Матем. Сборник
Том 117(159) (1982), Вып. 3

Math. USSR Sbornik
Vol. 45(1983), No. 3

DIFFEOMORPHISMS OF FUNCTION SPACES CORRESPONDING TO QUASILINEAR PARABOLIC EQUATIONS

UDC 517.946

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ABSTRACT. This paper considers a boundary value problem for a quasilinear parabolic equation. In terms of Sobolev and Besov spaces the author determines a solution space A and a space B of initial conditions and right hand members such that the operator corresponding to the boundary value problem is a diffeomorphism, analytic in the Fréchet sense, of the whole space A and a domain $\mathcal{C}$ in the space B. The behavior of the inverse operator of the problem around the boundary of $\mathcal{C}$ is studied, and it is shown that for different problems the domain $\mathcal{C}$ can coincide with the whole function space or be a strict subset of it.

Bibliography: 13 titles.

Introduction

In this paper we consider a boundary value problem for a quasilinear parabolic equation. We show that the operator corresponding to such a problem establishes an analytic diffeomorphism of the whole function space of solutions and a domain $\mathcal{C}$ in the function space of the initial and boundary conditions and right hand members. We study the behavior of the inverse operator of the problem near the boundary of $\mathcal{C}$, and we show that for different problems the domain $\mathcal{C}$ can coincide with the whole function space or be a strict subset of it. Our main results appeared earlier without proof in [1].

In §2 we consider a boundary value problem for the second order quasilinear parabolic equation

$$\dot{u}(t,x) - \sum_{i,j=1}^{n} F_{i,j}(\mathfrak{D}_1 u)\frac{\partial^2 u}{\partial x_i \partial x_j} + F(\mathfrak{D}_1 u) = v(t,x),$$
$$(t,x) \in Q_T = (0,T)\times\Omega, \quad u\Big|_{\Gamma_T=(0,T)\times\partial\Omega} = 0, \quad u\Big|_{t=0} = u_0(x), \tag{1}$$

where $\mathfrak{D}_1 u = (u, \partial u/\partial x_1, \dots, \partial u/\partial x_n)$, $0 < T < \infty$, Ω is a domain in $\mathbf{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, and F and $F_{i,j}$ are analytic functions on $\mathbf{R}^{n+1}$ such that $F(0) = 0$ and the condition of uniform parabolicity is satisfied:

$$\sum_{i,j=1}^{n} F_{i,j}(z)\xi_i\xi_j \geqslant \delta|\xi|^2, \qquad \delta > 0\ \forall \xi \in \mathbf{R}^n,\ z \in \mathbf{R}^{n+1}. \tag{2}$$

1980 *Mathematics Subject Classification*. Primary 35K55, 46E35, 46E15, 58C20.

We seek a solution of problem (1) in the Sobolev space

$$A = \left\{ u(t,x) \in W^{1,2}_{p,t,x}(Q_T) u \Big|_{\Gamma_T} = 0 \right\},$$

the function pairs $(u_0(x), v(t,x))$ being taken from the space $B = \mathring{B}^{p-2/p}_p(\Omega) \times L_p(Q_T)$, where $\mathring{B}^{p-2/p}_p(\Omega)$ is the Besov space of functions on Ω, which are equal to zero on $\partial\Omega$ (see §1). We define the operator of problem (1):

$$N\colon A \to B,\ u(t,x) \mapsto \left(u\Big|_{t=0},\ \dot{u} - \sum_{i,j=1}^{n} F_{i,j}(\mathfrak{D}_1 u)\frac{\partial^2 u}{\partial x_i \partial x_j} + F(\mathfrak{D}_1 u) \right).$$

In §2 we prove the following theorem.

THEOREM 1. *Let $p > n + 2$. Then the following assertions are true*:

1) *There exists a domain $\mathcal{O} \subset B$ such that problem* (1) *has a solution $u \in A$ if and only if $(u_0, v) \in \mathcal{O}$. Moreover, such a solution is unique.*

2) *The mapping $N\colon A \to \mathcal{O}$ is a diffeomorphism analytic in the Fréchet sense* (*see* [8]).

3) *Let the domain Ω be bounded and assume that $(u_0^{(j)}, v^{(j)}) \in \mathcal{O}$, $j = 1, 2, \ldots$, and $(u_0^{(j)}, v^{(j)}) \xrightarrow{B} (u_0^{(0)}, v^{(0)}) \in \partial\mathcal{O}$, where $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$ in $\bar{B}$. Then $\| N^{-1}(u_0^{(j)}, v^{(j)}); B \| \to \infty$.*

We employ the symbol $\|\cdot\,; B\|$ for the norm in B. From Theorem 1 we obtain the

COROLLARY. 1) *Let the space $\tilde{B}$ be continuously imbedded in B, let $(\tilde{u}_0, \tilde{v}) \in \tilde{B}$, $(\tilde{U}_0, \tilde{V}) \in \tilde{B}$ and assume that for $u_0 = \tilde{u}_0$ and $v = \tilde{v}$ there exists a solution $u \in A$ of problem* (1). *Then, if $|\lambda| < \delta \ll 1$, it follows that for $u_0 = \tilde{u}_0 + \lambda\tilde{U}_0$, $v = \tilde{v} + \lambda\tilde{V}$ there exists a solution of problem* (1) *of the form*

$$u(t,x) = \sum_{j=0}^{\infty} \lambda^j U_j, \qquad U_j = U_j\big(t, x; \tilde{u}_0, \tilde{U}_0, \tilde{v}, \tilde{V}\big) \in A, \tag{3}$$

and that for $|\lambda| < \delta$ the series (3) *converges in the norm of the space A uniformly with respect to λ.*

2) *If for a solution of problem* (1) *the a priori estimate $\|u; A\| \leqslant f(\|(u_0, v); B\|)$ holds, where $f(\cdot)$ is a continuous function, then $\mathcal{O} = B$.*

We remark that Corollary 1) can be considered as the basis of a perturbation method for problem (1), since the coefficients $U_j(t,x)$ are sought from recurrence formulas arising from the substitution of the series (3) in (1).

Problem (1) was studied earlier in a large number of papers by various authors. In some of these papers results were obtained which are related to ours. Thus, in the case of periodic boundary conditions problem (1) was considered by Vishik and Fursikov [2]. They showed that the operator N of problem (1) has an analytic inverse in some neighborhood of the origin of the space of function pairs (u_0, v). Conclusion 1) of the theorem was obtained earlier by Babin [3], and result 2) was obtained by Pokhozhaev [4].

We consider a particular case of problem (1):

$$\dot{u}(t,x) - \Delta u(t,x) + \varphi(u) = v(t,x), \qquad u\big|_{t=0} = u_0(x),\ u\big|_{\Gamma_T} = 0, \tag{4}$$

where $\varphi(u) = \varphi_1 u + \cdots + \varphi_M u^M$, $\varphi_M \neq 0$, $M \geqslant 2$. As an example, problem (4) shows that under the conditions of Theorem 1 it is possible to have both the case $\mathcal{O} = B$ and the case $\mathcal{O} \neq B$.

THEOREM 2. 1) *The conclusions of Theorem* 1 *are valid for problem* (4) *when*

$$p > \max\left\{\frac{3}{2}, \frac{M-1}{M}\frac{n+2}{2}\right\}.$$

Let the domain Ω *be bounded, and let* $p \geqslant 2$ *and* $p > (M-1)(n+2)/2M$. *Then*:

2) *If* $n = 2$ *and* $\varphi'(u) \geqslant 0\ \forall u \in \mathbf{R}$, *then* $\mathcal{C} = B$.

3) *If* $g(u) = -\int_0^u \varphi(\tau)\, d\tau$ *and an* $\alpha > 0$ *exists such that* $-u\varphi(u) \geqslant (2+\alpha)g(u) \geqslant 0$ $\forall u \in \mathbf{R}$, *then* $\mathcal{C} \neq B$.

As an example, the function $\varphi(u) = u^3$ satisfies the conditions of assertion 2) while $\varphi(u) = -u^3$ satisfies the conditions of 3).

At the conclusion of §2 we give an analog of Theorem 1 for the nonstationary Navier-Stokes system of equations.

In §3 we show by way of example of a boundary value problem for a monotonic parabolic equation that if the space of function pairs (u_0, v) is not contained in the space B or if the condition (2) of uniform parabolicity is not satisfied, then the operator of the boundary value problem can have a continuous inverse which is nowhere analytic.

In §4 we consider a boundary value problem for a quasilinear $2b$-parabolic equation for which we prove analogs of Theorem 1 and corollaries from it. Such problems were considered earlier in [3] and [4], wherein some of the results of §4 were obtained.

The author expresses his deep gratitude to M. I. Vishik for his constant interest in this work.

§1. Auxiliary results

1. *Notation*. Throughout this paper ε, δ and C represents various real positive constants; Ω is a domain in $\mathbf{R}^n$ with boundary $\partial\Omega$; $Q_T = (0, T) \times \Omega$, $\Gamma_T = (0, T) \times \partial\Omega$, $\Omega_0 = \{0\} \times \Omega$; here we always have $0 < T < \infty$. The notation $\partial\Omega \in C^m$ signifies that Ω is a domain with an m-times uniformly continuously differentiable boundary (see [5] and [6]). If X is a Banach space, we denote the norm in it by $\|\cdot\,; X\|$. We use the following notation for neighborhoods of elements from X: $\mathcal{O}_\varepsilon(u, X) = \{v \in X \mid \|v - u; X\| < \varepsilon\}$, $\mathcal{O}_\varepsilon(X) = \mathcal{O}_\varepsilon(0, X)$. If X is realized as a space of real functions on M, then $X \otimes \mathbf{R}^N$ is the space of functions $\varphi = (\varphi_1, \dots, \varphi_N)$: $M \to \mathbf{R}^N$, $\varphi_j \in X$, $j = 1, \dots, N$, with norm $\|\varphi; X \otimes \mathbf{R}^N\| = \sum_j \|\varphi_j; X\|$.

In denoting functions we often omit their arguments. Thus, $u(t, x)$ may be written as u or $u(t, \cdot)$. We use the following notation for writing partial derivatives:

$$\dot u(t, x) = \partial u/\partial t, \qquad u'_{x_l}(t, x) = \partial u/\partial x_l,$$

$$D_t^\mu u(t, x) = \frac{\partial^\mu}{\partial t^\mu} u, \qquad D_x^\nu u(t, x) = \frac{\partial^{\nu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x^{\nu_n}} u;$$

$\mathfrak{D}_m^{(2b)} u(t, x)$ is a row containing all the $N = N(m, 2b)$ partial derivatives of $u(t, x)$ of the form $D_t^\mu D_x^\nu u(t, x)$, where $[\mu, \nu] \equiv \mu + |\nu|/2b \leqslant m$ and $|\nu| = \nu_1 + \cdots + \nu_n$. Finally, we agree that the notation "if $\varepsilon \ll 1$" signifies "if $\varepsilon > 0$ is sufficiently small", while the notation "if $C \gg 1$" signifies "if C is sufficiently large".

2. *Function spaces*. In this paper we employ: the Lebesgue spaces $L_p(\Omega)$ and $L_p(Q_T)$, $\infty \geqslant p \geqslant 1$; the spaces $C(\Omega)$ and $C(Q_T)$ of functions uniformly continuous and bounded on $\bar\Omega$ and $\bar Q_T$, respectively; and the Sobolev spaces $W_p^l(\Omega)$ of functions on the domain Ω with the norm

$$\|u(x); W_p^l\| = \left(\sum_{|\alpha| \leqslant l} \|D_x^\alpha u; L_p(\Omega)\|^p\right)^{1/p},$$

here l is a positive integer and $p \geq 1$. In addition, we denote by $\dot{W}_p^l(\Omega)$ the closure in $W_p^l(\Omega)$ of functions smooth and finite on Ω; we also employ the anisotropic Sobolev spaces $W_p^{s,2bs}(Q_T)$ of functions on the cylinder Q_T with the norm

$$\|u; W_p^{s,2bs}\| = \left(\sum_{[\mu,\nu]\leq s} \|D_t^\mu D_x^\nu u; L_p\|^p \right)^{1/p},$$

where $[\mu, \nu] = |\nu|/2b + \mu$, $p \geq 1$, and b and s are positive integers. In addition, we employ Besov spaces of functions on Ω, Γ_T, and $\partial\Omega$, in terms of which we define the traces of functions from $W_p^{s,2bs}(Q_T)$. For the definition of these spaces we introduce the following norms:

$$[u]_{\rho,\Omega}^{(p)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+\rho p}} \, dx \, dy \right)^{1/p}, \qquad 0 < \rho < 1.$$

If S is a domain in $\mathbf{R}^{n-1}$ and $\Gamma' = (0, T) \times S$, we put

$$[u]_{\rho,t,\Gamma'}^{(p)} = \left(\int_0^T \int_S \int_0^T \frac{|u(t, x') - u(t', x')|^p}{|t - t'|^{1+\rho}} \, dt' \, dx' \, dt \right)^{1/p},$$

$$[u]_{\rho,x,\Gamma'}^{(p)} = \left(\int_0^T \int_S \int_S \frac{|u(t, x') - u(t, y')|^p}{|x' - y'|^{n-1+\rho p}} \, dy' \, dx' \, dt \right)^{1/p}.$$

Let $l > 0$ be an arbitrary nonintegral number. We define the space $B_p^l(\Omega)$ as the closure in the norm

$$\|u; B_p^l\| = \|u; W_p^{[l]}(\Omega)\| + \sum_{|\nu|=[l]} [D_x^\nu u]_{\{l\},\Omega}^{(p)}$$

of the set of smooth functions, equal to zero for large $|x|$, where $[l]$ is the integer part of l and $\{l\} = l - [l]$. If $r > 0$ is a real number such that $2br$ is nonintegral, we define the space $B_p^{r,2br}(\Gamma')$ as the closure in the norm

$$|u; B_p^{r,2br}\| = \sum_{[\mu,\nu]\leq r} \|D_t^\mu D_x^\nu u; L_p(\Gamma')\| + \sum_{[\mu,\nu]=[2br]/2b} [D_t^\mu D_x^\nu u]_{\{2br\},x',\Gamma'}^{(p)}$$
$$+ \sum_{r-1<[\mu,\nu]<r} [D_t^\mu D_x^\nu u]_{r-[\mu,\nu],t,\Gamma'}^{(p)}$$

of the set of smooth functions defined on Γ' and equal to zero for large $|x|$.

If $\partial\Omega$ is a sufficiently smooth surface ($\partial\Omega \in C^{[l]+1}$ in the sense of [5] and [6] or, correspondingly, $\partial\Omega \in C^{[2br]+1}$), then, with the help of the identity decomposition, we can define the spaces $B_p^l(\partial\Omega)$ and $B_p^{r,2br}(\Gamma_T)$. The reader can find a detailed study of Besov spaces in [8]. A definition of these spaces is given there for integral l. In this paper we employ spaces of integral complex-valued functions and also complex Sobolov and Besov spaces. These are denoted by $L_{p,\mathbf{C}}(Q_T)$, $W_{p,\mathbf{C}}^{s,2bs}(Q_T)$, $B_{p,\mathbf{C}}^l(Q_T)$, etc. Theorems concerning traces and imbeddings (see [6]–[8]) are valid for complex and real Sobolev and Besov spaces. We state them here for real spaces.

THEOREM 1.1. 1) *If* $[\mu, \nu] < s - 1/p$, *then the mapping*

$$W_p^{s,2bs}(Q_T) \to B_p^{2b(s-[\mu,\nu]-1/p)}(\Omega), \qquad u \mapsto D_t^\mu D_x^\nu u\,|_{t=0}$$

is continuous.

2) *If* $[\mu, \nu] < s - 1/2bp$, *the mapping*

$$W_p^{s,2bs}(Q_T) \to B_p^{l/2b,l}(\Gamma_T), \quad l = 2b(s - [\mu, \nu]) - 1/p, \quad u(t, x) \mapsto D_t^\mu D_x^\nu u \big|_{\Gamma_T}$$

is continuous.

3) *If* $\mu < r - 1/p$, *the mapping*

$$B_p^{r,2br}(\Gamma_T) \to B_p^{2b(r-\mu-1/p)}\partial(\Omega), \qquad u \mapsto D_t^\mu u \big|_{\{0\}\times\partial\Omega}$$

is continuous.

4) *If* $l > 1/p$, *the mapping*

$$B_p^l(\Omega) \to B_p^{l-1/p}(\partial\Omega), \qquad u \mapsto u \big|_{\partial\Omega}$$

is continuous.

THEOREM 1.2. *If*

$$M \geq p, \qquad \frac{1}{M} > \frac{1}{p} - 2b\frac{s - [\mu, \nu]}{n + 2b},$$

then the mapping

$$W_p^{s,2bs}(Q_T) \to L_M(Q_T), \qquad u \mapsto D_t^\mu D_x^\nu u$$

is continuous. If $p(s - [\mu, \nu]) > n/2b + 1$, *the mapping*

$$W_p^{s,2bs}(Q_T) \to C(Q_T), \qquad u \mapsto D_t^\mu D_x^\nu u$$

is continuous. If the domain Ω *is bounded, these mappings are completely continuous (recall that* $0 < T < \infty$).

We also need a lemma of Hadamard in the following form.

LEMMA 1.1. *Let* $u_j \in W_p^{s,2bs}(Q_T)$, $j = 1, 2$, *and let* $f(\cdot)$ *be a continuously differentiable function on* $\mathbf{R}^N$, $N = N(s_0, 2b)$. *If* $p(s - s_0) > n/2b + 1$, *then*

$$f\big(\mathfrak{D}_{s_0}^{(2b)}u_1\big) - f\big(\mathfrak{D}_{s_0}^{(2b)}u_2\big) = U(t, x)\cdot \mathfrak{D}_{s_0}^{(2b)}(u_1 - u_2),$$

where $U(t, x) \in C(Q_T) \otimes \mathbf{R}^N$.

PROOF. According to Theorem 1.2 we have $U_j(t, x) \equiv \mathfrak{D}_{s_0}^{(2b)}u_j \in C(Q_T) \otimes \mathbf{R}^N$, $j = 1, 2$. The lemma is therefore a consequence of the equation

$$f(U_1) - f(U_2) = \int_0^1 \nabla f(U_2 + \lambda(U_1 - U_2))\, d\lambda \cdot (U_1 - U_2).$$

3. *Analytic mappings*. Let A and B be Banach spaces. The mapping $N_j\colon A \to B$ is said to be *j-homogeneous* if $N_j(u) = \tilde{N}_j(u, \ldots, u)$, where $\tilde{N}_j\colon A \times \cdots \times A$ (j terms) $\to B$ is a linear mapping. If A and B are complex spaces, N_j is said to be *j-homogeneous over* $\mathbf{C}$ if N_j is $\mathbf{C}$-linear. Let Q_A and Q_B be nonempty domains in A and B and let $F\colon Q_A \to Q_B$.

DEFINITION 1. A) The mapping F is said to be *analytic*, and we write $F \in \Phi(Q_A, Q_B)$, if $\forall u \in Q_A\, \exists \varepsilon = \varepsilon(u) > 0$ such that $\forall v \in \mathcal{O}_\varepsilon(u, A)$

$$F(v) = F(u) + \sum_{j=1}^{\infty} F_j(u; v - u),$$

where $F_j(u;\cdot)$ is a j-homogeneous continuous mapping from A into B and

$$\sum_{j=1}^{\infty} \sup_{w \in \mathcal{O}_\varepsilon(A)} \| F_j(u; w); B \| < \infty.$$

In this connection, $F_1(u;\cdot) = dF(u)\colon A \to B$ is called the *differential* of F at the point u.

B) The mapping F is said to be *complex-analytic*, and we write $F \in \Phi(Q_A, Q_B)$, if A and B are complex spaces, $F \in \Phi(Q_A, Q_B)$ and $F_j(u;\cdot)$ is j-homogeneous over $\mathbf{C}$.

DEFINITION 2. The mapping $F \in \Phi(Q_A, Q_B)$ is called an *analytic diffeomorphism* if there exists an inverse mapping $F^{-1} \in \Phi(Q_B, Q_A)$, and it is called a *complex-analytic diffeomorphism* if the spaces A and B are complex and $F \in \Phi_{\mathbf{C}}(Q_A, Q_B)$, $F^{-1} \in \Phi_{\mathbf{C}}(Q_B, Q_A)$.

DEFINITION 3. The mapping $F \in \Phi(Q_A, Q_B)$ is called a *locally analytic diffeomorphism* if $\forall u \in Q_A$ $\exists \varepsilon = \varepsilon(u) > 0$ such that $F\colon \mathcal{O}_\varepsilon(u, A) \to F(\mathcal{O}_\varepsilon(u, A))$ is an analytic diffeomorphism.

Definition 1 is the definition of analyticity in the Fréchet sense (see [9] and [10]). We note that in the case of complex spaces A and B a mapping cannot belong to $\Phi_{\mathbf{C}}(Q_A, Q_B)$ and also belong to $\Phi(Q_A, Q_B)$.

The following proposition contains some properties of mappings analytic in the Fréchet sense. Its proof may be found in [9].

PROPOSITION 1.1. 1) *If $N_j\colon A \to B$ is a j-homogeneous continuous mapping, then $N_j \in \Phi(A, B)$.*

2) *If $N\colon \mathcal{O}_\varepsilon(u, A) \to B$ is defined by the series*

$$N(u+v) = N(u) + \sum_{\nu=1}^{\infty} N_\nu(v), \quad \|N_\nu(v); B\| \leqslant C\varepsilon_1^{-\nu}\|v; A\|^\nu, \qquad \varepsilon < \varepsilon_1,$$

of homogeneous continuous mappings, it follows that $N \in \Phi(\mathcal{O}_\varepsilon(u, A), B)$.

3) *If $N_1, N_2 \in \Phi(Q_A, B)$ and $a_1, a_2 \in \mathbf{R}$, it follows that $a_1N_1 + a_2N_2 \in \Phi(Q_A, B)$.*

4) *If E is a Banach space, Q_E a domain in E, and $N_1 \in \Phi(Q_A, Q_B)$, $N_2 \in \Phi(Q_B, Q_E)$, then $N_1 \circ N_2 \in \Phi(Q_A, Q_E)$.*

5) *If $N_1 \in \Phi(Q_A, B)$, $N_2 \in \Phi(Q_A, E)$ then $N_1 \times N_2 \in \Phi(Q_A; B \times E)$.*

6) *If A, B and E are complex spaces, then the complex analogs of* 1)–5) *are valid.*

We give a number of examples of mappings analytic in the Fréchet sense.

PROPOSITION 1.2. *Let $N = N(m, 2b)$, $F \in \Phi(\mathbf{R}^N, \mathbf{R})$ and $M \in \mathbf{N}$ be such that $p(M-m) > n/2b + 1$. Then the following assertions are true*:

1) *The mapping*

$$W_p^{M,2bM}(Q_T) \to C(Q_T), \qquad u \mapsto F\big(\mathfrak{D}_m^{(2b)}u\big)$$

is analytic.

2) *If $F(0) = 0$, the mapping*

$$W_p^{M,2bM}(Q_T) \to L_p(Q_T), \qquad u \mapsto F\big(\mathfrak{D}_m^{(2b)}u\big)$$

is analytic.

3) *If $F \in \Phi_{\mathbf{C}}(\mathbf{C}^N, \mathbf{C})$ and the Sobolev spaces are replaced by complex spaces, items* 1) *and* 2) *remain valid.*

PROOF. We first prove 2). Let $u(t,x) \in W_p^{M,2bM}(Q_T)$. Then, in accordance with Theorem 1.2, $\mathfrak{D}_m^{(2b)}u \in C(Q_T) \otimes \mathbf{R}^N$, $N = N(m, 2b)$. But since

$$F\big(\mathfrak{D}_m^{(2b)}u(t,x)\big) \leqslant C_1\,|\mathfrak{D}_m^{(2b)}u(t,x)| \sup_{|z| \leqslant C_2} |dF(z)|,$$

where $C_2 = C_2(u) = \|\mathcal{D}_m^{(2b)}u;\, C(Q_T) \times \mathbf{R}^N\|$, we have

$$\|F(\mathcal{D}_m^{(2b)}u);\, L_p(Q_T)\| \leqslant C_4(u)\|u;\, W_p^{M,2bM}\|.$$

Therefore, $F(\mathcal{D}_m^{(2b)}u) \in L_p(Q_T)$. Assume now that $v \in W_p^{M,2bM}(Q_T)$, $\|v;\, W_p^{M,2bM}\| \leqslant \varepsilon = \varepsilon(u)$. Then, if $\varepsilon \ll 1$, it follows that

$$F(\mathcal{D}_m^{(2b)}(u+v)) = \sum_{\beta \in \mathbf{Z}_+^N} \frac{1}{\beta!} D_z^\beta F(\mathcal{D}_m^{(2b)}u)(\mathcal{D}_m^{(2b)}v)^\beta, \tag{1.1}$$

where $|(\beta!)^{-1}D_z^\beta F(\mathcal{D}_m^{(2b)}u) \leqslant C_5(u)^{|\beta|}$, since

$$\|\mathcal{D}_m^{(2b)}u;\, C(Q_T) \otimes \mathbf{R}^N\| \leqslant C_6\|u;\, W_p^{M,2bM}\|.$$

Therefore, if $\varepsilon \ll 1$ and $u + v \in \mathcal{O}_\varepsilon(u;\, W_p^{M,2bM})$, the series (1.1) converges in the $L_p(Q_T)$ norm uniformly with respect to v, and the proof of 2) is complete.

Assertions 1) and 3) are proved in a similar way.

COROLLARY. *Under the conditions of Proposition* 1.2 *the mapping*

$$W_p^{M,2bM}(Q_T) \to L_p(Q_T), \qquad u \mapsto D_t^{\mu_0} D_x^{\nu_0} u F(\mathcal{D}_m^{(2b)}u) \tag{1.2}$$

is analytic, where $[\mu_0, \nu_0] \to M$.

PROOF. The mapping (1.2) is a composition of two other mappings:

$$W_p^{M,2bM}(Q_T) \to C(Q_T) \times L_p(Q_T), \qquad u \mapsto (F(\mathcal{D}_m^{(2b)}u),\, D_t^{\mu_0} D_x^{\nu_0} u),$$
$$C(Q_T) \times L_p(Q_T) \to L_p(Q_T), \qquad (v; w) \mapsto vw,$$

which are analytic mappings by virtue of Propositions 1.1 and 1.2. Therefore the mapping (1.2) is analytic in accordance with 4) of Proposition 1.1.

We give a theorem concerning the analytic inverse of a mapping in Banach spaces. Its proof may be found in [10].

THEOREM 1.3. 1) *Let* Q_A *and* Q_B *be domains in the Banach spaces* A *and* B, *let* $N \in \Phi(Q_A, Q_B)$, $u \in A$, *and let* $dN(u)$ *be an isomorphism of* A *and* B. *Then there exists an* $\varepsilon = \varepsilon(u) > 0$ *such that* $N(\mathcal{O}_\varepsilon(u, A))$ *is a domain in* Q_B *and* $N\colon \mathcal{O}_\varepsilon(u, A) \to N(\mathcal{O}_\varepsilon(u, A))$ *is an analytic diffeomorphism.*

2) *If* A *and* B *are complex spaces*, $N \in \Phi_{\mathbf{C}}(Q_A, Q_B)$, *and* $dN(u)$ *is an isomorphism of* A *and* B, *there exists an* $\varepsilon > 0$ *such that* $N\colon \mathcal{O}_\varepsilon(u, A) \to N(\mathcal{O}_\varepsilon(u, A))$ *is a complex-analytic diffeomorphism.*

COROLLARY. *Let* $F \in \Phi(Q_A, Q_B)$, *and for all* $u \in Q_A$ *let* $dF(u)$ *be an isomorphism of* A *and* B. *Then* $F(Q_A) = \tilde{Q}_B$ *is a domain in* Q_B *and* $F\colon Q_A \to \tilde{Q}_B$ *is a local analytic diffeomorphism. If, furthermore,* F *is an imbedding, then* F *is an analytic diffeomorphism of* Q_A *and* $\tilde{Q}_B$.

PROOF. Let $\varepsilon = \varepsilon(u)$ be as in Theorem 1.3. Then the assertion that $\tilde{Q}_B$ is a domain follows from the representation

$$\tilde{Q}_B = \bigcup_{u \in Q_A} N(\mathcal{O}_{\varepsilon(u)}(u, A)).$$

The remaining assertions follow immediately from the definitions.

§2. Second order quasilinear parabolic equations

We consider the first boundary value problem for the quasilinear parabolic equation

$$\dot{u}(t,x) - \sum_{i,j=1}^{n} F_{i,j}(\mathfrak{D}_1 u)\frac{\partial^2 u}{\partial x_i \partial x_j} + F(\mathfrak{D}_1 u) = v(t,x), \qquad (t,x) \in Q_T, \tag{2.1}$$

$$u\big|_{\Gamma_T} = 0, \qquad u\big|_{t=0} = u_0(x), \tag{2.2}$$

where Ω is the domain in $\mathbf{R}^n$, $\partial\Omega \in C^2$, and where for brevity we have employed the notation $\mathfrak{D}_1 u = \mathfrak{D}_1^{(2)} u = (u, \partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. We assume that F: $\mathbf{R}^{n+1} \to \mathbf{R}$, $F_{i,j}$: $\mathbf{R}^{n+1} \to \mathbf{R}$, $1 \leqslant i, j \leqslant n$, are analytic functions, $F(0) = 0$, and that the condition of uniform parabolicity is satisfied, namely,

$$\sum_{i,j=1}^{n} F_{i,j}(z)\xi_i\xi_j \geqslant \delta|\xi|^2, \qquad \delta > 0\ \forall \xi \in \mathbf{R}^n,\ z \in \mathbf{R}^{n+1}. \tag{2.3}$$

Sometimes we shall consider a special case of (2.1) and (2.3):

$$\dot{u}(t,x) - \Delta u(t,x) + \varphi(u) = v(t,x), \qquad (t,x) \in Q_T, \tag{2.1$'$}$$

where φ is an analytic function.

We seek a solution of problem (2.1)–(2.2) in the space $A = \{u(t,x) \in W_q^{1,2}(Q_T) \mid u|_{\Gamma_T} = 0\}$, while we take the function pairs $(u_0(x), v(t,x))$ from the space

$$B = Б \times L_q(Q_T), \qquad Б = \left\{u_0(x) \in B_q^{2-2/q}(\Omega) \mid u_0|_{\partial\Omega} = 0\right\},\ q > 3/2.$$

We define the operator of problem (2.1)–(2.2):

$$N\colon A \to B, \qquad u(t,x) \mapsto (u)\big|_{t=0},$$

$$\dot{u}(t,x) - \sum_{i,j=1}^{n} F_{i,j}(\mathfrak{D}_1 u(t,x)) + F(\mathfrak{D}_1 u(t,x)).$$

From Proposition 1.2 and its corollary we have

LEMMA 2.1. *Let $q > n+2$ or, in the case of equation (2.1$'$), $q > (n+2)/2$. Then $N \in \Phi(A, B)$ and*

$$dN(u)v = \left(v\big|_{t=0}, \dot{v} - \sum_{i,j=1}^{n} a_{i,j}(t,x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{n} w_j(t,x)\frac{\partial v}{\partial x_j} + w_0(t,x)v\right),$$

where $a_{i,j}(t,x) = F_{i,j}(\mathfrak{D}_1 u)$ and

$$w_l(t,x) = \sum_{i,j=1}^{r} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial}{\partial u'_{x_l}} F(\mathfrak{D}_1 u) + \frac{\partial}{\partial u'_{x_l}} F(\mathfrak{D}_1 u),$$

$$w_0(t,x) = \sum_{i,j=1} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial}{\partial u} F_{i,j}(\mathfrak{D}_1 u) + \frac{\partial}{\partial u} F(\mathfrak{D}_1 u).$$

Thus, $dN(u)$ corresponds to a boundary value problem for the linear parabolic equation

$$\dot{v}(t,x) - \sum a_{i,j}(t,x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum w_j(t,x)v'_{x_j} + w_0(t,x)v = \psi(t,x),$$

$$v\big|_{\Gamma_T} = 0, \qquad v\big|_{t=0} = \chi(x). \tag{2.4}$$

LEMMA 2.2. *Let the following conditions hold*:

a) $q > 3/2$, $\partial\Omega \in C^2$, $a_{i,j}(t, x) \in C(Q_T)$, $i, j = 1, \ldots, n$, *and the condition of uniform parabolicity is satisfied*:

$$\sum_{i,j=1}^{n} a_{i,j}(t, x)\xi_i\xi_j \geqslant \delta |\xi|^2, \qquad \delta > 0\ \forall (t, x) \in Q_T, \xi \in \mathbf{R}^n.$$

b) $w_j = w_j^{(1)} + w_j^{(2)}$, $j = 0, \ldots, n$, *where* $w_j^{(2)} \in C(Q_T)$, $j = 0, \ldots, n$, $w_0^{(1)} \in L_s(Q_T)$, $w_j^{(1)} \in L_p(Q_T)$, $j = 1, \ldots, n$, *with*

$$s \geqslant s_1 = \begin{cases} \max(q, (n+2)/2) & \text{for } q \neq (n+2)/2, \\ (n+2)/2 + \varepsilon_1 & \text{for } q = (n+2)/2, \end{cases}$$

$$p \geqslant p_1 = \begin{cases} \max(q, n+2) & \text{for } q \neq n+2, \\ n+2+\varepsilon_2 & \text{for } q = n+2, \end{cases}$$

where ε_1 *and* ε_2 *are positive quantities.*

Then for each pair $(\chi, \psi) \in B$ *of problem* (2.4) *there is a unique solution* $v \in A$. *Moreover*, $\|v; A\| \geqslant C\|(\chi, \psi); B\|$.

PROOF. We introduce the norm $|f|_r^{Q_T} = \sup_{q_T} \|f|_{q_T}; L_r(q_T)\|$, where the supremum is taken over all cylinders $q_T = (0, T) \times \omega$ in which the base ω is the intersection of the domain Ω with the arbitrary ball of measure one. In [5], p. 388 (transl. p. 341), it was shown that if the conditions a) are satisfied and also the conditions

c) *the norms* $|w_0|_{s_1}^{Q_T}$, $|w_j|_{p_1}^{Q_T}$, $j = 1, \ldots, n$, *are finite and*

$$\lim_{\tau \to +0} \left(|w_0|_{s_1}^{Q_{t,t+\tau}} + \sum_j |w_j|_{p_1}^{Q_{t,t+\tau}} \right) = 0 \quad \textit{uniformly with respect to } t \in [0, T),$$

where $Q_{t,t+\tau} = (t, t+\tau) \times \Omega$ and s_1 and p_1 are the same as in condition b), then the assertions of the lemma are valid. We verify that the conditions c) follow from b). Using Hölder's inequality, we find that, if $\infty > r \geqslant r_1 \geqslant 1$,

$$|f|_{r_1}^{Q_T} = \sup_\omega \left(\int_0^T \int_\omega |f(t, x)|^{r_1}\, dx\, dt \right)^{1/r_1}$$

$$\leqslant \sup_\omega T^{(r-r_1)/r} \left(\int_0^T \int_\omega |f(t, x)|^r\, dx\, dt \right)^{1/r} \leqslant T^{(r-r_1)/r} |f|_r^{Q_T}.$$

Therefore,

$$|w_0|_{s_1}^{Q_{t,t+\tau}} \leqslant |w_0^{(1)}|_{s_1}^{Q_{t,t+\tau}} + |w_0^{(2)}|_{s_1}^{Q_{t,t+\tau}}$$

$$\leqslant \tau^{(s-s_1)/s} \left(\int_t^{t+\tau} \int_\Omega |w_0^{(1)}(\tau, x)|^s\, dx\, dt \right)^{1/s} + \tau^{1/s_1} \|w_0^{(2)}; L_\infty(Q_T)\|,$$

$$|w_j|_{p_1}^{Q_{t,t+\tau}} \leqslant \tau^{(p-p_1)/p} \left(\int_t^{t+\tau} \int_\Omega |w_j^{(1)}(\tau, x)|^p\, dx\, d\tau \right)^{1/p} + \tau^{1/p_1} \|w_j^{(2)}; L_\infty\|. \tag{2.5}$$

Since $\|w_0^{(1)}; L_s(Q_T)\| < \infty$, $\|w_j^{(1)}; L_p(Q_T)\| < \infty$, the first of the conditions c) follows from (2.5) for $t = 0$ and $\tau = T$. The second condition also follows from (2.5). In fact, when $p > p_1$ this is obvious; but when $p = p_1$, by virtue of the absolute continuity of the Lebesgue integral, the integral $\int_t^{t+\tau} \|w_j^{(1)}(t, \cdot); L_p(\Omega)\|\, d\tau$ tends towards zero uniformly with respect to t as $\tau \to 0$. Thus the lemma is a consequence of the results obtained in [5].

THEOREM 2.1. *Let $q > n + 2$.*

1) *There exists a domain $\mathcal{C} \subset B$ such that problem* (2.1)–(2.3) *has a solution $u \in A$ if and only if $(u_0, v) \in \mathcal{C}$. Moreover, the mapping $N: A \to \mathcal{C}$ is an analytic diffeomorphism.*

2) *Let the domain $\mathcal{C}$ be bounded. If*

$$(V_j, U_j) \in \mathcal{C}, \qquad (V_j, U_j) \xrightarrow{B} (V, U) \in \partial\mathcal{C},$$

where $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$ in B, then $\| N^{-1}(V_j, U_j); A\| \to \infty$.

Let equation (2.1), (2.3) *have the form* (2.1′). *Then*:

3) *In* 1) *and* 2) *it is sufficient to take $q > (n+2)/2$.*

4) *If $\varphi(u) = \varphi_1 u + \cdots + \varphi_M u^M$, it is sufficient to take*

$$q > \max\left(\frac{3}{2}, \frac{M-1}{M} \cdot \frac{n+2}{2}\right).$$

Moreover, under the conditions of 2),

$$\| N^{-1}(V_j, U_j); L_r(Q_T)\| \to \infty, \quad \text{where } r = Mq.$$

In specific equations of mathematical physics we often have $n \leqslant 3$ and $\varphi(u)$ a polynomial of degree at most three. In such a case, thanks to 4), we can take $q = 2$, in which event A and B become Hilbert spaces and Besov space Б coincides with $\mathring{W}_2^1(\Omega)$.

PROOF OF THE THEOREM. 1) According to Lemma 2.1, $dN(u)$ corresponds to the boundary value problem (2.4). Moreover, according to Proposition 1.2, the terms forming the coefficients of problem (2.4) have the following smoothness:

$$\nabla_{\mathfrak{D}_1 u} F(\mathfrak{D}_1 u) \in C(Q_T) \otimes \mathbf{R}^{n+1}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \nabla_{\mathfrak{D}_1 u} F_{i,j}(\mathfrak{D}_1 u) \in L_q(Q_T) \otimes \mathbf{R}^{n+1},$$

$$F_{i,j}(\mathfrak{D}_1 u) \in C(Q_T), \quad \frac{\partial}{\partial u} F(\mathfrak{D}_1 u) \in C(Q_T), \quad \frac{\partial^2 u}{\partial x_i \partial x_j} F_{i,j}(\mathfrak{D}_1 u) \in L_q(Q_T).$$

Therefore, Lemma 2.2 is applicable to problem (2.4), and it follows that $dN(u)$ is an isomorphism of A and B.

We verify that the mapping N is an imbedding. Assume this is not so and $N(u) = N(v)$, $u, v \in A$, $u - v = h$. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j} F_{i,j}(\mathfrak{D}_1 u) &- \frac{\partial^2 v}{\partial x_i \partial x_j} F_{i,j}(\mathfrak{D}_1 v) \\ &= \frac{\partial^2 h}{\partial x_i \partial x_j} F(\mathfrak{D}_1 u) + \frac{\partial^2 v}{\partial x_i \partial x_j}\left(F_{i,j}(\mathfrak{D}_1 u) - F_{i,j}(\mathfrak{D}_1 v)\right).\end{aligned}$$

In accordance with a lemma of Hadamard (Lemma 1.1)

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$$F(\mathfrak{D}_1 u) - F(\mathfrak{D}_1 v) = d(t,x) \cdot \mathfrak{D}_1 h, \qquad d \in C(Q_T) \otimes \mathbf{R}^{n+1}.$$

But since $N(u) - N(v) = 0$, it follows that $h(t, x)$ satisfies a boundary value problem of the form (2.4), where $\chi = 0$ and $\psi = 0$. Hence, in accord with Lemma 2.2, $h = 0$ and $u = v$. Therefore, the corollary to Theorem 1.3 applies to the mapping N; this proves 1).

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$$F_{i,j}(\mathcal{D}_1 u) \in C(Q_T), \quad \frac{\partial}{\partial u} F(\mathcal{D}_1 u) \in C(Q_T), \quad \frac{\partial^2 u}{\partial x_i \partial x_j} F_{i,j}(\mathcal{D}_1 u) \in L_q(Q_T).$$

Therefore, Lemma 2.2 is applicable to problem (2.4), and it follows that $dN(u)$ is an isomorphism of A and B.

We verify that the mapping N is an imbedding. Assume this is not so and $N(u) = N(v)$, $u, v \in A$, $u - v = h$. Then

$$\frac{\partial^2 u}{\partial x_i \partial x_j} F_{i,j}(\mathcal{D}_1 u) - \frac{\partial^2 v}{\partial x_i \partial x_j} F_{i,j}(\mathcal{D}_1 v)$$
$$= \frac{\partial^2 h}{\partial x_i \partial x_j} F(\mathcal{D}_1 u) + \frac{\partial^2 v}{\partial x_i \partial x_j}\left(F_{i,j}(\mathcal{D}_1 u) - F_{i,j}(\mathcal{D}_1 v)\right).$$

In accordance with a lemma of Hadamard (Lemma 1.1)

$$\frac{\partial^2 u}{\partial x_i \partial x_j} F_{i,j}(\mathcal{D}_1 u) - \frac{\partial^2 v}{\partial x_i \partial x_j} F_{i,j}(\mathcal{D}_1 v)$$
$$= \frac{\partial^2 h}{\partial x_i \partial x_j} F(\mathcal{D}_1 u) + b_{i,j}(t, x)\cdot \mathcal{D}_1 h, \qquad b_{i,j} \in L_q(Q_T) \otimes \mathbf{R}^{n+1};$$

$$F(\mathcal{D}_1 u) - F(\mathcal{D}_1 v) = d(t, x)\cdot \mathcal{D}_1 h, \qquad d \in C(Q_T) \otimes \mathbf{R}^{n+1}.$$

But since $N(u) - N(v) = 0$, it follows that $h(t, x)$ satisfies a boundary value problem of the form (2.4), where $\chi = 0$ and $\psi = 0$. Hence, in accord with Lemma 2.2, $h = 0$ and $u = v$. Therefore, the corollary to Theorem 1.3 applies to the mapping N; this proves 1).

2) We assume the contrary. Then $\|u_j(t, x); A\| \leqslant C$, $j = 1, 2, \ldots$, where $N(u_j) = (V_j, U_j)$. We prove the following auxiliary lemma.

LEMMA 2.3. *If* $\|u_j; A\| \leqslant C$, $j = 1, 2, \ldots$, *then the sequence* $\{u_i\}$ *contains a subsequence converging weakly in* A.

PROOF. We consider the isometric imbedding

$$\mathfrak{D}_2^{(2)}\colon A \to L_q(Q_T) \otimes \mathbf{R}^N, \quad N = N(2, 2), \qquad u \mapsto \mathfrak{D}_2^{(2)} u.$$

Then $\|\mathfrak{D}_2^{(2)} u_j; L_q(Q_T) \otimes \mathbf{R}^N\| \leqslant C'$, but since $L_q(Q_T) \otimes \mathbf{R}^N$ is a reflexive Banach space, after going over to the subsequence we see that $\mathfrak{D}_2^{(2)} u_{\alpha(j)} \rightharpoonup \mathfrak{U}$ weakly in $L_q(Q_T) \otimes \mathbf{R}^{n^2+n+2}$. Since $\mathfrak{D}_2^{(2)}$ is an isometric imbedding, it follows that $\mathfrak{D}_2^{(2)}(A)$ is a closed subspace and $\mathfrak{U} = \mathfrak{D}_2 u$, $u \in A$. Consequently, $u_{\alpha(j)} \rightharpoonup u$ weakly in A, and the lemma is proved.

In view of the boundedness of Ω the imbeddings in Theorem 1.2 are completely continuous. Therefore, $\mathfrak{D}_1 u_{\alpha(j)} \to \mathfrak{D}_1 u$ in $C(Q_T) \otimes \mathbf{R}^{n+1}$; consequently,

$$F(\mathfrak{D}_1 u_{\alpha(j)}) \to F(\mathfrak{D}_1 u), \quad F_{k,l}(\mathfrak{D}_1 u_{\alpha(j)}) \to F_{k,l}(\mathfrak{D}_1 u), \qquad k, l = 1, \ldots, n,$$

in $C(Q_T)$. But then

$$\frac{\partial^2 u}{\partial x_k \partial x_l} F_{k,l}(\mathfrak{D}_1 u_{\alpha(j)}) \rightharpoonup \frac{\partial^2 u}{\partial x_k \partial x_l} F_{k,l}(\mathfrak{D}_1 u)$$

weakly in $L_q(Q_T)$ and $N(u_{\alpha(j)}) \rightharpoonup N(u)$ weakly in B, whence $(V, U) = N(u) \in \mathfrak{C}$. Thus we have a contradiction.

3) The proof is similar to that of 1).

4) In this case

$$N(u) = \left(u\big|_{t=0}, \dot{u} - \Delta u + \varphi_1 u + \cdots + \varphi_M u^M\right).$$

However, for $q > (M-1)(n+2)/2M$ we have, according to Theorem 1.2, the continuous imbeddings

$$W_q^{1,2}(Q_T) \hookrightarrow L_{qm}(Q_T), \qquad 1 \leqslant m \leqslant M. \tag{2.6}$$

Therefore the mapping

$$\tilde{\varphi}\colon A \to L_q(Q_T), \qquad u(t, x) \mapsto \varphi(u(t, x)) = \varphi_1 u + \cdots + \varphi_M u^M$$

is analytic. Hence, $N \in \Phi(A, B)$ and

$$dN(u)v = \left(v\big|_{t=0}, \dot{u} - \Delta v + \varphi'(u)v\right),$$

where $\varphi'(u) \in L_s(Q_T)$, $s = qM/(M-1)$, since $\varphi'(\cdot)$ is a polynomial of degree $M-1$. But $qM/(M-1) > \max(q, (n+2)/2)$; therefore, in accordance with Lemma 1.2, $dN(u)$ is an isomorphism of A and B.

We verify that N is an imbedding. Suppose that this is not so and that $N(u) = N(v)$, $u, v \in A$. Then if $h = u - v$, we have

$$\dot{h} - \Delta h + r(u, v)h = 0, \qquad h\big|_{\Gamma_T} = 0, h\big|_{t=0} = 0,$$

where $r(\cdot, \cdot)$ is a polynomial of degree $M-1$ in u and v and, therefore, $r(u, v) \in L_s(Q_T)$, $s = qM/(M-1)$. Hence, according to Lemma 2.2, $h = 0$. By the corollary to Theorem 1.3, N is an analytic diffeomorphism of A in the domain $\mathfrak{C} = N(A)$.

Let $N(u_j) = (V_j, U_j)$, and assume that $\|u_j, L_{Mq}(Q_T)\| \leqslant C_1$. Then, since Ω is bounded, $\|\varphi(u_j); L_q\| \leqslant C_2$. Therefore,

$$\dot{u}_j - \Delta u_j = W_j, \qquad u_j|_{t=0} = V_j, \qquad u_j|_{\Gamma_T} = 0,$$

where $W_j = U_j - \varphi(u_j)$ and $\|(V_j, W_j); B\| \leqslant C_3$. From this and Lemma 2.2 it follows that $\|u_j, A\| \leqslant C_4$. Hence, by Lemma 2.3, $u_{\alpha(j)} \to u$ weakly in A. Since the imbeddings (2.6) are completely continuous, it follows that

$$\|u_{\alpha(j)} - u; L_{qm}(Q_T)\| \to 0, \qquad 1 \leqslant m \leqslant M.$$

Therefore $\varphi(u_{\alpha(j)}) \to \varphi(u)$ in $L_q(Q_T)$ and $N(u_{\alpha(j)}) - N(u)$ weakly in B. Consequently, $(V, U) = N(u) \in \mathcal{O}$. Thus we have a contradiction.

COROLLARY 1. *If under the conditions of Theorem* 2.1, 2)

$$\|u; A\| \leqslant g_1(\|(u_0, v); B\|),$$

or if under the conditions of Theorem 2.1, 4)

$$\|u; L_{Mq}(Q_T)\| \leqslant g_2(\|(u_0, v); B\|),$$

where g_1 and g_2 are continuous functions, then $\mathcal{O} = B$.

PROOF. Assume that $\mathcal{O} \neq B$. Then $\partial\mathcal{O} \neq \varnothing$ and there exists a $(V, U) \in \partial\mathcal{O}$. We select a sequence of points (V_j, U_j) from $\mathcal{O}$ which converges to (V, U). For such a choice it follows from the conditions that $\|N^{-1}(V_j, U_j); A\| \leqslant C$, which contradicts Theorem 2.1, 2). The second conclusion is proved similarly with the use of part 4) of the theorem.

COROLLARY 2. *If in* (2.1) *we have $F_{i,j}(\mathcal{D}_1 u) \equiv \delta_{i,j}$, $|F(\mathcal{D}_1 u)| \leqslant C$ and the domain Ω is bounded, then $\mathcal{O} = B$.*

PROOF. In (2.1) we carry $F(\mathcal{D}_1 u)$ to the right side. Then according to Lemma 2.2 we have $\|u; A\| \leqslant C\|(u_0, v); B\| + C'$. In accordance with Corollary 1 we find that $\mathcal{O} = B$.

By the *solving operator of problem* (2.1)–(2.2) we shall mean the mapping Ψ which makes the solution $u(t, x)$ correspond to the given functions $u_0(x)$ and $v(t, x)$.

COROLLARY 3. *Let B' be a Banach space continuously imbedded in B, $(u_0', v') \in B'$, and assume that when $u_0 = u_0'$ and $v = v'$ there exists a solution $u'(t, x) \in A$ of problem* (2.1)–(2.3). *Then for some $\varepsilon > 0$ there exists a solving operator $\Psi \in \Phi(\mathcal{O}_\varepsilon((u_0', v'), B'), A)$ of problem* (2.1)–(2.3).

PROOF. Let $\mathcal{O}' = \mathcal{O} \cap B'$. Then $(u_0', v') \in \mathcal{O}'$ and $\mathcal{O}'$ is a domain in B'. We can therefore select an $\varepsilon > 0$ such that $\mathcal{O}_\varepsilon = \mathcal{O}_\varepsilon((u_0', v'), B') \subset \mathcal{O}'$. It remains then to put $\Psi = N^{-1}|_{\mathcal{O}_\varepsilon}$.

We consider equation (2.1′) with the polynomial nonlinearity

$$\varphi(u) = \varphi_1 u + \cdots + \varphi_M u^M, \qquad \varphi_M \neq 0,\ M \geqslant 2.$$

THEOREM 2.2. *Let Ω be a bounded domain, $q \geqslant 2$ and $q > (M-1)(n+2)/2M$.*

1) *If $n = 2$ and $\varphi'(u) \geqslant 0\ \forall u \in \mathbf{R}$, then $\mathcal{O} = B$.*

2) *If an $\alpha > 0$ exists such that $-u\varphi(u) \geqslant (2\alpha + 2)g(u) \geqslant 0\ \forall u \in \mathbf{R}$, where $g(u) = \int_0^u \varphi(\tau)\, d\tau$, then $\mathcal{O} \neq B$.*

PROOF. 1) We denote the $L_2(\Omega)$ norm by $\|\cdot\|_2$ and the inner product in $L_2(\Omega)$ by $\langle\cdot,\cdot\rangle$. Then, multiplying (2.1′) scalarly in $L_2(\Omega)$ by $-\Delta u(t,\cdot)$, we obtain

$$-\langle \dot{u}(t,\cdot), \Delta u(t,\cdot)\rangle + \|\Delta u(t,\cdot)\|_2^2 - \langle \varphi(u(t,\cdot)), \Delta u(t,\cdot)\rangle = -\langle v(t,\cdot), \Delta u(t,\cdot)\rangle.$$

Since $\varphi'(u) \geqslant 0$, we have

$$\langle \varphi(u(t,\cdot)), \Delta u(t,\cdot)\rangle \leqslant 0 \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}\|\nabla u(t,\cdot)\|_2^2 \leqslant \|v(t,\cdot)\|_2^2.$$

Therefore, by Gronwall's lemma,

$$\|\nabla u(t,\cdot)\|_2^2 \leqslant \|\nabla u_0\|_2^2 \exp 2\int_0^t \|v(\tau,\cdot)\|_2^2\, d\tau,$$

from which it follows that

$$\sup_{0\leqslant t\leqslant T} \|u(t,\cdot);\, \mathring{W}_2^1(\Omega)\| \leqslant C_1\|u_0;\, B_q^{2-2/q}(\Omega)\| \exp C_2\|v;\, L_q(Q_T)\|^2,$$

and, by a Sobolev imbedding theorem,

$$\|u;\, L_{Mq}(Q_T)\| \leqslant C_3\|u_0;\, B_q^{2-2/q}(\Omega)\| \exp C_2\|v;\, L_q(Q_T)\|^2.$$

It then remains to apply Corollary 1 of Theorem 2.1.

2) We assume, conversely, that $\mathcal{O} = B$. Then problem (2.1')–(2.2) has for $v = 0$ and for an arbitrary $u_0 \in Б$ a solution $u(t, x)$ such that

$$u \in W_2^{1,2}(Q_T), \qquad \varphi(u) \in L_2(Q_T). \tag{2.7}$$

LEMMA 2.4 (LEVINE [11]). *Let the conditions of Theorem 2.2, 2) be satisfied, and let $u(t, x)$ be a solution of problem (2.1')–(2.2) for $v = 0$ such that conditions (2.7) are satisfied. Then*

$$T \leqslant T(u_0) = \frac{\|u_0\|_2^2}{\alpha^2\int\left(g(u_0(x)) - \frac{1}{2}|\nabla u_0(x)|^2\right)dx}, \tag{2.8}$$

if the denominator of this expression is positive.

Assertion 2) of the theorem follows very simply from Lemma 2.4. Indeed, let us put $u_0(x) = \lambda v(x)$, where $v \in C_0^\infty(\Omega)$, $v \geqslant 0$, $\int v^2(x)\,dx > 0$. Then (since the denominator of $T(\lambda v)$ is a polynomial in λ of degree $M + 1 \geqslant 3$ with the coefficient of λ^{M+1} positive) we have $T(\lambda v) > 0$ for large λ and $\lim_{\lambda\to\infty} T(\lambda v) = 0$, which contradicts (2.8).

In [11] a somewhat greater smoothness than that in (2.7) was required of the solution $u(t, x)$. Therefore, although under the conditions of Lemma 2.4 the proof from [11] is even simplified, we give it here in its entirety.

PROOF OF THE LEMMA (LEVINE). We define a function

$$p(t) = \int_0^t \|u(\eta)\|_2^2\, d\eta + r(t+\tau)^2 + (T_0 - t)\|u_0\|_2^2,$$

where the constants $\tau, r > 0$ and $0 < T_0 \leqslant T$ are chosen below. Then

$$\dot p(t) = \|u(t)\|_2^2 - \|u_0\|_2^2 + 2r(t+\tau) = 2\int_0^t \langle u(\eta), \dot u(\eta)\rangle\, d\eta + 2r(t+\tau),$$

$$\ddot p(t) = 4(\alpha+1)I_1(t) + 2I_2(t),$$

where

$$I_1(t) = \int_0^t \|\dot u(\eta)\|_2^2\, d\eta + r; \quad I_2(t) = \langle u, \dot u\rangle - 2(\alpha+1)\int_0^t \|\dot u(\eta)\|_2^2\, d\eta - (2\alpha+1)r.$$

For convenience we let $p(t) = a(t) + b(t)$, where

$$a(t) = \int_0^t \|u(\eta)\|_2^2\, d\eta + r(t+\tau)^2, \qquad b(t) = (T_0 - t)\|u_0\|_2^2.$$

From (2.7) it follows that $p(t) \in W_1^2(0, T)$. Therefore, $p^{-\alpha}$ is determined when $0 \leqslant t \leqslant T$ and $p > 0$. We show below that we can arrange the constants τ, r and T_0 so that

$$(p^{-\alpha})''_{tt}\big|_{[0,T_0]} \leqslant 0, \qquad p(0)/\alpha\dot{p}(0) = T_0.$$

But then $p^{-\alpha}(t) \leqslant p^{-\alpha}(0) + t(p^{-\alpha}(0))'_t$ and the function $p^{-\alpha}$ vanishes on $[0, T_0]$. Therefore $T(u_0) \leqslant T_0$.

The inequality $(p^{-\alpha}(t))''_{tt} \leqslant 0$ is equivalent to

$$p\ddot{p} - (\alpha + 1)\dot{p}^2 \equiv 4(\alpha + 1)(\alpha I_1 - \dot{p}^2/4) + bI_1 + pI_2 \geqslant 0. \tag{2.9}$$

We verify that all the terms in (2.9) are positive.

1) We verify that $aI_1 - \dot{p}^2/4 \geqslant 0$. This inequality is a consequence of two other inequalities, namely,

$$\left(\int_0^t \langle u(\eta), \dot{u}(\eta)\rangle\, d\eta\right)^2 \leqslant \int_0^t \|\dot{u}(\eta)\|_2^2\, d\eta \int_0^t \|u(\eta)\|_2^2\, d\eta,$$

$$2\int_0^t \langle u(\eta), \dot{u}(\eta)\rangle\, d\eta\, r(t+\tau) \leqslant r\int_0^t \|u(\eta)\|_2^2\, d\eta + r(t+\tau)^2 \int_0^t \|\dot{u}(\eta)\|_2^2\, d\eta.$$

2) Positiveness in I_1 is obvious.

3) We verify that $I_2(t) \geqslant 0$:

$$\begin{aligned} I_2(t) &= \langle u(t), \dot{u}(t)\rangle - (2\alpha+1)r - 2(\alpha+1)\int_0^t \langle \dot{u}(\eta), g'(u(\eta))\rangle\, d\eta \\ &\quad -2(\alpha+1)\int_0^t \langle \dot{u}(\eta), \Delta u(\eta)\rangle\, d\eta \\ &= -\alpha\langle u(t), \Delta u(t)\rangle + \left(\langle u(t), g'(u(t))\rangle - 2(\alpha+1)\int g(u(t,x))\, dx\right) \\ &\quad + 2(\alpha+1)\left(\left(\int g(u_0(x))\, dx - \tfrac{1}{2}\langle u_0, \Delta u_0\rangle\right) - \frac{2\alpha+1}{2\alpha+2} r\right). \end{aligned}$$

The first term is obviously nonnegative. The second is nonnegative according to a condition of the theorem. Therefore, if we choose

$$r = \int g(u_0(x)) - \tfrac{1}{2}|\nabla u_0(x)|^2\, dx$$

(we note that for such a choice $r > 0$), we have $L_2 \geqslant 0$ and (2.9) is satisfied.

We put $T_0 = \|u_0\|_2^2/\alpha^2 r$, $\tau = \|u_0\|_2^2/\alpha r$. Then $p(0)/\alpha\dot{p}(0) = T_0$ and

$$T(u_0) \leqslant T_0 = \frac{\|u_0\|_2^2}{\alpha^2\left(\int g(u_0(x)) - \frac{1}{2}|\nabla u_0(x)|^2\, dx\right)},$$

which establishes the lemma.

We can consider problem (2.1)–(2.2) in a space of complex functions. For this we assume that $F, F_{ij}\colon \mathbf{C}^{n+1} \to \mathbf{C}$ are real-analytic functions; we define $A_{\mathbf{C}}$, $Б_{\mathbf{C}}$ and $B_{\mathbf{C}}$ as complex versions of the spaces A, $Б$ and B, and we specify the mapping

$$N_{\mathbf{C}}\colon A_{\mathbf{C}} \to B_{\mathbf{C}}, \quad u \mapsto \left(u\big|_{t=0}, \dot{u} - \sum F_{i,j}(\mathfrak{D}_1 u)\frac{\partial^2 u}{\partial x_i \partial x_j} + F(\mathfrak{D}_1 u)\right). \tag{2.10}$$

THEOREM 2.3. 1) *There exists a domain $\mathcal{O}_{\mathbf{C}} \subset B_{\mathbf{C}}$ such that the assertions of Theorem 2.1 are valid upon replacing A by $A_{\mathbf{C}}$, B by $B_{\mathbf{C}}$, $\mathcal{O}$ by $\mathcal{O}_{\mathbf{C}}$, and N by $N_{\mathbf{C}}$.*

2) *If $F_{i,j}(z) \equiv \delta_{i,j}$ and F is a holomorphic function, then $N_{\mathbf{C}}\colon A_{\mathbf{C}} \to \mathcal{O}_{\mathbf{C}}$ is a complex-analytic diffeomorphism.*

The proof of this theorem does not differ from the proof of Theorem 2.1. It is necessary only to take into account the fact that the complex analog of Lemma 2.2 holds by virtue of the results obtained in [5]–[7].

REMARK. Let us assume, under the conditions of Theorem 2.3, 2), that $F(\mathbf{R}^{n+1}) \subset \mathbf{R}$. Then $\mathcal{O} = \mathcal{O}_{\mathbf{C}} \cap B$. In particular, Theorem 2.2, 2) furnishes examples of polynomial diffeomorphisms of the whole complex Banach space onto a strict subdomain in another space. We note that in the case of finite-dimensional spaces $\mathbf{C}^N$, $N \geq 2$, the existence of such polynomial diffeomorphisms is a problem.

A result analogous to Theorem 2.1 can be proved for the Navier-Stokes system of equations, which describes the flow of a fluid in a bounded three-dimensional domain Ω, $\partial\Omega \in C^2$ (see [10]):

$$\dot{u}(t, x) - \Delta u + \sum_{j=1}^{3} u^j \frac{\partial u}{\partial x_j} + \nabla p = f, \quad \operatorname{div} u = 0, \qquad (t, x) \in Q_T, \tag{2.11}$$

$$u\,|_{t=0} = u_0(x), \qquad u\,|_{\Gamma_T} = 0, \tag{2.12}$$

where $u(t, x) = (u^1, u^2, u^3)$, $f(t, x) = (f^1, f^2, f^3)$. We define the following function spaces:

$\mathcal{V} = \{u(x) \in C_0^\infty(\Omega) \otimes \mathbf{R}^3 : \operatorname{div} u = 0\}$,

$\mathcal{H}$ is the closure of $\mathcal{V}$ in $L_2(\Omega) \otimes \mathbf{R}^3$;

$\mathcal{H}'$ is the closure of $\mathcal{V}$ in $W_2^1(\Omega) \otimes \mathbf{R}^3$;

$\mathcal{H}^{1,2} = \{u(t, x) \in W_2^{1,2}(Q_T) \otimes \mathbf{R}^3 \mid \operatorname{div} u = 0, u\,|_{\Gamma_T} = 0\}$,

$L_{2,g}$ is the orthogonal complement to $\mathcal{H}$ in $L_2(\Omega) \otimes \mathbf{R}^3$;

$\bar{A} = \mathcal{H}^{1,2} \times L_2(0, T; L_{2,g})$, $\bar{B} = \mathcal{H}^1 \times (L_2(\mathcal{O}_T) \otimes \mathbf{R}^3)$.

It is well known that if $\zeta \in L_2(0, T; L_{2,g})$, then

$$\zeta = \nabla p(t, x), \qquad p(t, x) \in L_2(0, T; W_2^1(\Omega)).$$

We shall therefore write elements from $L_2(0, T; L_{2,g})$ as ∇p. We define the operator of problem (2.11)–(2.12):

$$\bar{N} : \bar{A} \to \bar{B}; \quad (u, \nabla p) \mapsto \left(u\,|_{t=0}, \dot{u} - \Delta u + \sum_{j=1}^{3} u^j \frac{\partial u}{\partial x_j} + \nabla p \right).$$

THEOREM 2.4. *There exists a domain $\bar{\mathcal{O}} \subset \bar{B}$ such that $\bar{N} : \bar{A} \to \bar{\mathcal{O}}$ is an analytic diffeomorphism. Moreover, if $(u_0^{(j)}, f^{(j)}) \in \bar{\mathcal{O}}$ and $(u_0^{(j)}, f^{(j)}) \xrightarrow{\bar{B}} (u_0^{(0)}, f^{(0)}) \in \partial\bar{\mathcal{O}}$, then $\| N^{-1}(u_0^{(j)}, f^{(j)}); \bar{A} \| \to \infty$.*

The proof is analogous to the proof of Theorem 2.1 and will not be supplied here. We note that the second assertion of the theorem bears a conditional character since we do not know whether it is true that $\partial\bar{\mathcal{O}} \neq \varnothing$.

§3. Examples of continuous but not analytic solving operators of problems of the form (2.1)–(2.2)

We recall that a *solving operator* of problem (2.1)–(2.2) is a mapping which makes the solution $u(t, x)$ correspond to the function pair $(u_0(x), v(t, x))$. We proved above that if the uniform parabolicity condition (2.3) is satisfied, and if the space B' is continuously imbedded in the space B, then in a neighborhood of the zero of B' there exists an analytic solving operator Ψ. We now prove that if the space B'' of the function pairs $(u_0(x), v(t, x))$ is less smooth than B, or if condition (2.3) is not satisfied, then problem (2.1)–(2.2) can then have a continuous, but not analytic, solving operator. For this we consider the family of problems depending on the parameter $\lambda \geqslant 0$:

$$\dot{u}(t, x) - \lambda \Delta u - \mathfrak{F}(u) = v(t, x), \qquad u\,|_{t=0} = u_0(x),\ u\,|_{\Gamma_T} = 0, \tag{3.1}$$

where $(t, x) \in Q_T = (0, T) \times \Omega$, Ω is a bounded domain in $\mathbf{R}^n$, $\partial\Omega \in C^2$, and

$$\mathfrak{F}(u) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\right)^3.$$

We define the spaces

$$A'' = \left\{u(t, x) \in L_4\big(0, T; \dot{W}_4^1(\Omega)\big) \mid \dot{u} \in L_{4/3}\big(0, T; W_{4/3}^{-1}(\Omega)\big)\right\},$$
$$B'' = L_2(\Omega) \times L_{4/3}\big(0, T; W_{4/3}^{-1}(\Omega)\big),$$

and we specify the operator of problem (3.1):

$$N_\lambda\colon A'' \to B'', \quad u(t, x) \to (u)_{t=0}, \quad \dot{u} - \lambda\Delta u - \mathfrak{F}(u).$$

It is easy to see that $N_\lambda \in \Phi(A'', B'')$. In addition, as Vishik [12] proved, the mappings N_λ are homeomorphisms of A'' and B''. However, $B'' \not\subset B$ and, furthermore, when $\lambda = 0$ the uniform parabolicity condition (2.3) is not satisfied for problem (3.1). It turns out that the operators N_λ^{-1} are not analytic.

A much stronger result is valid.

THEOREM 3.1. *There exists no domain $\mathcal{O}_B \subset B''$ such that for some $\lambda \geqslant 0$ the mapping $N_\lambda^{-1}\colon \mathcal{O}_B \to A''$ is continuously differentiable in the Fréchet sense.*

PROOF. We assume that the domain $\mathcal{O}_B$ and $\lambda \geqslant 0$ exist. Assume then that $N_\lambda^{-1}(\mathcal{O}_B) = \mathcal{O}_A$. We put

$$A_0'' = \left\{v(t, x) \in A'' \mid \operatorname{supp} v \in [0, T] \times \Omega_v,\ \Omega_v \Subset \Omega,\ \partial\Omega_v \in C^2\right\}.$$

The space A_0'' is dense in A''. For the proof it is sufficient, with the aid of a partition of unity, to rectify $\partial\Omega$ and to note that if $\Omega = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 > 0\}$ and $v(t, x) \in A$, we can then put

$$v_m(t, x) = \tilde{v}(t, x_1 - 1/m, x_2, \ldots, x_n),$$

where $\tilde{v}$ is the function v continued through the origin into $\{(t, x) \mid t \geqslant 0, x_1 \leqslant 0\}$. For such a choice $v_m \in A_0''$ and $v_m \xrightarrow{A''} v$.

Hence there exists a $\chi(t, x) \in \mathcal{O}_A \cap A_0''$. We define $\Omega' = \Omega \setminus \overline{\Omega}_\chi$ and we consider

$$\mathfrak{N}_\lambda = \big(\pi_2 \circ dN_\lambda(\chi)u(t, x)\big)\,|_{(0,T)\times\Omega'},$$

where $\pi_2(u_0(x), v(t, x)) = v(t, x)$ and the restriction on $(0, T) \times \Omega'$ is considered as a continuous operator from $\mathfrak{D}'(0, T; W_4^{-1}(\Omega))$ into $\mathfrak{D}'(0, T; W_4^{-1}(\Omega'))$. Then, since $\Delta\colon W_4^1(\Omega') \to W_4^{-1}(\Omega')$ is a continuous mapping, we have

$$\mathfrak{N}_\lambda(A) \subset \mathfrak{D}'\big(0, T; W_4^{-1}(\Omega')\big). \tag{3.2}$$

However, by assumption, $N_\lambda\colon \mathcal{O}_A \to \mathcal{O}_B$ has a continuously differentiable inverse mapping N_λ^{-1}. Therefore $dN_\lambda(\chi)\colon A'' \to B''$ is an isomorphism. In particular, $\pi_2 \circ dN_\lambda(\chi)(A) = \pi_2(B)$. From this and from (3.2) we find that

$$L_{4/3}\big(0, T; W_{4/3}^{-1}(\Omega)\big)\,|_{(0,T)\times\Omega'} \subset \mathfrak{D}'\big(0, T; W_4^{-1}(\Omega')\big),$$

or, considering this imbedding on generalized functions independent of t,

$$W^{-1}_{4/3}(\Omega') \subset W^{-1}_4(\Omega). \tag{3.3}$$

However, since Ω' is bounded, we have the strict imbedding $\mathring{W}^{-1}_4(\Omega') \underset{\neq}{\subset} \mathring{W}^{1}_{4/3}(\Omega')$, whence $W^{-1}_{4/3}(\Omega) \underset{\neq}{\supset} W^{-1}_4(\Omega)$, which contradicts (3.3). This completes the proof of the theorem.

It is interesting to compare Theorem 3.1 with the finite-dimensional situation. Let $\mathfrak{N}$: $\mathbf{R}^N \to \mathbf{R}^N$ be an analytic mapping with the continuous inverse $\mathfrak{N}^{-1}$. Then by Sard's lemma there exists a domain $\mathfrak{D} \subset \mathbf{R}^N$ such that $(\mathbf{R}^N \setminus \mathfrak{D}) = 0$ and $\mathfrak{N}^{-1}|_{\mathfrak{D}}$ is analytic. Thus Theorem 3.1 supplies a counterexample to Sard's lemma in the infinite-dimensional case.

We remark that N_λ does not satisfy Sard's lemma because it is not a Fredholm mapping. In this connection, see [13], where it is shown that Sard's lemma is satisfied for Fredholm mappings of Banach spaces. We recall that a continuously differentiable mapping F: $A \to B$ is called a *Fredholm mapping* if $\forall u \in A$ the mapping $dF(u)$: $A \to B$ is a linear Fredholm mapping.

If $\lambda > 0$, problem (3.1) satisfies condition (2.3). Therefore, although the mapping N_λ^{-1}: $B'' \to A''$ is not analytic, we can select, in accordance with Corollary 3 of Theorem 2.1, $\delta > 0$, $p > 1$ and $m_1, m_2 \in \mathbf{N}$ such that $N_\lambda^{-1}|_{\mathcal{O}_\delta(\tilde{B})} \in \Phi(\mathcal{O}_\delta(\tilde{B}), A'')$, where $\tilde{B} = \mathring{W}_p^{m_1}(\Omega) \times W_p^{m_2, 2m_2}(Q_T)$. That is, when $\lambda > 0$ the operator N_λ^{-1} becomes analytic after a restriction to a subspace $\tilde{B}$ of smoother functions. We prove now that when $\lambda = 0$ no such space $\tilde{B}$ exists in the scale of Sobolev spaces.

THEOREM 3.2. *There exist no $p > 1$, $m \in \mathbf{N}$ and $\delta > 0$ such that*

$$N_0^{-1}|_{\mathcal{O}_\delta(\mathring{W}_p^m(\Omega)) \times \{0\}} \in \Phi\big(\mathcal{O}_\delta\big(\mathring{W}_p^m(\Omega)\big), A''\big).$$

PROOF. We suppose the contrary to be true and we let $S(u_0) = N_0^{-1}(u_0, 0)$. Then

$$S(u_0) = \sum_{l=1}^{\infty} S_l(u_0) \quad \forall u_0 \in \mathcal{O}_\delta\big(\mathring{W}_p^m(\Omega)\big). \tag{3.4}$$

Since $N_0 \in \Phi(A'', B'')$, we can apply N_0 to the series (3.4) and in the equation $N_0 \circ S(u_0) \equiv (u_0, 0)$ we can equate mappings that are homogeneous and of the same degree. We obtain a sequence of equations, where

$$\mathfrak{F}(u, v, w) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(u'_{x_j} v'_{x_j} w'_{x_j}\big):$$

$$S_1(u_0)|_{t=0} = u_0, \quad \frac{d}{dt} S_1(u_0) = 0,$$

$$S_l(u_0)_{t=0} = 0, \quad \frac{d}{dt} S_l(u_0) = \sum_{l_1 + l_2 + l_3 = l} \mathfrak{F}\big(S_{l_1}(u_0), S_{l_2}(u_0), S_{l_3}(u_0)\big), \qquad l \geqslant 2,$$

from which we obtain by induction that $S_{2m}(u_0) \equiv 0$ and that $S_{2l+1}(u_0)$ contains the term

$$t^\nu\big(\partial^{2l} u_0 / \partial x_1^{2l}\big)\big(\partial u_0 / \partial x_1\big)^2,$$

with a nonzero factor, where $\nu = \nu(t) \in \mathbf{N}$ and the derivatives of u_0 with respect to x_1 of order less than $2l$ do not appear. We can assume that $0 \in \Omega$. The if we consider the function

$$\tilde{u}_0(x) = \kappa(x) \times \begin{cases} x_1 + x_1^{m+1/2}, & x_1 > 0, \\ 0, & x_1 \leqslant 0, \end{cases}$$

where $\kappa \in C_0^\infty(\Omega)$, $\kappa(0) = 1$, then $\tilde{u}_0 \in \mathring{W}_p^m(\Omega)$ and $S_{2l+1}(u_0) \notin A''$, if $l \gg 1$. The resulting contradiction establishes the theorem.

§4. Quasilinear $2b$-parabolic equations

We consider in the cylinder $Q_T = (0, T) \times \Omega$, $0 < T < \infty$, the quasilinear $2b$-parabolic equation

$$Au(t, x) \equiv \sum_{[\mu,\nu]=M} a_{\mu,\nu}\left(\mathfrak{D}_m^{(2b)}u\right) D_t^\mu D_x^\nu u(t, x) + a\left(\mathfrak{D}_m^{(2b)}u\right) = f(t, x),$$

$$(t, x) \in Q_T, \quad (4.1)$$

where $[\mu, \nu] = |\nu|/2b + \mu$, $m, M, b \in \mathbf{N}$, $m < M$, $\mathfrak{D}_m^{(2b)} u$ is a row of all the $N = N(m, 2b)$ partial derivatives of the form $D_t^\mu D_x^\nu u$, $[\mu, \nu] \leqslant m$, and $a_{\mu,\nu}(\cdot)$ and $a(\cdot)$ are analytic functions on $\mathbf{R}^N$, $a(0) = 0$. We assume the following uniform parabolicity condition to be satisfied:

A) The roots of the polynomial

$$\mathfrak{A}(z; \xi, p) \equiv \sum a_{\mu,\nu}(z) p^\mu (i\xi)^\nu = 0$$

for arbitrary $Z \in \mathbf{R}^N$ satisfy the inequality $\operatorname{Re} p \leqslant -\delta|\xi|^{2b}$, $\delta > 0$.

For equation (4.1) we pose the following boundary and initial conditions:

$$G_j\varphi \equiv \sum_{[\mu,\nu]\leqslant r_j} g_{j\mu,\nu} D_t^\mu D_x^\nu u\,|_{\Gamma_T} = \varphi_j, \qquad 1 \leqslant j \leqslant Mb, \quad (4.2)$$

where $r_j > 0$, $2br_j \in \mathbf{N}$, $j = 1, \ldots, Mb$;

$$D_t^k u\,|_{t=0} = u_k(x), \qquad 0 \leqslant k \leqslant M - 1. \quad (4.3)$$

We assume that problem (4.1)–(4.3) satisfies the Lopatinskiĭ condition:

B) Let $x' \in \partial\Omega$, let ζ be a vector tangent to $\partial\Omega$ at x', and let τ be the normal to $\partial\Omega$ at x'. Then the equation $\mathfrak{A}(z; (\zeta + y\tau), p) = 0$ in y has for $|\zeta| + |p| \neq 0$ exactly Mb roots $\tau_1^+, \ldots, \tau_{Mb}^+$ with positive imaginary part, and the polynomials

$$\sum_{[\mu,\nu]\leqslant r_j} g_{j\mu\nu} p^\mu (i(\zeta + \tau y))^\nu, \qquad j = 1, \ldots, Mb,$$

are linearly independent with respect to the modulus of the polynomial $\prod_s(y - \tau_s^+(z; \zeta, p))$ when $\operatorname{Re} p > -\delta_1\zeta^{2b}$, $|p| + |\zeta| \neq 0$.

We define the following spaces: $A = W_p^{M,2bM}(Q_T)$,

$$Б = \left\{(\varphi_1, \ldots, \varphi_{Mb}; u_0, \ldots, u_{m-1}) \in \prod_{k=1}^{Mb} B_p^{l_k, 2bl_k}(\Gamma_T) \times \prod_{j=0}^{M-1} B_p^{2b(M-j-1/p)}\Big|_{l_k} \right.$$

$$\left. = M - r_k - \frac{1}{2bp}, \sum_{[\mu,\nu]<r_j} b_{j\mu,\nu} D_x^\nu u_\mu\,|_{\partial\Omega} = \varphi_j\,|_{\{0\}\times\partial\Omega};\ 2br_j + \frac{2b+1}{p} < 2Mb \right\},$$

$B = Б \times L_p(Q_T)$ (the space Б is correctly defined according to Theorem 1.1).

We seek a solution of problem (4.1)–(4.3) in A, assuming that the given functions $(\varphi_1, \ldots, u_{m-1}; f)$ belong to B. We specify the operator of problem (4.1)–(4.3):

$$N: A \to B, \qquad u(t, x) \mapsto (K(u), A(u)),$$

where

$$K: A \to Б, \qquad u(t, x) \mapsto \left(G_1(u), \ldots, G_{Mb}(u); u\,|_{t=0}, \ldots, D_t^{M-1}u\,|_{t=0}\right).$$

If $p > (n + 2b)/2b(M - m)$, $p > 1$, then, according to Theorem 1.1 and Proposition 1.2, $N \in \Phi(A, B)$. Moreover, an analog of Theorem 2.1 holds for N.

THEOREM 4.1. *Let* $p > \max((n + 2b)/2b(M - m), 1)$, *let conditions* A) *and* B) *be satisfied, and let* $\partial\Omega \in C^{2Mb}$. *Then the following assertions are true*:

1) *There exists a domain* $\mathcal{O} \subset B$ *such that* $N: A \to \mathcal{O}$ *is an analytic diffeomorphism.*

2) *If* Ω *is a bounded domain and* $\Theta_j \in \mathcal{O}$, $j = 1, 2, \ldots, \Theta_j \xrightarrow{B} \Theta \in \partial\mathcal{O}$, *then* $\|N^{-1}(\Theta_j); A\| \to \infty$.

We sketch the proof of this theorem. It follows immediately from the definition of the mapping N that $dN(u)v = (K(v), \mathfrak{L}_u(v))$,

$$\mathfrak{L}_u(v) = \sum_{[\mu,\nu]=M} a_{\mu,\nu}\big(\mathfrak{D}_m^{(2b)}u\big)D_t^\mu D_x^\nu v + \sum_{[\mu,\nu]=M}\Big(D_t^\mu D_x^\nu u \nabla_{\mathfrak{D}_m^{(ab)}u} a_{\mu,\nu}\big(\mathfrak{D}_m^{(2b)}u\big) + \nabla_{\mathfrak{D}_m^{(ab)}u} a\big(\mathfrak{D}_m^{(2b)}u\big)\Big)\mathfrak{D}_m^{(2b)}v,$$

where $\nabla_{\mathfrak{D}_m^{(ab)}u} a_{\mu,\nu}(\mathfrak{D}_m^{(2b)}u)$ is the row of all the $N(m, 2b)$ partial derivatives

$$\partial a_{\mu,\nu}(z)/\partial z_j\big|_{z=\mathfrak{D}_m^{(2b)}u}, \qquad j = 1, \ldots, N.$$

According to Theorem 1.2 we have $\mathfrak{D}_m^{(2b)}u \in C(Q_T) \times \mathbf{R}^N$, and since conditions A) and B) are satisfied, it follows from results of Solonnikov [6], [7] (see also [5], Chapter VII, §10) that $dN(u)$ is an isomorphism of A and B.

We verify that N is an imbedding. Assume this is not so and that $N(u_1) = N(u_2)$, $\mathfrak{D}_m^{(2b)}u_j = \mathfrak{U}_j$, $j = 1, 2$, $u_1 - u_2 = v$. Then

$$\big(K(v), \mathfrak{L}_{u_1 u_2}(v)\big) = 0, \tag{4.4}$$

where

$$\mathfrak{L}_{u_1,u_2}(v) = \sum_{[\mu,\nu]\leq M}\Big(a_{\mu,\nu}(\mathfrak{U}_1)D_t^\mu D_x^\nu v + D_t^\mu D_x^\nu u_2\big(a_{\mu,\nu}(\mathfrak{U}_1) - a_{\mu,\nu}(\mathfrak{U}_2)\big)\Big) + a(\mathfrak{U}_1) - a(\mathfrak{U}_2).$$

Since $p(M - m) > 1 + n/2b$, it follows from Hadamard's lemma (Lemma 1.1) that

$$a_{\mu,\nu}(\mathfrak{U}_1) - a_{\mu,\nu}(\mathfrak{U}_2) = \mathfrak{D}_m^{(2b)}v \cdot \mathfrak{A}_{\mu,\nu}(t, x), \quad a(\mathfrak{U}_1) - a(\mathfrak{U}_2) = \mathfrak{D}_m^{(2b)}v \cdot \mathfrak{A}(t, x),$$

where $\mathfrak{A}_{\mu,\nu}$, $\mathfrak{A} \in C(Q_T) \in \mathbf{R}^N$. Therefore, since $D_t^\mu D_x^\nu u_2 \in L_p(Q_T)$ for $[\mu, \nu] \leq M$, it follows that

$$\mathfrak{L}_{\mu_1,\mu_2}(v) = \sum_{[\mu,\nu]=M} a_{\mu,\nu}(U_1)D_t^\mu D_x^\nu v + \sum_{[\mu,\nu]\leq m}\big(\mathfrak{A}_{\mu,\nu}^{(1)}(t, x) + \mathfrak{A}_{\mu,\nu}^{(2)}(t, x)\big)D_t^\mu D_x^\nu v,$$

where $\mathfrak{A}_{\mu,\nu}^{(1)} \in C(Q_T)$, $\mathfrak{A}_{\mu,\nu}^{(2)} \in L_p(Q_T)$ and $a_{\mu,\nu}(\mathfrak{U}_1) \in C(Q_T)$. It follows now from (4.4), conditions A) and B), and the results obtained in [6] and [7] that $v = 0$. Consequently, $u_1 = u_2$; N is an imbedding, and assertion 1) of the theorem holds by virtue of the corollary to Theorem 1.3.

Assertion 2) is proved in the same way as in Theorem 2.1.

COROLLARY. *If under the conditions of Theorem* 4.1, 2) *the a priori estimate*

$$\|u; A\| \leq h(\|(\varphi_1, \ldots, f); B\|)$$

is obtained, where h is a continuous function, then $\mathcal{O} = B$.

REMARK. Results analogous to those obtained above are valid for a boundary value problem for a system of quasilinear equations, $2b$-parabolic in the Solonnikov sense (see [6] and [7]).

We note that problem (4.1)–(4.3) was considered earlier in [3] and [4]. In [3] it was shown that the set of those $(\varphi_1,\ldots,f)\in B$, for which problem (4.1)–(4.3) is solvable forms a domain in B. A proof of the above corollary to Theorem 4.1 is given in [4].

Moscow Recieved 4/FEB/81

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Translated by J. F. HEYDA