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Derivation of a wave kinetic equation from the resonant-averaged stochastic NLS equation

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HIGHLIGHTS

- A new derivation of a wave kinetic equation for the NLS is presented.
- It applies to the stochastically forced equation with dissipation.
- It is part of a two steps procedure, the first step was already rigorously proved.
- We show here how to perform the second step, getting the kinetic equation.

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ABSTRACT

We suggest a new derivation of a wave kinetic equation for the spectrum of the weakly nonlinear Schrödinger equation with stochastic forcing. The kinetic equation is obtained as a result of a double limiting procedure. Firstly, we consider the equation on a finite box with periodic boundary conditions and send the size of the nonlinearity and of the forcing to zero, while the time is correspondingly rescaled; then, the size of the box is sent to infinity (with a suitable rescaling of the solution). We report here the results of the first limiting procedure, analysed with full rigour in [8], and show how the second limit leads to a kinetic equation for the spectrum, if some further hypotheses (commonly employed in the weak turbulence theory) are accepted. Finally we show how to derive from these equations the Kolmogorov–Zakharov spectra.

0. Introduction

0.1. Weak turbulence and spectra

The theory of the weak turbulence (WT) studies weakly nonlinear PDEs, focusing in particular on the distribution and exchange of energy among the normal modes of oscillations (in most cases, these are Fourier modes of the solutions). The weakness of the nonlinearity allows to consider the interaction between different modes (or, in a more physical language, different waves) as a small perturbation to the linear flow, so that solutions of the equations can be approximated by suitable power series expansions. Usually, the lowest order nontrivial approximation for a solution is considered and the attention is payed to its statistical properties on long time intervals. That is, one deals with averaged values of certain quantities, taken with respect to some probability measure. The latter can be introduced either as the probability of a given configuration of initial data, or as the probability of a realisation of a stochastic forcing which is added to the system, together with a damping to dissipate the energy pumped by the forcing.

The most important object of the study is the distribution of energy among the modes, that is the energy spectrum

$$n_{\mathbf{k}}(t) = \mathbf{E}\left[|v_{\mathbf{k}}(t)|^2\right],$$

where $|v_{\mathbf{k}}(t)|$ denotes the amplitude of $\mathbf{k}$-th mode ($\mathbf{k} \in \mathbb{R}^d$ or $\mathbf{k} \in \mathbb{Z}^d$) at time t and $\mathbf{E}$ is the expected value with respect to the probability measure. Most of the predictions of the WT concern the behaviour of $n_{\mathbf{k}}$ as a function of $\mathbf{k}$. Possibly the most remarkable among them is the existence of stationary solutions with spectra decaying as a power law of $|\mathbf{k}|$, when the dispersion relation of the linearised at zero equation is homogeneous in $|\mathbf{k}|$. In the case when the system is stirred by a forcing, acting significantly only on some modes (think, for instance, of the set $|\mathbf{k}| \leq r_1$) and is

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subject to a dissipation having sensible effects only on a set of modes well separated from the first (for example, $|\mathbf{k}| \geq r_2$ with $r_2 \gg r_1$), these solutions correspond[1] to a constant flux of a quantity (typically, the energy) through the modes $\mathbf{k}$ such that $|\mathbf{k}| \in [r_1, r_2]$. If this happens, one says that a *cascade (of energy, etc.) occurs*. The segment $[r_1, r_2]$ is called the *inertial interval*, and the collection of modes $\mathbf{k}$ such that $|\mathbf{k}| \in [r_1, r_2]$ is the *inertial range*, see in [3].

The stationary energy spectra $n_{\mathbf{k}}$ which behave as a power law of $|\mathbf{k}|$ for $\mathbf{k}$ in the inertial range are called the *Kolmogorov–Zakharov (KZ) spectra.*

0.2. The wave kinetic equation

The main tool used to study the spectra is the wave kinetic (WK) equation (or, better the class of equations). It can be written as

$$\frac{d}{dt} n_{\mathbf{k}} = f_{\mathbf{k}}(\{n.\}),$$

where $f_{\mathbf{k}}$ is a function of the whole spectrum $\{n_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d\}$, constructed in terms of the original weakly nonlinear PDE. If it is shown that the limiting energy spectrum satisfies a WK equation, then the problem of finding stationary spectra reduces to that of finding spectra which make f vanish. See [4,3] for derivation of the WK equations and their discussion, and see [5] for the study of the Cauchy problem for the WK equation, obtained from cubic nonlinear Schrödinger equation.

However, known derivations of the WK equation all are heuristic, and serious doubts always existed concerning their validity. These concerns have become even more serious after the appearance of some results which seem to be in contradiction with the prediction of WT (see, for instance [6,7]).

Our work [8] suggests a program to study the WT and verify its postulates, in the frame of stochastic PDEs, following [9]. The scheme consists in considering a nonlinear PDE with stochastic forcing and damping on a torus of size L and performing, in sequence, two limiting procedures:

$$\begin{aligned} &(1)\ \text{the limit (on long times) when the nonlinearity goes} \\ &\quad \text{to zero together with the forcing;} \\ &(2)\ \text{the limit for } L \to \infty \text{ (with a possible scaling of} \\ &\quad \text{the size of the solutions).} \end{aligned} \tag{I}$$

In the works on WT limiting procedures are considered, when both the size of the nonlinearity goes to zero and the period L goes to infinity. But the order of the limits often is not made precise, and if it is precise, sometimes it is opposite to the order above. In particular, the order of the limits is clearly made in the book by Nazarenko [3], and there it is opposite to our choice.

The two limits in (I) correspond to two specific steps:

Step 1. Prove that the evolution of the spectrum on long times is governed by certain stochastic *effective equation*, built from the resonant terms of the nonlinearity.

Step 2. Under some assumptions on the form of the damping, prove that when $L \to \infty$, the energy spectrum of solutions for effective equations converges to a solution of certain WK equation, and derive from this that spectra of stationary solutions converge to some KZ spectra.

In [8], Step 1 is done rigorously, using the method of resonant stochastic averaging in the spirit of Khasminskii. The goal of this work is to perform Step 2 on the level of accuracy usual for WT. To do this we first in Section 2 use some approximation, commonly used in the WT theory, and derive that the limiting (as $L \to \infty$) energy spectrum satisfies the WK equation (2.12). Next in Section 3 we evoke the classical Zakharov argument to show that stationary solutions of this equation have KZ spectra with the exponents, specified in (3.4).

1. The limit of the weak nonlinearity and the effective equation

Now we briefly sum up the main results in [8]. There it was considered the Schrödinger equation on the torus $\mathbb{T}^d_L = \mathbb{R}^d/(2\pi L\mathbb{Z}^d)$,

$$u_t(t,x) - i\Delta u(t,x) = 0, \quad x \in \mathbb{T}^d_L, \tag{1.1}$$

stirred by a perturbation, which comprises a Hamiltonian term, a linear damping and a random force. That is, we have considered the equation

$$\begin{aligned} u_t - i\Delta u &= -i\varepsilon^{2q_*}|u|^{2q_*}u - \nu f(-\Delta)u \\ &\quad + \sqrt{\nu}\frac{d}{dt}\sum_{\mathbf{k}\in\mathbb{Z}^d_L} b_{\mathbf{k}}\boldsymbol{\beta}^{\mathbf{k}}(t)e^{i\mathbf{k}\cdot x}, \end{aligned} \tag{1.2}$$

$$u = u(t,x), \quad x \in \mathbb{T}^d_L,$$

where $q_* \in \mathbb{N}$ and $\varepsilon, \nu > 0$ are two small parameters, controlling the size of the perturbation, while $\mathbb{Z}^d_L$ denotes the set of vectors of the form $\mathbf{k} = \mathbf{l}/L$ with $\mathbf{l} \in \mathbb{Z}^d$. The damping $-f(-\Delta)$ is the selfadjoint linear operator in $L_2(\mathbb{T}^d_L)$ which acts on the exponents $e^{i\mathbf{k}\cdot x}$, $\mathbf{k} \in \mathbb{Z}^d_L$, according to

$$f(-\Delta)e^{i\mathbf{k}\cdot x} = \gamma_{\mathbf{k}}e^{i\mathbf{k}\cdot x}, \quad \gamma_{\mathbf{k}} = f(\lambda_{\mathbf{k}}) \text{ where } \lambda_{\mathbf{k}} = |\mathbf{k}|^2. \tag{1.3}$$

The function f is real positive and continuous. To avoid technicalities, we assume that $f(t) \geq C_1|t| + C_2$ for all t, for suitable positive constants C_1, C_2 (for example, $f(-\Delta)u = -\Delta u + u$). The processes $\boldsymbol{\beta}^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d_L$, are standard independent complex Wiener processes. The real numbers $b_{\mathbf{k}}$ are all non-zero and decay fast when $|\mathbf{k}| \to \infty$. That is, the equation is stirred by a random force, which is smooth in x, while as a function of time t it is a white noise.

Eq. (1.2) with small ν and ε is important for physics and mathematical physics, where it serves as a universal model, see [1,4,2]. The parameters ν and ε measure, respectively, the inverse time-scale of the forced oscillations under consideration and their amplitude. We consider the regime in which

$$\varepsilon^{2q_*} = \rho\nu,$$

where $\rho > 0$ is a constant. This assumption is in agreement with the requirement that one should consider the dynamics on a time-scale which becomes longer and longer as the amplitude goes to zero (see [3] for the actual bounds imposed to the relation between ε and ν). Passing to the slow time $\tau = \nu t$ and writing $u(\tau, x)$ as Fourier series, $u(\tau,x) = \sum_{\mathbf{k}} v_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot x}$, we get the system

$$\dot{v}_{\mathbf{k}} + i\nu^{-1}\lambda_{\mathbf{k}}v_{\mathbf{k}} = -\gamma_{\mathbf{k}}v_{\mathbf{k}} + 2\rho\, i\,\frac{\partial\mathcal{H}(v)}{\partial\bar{v}_{\mathbf{k}}} + b_{\mathbf{k}}\dot{\beta}^{\mathbf{k}}(\tau), \qquad \mathbf{k} \in \mathbb{Z}^d_L. \tag{1.4}$$

Here $v_{\mathbf{k}} = v_{\mathbf{k}}(\tau)$, the dot $\dot{}$ stands for $\frac{d}{d\tau}$, and $\mathcal{H}(v)$ is the Hamiltonian of the nonlinearity, expressed in terms of the Fourier coefficients $v = (v_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d_L)$:

$$\begin{aligned} \mathcal{H}(v) &= \frac{1}{2q_*+2}\sum_{\mathbf{k}_1,\ldots\mathbf{k}_{2q_*+2}\in\mathbb{Z}^d_L} v_{\mathbf{k}_1}\cdots v_{\mathbf{k}_{q_*+1}}\bar{v}_{\mathbf{k}_{q_*+2}}\cdots\bar{v}_{\mathbf{k}_{2q_*+2}} \\ &\qquad \times \delta^{1\ldots q_*+1}_{q_*+2\ldots 2q_*+2}, \end{aligned} \tag{1.5}$$

[1] See [1] for the general case and [2] for the case of NLS.

where we used the notation (see [3]):

$$\delta^{1\ldots q_*+1}_{q_*+2\ldots 2q_*+2} = \begin{cases} 1 & \text{if } \mathbf{k}_1+\cdots+\mathbf{k}_{q_*+1}-\mathbf{k}_{q_*+2}-\cdots-\mathbf{k}_{2q_*+2}=0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

As before we are interested in the limit $\nu \to 0$, corresponding to small oscillations in the original non-scaled equation.

The limiting procedure rests on the stochastic averaging theorem for resonant systems with an infinite number of degrees of freedom (see Introduction in [8]). Let us consider the equation

$$\dot v_{\mathbf{k}} = -\gamma_{\mathbf{k}} v_{\mathbf{k}} + 2\rho\, i\, \frac{\partial \mathcal{H}^{\mathrm{res}}(v)}{\partial \bar v_{\mathbf{k}}} + b_{\mathbf{k}} \dot\beta^{\mathbf{k}}(\tau), \quad \mathbf{k}\in\mathbb{Z}^d_L, \tag{1.7}$$

where $\mathcal{H}^{\mathrm{res}}$ is obtained as the resonant average of the Hamiltonian $\mathcal{H}(v)$:

$$\mathcal{H}^{\mathrm{res}}(v) = \frac{1}{2q_*+2} \sum_{\mathbf{k}_1,\ldots \mathbf{k}_{2q_*+2}\in\mathbb{Z}^d_L} v_{\mathbf{k}_1}\cdots v_{\mathbf{k}_{q_*+1}} \bar v_{\mathbf{k}_{q_*+2}}\cdots \bar v_{\mathbf{k}_{2q_*+2}} \times \delta^{1\ldots q_*+1}_{q_*+2\ldots 2q_*+2}\, \delta(\lambda^{1\ldots q_*+1}_{q_*+2\ldots 2q_*+2}), \tag{1.8}$$

and we use another standard notation:

$$\delta(\lambda^{1\ldots q_*+1}_{q_*+2\ldots 2q_*+2}) = \begin{cases} 1 & \text{if } \lambda_{\mathbf{k}_1}+\cdots+\lambda_{\mathbf{k}_{q_*+1}}-\lambda_{\mathbf{k}_{q_*+2}}-\cdots-\lambda_{\mathbf{k}_{2q_*+2}}=0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

That is, Eq. (1.7) is obtained from the system (1.4) by a simple procedure: we remove fast terms $i\nu^{-1}\lambda_{\mathbf{k}} v_{\mathbf{k}}$ and replace the Hamiltonian $\mathcal{H}$ by its resonant average $\mathcal{H}^{\mathrm{res}}$.

If $v(\tau)$ is a solution of (1.4) or (1.7) and $I(v(\tau))$ is its vector of energies, $I(v(\tau)) = \{|v_{\mathbf{k}}(\tau)|^2, \mathbf{k}\in\mathbb{Z}^d_L\}$, then the main result of [8] is the following theorem, which shows that the evolution of the energy vector is given by the effective equation (1.7). Moreover, in the stationary regime the effective equation completely controls the limiting distribution of solutions:

Theorem 1.1. *Let $v_0 := \{v_{0\mathbf{k}}, \mathbf{k}\in\mathbb{Z}^d_L\}$ be a sufficiently smooth initial condition and $v^\nu(\tau)$ be a solution for (1.4) with $v^\nu_0 = v_0$. When $\nu\to 0$, we have the weak convergence of distributions of solutions*

$$\mathcal{D}(I(v^\nu(\tau))) \rightharpoonup \mathcal{D}(I(v^0(\tau))), \quad 0\le\tau\le T,$$

where $v^0(\tau)$ is a solution of Eq. (1.7) such that $v^0(0)=v_0$.

Moreover, if Eq. (1.7) has a unique stationary measure[2] μ^0 and $v^\nu(\tau)$ is a stationary (in time) solution of Eq. (1.4), then

$$\mathcal{D}(v^\nu(\tau)) \rightharpoonup \mu^0 \quad \text{as } \nu\to 0.$$

Motivated by this result, we call (1.7) the *effective equation* (for (1.4)).

2. The limit $L \to \infty$

The effective equation enlightens the relevance of exact resonant terms, as predicted in the studies of weak turbulence in bounded domains (also called discrete WT, see, for instance [10]). We intend to investigate here the behaviour of its solutions, as the size of the domain goes to infinity, exploring in some sense the transition from discrete WT to classical WT.

From the point of view of mathematics, the limit in Eq. (1.7) in which the size L of the torus tends to infinity (which, as we have seen, determines the spectrum) presents serious problems, in particular for what concerns the definition of the stochastic forcing. We prefer instead to study for finite L some averaged quantities, calculated for solutions of the equation (i.e., their moments), and then pass to the limit as $L\to\infty$ only for them.

For the sake of simplicity, we will consider the case of cubic nonlinearity, i.e., we choose $q_* = 1$ in (1.2). Then the effective equation takes the form

$$dv_{\mathbf{k}}(\tau) = \left(-\gamma_{\mathbf{k}} v_{\mathbf{k}} - i\rho \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\in\mathbb{Z}^d_L} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar v_{\mathbf{k}_3} \delta^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}} \delta(\lambda^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}_3\mathbf{k}})\right) d\tau + b_{\mathbf{k}} d\boldsymbol{\beta}_{\mathbf{k}}, \quad \mathbf{k}\in\mathbb{Z}^d_L. \tag{2.1}$$

The moment $M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}(\tau)$ of $v(\tau)$ of order n_1+n_2 is defined as

$$M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}(\tau) = \mathbf{E}_\tau\left(v_{\mathbf{k}_1}\cdots v_{\mathbf{k}_{n_1}} \bar v_{\mathbf{k}_{n_1+1}}\cdots \bar v_{\mathbf{k}_{n_1+n_2}}\right), \tag{2.2}$$

where $\mathbf{E}_\tau$ denotes the expected values at time τ, i.e., $\mathbf{E}_\tau[f(v)] = \mathbf{E}[f(v(\tau))]$ for any measurable function $f(v)$. Note that $M^{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}_{\mathbf{k}_1\ldots\mathbf{k}_{n_1}} = \bar M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}$. In order to write the evolution equation for the moments we set

$$AM^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}} = -\left(\sum_{l=1}^{n_1+n_2}\gamma_{\mathbf{k}_l}\right) M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}},$$

and for any $l\in\mathbb{Z}^d_L$ introduce the operator $\Gamma_{\mathbf{k}_l}$ which erases the index $\mathbf{k}_l$ (in lower or upper position) according to

$$\Gamma_{\mathbf{k}_l} M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}} = \begin{cases} M^{\mathbf{k}_1\ldots\cancel{\mathbf{k}_l}\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}} & \text{if } l\le n_1 \\ M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\cancel{\mathbf{k}_l}\ldots\mathbf{k}_{n_1+n_2}} & \text{if } l> n_1, \end{cases}$$

where $\cancel{\mathbf{k}_l}$ signifies that the index $\mathbf{k}_l$ is omitted. So, by making again use of Ito's formula, we get

$$\begin{aligned} \frac{dM^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}}{d\tau} &= AM^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}} \\ &\quad - i\rho\left(\sum_{l=1}^{n_1}\sum_{\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3} \Gamma_{\mathbf{k}_l} M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}\mathbf{k}'_1\mathbf{k}'_2}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}\mathbf{k}'_3} \delta^{\mathbf{k}'_1\mathbf{k}'_2}_{\mathbf{k}'_3\mathbf{k}_l} \delta(\lambda^{\mathbf{k}'_1\mathbf{k}'_2}_{\mathbf{k}'_3\mathbf{k}_l})\right. \\ &\quad \left. - \sum_{l=n_1+1}^{n_1+n_2}\sum_{\mathbf{k}'_1,\mathbf{k}'_2\mathbf{k}'_3} \Gamma_{\mathbf{k}_l} M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}\mathbf{k}'_3}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}\mathbf{k}'_1\mathbf{k}'_2} \delta^{\mathbf{k}'_3\mathbf{k}_l}_{\mathbf{k}'_1\mathbf{k}'_2} \delta(\lambda^{\mathbf{k}'_3\mathbf{k}_l}_{\mathbf{k}'_1\mathbf{k}'_2})\right) \\ &\quad + 2\sum_{l=1}^{n_1}\sum_{m=n_1+1}^{n_1+n_2} b^2_{\mathbf{k}_l}\delta^{\mathbf{k}_l}_{\mathbf{k}_m}\Gamma_{\mathbf{k}_l}\Gamma_{\mathbf{k}_m} M^{\mathbf{k}_1\ldots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1}\ldots\mathbf{k}_{n_1+n_2}}. \end{aligned} \tag{2.3}$$

This equation expresses the time derivative of a moment of order n_1+n_2 as a function of the moments of order n_1+n_2-2 and those of order n_1+n_2+2. The coupled system containing the equations for all moments is called the chain of moments equation (see [11]).[3] Systems of this kind are usually treated by approximating moments of high order by suitable functions of lower order moments in order to get a closed system of equations. We will show that if the quasi-stationary and quasi-Gaussian approximations (see below) are chosen to close the system of

[2] This happens, e.g. if $q_*=1$, $d\le 3$ and $f(\lambda)=C_1|\lambda|+C_2$, or if $q_*=1$, d is any, but the function $f(\lambda)$ growth with λ sufficiently fast, see [8].

[3] Notice that, due to our choice of the degree of the nonlinearity, in our case the equations for moments of even order are decoupled from those for moments of odd order.

moment equations, then under the limit $L \to \infty$ we recover a modified version of the WK equation.

We start from Eq. (2.3) for $M_{\mathbf{k}}^{\mathbf{k}}$ with a fixed L, which gives

$$\dot M_{\mathbf{k}}^{\mathbf{k}} = -2\gamma_{\mathbf{k}} M_{\mathbf{k}}^{\mathbf{k}} + 2b_{\mathbf{k}}^2 + 2\rho \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \operatorname{Im} M_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}). \tag{2.4}$$

To study the sum on the r.h.s., we notice that if the Krönecker deltas are different from zero because $\mathbf{k}$ equals to one among $\mathbf{k}_1, \mathbf{k}_2$ and $\mathbf{k}_3$ is equal to another, then the moment is real and does not contribute to the sum. So we may assume that $\mathbf{k} \neq \mathbf{k}_1, \mathbf{k}_2$, $\mathbf{k}_3 \neq \mathbf{k}_1, \mathbf{k}_2$. In this case to calculate the fourth order moments on the r.h.s. of (2.4) we consider corresponding Eq. (2.3) of order four. Due to the just mentioned restriction on $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, the last double sum on the r.h.s. of (2.3) vanishes, and we get:

$$\begin{aligned} \dot M_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} &= -(\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}) M_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \\ &\quad + i\rho \sum_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \Big(M_{\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4} \delta_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}\mathbf{k}_4} \delta(\lambda_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}\mathbf{k}_4}) \\ &\quad + M_{\mathbf{k}\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4} \delta_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_3\mathbf{k}_4} \delta(\lambda_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_3\mathbf{k}_4}) - M_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}^{\mathbf{k}_2\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_1\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6} \delta(\lambda_{\mathbf{k}_1\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6}) \\ &\quad - M_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}^{\mathbf{k}_1\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_2\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6} \delta(\lambda_{\mathbf{k}_2\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6}) \Big). \end{aligned} \tag{2.5}$$

We make now a first approximation by neglecting the term containing the time derivative at the l.h.s. of (2.5). This can be justified, if τ is large enough, by the quasi-stationary approximation (cf. Section 2.1.3 in [4]). Namely, let us write Eq. (2.5) as

$$\left(\frac{d}{d\tau} + (\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}) \right) M_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} = f(\mathbf{k}).$$

Notice that since all $\gamma_{\mathbf{k}}$'s are positive, then the linear differential equation on the l.h.s. is exponentially stable. Assume that $f(\mathbf{k})$ as a function of τ is almost constant during time-intervals, sufficient for relaxation of the differential equation. Then

$$M_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \approx \frac{f(\mathbf{k})}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}}.$$

We can finally insert this in (2.4) and get

$$\begin{aligned} \dot M_{\mathbf{k}}^{\mathbf{k}} &\approx -2\gamma_{\mathbf{k}} M_{\mathbf{k}}^{\mathbf{k}} + 2b_{\mathbf{k}}^2 + 2\rho^2 \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \frac{1}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}} \\ &\quad \times \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}) \operatorname{Re} \left(\sum_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \Big(M_{\mathbf{k}_3\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4} \delta_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}\mathbf{k}_4} \delta(\lambda_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}\mathbf{k}_4}) \right. \\ &\quad + M_{\mathbf{k}\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_4} \delta_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_3\mathbf{k}_4} \delta(\lambda_{\mathbf{k}_5\mathbf{k}_6}^{\mathbf{k}_3\mathbf{k}_4}) - M_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}^{\mathbf{k}_2\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_1\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6} \delta(\lambda_{\mathbf{k}_1\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6}) \\ &\quad \left. - M_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}^{\mathbf{k}_1\mathbf{k}_5\mathbf{k}_6} \delta_{\mathbf{k}_2\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6} \delta(\lambda_{\mathbf{k}_2\mathbf{k}_4}^{\mathbf{k}_5\mathbf{k}_6}) \Big) \right). \end{aligned} \tag{2.6}$$

We then apply a second approximation, generally accepted in the WT (see [4,1,3]) which enables us to transform the previous relation to a closed equation for the second order moments. This consists in the quasi-Gaussian approximation, i.e., in the assumption that the higher-order moments (2.2) can be approximated by polynomials of the second-order moments, as if the random variables $v_{\mathbf{k}}$ were independent complex Gaussian variables. So, in particular,

$$\begin{aligned} M_{\mathbf{l}_4\mathbf{l}_5\mathbf{l}_6}^{\mathbf{l}_1\mathbf{l}_2\mathbf{l}_3} &\approx M_{\mathbf{l}_1}^{\mathbf{l}_1} M_{\mathbf{l}_2}^{\mathbf{l}_2} M_{\mathbf{l}_3}^{\mathbf{l}_3} \left(\delta_{\mathbf{l}_4}^{\mathbf{l}_1} (\delta_{\mathbf{l}_5}^{\mathbf{l}_2} \delta_{\mathbf{l}_6}^{\mathbf{l}_3} + \delta_{\mathbf{l}_6}^{\mathbf{l}_2} \delta_{\mathbf{l}_5}^{\mathbf{l}_3}) \right. \\ &\quad \left. + \delta_{\mathbf{l}_5}^{\mathbf{l}_1} (\delta_{\mathbf{l}_4}^{\mathbf{l}_2} \delta_{\mathbf{l}_6}^{\mathbf{l}_3} + \delta_{\mathbf{l}_6}^{\mathbf{l}_2} \delta_{\mathbf{l}_4}^{\mathbf{l}_3}) + \delta_{\mathbf{l}_6}^{\mathbf{l}_1} (\delta_{\mathbf{l}_4}^{\mathbf{l}_2} \delta_{\mathbf{l}_5}^{\mathbf{l}_3} + \delta_{\mathbf{l}_5}^{\mathbf{l}_2} \delta_{\mathbf{l}_4}^{\mathbf{l}_3}) \right). \end{aligned} \tag{2.7}$$

At this point we pass in Eq. (2.6), closed using the relation (2.7), to the limit $L \to \infty$. To do it we pass to a limit in the sum $S_{\mathbf{k}}$ on the r.h.s. of (2.6), by replacing the summation by integration. It is not hard to see that the sum in (2.6), transformed using (2.7), splits into a finite number of sums like

$$S_{\mathbf{k}}^j = \sum_{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6) \in \mathbb{Z}_L^{6d} \cap \Sigma_{\mathbf{k}}^j} F_{\mathbf{k}}^j(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6),$$

where $\Sigma_{\mathbf{k}}^j$ is a manifold in $\mathbb{R}^{6d}$ defined by

$$\begin{aligned} \Sigma_{\mathbf{k}}^j = \Big\{ &(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6) : \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{k} + \mathbf{x}_3, \\ &|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 = |\mathbf{k}|^2 + |\mathbf{x}_3|^2, \mathbf{x}^j + \mathbf{x}_4 = \mathbf{x}_5 + \mathbf{x}_6, \\ &|\mathbf{x}^j|^2 + |\mathbf{x}_4|^2 = |\mathbf{x}_5|^2 + |\mathbf{x}_6|^2, \mathbf{x}_1^j = \mathbf{x}_2^j, \mathbf{x}_3^j = \mathbf{x}_4^j, \mathbf{x}_5^j = \mathbf{x}_6^j \Big\}, \end{aligned}$$

where $\mathbf{x}^j$ stands for one among $\mathbf{k}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\{\mathbf{x}_1^j, \ldots, \mathbf{x}_6^j\}$—for a permutation of the set $\{\mathbf{k}, \mathbf{x}_1, \ldots, \mathbf{x}_6\} \setminus \{\mathbf{x}^j\}$.[4] In passing from sums to integrals in the limit for $L \to \infty$, it is easy to see that, if F^j is smooth enough, each term $S_{\mathbf{k}}^j$ depends on L as L^m, where m is the dimension of the manifold $\Sigma_{\mathbf{k}}^j$. A detailed analysis of all cases shows that the terms of the highest order in L in the integral correspond to terms of the form

$$S_{\mathbf{k}}^j = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} F_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2})$$

in the sum $S_{\mathbf{k}}$, where $\vec{k} := (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathbb{Z}_L^{3d} =: \mathcal{M}$. Denote

$$\begin{aligned} \Sigma_{\mathbf{k}} = \Big\{ &\vec{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}^{3d} : \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{k} + \mathbf{x}_3, \\ &|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 = |\mathbf{k}|^2 + |\mathbf{x}_3|^2 \Big\}. \end{aligned}$$

This is a manifold of dimension $3d - d - 1 = 2d - 1$, smooth outside the origin. The latter lies outside $\Sigma_{\mathbf{k}}$ if $\mathbf{k} \neq 0$, and is a singular point of $\Sigma_{\mathbf{k}}$ if $\mathbf{k} = 0$. For any non-zero point $\vec{x} \in \Sigma_{\mathbf{k}}$ denote by $\pi_{\mathbf{k}}(\vec{x})$ the tangent space $T_{\vec{x}}\Sigma_{\mathbf{k}}$, regarded as a subspace of $\mathbb{R}^{3d}$, and denote by $r^{2d-1}\varphi_{\mathbf{k}}(\vec{x})$ the $(2d-1)$-area of the intersection of $\pi_{\mathbf{k}}(\vec{x})$ with the r-cube, centred at $\vec{x}$ (with the sides parallel to the axes of $\mathbb{R}^{3d}$). Clearly, $\varphi_{\mathbf{k}}(\vec{x})$ is a smooth function on $\Sigma_{\mathbf{k}}$ outside zero, such that

$$V_1 \le \varphi_{\mathbf{k}}(\vec{x}) \le V_1 (3d)^{d-1/2}, \tag{2.8}$$

where V_1 is the volume of the 1-ball in $\mathbb{R}^{2d-1}$. Moreover, since in view of the homogeneity of the relation which defines Σ we have $\Sigma_{m\mathbf{k}} = m\Sigma_{\mathbf{k}}$, then $\pi_{m\mathbf{k}}(m\vec{x}) = \pi_{\mathbf{k}}(\vec{x})$. Accordingly,

$$\varphi_{\mathbf{k}}(\vec{x}) := \varphi_{m\mathbf{k}}(m\vec{x}). \tag{2.9}$$

Since the surface $\Sigma_{\mathbf{k}}$ is invariant under the transposition $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mapsto (\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3)$, then $\varphi_{\mathbf{k}}(\vec{x})$ as well is invariant with respect to it. Similarly, $\varphi_{\mathbf{k}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \varphi_{\mathbf{x}_3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{k})$. Let us put $r = L^{-1}$ and write S as

$$S_{\mathbf{k}} = L^{2d-1} \sum_{\substack{\vec{x} \in \Sigma_{\mathbf{k}} \cap \mathcal{M} \\ \vec{x} \neq 0}} \left(\frac{F_{\mathbf{k}}(\vec{x})}{\varphi_{\mathbf{k}}(\vec{x})} \right) \varphi_{\mathbf{k}}(\vec{x}) r^{2d-1}.$$

The sum on the r.h.s. is the Riemann sum for the integral $\int_{\Sigma_{\mathbf{k}} \setminus \{0\}} \frac{F_{\mathbf{k}}(\vec{x})}{\varphi_{\mathbf{k}}(\vec{x})} d\vec{x}$. So

$$S_{\mathbf{k}} \approx L^{2d-1} \int_{\Sigma \setminus \{0\}} \frac{F_{\mathbf{k}}(\vec{x})}{\varphi_{\mathbf{k}}(\vec{x})} d\vec{x},$$

where $\varphi_{\mathbf{k}}(\vec{x})$ is a smooth function satisfying (2.8) and (2.9).

[4] The relations defining Σ_k^j are not independent.

After some calculations we get the limiting (as $L \to \infty$) equation in the form

$$\dot{M}_{\mathbf{k}}^{\mathbf{k}} \approx -2\gamma_{\mathbf{k}} M_{\mathbf{k}}^{\mathbf{k}} + 2b_{\mathbf{k}}^2 + 4\rho^2 L^{2d-1} \times \int_{\mathbb{R}^{3d}} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}} \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}\delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}) \times \Big(M_{\mathbf{k}_1}^{\mathbf{k}_1} M_{\mathbf{k}_2}^{\mathbf{k}_2} M_{\mathbf{k}_3}^{\mathbf{k}_3} + M_{\mathbf{k}}^{\mathbf{k}} M_{\mathbf{k}_1}^{\mathbf{k}_1} M_{\mathbf{k}_2}^{\mathbf{k}_2} - M_{\mathbf{k}}^{\mathbf{k}} M_{\mathbf{k}_2}^{\mathbf{k}_2} M_{\mathbf{k}_3}^{\mathbf{k}_3} - M_{\mathbf{k}}^{\mathbf{k}} M_{\mathbf{k}_1}^{\mathbf{k}_1} M_{\mathbf{k}_3}^{\mathbf{k}_3} \Big).$$

Finally, we define

$$n_{\mathbf{k}} = L^d M_{\mathbf{k}}^{\mathbf{k}}/2, \qquad \tilde{b}_{\mathbf{k}} = L^{d/2} b_{\mathbf{k}}, \tag{2.10}$$

(so that $\sum_{\mathbf{k}} M_{\mathbf{k}}^{\mathbf{k}}/2 \to \int n_{\mathbf{k}}$ and $\sum_{\mathbf{k}} b_{\mathbf{k}}^2 \to \int \tilde{b}_{\mathbf{k}}^2$ as L goes to infinity), choose

$$\rho = \tilde{\varepsilon}^2 L^{1/2} = \frac{\varepsilon^{2q*}}{\nu}\tilde{\varepsilon}^2 L^{1/2}, \tag{2.11}$$

for some $\tilde{\varepsilon} > 0$, and get

$$\dot{n}_{\mathbf{k}} = -2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \tilde{b}_{\mathbf{k}}^2 + 16\tilde{\varepsilon}^4 \int_{\mathbb{R}^{3d}} d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}\delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}) \times \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}} \times \Big(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} \Big), \tag{2.12}$$

where the smooth outside the origin function $\varphi_{\mathbf{k}}(\vec{x})$ satisfies (2.8), (2.9). We have thus shown that, with a proper scaling of ρ and b (see (2.10)–(2.11)), we get an equation, similar to the WK equation for the NLS (see, for instance, formula (6.81) of [3], where $d = 2$). The differences are two: obviously in our case there appear the forcing and the dissipation, which are absent in his case in the traditional WK equations; more interestingly, the nonvanishing denominator $\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}$ appears in the integral, which modifies the spectra. The denominator goes to infinity with the norm of the vector $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ thus regularising the kinetic equation and improving its analytical properties.

3. Kolmogorov–Zakharov spectra

We show here how to deduce a power law spectrum from Eq. (2.12), following the well known Zakharov argument (see [4,3]).

First of all, we have to restrain our analysis to the inertial interval, i.e., to the spectral interval, where the damping and the forcing are negligible. This means that we have to suppose that damping and forcing are such that, for wave-vectors $\mathbf{k}$ belonging to a sufficiently large spectral region, the first two terms at the r.h.s. of (2.12) can be neglected if compared to the third. Clearly this happens, e.g., if a solution $\{n_{\mathbf{k}}\}$ is of order one, while $b_{\mathbf{k}} \ll 1$ and $\gamma_{\mathbf{k}} \ll 1$ (i.e., the damping and the dissipation are small at that spectral region). In the inertial interval we end up with the equation

$$\dot{n}_{\mathbf{k}} \approx 16\,\tilde{\varepsilon}^4 \int_{\mathbb{R}^{3d}} d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}\delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}) \times \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}} \times \Big(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3} \Big). \tag{3.1}$$

Notice that, while in the inertial interval we can simply approximate $\tilde{b}_{\mathbf{k}}$ with zero, this cannot be done with $\gamma_{\mathbf{k}}$, as it appears in the denominator of the integral at the r.h.s. of (2.12) (the so-called collision term), and can play an essential role a determination of the spectrum.

The previous equation has the form of the four-wave kinetic equation (see, for instance, formula 2.1.29 of [4]). It is well known (see [4,3]) how to solve such an equation for stationary spectra with the aid of the Zakharov transformations, if the terms

$$\mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_3} = \frac{\varphi_{\mathbf{k}}^{-1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}{\gamma_{\mathbf{k}}+\gamma_{\mathbf{k}_1}+\gamma_{\mathbf{k}_2}+\gamma_{\mathbf{k}_3}}$$

satisfy some conditions of symmetry and homogeneity. Namely, one should have that

$$\mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_3} = \mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k}_3,\mathbf{k}} = \mathcal{T}_{\mathbf{k}_2,\mathbf{k}_1}^{\mathbf{k},\mathbf{k}_3} = \mathcal{T}_{\mathbf{k},\mathbf{k}_3}^{\mathbf{k}_1,\mathbf{k}_2},$$
$$\mathcal{T}_{\lambda\mathbf{k}_1,\lambda\mathbf{k}_2}^{\lambda\mathbf{k},\lambda\mathbf{k}_3} = \lambda^m \mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_3},$$

for some $m \in \mathbb{R}$. For the sake of simplicity, we confine ourselves to the isotropic case when $n_{\mathbf{k}}$ is a function of $k = |\mathbf{k}|$ only.

Since φ is a homogeneous function of degree 0 due to (2.8) and (2.9), the requirements above are met if $\gamma_{\mathbf{k}}$ can be approximated by a homogeneous function of the form $\gamma_{\mathbf{k}} = \varepsilon'|\mathbf{k}|^m$, where $\varepsilon' \ll 1$ is a parameter that guarantees that the dissipation term indeed is negligible, and $m \in \mathbb{R}$.[5]

We continue (following [4, Sec. 3.1.3]; see also [3, Sec. 9.2.2]), by integrating equation (3.1) over the angles, and obtain that

$$\dot{n}_k = C\tilde{\varepsilon}^4 \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \mathfrak{T}_{12}^{k3} \times n_1 n_2 n_3 n_k (k_1 k_2 k_3)^{d-1} dk_1\, dk_2\, dk_3,$$

where

$$\mathfrak{T}_{12}^{k3} = 4\left(\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2}\right)\delta(\lambda_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2}) \times \int \delta_{\mathbf{k}\mathbf{k}_3}^{\mathbf{k}_1\mathbf{k}_2} \mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_3} d\Omega_1 d\Omega_2 d\Omega_3,$$

C denotes a suitable positive constant and the integration is taken on the d-dimensional solid angles $\Omega_i = \Omega(\mathbf{k}_i)$. Due to the symmetries of $\mathfrak{T}_{12}^{k3}$ (inherited from those of $\mathcal{T}_{\mathbf{k}_1,\mathbf{k}_2}^{\mathbf{k},\mathbf{k}_3}$), by a proper renaming of mute integration variables we can rewrite the previous equation as

$$\dot{n}_k = \frac{C}{4}\tilde{\varepsilon}^4 \int \left(\mathfrak{T}_{12}^{k3} + \mathfrak{T}_{12}^{3k} - \mathfrak{T}_{k2}^{13} - \mathfrak{T}_{1k}^{23}\right) \times n_1 n_2 n_3 n_k (k_1 k_2 k_3)^{d-1} dk_1\, dk_2\, dk_3 =: I_1 + I_2 + I_3 + I_4. \tag{3.2}$$

We introduce then as an ansatz for a solution the power law $n_k \propto k^\nu$, with some real ν, and, following Zakharov, make in the integrals I_2, I_3, I_4 the following substitutions: in I_2 we put

$$k_1 = \frac{kk_1'}{k_3'}, \qquad k_2 = \frac{kk_2'}{k_3'}, \qquad k_3 = \frac{k^2}{k_3'},$$

in I_3 put

$$k_1 = \frac{k^2}{k_1'}, \qquad k_2 = \frac{kk_2'}{k_1'}, \qquad k_3 = \frac{kk_3'}{k_1'},$$

and in I_4

$$k_1 = \frac{kk_1'}{k_2'}, \qquad k_2 = \frac{k^2}{k_2'}, \qquad k_3 = \frac{kk_3'}{k_2'}.$$

[5] This agrees with the hypotheses on the dissipation (see (1.3)), if, for instance, we choose $\gamma_{\mathbf{k}} = \varepsilon_1 + \varepsilon_2|\mathbf{k}|^\beta$, where $\varepsilon_1, \varepsilon_2 \ll 1$, and either $\varepsilon_1 \gg \varepsilon_2$, which gives $m = 0$, or vice versa, which gives $m = \beta$.

Then we re-denote the variables k'_j back to k_j and sum up the four integrals. By the homogeneity of $\mathcal{T}^{\mathbf{k},\mathbf{k}_3}_{\mathbf{k}_1,\mathbf{k}_2}$, one gets that the integral in (3.2) is proportional to

$$\int \mathfrak{T}^{k3}_{12}\left[1+\left(\frac{k_3}{k}\right)^{\kappa}-\left(\frac{k_1}{k}\right)^{\kappa}-\left(\frac{k_2}{k}\right)^{\kappa}\right] \times n_1 n_2 n_3 n_k (k_1 k_2 k_3)^{d-1} dk_1\, dk_2\, dk_3, \tag{3.3}$$

where

$$\kappa = 2 - 3\nu - m - 3d.$$

The stationary solutions are found by looking for ν for which the integral in (3.3) vanishes. In addition to the equilibrium solutions $n_k = C$ and $n_k = C/k^2$, which correspond, respectively, to the equipartition of the wave action and of the quadratic energy (Rayleigh–Jeans distribution), two nontrivial power law stationary distributions appear by equating to zero the term in square brackets in (3.3), corresponding to $\kappa = 0$ and $\kappa = 2$.[6] These are the Kolmogorov–Zakharov solutions:

$$n_k \propto k^{-(m+3d-2)/3}, \qquad n_k \propto k^{-(m+3d)/3}. \tag{3.4}$$

They coincide with the Kolmogorov–Zakharov spectra for the NLS equation without dissipation (for $d \geq 2$) if $m = 0$, but the dissipation modifies the power law of the decay if $m \neq 0$.

4. Conclusions

Despite the theory of WT deals with well defined mathematical objects (nonlinear PDEs), and gives for their solutions explicit analytical predictions, some of which are well checked numerically, it seems that there is only one work where the convergence to a WK equation in the spirit of WT is rigorously established—the paper [12]. That work deals with the NLS equation discretised on the lattice $\mathbb{Z}^d$, where the randomness comes through the initial data, distributed in accordance with the Gibbs measure.[7] The approach of [12] does not seem to be applicable to PDEs.

Our work suggests to justify the WT limits for damped and driven Hamiltonian PDEs in the two steps (I), where the first step is rigorously justified in our previous work [8], while the second is verified now on the level of accuracy, used in WT. The two steps together lead to a new WK equation, similar to those which appear in the WT theory and, as a consequence, to a KZ spectra. We strongly believe that the method proposed applies to a quite general class of PDEs and that the heuristic argument, used to verify the second step, can be translated to the rigorous mathematical language, thus converting the approach (I) to a complete proof of the main predictions the WT theory.

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[6] Notice that the second solution appears because the integration is restricted to the surface $k^2 + k_3^2 - k_1^2 - k_2^2 = 0$, due to the factor $\delta(\lambda^{\mathbf{k}_1\mathbf{k}_2}_{\mathbf{k}\mathbf{k}_3})$.

[7] The obtained WK equation is derived near the statistical equilibrium and differs from those, usual for the WT. It is unclear how to use them to derive the KZ spectra.