# On reducibility of Schrödinger equations with quasiperiodic in time potentials

Håkan L. Eliasson and Sergei B. Kuksin

**Abstract**

We prove that a linear $d$-dimensional Schrödinger equation with an $x$-periodic and $t$-quasiperiodic potential reduces to an autonomous equation for most values of the frequency vector. The reduction is made by means of a non-autonomous linear transformation of the space of $x$-periodic functions. This transformation is a quasiperiodic function of $t$.

-

## 1 Results.

We consider a linear Schrödinger equation on a $d$-dimensional torus with a non-autonomous potential which is a quasiperiodic function of time:

$$\dot{u} = -i\big(\Delta u - \varepsilon V(\varphi_0 + t\omega, x; \omega)u\big), \qquad u = u(t,x),\ x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d. \quad (1.1)$$

Here $0 \le \varepsilon \le 1$ and the frequency vector $\omega$ is regarded as a parameter: $\omega \in U \subset \mathbb{R}^d$, where $U$ is an open subset of the cube $\{y \in \mathbb{R}^d \mid |y| \le C\}$. The function $V(\varphi, x; \omega)$, $(\varphi, x, \omega) \in \mathbb{T}^n \times \mathbb{T}^d \times U$, is $C^1$-smooth in all its variables and is analytic in $(\varphi, x)$. For some $\rho > 0$ it analytically in $\varphi, x$ extends to the domain

$$\mathbb{T}^n_\rho \times \mathbb{T}^d_\rho \times U, \qquad \mathbb{T}^n_\rho = \{(a+ib) \in \mathbb{C}^n/2\pi\mathbb{Z}^n \mid |b| < \rho\},$$

where it is bounded by $C_1$, as well as its gradient in $\omega$. We regard (1.1) as a linear non-autonomous equation in the complex Hilbert space $L^2(\mathbb{T}^d) = L^2(\mathbb{T}^d;\mathbb{C})$. By $\langle\cdot,\cdot\rangle$ we denote the Hermitian $L^2$-scalar product in $L^2(\mathbb{T}^d)$.

In this work we prove that eq. (1.1) reduces to constant coefficients for 'most values of the parameter $\omega$'. The result is stated in the theorem below. There by $H^p(\mathbb{T}^d)$ and $H^p(\mathbb{T}^d;\mathbb{R})$, $p \in \mathbb{R}$, we denote the complex and real Sobolev spaces with the norm $\|\cdot\|_p$, where

$$\|u\|_p^2 = \int |(-\Delta+1)^{p/2}u(x)|^2\,dx = \langle(-\Delta+1)^p u,\ u\rangle,$$

and by $\|\cdot\|_{p,p}$ denote the norm in the space of linear operators in $H^p$. The exponential functions $\{e_s \mid s \in \mathbb{Z}^d\}$, $e_s(x) = (2\pi)^{-d/2}e^{is\cdot x}$, form a Hilbert basis of the space $L^2(\mathbb{T}^d)$ and form an orthogonal basis of each Sobolev space. For any linear operator $B$ between Sobolev spaces (real or complex) we denote by $(B_{ab}, a, b \in \mathbb{Z}^d)$ its matrix with respect to this basis. By $|\cdot|$ we denote the Euclidean norm and the operator-norms of finite-dimensional matrices.

**Theorem 1.1.** *For any $0 < \varepsilon \le \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small, there exists a Borel set $U_\varepsilon \subset U$, $\mathrm{mes}(U \setminus U_\varepsilon) \le K\varepsilon^{\varkappa}$, such that for $\omega \in U_\varepsilon, \varphi \in \mathbb{T}^n$ in the space $L^2(\mathbb{T}^d)$ exist a complex-linear isomorphism $\Psi(\varphi) = \Psi(\varphi)_{\varepsilon,\omega}$ which analytically depends on $\varphi \in \mathbb{T}^n_{\rho/2}$ and a bounded Hermitian operator $Q = Q_{\varepsilon,\omega}$ with the following property: a curve $v(t) = v(t,\cdot) \in L^2(\mathbb{T}^d)$ satisfies the autonomous equation*

$$\dot v = -i\Delta v + i\varepsilon Q v \tag{1.2}$$

*if and only if $u(t,\cdot) = \Psi(\varphi_0 + t\omega)v(t,\cdot)$ is a solution of (1.1).*

*The matrix $(Q_{ab})$ of operator $Q$ satisfies*

$$Q_{ab} = 0 \quad \text{if} \quad |a| \neq |b|. \tag{1.3}$$

*For any $p \in \mathbb{N}$ operators $Q$ and $\Psi(\varphi)$ meet the estimates*

$$\|Q\|_{p,p} = \|Q\|_{0,0} \le K_1\,, \tag{1.4}$$

$$\|\Psi(\varphi) - \mathrm{id}\,\|_{p,p} \le \varepsilon K_2 \quad \forall\, \varphi \in \mathbb{T}^n_{\rho/2}. \tag{1.5}$$

*Moreover, $Q_{\varepsilon,\omega}$ and $\Psi(\varphi)_{\varepsilon,\omega}$ are operator-valued Lipschitz functions of $\omega \in U_\varepsilon$ and*

$$\|\nabla_\omega Q\|_{p,p} \le K_1 \quad , \quad \|\nabla_\omega \Psi(\varphi)\|_{p,p} \le \varepsilon K_2, \tag{1.6}$$

*for all $\varphi \in \mathbb{T}^n_{\rho/2}$ and a.a. $\omega \in U_\varepsilon$.*

*The positive constants $\varepsilon_0$, $K$ and $k$ depend only on $n, d, C, C_1$ and $\rho$, while $K_1$ and $K_2$ also depend on $\omega$ and $K_2$ depends on $p$.*

Since operator $Q$ is Hermitian and satisfies (1.3), then spectrum of the linear operator in the r.h.s. of (1.2) is pure point and imaginary. So all solutions $v(t) \in L^2(\mathbb{T}^d)$ of (1.2) are almost-periodic functions of $t$. Estimates (1.4), (1.5) imply that these solutions are well localised in the Fourier presentation:

**Corollary 1.2.** *For any $p$ there exists $\varepsilon_0' > 0$ and $K_3 > 0$ such that for $\omega \in U_\varepsilon$ every solution $u(t)$ of (1.1) with $\varepsilon \le \varepsilon_0'$ satisfies*

$$(1 - K_3\varepsilon)\|u(0)\|_p \le \|u(t)\|_p \le (1 + K_3\varepsilon)\|u(0)\|_p \qquad \forall\, t\,. \tag{1.7}$$

*Apart from $p$, the constant $\varepsilon_0'$ depends on $n, d, C, C_1$ and $\rho$, while $K_3$ also depends on $\omega$.*

In particular, if $u(0) = u(0,x)$ is a finite trigonometrical polynomial and $u(t,x) = \sum u_s(t)e^{is\cdot x}$, then

$$\sup_t |u_s(t)| \le C_p|s|^{-p} \qquad \forall\, s,\ \forall\, p\,. \tag{1.8}$$

Such behaviour of solutions for a dynamical equation is called *dynamical localisation.*

*Remark 1.* The linear operators in the r.h.s. of linear Hamiltonian equations (1.1) and (1.2) are complex-linear Hermitian transformations. So the flow-maps of these equations are complex-linear, symplectic and unitary. The conjugating transformations $\Psi(\varphi)$ are complex linear. It can be shown that they also are symplectic. Hence, they are unitary. So the conjugations respect all the three structures, preserved by equations (1.1) and (1.2).

*Remark 2.* In fact, the constant $K_1$ does not depend on $\omega$. Moreover, if we replace (1.5) by the weaker estimate

$$\|\Psi(\varphi) - \mathrm{id}\,\|_{p,p} \le \sqrt{\varepsilon}\, K_2' \quad \forall\, \varphi \in \mathbb{T}^n_{\rho/2},$$

and similar with (1.6), then the constant $K_2'$ can be chosen $\omega$-independent. See below footnote 3.

*Remark 3.* The estimates (1.5)–(1.7) remain true with arbitrary $p \ge 0$ if we replace the Sobolev norms $\|\cdot\|_p$ and the operator norms $\|\cdot\|_{p,p}$ by the stronger norm $[\cdot]_p$, where

$$[u]_p^2 = \sum_{s\in\mathbb{Z}^d} |u_s|^2 e^{2\left(\ln(|s|+1)\right)^p}, \qquad u(x) = \sum_{s\in\mathbb{Z}^d} u_s e^{is\cdot x}\,,$$

and by the corresponding operator-norm $[\cdot]_{p,p}$. Again the constants $K_2, K_3$ and $\varepsilon'$ depend on $p$. In particular, (1.8) remains true if we replace its r.h.s. by $C_p' \exp\left(-\left(\ln(|s|+1)\right)^p\right)$ ($p > 0$ is any).

In the next section we derive Theorem 1.1 from an abstract theorem in [EK], prove Corollary 1.2 and discuss Remark 3.

*Related results.* It was observed by N. Bogolyubov in 1960's (see in [BMS69]) that KAM-technique apply to prove reducibility of non-autonomous finite-dimensional linear systems to constant coefficient equations. Such results are also contained in [Mos67]. Since then establishing the reducibility of finite-dimensional systems by means of the KAM tools is an active field of research. For the case of partial differential equations the techniques from 'KAM for PDE' theory were used by Bambusi and Graffi in [BG01] to prove reducibility of one-dimensional Schrödinger equation (1.1) to constant in time coefficients. Their results are similar to those in Theorem 1.1 with $d = 1$.

The problem of growth of solutions for linear Schrödinger equation with time-quasiperiodic and with smooth bounded potentials was considered by J. Bourgain in [Bou99a] and [Bou99b], respectively. In the first work it is shown that for a Diophantine frequency vector $\omega$ Sobolev norms of any solution for (1.1) growth with $t$ at most logarithmically, while results of the second work imply that for any $\omega$ each Sobolev norm growths slower than any positive degree of $t$. Corollary 1.2 specify these results for 'typical' vectors $\omega$.

Corollary 1.2 shows that Sobolev norms of solutions for eq. (1.1) remain bounded in time, provided that the frequency vector $\omega$ is 'typical'. In particular, it should be non-resonant with the numbers $\{|s|^2 \mid s \in \mathbb{Z}^d\}$, forming the spectrum of the operator $-\Delta$. It turns out that the norms of the solutions may stay bounded also in the opposite case when $\omega$ is completely resonant with the spectrum. Namely, W.-M. Wang [Wan07] proved this for eq. (1.1) where $n = d = 1$ and $\omega = 1$.

# 2 Proofs.

*Proof of Theorem 1.1.* The operator $\Delta$ on torus has zero in its spectrum. This is inconvenient for some technical reasons. So we make the substitution $u := e^{-it/2}u$ and re-write eq. (1.1) as

$$\dot{u} = -i\Big(\big(\Delta - \tfrac{1}{2}\big)u - \varepsilon V(\varphi_0 + t\omega, x; \omega)u\Big). \tag{2.1}$$

Below we usually do not indicate dependence of functions on the parameter $\omega$.

Firstly we re-interpret eq. (2.1) as an autonomous Hamiltonian system in an extended phase-space. To do this we write $u(x) = \big(\xi(x) + i\eta(x)\big)/\sqrt{2}$, where $\xi$ and $\eta$ are real functions. Then (2.1) becomes

$$\begin{aligned} \dot{\xi} &= -\big((-\Delta + \tfrac{1}{2})\eta + \varepsilon V(\varphi_0 + t\omega, x)\eta\big)\,, \\ \dot{\eta} &= \big(-\Delta + \tfrac{1}{2}\big)\xi + \varepsilon V(\varphi_0 + t\omega, x)\xi\,. \end{aligned} \tag{2.2}$$

Let us consider the space

$$\mathcal{Z} = H^1(\mathbb{T}^d; \mathbb{R}) \times H^1(\mathbb{T}^d; \mathbb{R}) \times \mathbb{T}^n \times \mathbb{R}^n = (\xi, \eta, \varphi, r).$$

We provide it with a symplectic structure, given by the two-form $\alpha_2 \oplus (dr \wedge d\varphi)$, where $\alpha_2[(\xi_1, \eta_1), (\xi_2, \eta_2)] = \langle \eta_1, \xi_2 \rangle - \langle \xi_1, \eta_2 \rangle$ and $\langle \cdot, \cdot \rangle$ stands for the usual $L^2$-scalar product.

The function $h^\varepsilon_\omega(\xi, \eta, \varphi, r)$,

$$h^\varepsilon_\omega = \omega \cdot r + \frac{1}{2}\int \big((|\nabla\xi|^2 + |\nabla\eta|^2) + \tfrac{1}{2}(|\xi|^2 + |\eta|^2) + \varepsilon V(\varphi, x)(\xi^2 + \eta^2)\big)\, dx \tag{2.3}$$

is analytic in $\mathcal{Z}$. The symplectic structure above corresponds to the function $h^\varepsilon_\omega$ the Hamiltonian equation

$$\begin{aligned} \dot{\xi} &= -\nabla_\eta h^\varepsilon_\omega = -\big((-\Delta + \tfrac{1}{2})\eta + \varepsilon V(\varphi)\eta\big), \\ \dot{\eta} &= \nabla_\xi h^\varepsilon_\omega = \big((-\Delta + \tfrac{1}{2})\xi + \varepsilon V(\varphi)\xi\big), \\ \dot{\varphi} &= \nabla_r h^\varepsilon_\omega = \omega, \\ \dot{r} &= -\nabla_\varphi h^\varepsilon_\omega. \end{aligned} \tag{2.4}$$

The first three equations are independent from $r$ and are equivalent to eq. (2.2).

The Hamiltonian $h_\omega^\varepsilon$ is a perturbation of the integrable Hamiltonian $h_\omega^0$ (which corresponds to the Schrödinger equation $i\dot u = (\Delta - 1/2)u$) by the quadratic in $(\xi, \eta)$ function $\varepsilon f$. The function $f$ is the quadratic form, corresponding to the linear operator $\frac{1}{2} F_\varphi$, where

$$F_\varphi : (\xi(x), \eta(x)) \mapsto (V(\varphi, x)\xi(x), V(\varphi, x)\eta(x))$$

(this operator depends on the parameter $\omega$). Write $V(\varphi, x)$ as $V = \sum V_s(\varphi)e^{is\cdot x}$. Then $F_\varphi$, regarded as an operator on vectors $\zeta = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in L^2(\mathbb{T}^d;\mathbb{R})\times L^2(\mathbb{T}^d;\mathbb{R})$ (or on complex vectors $\zeta \in L^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)$) has a matrix, formed by $2\times 2$-blocks $F_{ab}(\varphi) = V_{b-a}(\varphi) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By the analyticity assumption,

$$|V_s(\varphi)|,\ |\nabla_\omega V_s(\varphi)| \le C_1 e^{-\rho|s|} \qquad \forall\, s,\ \forall\, \varphi \in \mathbb{T}^n_\rho,\ \forall\, \omega \in U.$$

We see that $F = (F_{ab})$ is a Töplitz matrix, formed by diagonal $2 \times 2$-blocks, which has finite exponential norm $|F|_\rho$,

$$|F|_\rho = \sup_{a,b} \left| e^{\rho|a-b|} |F_{ab}| \right|. \tag{2.5}$$

In the space of complex $2 \times 2$-matrices, provided with the scalar product $\mathrm{Tr}\,({}^t\bar{A}B)$, consider the orthogonal projection $\pi$ on the subspace, generated by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For a matrix $G$, formed by $2 \times 2$-blocks $G_{ab}$, we define $\pi G$ as the matrix

$$(\pi G)_{ab} = (\pi G_{ab}).$$

Note that a real matrix $G$, operating on vectors $\zeta = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, corresponds to a complex-linear transformation, operating on complex vectors $u = (\xi + i\eta)/\sqrt{2}$, if and only if $\pi G = G$. In particular, the matrix $F$ satisfies $\pi F = F$.

These properties of matrix $F$ imply that it is a special case of the Töplitz–Lipschitz matrices, defined in [EK08, EK], and that for any $\Lambda \in \mathbb{N}$ its Töplitz–Lipschitz norm [1] satisfies the estimates

$$\langle F \rangle_{\Lambda,\rho},\ \langle \nabla_\omega F \rangle_{\Lambda,\rho} \le C_1 \qquad \forall \varphi \in \mathbb{T}^n_\rho,\ \omega \in U. \tag{2.6}$$

[1]For reader's convenience we now define the Töplitz-Lipschitz norm $\langle X \rangle_{\Lambda,\rho}$ of a matrix $X$, assuming for simplicity that $d = 2$ and $X$ satisfies $\pi X = X$. A matrix $X$ is called Töplitz at $\infty$ if the limit $X_{ab}(c) = \lim_{t\to\infty} X_{a+tc,\, b+tc}$ exists for all $a, b, c \in \mathbb{Z}^d$. Let $\mathcal{D}_\Lambda(c)$ be the set of all $(a, b) \in \mathbb{Z}^d \times \mathbb{Z}^d$ such that

$$|a = a' + tc| \ge \Lambda(|a'| + |c|)|c|\,, \quad |b = b' + tc| \ge \Lambda(|b'| + |c|)|c|$$

and $\frac{|a|}{|c|}, \frac{|b|}{|c|} \ge 2\Lambda^2$. If $X$ is Töplitz at $\infty$, we define

$$\langle X \rangle_{\Lambda,\rho} = \sup_{c\neq 0} \sup_{(a,b)\in\mathcal{D}_\Lambda(c)} |X_{ab} - X_{ab}(c)| \cdot \max\left(\frac{|a|}{|c|}, \frac{|b|}{|c|}\right) e^{\rho|a-b|} + |X|_\rho.$$

Note that if $X$ is Töplitz, then it is Töplitz at infinity and the first term in the r.h.s. vanishes. So in this case $\langle X \rangle_{\Lambda,\rho} = |X|_\rho$.

In [EK] we study nonlinear Hamiltonian perturbations of infinite-dimensional linear systems. Results of that work apply to perturbations of Hamiltonian $h^0_\omega$ of the form

$$H^\varepsilon_\omega(\zeta,\varphi,r;\omega) = h^0_\omega(\zeta,r) + \varepsilon f(\zeta,\varphi,r;\omega), \quad \zeta = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The real valued function $f$ is $C^1$-smooth in $(\zeta,\varphi,r;\omega)$, is analytic in $\mathfrak{h}=(\zeta,\varphi,r)$ and analytically in $\mathfrak{h}$ extends to the complex domain $\mathcal{O}^0(\sigma,\rho)$, where it is bounded by a constant $C_1$. Here for $\kappa\geq 0$ and $\sigma,\rho>0$ we denote

$$\mathcal{O}^\kappa(\sigma,\rho) = \{\mathfrak{h} \mid \|\zeta\|'_\kappa < \sigma\,,\ |\mathrm{Im}\,\varphi| < \rho\,,\ |r| < \sigma^2\},$$

where $\|\eta = \sum \eta_s e_s\|'_\kappa = \big(\sum |\eta_s|^2 e^{2\kappa|s|}\langle s\rangle^2\big)^{1/2}$ with $\langle s\rangle = \max\{|s|,1\}$. It is assumed that there exists $\gamma>0$ such that for any $0\leq\gamma'\leq\gamma$ and any $\mathfrak{h}\in\mathcal{O}^{\gamma'}(\sigma,\rho)$ we have $\|\nabla_\zeta f(\mathfrak{h};\omega)\|_{\gamma'}\leq C_1$ and that the Hessian $\nabla^2_\zeta f$ satisfies $\langle\nabla^2_\zeta f\rangle_{\Lambda,\gamma'}\leq C_1$ for some $\Lambda\geq 3$. Moreover it is also assumed that each component of the gradient $\nabla_\omega f$ possesses the same properties.

For Hamiltonians of the form $H^\varepsilon_\omega$ the results of Theorem 7.1 in [EK] may be stated as follows:[2]

**Theorem 2.1.** *There is $\varepsilon_0>0$ and for every $\varepsilon\leq\varepsilon_0$ there is a Borel set $U_\varepsilon\subset U$, satisfying mes $(U\backslash U_\varepsilon)\leq K\varepsilon^{\varkappa}$, such that for all $\omega\in U_\varepsilon$ the following holds: there exists an analytical symplectic diffeomorphism $\Phi:\mathcal{O}^0(\sigma/2,\rho/2)\to\mathcal{O}^0(\sigma,\rho)$ and a vector $\omega'$ such that $(h^0_\omega+\varepsilon f)\circ\Phi$ equals (modulo a constant)*

$$h^0_{\omega'}(\zeta,r) + \frac{1}{2}\varepsilon\langle\tilde H(\omega')\zeta,\zeta\rangle + f'(\mathfrak{h},\omega') =: \tilde h^\varepsilon_{\omega'}\,.$$

*Here*

$$\nabla_\zeta f' = \nabla_r f' = \nabla^2_\zeta f' = 0 \quad \textit{for} \quad \zeta = r = 0\,, \tag{2.7}$$

*and $\tilde H = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^t & Q_1 \end{pmatrix}$, where the operator $Q=Q_1+iQ_2$ is a Hermitian operator in the space $L^2(\mathbb{T}^d)$ such that its matrix satisfies (1.3). The transformation $\Phi=(\Phi_\zeta,\Phi_\varphi,\Phi_r)$ satisfies*

$$\|\Phi_\zeta-\zeta\|'_0 + |\Phi_\varphi-\varphi| + |\Phi_r-r| \leq \beta\varepsilon \tag{2.8}$$

*for all $\mathfrak{h}\in\mathcal{O}^0(\sigma/2,\rho/2)$, and*

$$\|\tilde H\|_{0,0}\leq\beta.$$

[2] The theorem is applied with $H=0$, $|\mathcal{A}|=n$, $\mathcal{L}=\mathbb{Z}^d$, $\Omega_a(\omega)=|a|^2+\frac{1}{2}$, $m_*=1$ and $\mu=\sigma^2$. It is assumed in Theorem 7.1 that the eigenvalues $\Omega_a$ are exponentially close to the numbers $|a|^2$, but all arguments in the proof hold if they are close to the numbers $|a|^2+$const (e.g., to the numbers $|a|^2+\frac{1}{2}$). Indeed, the assumption is needed to examine cluster properties of the eigenvalues $\{\Omega_a\}$ and to control their growth as $|a|\to\infty$. For both these goals the assumption $\Omega_a\sim|a|^2+$const is as good.

*The positive constants $\varepsilon_0, \varkappa, K$ depend on $n, d, C, C_1, \sigma$ and $\rho$, while $\beta$ also depends on $\omega$.*[3]

The transformation $\Phi$ is obtained as a composition of infinitely many symplectic transformations $\Phi^j : \mathfrak{h} \mapsto \mathfrak{h}$ which iteratively put the Hamiltonian $H^\varepsilon_\omega$ to forms, more and more close to $\tilde h^\varepsilon_{\omega'}$, and change a bit the original frequency vector $\omega$. Each transformation $\Phi^j(\mathfrak{h}) = (\Phi^j_\zeta(\mathfrak{h}), \Phi^j_\varphi(\mathfrak{h}), \Phi^j_r(\mathfrak{h})$ has the form

$$\begin{aligned} \Phi^j_\zeta(\mathfrak{h}) &= z^j(\varphi) + D^j(\varphi)\zeta\,, \\ \Phi^j_\varphi(\mathfrak{h}) &= a^j(\varphi)\,, \\ \Phi^j_r(\mathfrak{h}) &= b^j(\zeta,\varphi) + c^j(\varphi)r\,, \end{aligned} \tag{2.9}$$

where $b(\zeta, \varphi)$ is quadratic in $\zeta$ and $D^j(\varphi)$ and $c^j(\varphi)$ are linear operators which are real for real $\varphi$. The composition $\Phi = \Phi^1 \circ \Phi^2 \circ \dots$ also has the form (2.9). So

$$\Phi_\zeta(\mathfrak{h}) = z(\varphi) + D(\varphi)\zeta.$$

Estimate (2.8) implies that $z(\varphi) \in H^1(\mathbb{T}^d;\mathbb{R})$ and that $D(\varphi)$ is a bounded linear operator in $H^1$ (note that the norm $\|\cdot\|'_0$ is equivalent to the Sobolev norm $\|\cdot\|_1$). In fact, $z(\varphi)$ and $D(\varphi)$ are smoother than that:

**Lemma 2.2.** *For any integer $p \ge 0$ there exists $\mathcal{K} = \mathcal{K}(p)$ (depending on $\omega$) such that for any $\varphi \in \mathbb{T}^n_{\rho/2}$ the maps $z(\varphi)$ and $D(\varphi)$ from the representation* (2.9) *for the map $\Phi$ satisfy*

$$\|z(\varphi)\|_p,\ \ \|D(\varphi) - \mathrm{id}\,\|_{p,p},\ \ \|\pi D(\varphi) - \mathrm{id}\,\|_{p,p} \le \mathcal{K}\varepsilon; \tag{2.10}$$

*and*

$$\|\nabla_\omega z(\varphi)\|_p,\ \ \|\nabla_\omega D(\varphi)\|_{p,p},\ \ \|\nabla_\omega \pi D(\varphi)\|_{p,p} \le \mathcal{K}\varepsilon\,. \tag{2.11}$$

*Remarks.* 4) As in Theorem 2.1, if we replace the r.h.s's of the two estimates by $\mathcal{K}\sqrt{\varepsilon}$, then $\mathcal{K}$ may be chosen $\omega$-independent.

5) Due to (2.7) the analytical torus $\Phi(\{0\}\times\mathbb{T}^n\times\{0\}) \subset \mathcal{O}^0(\sigma,p)$ is invariant for the Hamiltonian system with the Hamiltonian $H^\varepsilon_\omega$. Since (2.10) holds for any $p \in \mathbb{N}$, then this torus is smooth in $x$. That is, it lies in $C^\infty(\mathbb{T}^d;\mathbb{R}^2)\times\mathbb{T}^n\times\mathbb{R}^n$.

*Proof of the lemma.* The maps $D^j(\varphi), z^j(\varphi)$ and other maps, entering the decomposition (2.9) for $\Phi^j$ are analysed in Proposition 8.1, Corollary 8.2 and Proposition 8.4 of [EK]. Let us define inductively the sequences $\varepsilon_j \to 0$, $\sigma_j \to 0$, $\rho_j \to \rho/2$ and $\gamma_j \to 0$ as follows

$$\varepsilon_1 = \varepsilon,\ \ \sigma_1 = \sigma,\ \ \rho_1 = \rho,\ \ \gamma_1 = \gamma := \rho/2$$

and for $j \ge 1$

$$\varepsilon_{j+1} = \exp(-\tau\big(\log\varepsilon_j^{-1})^2\big)\,,\ \sigma_{j+1} = \varepsilon_{j+1}^{1/3+\tau}\sigma_j\,,$$
$$\gamma_{j+1} = (\log\varepsilon_j^{-1})^{-c_1}\gamma_j^{c_2}\,,\ \rho_j = (2^{-1}+2^{-j})\rho\,,$$

[3] $\beta$ may be chosen $\omega$-independent if in the r.h.s.'s of (1.8) we replace $\beta\varepsilon$ by $\beta\sqrt{\varepsilon}$. This is a well known property of the KAM arguments and it follows directly from the proof in [EK].

where $\tau = 1/30$ and $c_1, c_2$ are some positive constants. Also for $j \geq 1$ we set $\Lambda_j = \text{const}\,\gamma_j^{-2}$. It is equivalent, up to constant factors, to the definition of these constants in Section 8.3 of [EK], where $\mu_j = \sigma_j^2$ for all $j$. These relations easily imply that for any $M \in \mathbb{N}$ and $k > 0$ we have

$$\exp(\ln \gamma_j^{-1})^M \leq C(M,k)\varepsilon_j^{-k} \qquad \forall\, j \geq 1, \tag{2.12}$$

We want to estimate the maps $\Phi_\zeta^j$.

For any $j \geq 1$ the map $\Phi^j$ is constructed in Proposition 8.1 of [EK] as a composition of $n_j = [\log \varepsilon_j^{-1}]$ canonical transformations which are time-1-maps for additional Hamiltonians $s_l(\mathfrak{h})$, $l = 1, \ldots, n_j$. The Hamiltonians are functions of $\mathfrak{h}$, quadratic in $\zeta$. Norms of these functions, of their gradients and Hessians in $\zeta$ are estimated in Proposition 8.1. The $\zeta$-components of flow-maps of such Hamiltonians are affine functions of $\zeta$ and are studied in Section 8.1 of [EK] (see there estimates (49) and (50)). Combining of these results implies that the map $\Phi_\zeta^j(\mathfrak{h}) = z^j(\varphi) + D^j(\varphi)\zeta$ satisfies

$$\|z^j(\varphi)\|'_{\gamma_j} \leq \text{const}\,\gamma_j^{-1}\sigma_j^{-1}\varepsilon_j \leq \begin{cases} C\varepsilon_j, \ j = 1, \\ C\varepsilon_j^{1/2}, j \geq 2, \end{cases} \tag{2.13}$$

and

$$\left|D^j(\varphi) - \text{id}\right|_{\frac{1}{2}\gamma_j} \leq \text{const}\,\Lambda_j^2\gamma_j^{-1}\sigma_j^{-2}\varepsilon_j \leq \begin{cases} C\varepsilon_j, \ j = 1, \\ C\varepsilon_j^{1/4}, j \geq 2, \end{cases} \tag{2.14}$$

for any $\varphi \in \mathbb{T}^n_{\rho/2}$ (we use (2.12) and notation (2.15)). The matrix-norm $|\cdot|_\gamma$ majorises the Sobolev operator-norms up to a factor:

$$\|G\|_{m,m} \leq C_m \gamma^{-m-d}|G|_\gamma \qquad \forall m \geq 0 \tag{2.15}$$

(see [EK08] and estimate (2) in [EK], where $\gamma' = 0$). Combining (2.14), (2.12) and the last inequality we get that

$$\|D_j(\varphi) - \text{id}\,\|_{p,p} \leq \begin{cases} C_p\varepsilon_j, \ j = 1, \\ C_p\varepsilon_j^{1/5}, j \geq 2, \end{cases} \tag{2.16}$$

for any $\varphi \in \mathbb{T}^n_{\rho/2}$. Since $\|u\|_p \leq C_p\gamma^{2(1-p)}\|u\|'_\gamma$ for any $\gamma > 0$, then, similar,

$$\|z^j(\varphi)\|_p \leq \begin{cases} C_p\varepsilon_j, \ j = 1, \\ C_p\varepsilon_j^{1/3}, j \geq 2, \end{cases} \tag{2.17}$$

for any $\varphi \in \mathbb{T}^n_{\rho/2}$. Since clearly $|\pi G|_\gamma \leq C|G|_\gamma$, then the matrix $\pi D_j(\varphi)$ also satisfies estimates (2.16).

As

$$\Phi_\zeta(\mathfrak{h}) = z(\varphi) + D(\varphi)\zeta = \Phi_\zeta^1 \circ \Phi_\zeta^2 \circ \ldots, \quad \Phi_\zeta^j = z^j(\varphi) + D^j(\varphi)\zeta$$

and

$$\pi(AB) = \pi A \pi B + (1-\pi)A(1-\pi)B, \tag{2.18}$$

then (2.16), its analogy for $\pi D^j(\varphi)$ and (2.17) imply (2.10). The maps $D^j(\varphi)$ and the map $D(\varphi)$ are real for real $\varphi$.

Relations (2.11) follow from similar estimates on $\nabla_\omega z^j$ and $\nabla_\omega D^j(\varphi)$ which can be derived from the corresponding results in [EK] in the same way as above. □

It was pointed out in a remark to Theorem 7.1 in [EK] that if the perturbation $f$ is independent from $r$ and is quadratic in $\zeta$ (e.g. if $H_\omega^\varepsilon = h_\omega^\varepsilon$, see (2.3)), then

i) vector $\omega$ stays constant during the transformations $\Phi^j$;

ii) in formula (2.9) for $\Phi^j$ we have $z^j = 0$, $a^j = 0$ and $c^j = 0$. So each transformation $\Phi^j$ has the form

$$(\zeta, \varphi, r) \mapsto \big(D_\omega(\varphi)\zeta, \varphi, r + \frac{1}{2}\langle\zeta, B_\omega(\varphi)\zeta\rangle\big) \tag{2.19}$$

with suitable linear operators $D_\omega(\varphi)$ and $B_\omega(\varphi)$.

Accordingly the limiting transformation $\Phi = \Phi^1 \circ \Phi^2 \circ \dots$ also has the form (2.19) and $\omega' = \omega$. So the transformed Hamiltonian $\tilde h_\omega^\varepsilon = \tilde h_{\omega'}^\varepsilon$, as well as the original Hamiltonian $H_\omega^\varepsilon$, is linear in $r$ and quadratic in $\zeta$. Hence, in the expression for $h_\omega^\varepsilon$ we have $f' = 0$.

The equation with the Hamiltonian $\tilde h_\omega^\varepsilon$ implies for $v(t) = \big(\xi(t) + i\eta(t)\big)/\sqrt{2}$ equation

$$\dot v = -i(\Delta - \tfrac{1}{2})v + i\varepsilon Q v\,. \tag{2.20}$$

That is, we established reducibility of eq. (2.1) to equation (2.20) by means of the linear over real numbers operator $\Psi^0(\varphi)$, defined as the composition

$$\Psi^0(\varphi):\ u(x) = \frac{\xi + i\eta}{\sqrt{2}} \mapsto D_\omega(\varphi)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi'\\ \eta'\end{pmatrix} \mapsto \frac{\xi' + i\eta'}{\sqrt{2}} = v(x).$$

Next we replace the maps $\Psi^0(\varphi)$ by complex-linear transformations which still conjugate equations (2.20) and (2.1). Let us rewrite these two equations as

$$\dot X = \mathcal{Q}X$$

and

$$\dot Y = \mathcal{P}_t Y,$$

respectively. Now we regard them as equations on operator-valued curves $X(t)$ and $Y(t)$, formed by linear isomorphisms of the space $L^2(\mathbb{T}^d)$. Consider the third equation

$$\dot W = \mathcal{P}W - W\mathcal{Q}. \tag{2.21}$$

Let $X, Y, W$ be three operator-valued curves, formed by isomorphisms of $L^2(\mathbb{T}^d)$, satisfying

$$YX^{-1} = W.$$

Then if any two of them satisfy the corresponding equations, then the third one satisfies the third equation.

Let $X(t)$ be the fundamental solution of the first equation (i.e., $X(0) = \mathrm{id}$) and $W^0(t) = \Psi^0(\varphi_t)$, where $\varphi_t = \varphi_0 + \omega t$. Then $Y = W^0 X$ satisfies the second equation. So $W^0$ satisfies (2.21). Let us apply operator $\pi$ (written in terms of the complex variable $u = (\xi + i\eta)/\sqrt{2}$) to (2.21). Since the operators $\mathcal{Q}$ and $\mathcal{P}_t$ are complex linear, then $\pi\mathcal{Q} = \mathcal{Q}, \pi\mathcal{P}_t = \mathcal{P}_t$ and (2.18) implies that the complex-linear operator

$$W(t) := \pi W^0(t) = \pi\Psi^0(\varphi_0 + \omega t)$$

also satisfies (2.21). Relations (2.10) imply that the operator $\Psi(\varphi) = \pi\Psi^0(\varphi)$ satisfies (1.5) for any integer $p \geq 0$. In particular, the operator $\Psi(\varphi) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ is invertible since $\varepsilon$ is small. We have seen that $\Psi(\varphi)$ is a complex-linear transformation which reduces equation (2.1) to (2.20). Inverting the substitution $u := e^{-it/2}u$ we see that $\Psi(\varphi)$ also reduces (1.1) to (1.2).

Estimate (1.4) with $p = 0$ follows from the estimate for $\tilde{H}$ in Theorem 2.1. Since the operator $Q$ satisfies (1.3), then $\|Q\|_{p,p} = \|Q\|_{0,0}$ for each $p$ and (1.4) follows.

The estimates for $\nabla_\omega Q$ and $\nabla_\omega \Psi$ follow from Theorem 2.1 by the same arguments. □

*Proof of Corollary 1.2.* For any $v = \sum v_s e_s \in L^2(\mathbb{T}^d)$ and $k = 0, 1, 2, \ldots$ denote $V_k = \sum_{|s|^2=k} v_s e_s$ (if $d \leq 2$, then $V_k = 0$ for some $k$). Then $v = \sum V_k$ and

$$\|v\|_p^2 = \sum_{k=0}^{\infty} (1+k)^p \|V_k\|_0^2$$

for each $p$. Since the operator $Q$ is block-diagonal, then

$$\langle QV_k, V_l\rangle = 0 \qquad \text{if} \quad k \neq l\,. \tag{2.22}$$

Let $v(t)$ be a solution of (1.2) and $u(t) = \Psi(\varphi_0 + t\omega)v(t)$ be the corresponding solution of (1.1). Take the $\langle\cdot,\cdot\rangle$–scalar product of (1.2) with $V_k$. The imaginary part of the obtained relation implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V_k\|_0^2 &= \mathrm{Im}\left(\langle \Delta v, V_k\rangle - \varepsilon \sum_s \langle QV_s, V_k\rangle\right)\\ &= \mathrm{Im}\left(\langle \Delta V_k, V_k\rangle - \varepsilon\langle QV_k, V_k\rangle\right)\end{aligned}$$

(we use (2.22)). Since the operators $\Delta$ and $Q$ are Hermitian, then the r.h.s. vanishes. So $\|V_k(t)\|_0 = \text{const}$ for each $k$. Accordingly $\|v(t)\|_p = \text{const}$ for each $p$ and (1.7) follows from (1.5) if we choose $\varepsilon_0' \leq 1/2K_2$. □

*On Remark 3*. Estimate (1.4) is valid for the norm $[\cdot]_{p,p}$ since the operator $Q$ is block-diagonal. Estimate (1.5) holds for the same reason as before if instead of inequality (2.15) we use its counterpart for the norms $[\cdot]_{p,p}$:

$$[A]_{p,p} \leq c_1 \exp\left(c_2 (\ln \gamma^{-1})^p\right) |A|_\gamma \qquad \forall\, p\,,\ \gamma > 0,$$

where $c_1, c_2$ are independent from $\gamma$. Finally, estimate (1.7) follows from (1.4) (1.5).

# References


[BG01] D. Bambusi and S. Graffi, *Time quasi-periodic unbounded perturbations of Shrödinger operators and KAM method*, Comm. Math. Physics (2001), 465–480.

[BMS69] N. Bogoliubov, Yu. Mitropolsky, and A. Samoilenko, *The method of rapid convergence in nonlinear mechanics*, Naukova Dumka, Kiev, 1969 (Russian), English translation: Springer Verlag, 1976.

[Bou99a] J. Bourgain, *Growth of Sobolev norms in linear Schrödinger equation with quasi-periodic potential*, Comm. Math. Phys. **204** (1999), 207–247.

[Bou99b] ______, *On growth of Sobolev norms in linear Schrödinger equation with time dependent potential*, J. Anal. Math. **77** (1999), 315–348.

[EK] H. L. Eliasson and S. B. Kuksin, *KAM for the non-linear Schrödinger equation*, Annals of Mathematics, to appear, see http://annals.math.princeton.edu/issues/2007/FinalFiles/EliassonKuksinFinal.pdf.

[EK08] ______, *Infinite Töplitz–Lipschitz matrices and operators*, ZAMTP **59** (2008), 24–50.

[Mos67] J. Moser, *Convergent series expansions for quasiperiodic motions*, Math. Ann. **169** (1967), 136–176.

[Wan07] W.-M. Wang, *Bounded Sobolev norms for linear Schrödinger equations under resonant perturbations*, Preprint (2007).