finite-gap KdV-solutions persist as time-quasiperiodic solutions of (31). Moreover, these solutions are smooth and linearly stable. □

This result was first proved in [**Kuk89**] with a number of omissions. Two most serious ines are that Theorem 3.1, proved then only for semilinear equations, was used in a quasilinear case, and that the non-degeneracy assumptions (29) and (17) were taken for granted. These omissions were filled in later. The quasilinear version of Theorem 3.1 was proved in [**Kuk98**] (preprint of this paper appeared in 1995), and the non-degeneracy conditions were verified in [**BK91**]. The verification goes along the following scheme: the finite-gap manifold $\mathcal{T}_V^{2n}$ can be uniformized near its singular set $\{r_j = 0 \,\text{for some}\, j\}$ by means of coordinates $(R,\, q)$, $R \in \overline{\mathbb{R}^N_+}$, $q \in \mathbb{T}^n$, such that $R_j = 0$ iff $r_j = 0$, $j = 1, \dots, n$. The functions ω_r and λ_j all are q-independent and analytic in R, and their 1-jets at $R = 0$ can be calculated explicitly. Assuming that the left-hand side of (17) vanishes identically, taking its 1-jet at $R = 0$ and using elementary Diophantine arguments, we arrive at a contradiction. For another proof see [**Kuk00**], Section 6.2.1. The arguments in [**BK91, Kuk00**] are general and applies to other equations.

For a complete proof of 'KAM for KdV' see [**Kuk00, EKMY02, KP03**]. □

The Metatheorem (in its rigorous form as in [**Kuk00, EKMY02, KP03**]), applies to quasilinear Hamiltonian perturbations of higher equations from the KdV-hierarchy, provided that the non-degeneracy relations are checked for these equations. This can be done in the same way as in Example 4.2. See [**KP03**], where the nondegeneracy of the second KdV equation is verified.

EXAMPLE 4.3 (perturbed SG equation). Consider the equation

$$\ddot{u} = u_{xx} - \sin u + \varepsilon f(u, x), \quad u(t, 0) = u(t, \pi) = 0, \tag{32}$$

where $f(0, x) \equiv 0$ (and $f \in C^\infty$ is analytic in u). The Metatheorem applies to prove persistence most of finite-gap solutions of the SG-equation, see [**BK93, Kuk00, EKMY02**]. In general, due to the phenomenon explained in Example 2.7, the persisted solutions are only H^2-smooth in x. But if f is x-independent and odd in u, then these solutions are smooth.

In contrast with the KdV-case, large amplitude finite-gap SG-solutions, as well as the corresponding persisted solutions of (32), in general are not linearly stable. □

To end this section we note that since the persisted solutions $u_\varepsilon(t)$ have the form

$$u_\varepsilon(t) = \Phi_\varepsilon(p, q + t\omega_\varepsilon(p)) = \Phi_0(p, q + t\omega_\varepsilon(p)) + O(\sqrt{\varepsilon}),$$

to calculate them with accuracy $\sqrt{\varepsilon}$ for all values of time t, we can use the "finite gap map" Φ_0 with the corrected frequency vector. Moreover, $\omega_\varepsilon(p) = \nabla h(p) + \varepsilon W_1(p) + O(\varepsilon^2)$, where the vector $W_1(p)$ can be obtained by averaging over the corresponding finite-gap torus of some explicit quantity, see [**Kuk00**, p. 147].

5. Small amplitude solutions of HPDEs

Let us consider the nonlinear string equation

$$u_{tt} = u_{xx} - mu + f(u), \;\; u = u(t, x), \; 0 \le x \le \pi; \quad u(t, 0) = u(t, \pi) = 0. \tag{33}$$

Here $m > 0$ and f is an odd analytic function of the form

$$f(u) = \kappa u^3 + O(u^5), \quad \kappa > 0.$$

Since $m, \kappa > 0$, then constants $a, b > 0$ can be found such that $-mu + f(u) = -a \sin bu$. Hence, the equation (33) can be written as

$$u_{tt} = u_{xx} - a \sin bu + O(|u|^5).$$

After the scaling $u = \varepsilon w$, $\varepsilon \ll 1$, the higher-order perturbation transforms to a small one, and we can apply the Metatheorem (cf. Example 4.3) to prove that small-amplitude parts of the finite-gap manifolds $\mathcal{T}^{2n}$, $n = 1, 2, \ldots$, for the SG equation $u_{tt} = u_{xx} - a \sin bu$ with the Dirichlet boundary conditions mostly persist in (33). To put this scheme through, small-amplitude parts of the manifolds $\mathcal{T}^{2n}$,

$$\mathcal{T}^{2n}_\delta = \{(u, \dot{u}) \in \mathcal{T}^{2n} \mid \|u\| + \|\dot{u}\| < \delta\}, \quad 0 < \delta \ll 1,$$

have to be studied in details. This was accomplished in [**BK95b**], where the following results were proved:

i) the sets $\overline{\mathcal{T}^{2n}_\delta}$ are smooth manifolds,

ii) they are in one-to-one correspondence with their tangent spaces at the origin,

iii) these tangent spaces are invariant spaces for the Klein–Gordon equation $u_{tt} = u_{xx} - (ab)u$.

Another proof of i)–iii) was suggested in [**Kuk00**]. It is based on some ideas from [**Kap91**] and applies to other integrable equations. After i)–iii) are obtained, a version of the Metatheorem (or a version of Theorem 3.1) applies to prove that most of finite-gap solutions from a manifold $\mathcal{T}_\delta^{2n}$ persist in (33) in the following sense: the $2n$-dimensional Hausdorff measure of the persisted part of the manifold, divided by a similar measure of $\mathcal{T}_\delta^{2n}$, converges to one as $\delta \to 0$. See [**BK95a**] for a proof and [**Kuk94**] for discussion.

Similar results hold for the NLS equation

$$i\dot{u} = u_{xx} + mu + f(|u|^2)u, \tag{34}$$

where $f(0) = 0$, $f'(0) = \gamma \neq 0$, since it is a higher-order perturbation of the Zakharov–Shabat equation $i\dot{u} = u_{xx} + mu + \gamma|u|^2u$. But it turns out that it is easier to approximate (34) near the origin by its partial Birkhoff normal form. The latter is an integrable infinite-dimensional Hamiltonian system (which is not an HPDE), and a sibling of the Metatheorem applies to prove that most of its time-quasiperiodic solutions persist in (34), see [**KP96**]. More on the techniques of Birkhoff normal forms in HPDE see in [**Pös96b**] and [**KP03**]. The classical reference for finite-dimensional Birkhoff normal forms is the book [**MS71**].

6. Around the Nekhoroshev theorem

The classical Nekhoroshev theorem [**Nek77**] deals with nearly-integrable Hamiltonian systems with analytic hamiltonians $H_\varepsilon(p, q) = h(p) + \varepsilon H(p, q)$ on the phase-space $P \times \mathbb{T}^n$, $P \subset \mathbb{R}^n$, given the usual symplectic structure $dp \wedge dq$. Under the assumption that the hamiltonian $h(p)$ satisfies a mild non-degeneracy condition, called *the steepness*, the theorem states that the action variables change exponentially slow along trajectories of the system. Namely, there exist constants $a, b \in (0, 1)$ such that for any trajectory $(p(t)), q(t))$ of the system we have

$$|p(t) - p(0)| \leq C\varepsilon^a \ \text{ if } \ |t| \leq \exp(\varepsilon^{-b}). \tag{35}$$

Strictly convex functions $h(p)$ form an important class of the steep hamiltonians. An alternative proof of the theorem which applies in the convex case was suggested by Lochak [**Loc92**]. It is based on clever approximation of a trajectory $(p(t), q(t))$ by a time-periodic solution of the equation which is a high-order normal form for H_ε. So rational frequency-vectors play a very important role in in the Lochak approach.

Original Nekhoroshev's proof contains two parts, analytical and geometrical. The techniques developed in the analytical part of the proof, allow one to get the following result, which we call below the quasi-Nekhoroshev theorem: Let us consider the hamiltonian H_ε, depending on an additional vector-parameter $\omega \in \Omega \subset \mathbb{R}^n$, $H_\varepsilon = p \cdot \omega + \varepsilon H(p, q)$. Then for any $\gamma > 0$ there exists a Borel subset $\Omega_\gamma \subset \Omega$ ('the Diophantine subset') such that $\text{meas}(\Omega \backslash \Omega_\gamma) < \gamma$, and (35) with $C = C_\gamma$ holds if $\omega \in \Omega_\gamma$. Note that in the Cartesian coordinates (x, y), corresponding to the action-angle variables (p, q) (i.e., $x_j = \sqrt{2p_j} \cos q_j$, $y_j = \sqrt{2p_j} \sin q_j$), the hamiltonian H_ε reeds as

$$H_\varepsilon = \frac{1}{2} \sum \omega_j (x_j^2 + y_j^2) + \varepsilon H(x, y).$$

That is, H_ε is a perturbation of the quadratic hamiltonian H_0. Therefore the quasi-Nekhoroshev theorem should apply to the study of the dynamics in the vicinity of the origin of an analytical hamiltonian

$$H(x,\,y)=H_0+h,\quad h=O(|(x,\,y)|^3). \tag{36}$$

To get a corresponding theorem which applies to all small initial data is a nontrivial task, resolved by Niederman [**Nie98**] using the Lochak approach.

No analogy of the Nekhoroshev theorem for HPDEs is known yet, but a number of *ad hoc* quasi-Nekhoroshev theorems for HPDEs were proved, mostly by Bourgain and Bambusi, see [**Bam99a, Bam99b, Bou00**] and references therein. These works discuss the stability of the equilibrium for HPDEs (mostly 1D) with hamiltonians of the form (36). Under some restrictions on the quadratic part H_0 and on the higher-order part h, it is proved that if the initial data u_0 is an ε-small and 'very' smooth function, then a solution stays very close to the corresponding invariant torus of the linear system with the hamiltonian H_0 during the time which is polinomially large in ε^{-1}, or even exponentially large. This result is obtained either under the 'quasi-Nekhoroshev' condition that the spectrum of the operator A is 'highly non-resonant', or under the opposite assumption (needed to apply the Lochak–Niederman technique) that the spectrum is 'very resonant'. In particular, the following result is proved in [**Bam99a**] (also see [**Pös99, Bou00**]). Let us consider the NLS equation

$$\dot u=i(-u_{xx}+\varphi(|u|^2)u),\quad \varphi'(0)\neq 0, \tag{37}$$

where φ is an analytic function and the equation is being studied in the scale $\{H_0^s(0,\,\pi)\}$ of odd 2π-periodic functions. Assume that $u_0(x)=\sum_{k=1}^N u_{k0}\sin kx$, denote $\varepsilon=|u_0(x)|_{L_2}\ll 1$ and write the solution $u(t,\,x)$ of (37) as $u=\sum u_k(t)\sin kx$. Then there exist $\varepsilon_*>0$ and constants $C_1,C_2>0$ such that for $\varepsilon<\varepsilon_*$ and $|t|\le C_1\exp(\varepsilon_*/\varepsilon)^{1/N}=:T_\varepsilon$ we have

$$\sum_{k=1}^{\infty}\bigl(|u_k(t)|^2-|u_{k0}|^2\bigr)^2\le C_2\varepsilon^{4+1/N}. \tag{38}$$

Let us set $T^N=\{u(x)=\sum_{k=1}^N u_k\sin kx\mid |u_k|=|u_{k_0}|\}$. This is an n-torus of diameter $\sim\varepsilon$ and (38) implies that

$$\operatorname{dist}_{H_0^s}(u(t),\,T^n)\le C_s\varepsilon^{1+1/N}\quad\forall\,|t|\le T_\varepsilon,$$

if $s<-1/4$. Thus, during the time T_ε the trajectory $u(t)$ remains very close to its projection to T^N. The latter is a trajectory of an N-dimensional dynamical system, so the time of its return to a $\rho\varepsilon$-neighbourhood ($\rho\ll 1$) of the initial point 'should' be of order ρ^{-N}. The same is true for the trajectory $u(t)$, if ε is small in terms of ρ. The phenomenon of the pathologically good recurrence properties of small-amplitude trajectories of some 1D HPDEs is well known from numerics.[2] We have seen that quasi-Nekhoroshev theorems as above explain it to some extend.

[2] I observed it in late 80's in numerics which I was running in the Institute for Control Sciences (Moscow) to get numerical evidence for the persistence-result described in Section 3.3. Instead of verifying that a solution for a non-linear HPDE is quasiperiodic, I checked that it has good recurrence property. To my surprise, not only some solutions (those close to the persisted tori) were recurrent, but practically all of them were. With the 15-years delay I thank Rauf Izmailov for helping me with the numerics.

7. Open problems

In this section we state problems related to qualitative properties of nearly-integrable and nearly-linear HPDEs which we consider as important.

The KAM-theory for 1D HPDEs is now reasonably complete (see Sections 3.1–3.4 and 4). The main open problem here is the following.

Problem 7.1 (infinite-dimensional tori). What happens under perturbations to infinite-dimensional tori (15) with $n = \infty$, where the actions $I_j > 0$ decays as j^{-M} for some $M > 0$? □

Perturbation theory for infinite-dimensional tori with very fast decaying actions is similar to that for finite-dimensional ones, see [**Bou96, Pös02**].

For nD HPDEs with $n \geq 2$ the only published KAM-result is [**Bou98**]. The main problems in the nD-case seem to be the following.

Problem 7.2 (nD equations). i) Find an abstract theorem which applies to various nD equations (cf. Theorem 3.1 and its applications in section 3.3).

ii) Prove a KAM-theorem for some nD HPDE ($n \geq 2$) which is quasilinear but not semilinear.

iii) Analyse linearized stability of the persisted solutions.

iv) (a counter-example). Find a parameter-depending HPDE (11) such that H1) holds, but the n-torus (15) does not persist in (11) for arbitrarily small ε, for all parameters from a set of positive ε-independent measure. □

Most likely, the answer to question iii) is that the persisted solutions may be linearly unstable. The reason is that for the nD equations the assumption (17) is violated for $s = 0$ and $l = (0, \dots, -1, 1)$, asymptotically as $M_2 \to \infty$ along some sequence (cf. Problem 4 in [**Kuk94**], Section 5). To get a corresponding example one first has to show that the quasiperiodic solutions of finite-dimensional systems, constructed in [**Bou97**], may be linearly unstable.

KAM-theory for perturbations of integrable HPDEs is rather complete since 'almost all' of these equations are 1D. A physically important open problem here is to prove that the majority of large-amplitude finite-gap solutions persist under 'relatively small' perturbations. Below we repeat Problem 9 from [**Kuk94**].

Problem 7.3 (relatively small perturbations). Prove that the majority of large-amplitude finite-gap solutions of the defocusing Zakharov-Shabat equation $i\dot{u} = u_{xx} - |u|^2 u$ persist in the equation $i\dot{u} = u_{xx} + V(x)u - |u|^2 u$, where $V(x)$ is a real potential.

An easier form of this question is obtained when the perturbation $V(x)u$ is replaced by $f(x)$, where f is a real function. □

The Birkhoff normal form from [**GK03**] may be userful here.

The Kadomtsev–Petviashvili (KP) equation (see, e.g., [**Dub81, BBE$^+$94**]) is a notable example of an integrable HPDE that is not 1D:

Problem 7.4 (perturbed KP equation). Analyse KAM-stability of finite-gap solutions for the KP equation under Hamiltonian perturbations of the equation. □

Nothing is known about small-amplitude time-quasiperiodic solutions for a non-linear HPDE that is not a perturbation of an integrable equation by a higher-order term (cf. Section 5). In particular we have:

PROBLEM 7.5 (φ^4-equation). Construct small-amplitude time-quasiperiodic solutions for the zero mass φ^4-equation $u_{tt} = u_{xx} - u^3$ under, say, odd periodic boundary conditions (see [**ZIS79**] for related numerics). □

In the Nekhoroshev theory it is desirable to get an abstract quasi-Nekhoroshev theorem which would apply to various HPDEs. Still the main challenge here is to prove or disprove the Nekhoroshev theorem:

PROBLEM 7.6 (Nekhoroshev for PDEs). Prove or disprove the following statement: For any $m \geq 1$ and $K > 0$ there exist a, $b \in (0, 1)$ and constants $C_1, \dots, C_m$ such that if $u(t)$ is a solution for the perturbed KdV (31) with $u(0) = u_0$, and

$$H_1(u_0) \leq K, \dots, H_m(u_0) \leq K,$$

then for $0 \leq t \leq \exp \varepsilon^{-b}$ and $j = 1, \dots, \tilde{m}$ we have $|H_j(u(t)) - H_j(u_0)| \leq \varepsilon^a$. Here $\tilde{m} = \tilde{m}(m) \leq m$ goes to infinity with m and H_m is the m-th hamiltonian from the KdV-hierarchy. □

If this statement is true, then it might be easier to get it under the assumption that the function $u_0(x)$ is analytic (and $|u_0(x)| \leq K$ in a fixed complex strip).

As usual, closely related is the problem of diffusion. It can be understood differently. For example,

PROBLEM 7.7 (diffusion). For any $\varepsilon > 0$ find a smooth solution $u(t)$ of (31) such that $|H_j(u(T_\varepsilon)) - H_j(u(0))| \geq \delta > 0$ for some $T_\varepsilon > 0$, where j and δ are ε-independent. □

PROBLEM 7.8 (diffusion). Find a quasilinear nD HPDE ($n \geq 1$) in a suitable Sobolev scale $\{X_s\}$ such that for all sufficiently large s, for any $\varepsilon > 0$ and for a 'typical' initial condition $u(0) \in \{\|u\|_s = \varepsilon\}$, the corresponding solution $u(t)$ satisfies the condition $\|u(t)\|_s \to \infty$ (or $\limsup \|u(t)\|_s = \infty$) as $t \to \infty$. Does this property hold for a typical quasilinear HPDE? □

Concerning the last problem see [**Bou00**] and p. 188 in [**EKMY02**].

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