10. R. S. Asherova, Yu. F. Smirnov, and V. N. Tolstoi, "A description of a certain class of projection operators for semisimple complex Lie algebras," Mat. Zametki, 26, No. 1, 15-26 (1979).
11. M. Demazure, "Une nouvelle formule des characteres," Bull. Sci. Math., 98, 163-172 (1974).
12. A. Joseph, "On the Demazure characters formula," Preprint, Univ. de P. et M. Curie, Paris (1985).
13. I. V. Cherednik, "Factoring particles on the half line, and root systems," Teor. Mat. Fiz., 61, No. 1, 35-44 (1984).

HAMILTONIAN PERTURBATIONS OF INFINITE-DIMENSIONAL LINEAR SYSTEMS WITH AN IMAGINARY SPECTRUM

S. B. Kuksin

UDC 517.957

We consider the linear equation in a Hilbert space Q

$$\dot{w} = J_a^0 w, \ w \in Q. \tag{0.1}$$

Here a is an n-dimensional parameter, and J_a^0 is an anti-self-adjoint operator with discrete spectrum $\pm i\lambda j(a)$, $\lambda_j \sim Cj^d$. We prove that if $d \geq 1$, then for most values of the parameter a the quasiperiodic solutions of (0.1) with n basic frequencies are preserved under small Hamiltonian perturbations of the form

$$\dot{w} = J_a^0 (w + \varepsilon \nabla H(w)), \ 0 < \varepsilon \ll 1. \tag{0.2}$$

Equations of form (0.2) arise in the description of one-dimensional conservative physical systems [1], in particular, the nonlinear string; the corresponding equation is studied as an example.

The theorem proved in this paper has well-known finite-dimensional analogs. So, if Q = 2N < ∞ and N = n, the results of our work follow from more general theorems of Kolmogorov–Arnol'd–Moser (see, for instance, [2, 3]), and if $N \geq n$, they follow from the works of Mel'nikov [4, 12], Graff [13], and Moser [5].

We remark the connection of our results with the works of Nikolenko [6, 7], where the existence of conditionally periodic solutions for nonlinear perturbations of (0.1) is studied with no Hamiltonian assumptions (but under other fairly hard restrictions).

1. Statement of the Main Results

Let Q be a real Hilbert space with inner product $\langle\langle\cdot,\cdot\rangle\rangle$, and let W_a, $a \in R^n$, be a family of self-adjoint operators in Q such that $W_0 = I$ and $K \geq W_a \geq K^{-1}\ \forall a$ (here and further K, K_0, K_1, ... are positive constants). Let us denote by Q_a the space Q with the inner product $\langle\langle u_1, u_2\rangle\rangle_a = \langle\langle W_a u_1, u_2\rangle\rangle$. Let J_a^0 be an unbounded anti-self-adjoint operator in Q_a such that for some orthonormal basis $\{w_j^{\pm}(a) \mid j \geq -n+1\} \subset Q_a$ we have

$$J_a^0 w_j^{\pm}(a) = \mp\lambda_j(a) w_j^{\mp}(a), \ \lambda_j(a) \geq 0 \ \forall j \geq -n+1. \tag{1}$$

Let us assume that locally the n-dimensional family $\{W_a, J_a^0\}$ may be parametrized by the vector $\omega = (\lambda_{-n+1}, \lambda_{-n+2}, \ldots, \lambda_0)$,

$$\omega \in \Omega_0 = \{\omega \in R^n \mid |\omega - \omega_*| \leq K_0\}, \ 0 < K_0 \leq 1/2, \ |\omega_*| \leq K, \tag{2}$$

and $a(\omega_*) = 0$. Assume that for $\omega \in \Omega_0$ a continuously differentiable function $H_\Delta^0(\cdot;\omega)$: $Q_\omega \to R$ is given, and let $\nabla^\omega H_\Delta^0(\cdot;\omega)$: $Q_\omega \to Q$ be its gradient. Let us consider the family of Hamiltonian equations in Q_ω with Hamiltonians $H_0(u;\omega) = \langle\langle u, u\rangle\rangle_\omega/2 + \varepsilon_0 H_\Delta^0(u;\omega)$, $0 < \varepsilon_0 < 1$:

Institute of Control Problems, Academy of Sciences of the USSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 21, No. 3, pp. 22-37, July-September, 1987. Original article submitted July 2, 1986.

 0016-2663/87/2103-0192$12.50

$$\dot{u} = J^0_\omega \, (u + \varepsilon_0 \nabla^\omega H^0_\Delta \, (u; \omega)). \tag{3}$$

Let us make in (3) the substitution $u = U_\omega(w)$, where U_ω: $Q \to Q_\omega$ is a unitary map, taking $w_j^\pm = w_j^\pm(\omega_*)$ into $w_j^\pm(\omega)$, $j \geq -n + 1$. Then

$$W_\omega^{-1} \nabla_u H^0_{\Delta|} \, (u; \omega) = \nabla_u^\omega H^0_\Delta \, (u; \omega) = U_\omega \nabla_w H^0_\Delta \, (U_\omega \, (w); \omega), \tag{4}$$

where $\nabla = \nabla^{\omega*}$. Therefore

$$\dot{w} = J^1_\omega \nabla \, (\langle\langle w, w \rangle\rangle/2 + \varepsilon_0 H_\Delta \, (w; \omega)), \tag{5}$$

$$J^1_\omega = U^*_\omega J^0_\omega U_\omega, \quad H_\Delta \, (w; \omega) = H^0_\Delta \, (U_\omega \, (w); \omega).$$

By (1),

$$J^1_\omega w_j^\pm = \mp \lambda_j(\omega) w_j^\mp \quad \forall j \geqslant -n + 1; \quad \lambda_{l-n}(\omega) = \omega_l, \quad l = 1, \ldots, n.$$

Let us decompose Q into the direct sum $Q = Q_0 \oplus Z_R$, where $Q_0 \simeq \mathbf{R}^{2n}$ is the linear span of the vectors $\{w_j^\pm \mid j \leq 0\}$, Z_R is the closure of the linear span of the vectors $\{w_j^\pm \mid j \geq 1\}$. Let us denote $\lambda_{j*} = \lambda_j(\omega_*)$, $J(\omega) = J^1(\omega) \mid_{Z_R}$, $J_* = J(\omega_*)$ and let us assume that

$$2 \mid \lambda_{j*} \mid \geqslant \mid \lambda_j (\omega) \mid \geqslant K^{-1} \quad \forall \omega \in \Omega_0 \quad \forall j \geqslant 1. \tag{6}$$

Let $\langle \cdot, \cdot \rangle$ be the inner product in Z_R (induced by the inner product in Q) and let Y_R^s, $s \geq 0$, be the domain of the operator $|J_*|^s$, endowed with the norm $\| y \|_s^2 = \langle | J_* |^s_y, | J_* |^s y \rangle$. In particular, $Y_R^0 = Z_R$, $\| \cdot \|_0$ is the norm in Z_R and if $\mathfrak{h} = ((r, p), y) \in Q_0 \oplus Z_R = Q$, then $| \mathfrak{h} |_Q^2 = | r |^2 + | p |^2 + \| y \|_0^2$. We denote by Y_R^{-s} the space orthogonal to Y_R^s with respect to the inner product $\langle \cdot, \cdot \rangle$. Then

$$\| y \|_s^2 = \Sigma \lambda_{j*}^{2s} \mid y_j^\pm \mid^2 \quad \forall y = \Sigma y_j^\pm w_j^\pm \; (y_j^\pm \in \mathbf{R}), \quad \forall s \in \mathbf{R}. \tag{7}$$

Let

$$Y^s = Y^s_R \underset{R}{\otimes} \mathbf{C}, \quad \mathcal{Y}^s = (Q_0 \underset{R}{\otimes} \mathbf{C}) \times Y^s \; \forall s \in \mathbf{R}; \quad Q_c = Q \underset{R}{\otimes} \mathbf{C}.$$

Let us extend $\langle \cdot, \cdot \rangle$ to bilinear pairings $Y^s \times Y^{-s} \to \mathbf{C}$, $s \in \mathbf{R}$ over $\mathbf{C}$.

For $\varepsilon_0 = 0$ and all ω the space Q_0 is invariant for Eq. (5) and is foliated by the invariant n-dimensional tori

$$T^n(I) = \{\alpha_1^+ w_{1-n}^+ + \alpha_1^- w_{1-n}^- + \ldots + \alpha_n^- w_0^- \mid \alpha_j^{+2} + \alpha_j^{-2} = 2I_j, \; j = 1, \ldots, n\}.$$

On the torus $T^m(I)$ there occurs the quasiperiodic motion $\dot{q}_j = \omega_j$, $\dot{\xi}_j = 0$, $j = 1, \ldots, n$, where

$$q_j = \operatorname{Arg} \, (\alpha_j^+ + i\alpha_j^-), \quad \xi_j = (\alpha_j^{+2} + \alpha_j^{-2})/2 - I_j. \tag{8}$$

Let us consider the family of tori $T^n(I)$, $I \in \mathcal{J}$ ($\mathcal{J}$ is a measurable set in $\mathbf{R}^n$). Let us assume that the set $\bigcup \{T^n(I) \mid I \in \mathcal{J}\} \subset Q$ together with its complex neighborhood of radius K^{-1} is contained in a bounded domain $O \subset Q_C$ and that

$$K_1 \geqslant I_k \geqslant K_1^{-1} \quad \forall k = 1, \ldots, n, \quad \forall I = (I_1, \ldots, I_n) \in \mathcal{J}. \tag{9}$$

For a set $M = \{\mu\} \subset \mathbf{R}^n$, a Banach space B and a map h: $M \to B$ we write

$$| h(\cdot) |_B^{\mu_1} = \sup_{\mu_1 \neq \mu_2} | h(\mu_1) - h(\mu_2) |_B / | \mu_1 - \mu_2 | + \sup_\mu | h(\mu) |_B.$$

THEOREM 1. Assume that conditions (6) and (9) hold and that:

1) the maps H_Δ and ∇H_Δ may be extended to Frechet complex-analytic maps H_Δ: $O \to \mathbf{C}$, ∇H_Δ: $O \to Q_C$, and

$$| H_\Delta \, (w, \cdot) |^\omega \leqslant K, \; | \nabla H_\Delta \, (w, \cdot) |^\omega_{\mathcal{Y}^2} \leqslant K \quad \forall w \in O \; (\omega \in \Omega_0); \tag{10}$$

2)

$$K^{-1} j^d \leqslant \lambda_{j*} \leqslant K j^d, \; | \lambda_j \, (\cdot) - \lambda_{j*} |^\omega \leqslant K \quad \forall j \geqslant 1, \tag{11}$$

$$K^{-1} \mid k^d - j^d \mid \leqslant \mid \lambda_{j*} - \lambda_{k*} \mid \quad \forall j, \, k \geqslant 1, \tag{12}$$

where $d \geq 1$, and if $d = 1$, in addition there exist $K_2 > 0$, $d_1 \in (0, 1]$ and a function ζ: $\mathbf{N} \to \mathbf{R}$ such that

$$|\lambda_j(\cdot) - K_2 j - \zeta(j)|^{\omega} \leqslant K j^{-d_1} \qquad \forall j \geqslant 1,$$
$$|\zeta(k+l) - \zeta(k)| \leqslant K k^{-d_1} l \quad \forall k, l \geqslant 1. \tag{13}$$

Then there exist natural numbers j_1 and M_1 depending on K, K_1, d and n (or on K, K_1, K_2, n and d_1, if d = 1) such that if

3)
$$|s\cdot\omega_* + \gamma_1\lambda_{j_*} + \gamma_2\lambda_{k_*}| \geqslant 3K_3 > 0, \tag{14}$$
$$|s\cdot\omega_* + \gamma_1\lambda_{j_*}| \geqslant 3K_3 \tag{15}$$

for every $s \in Z^n$, $|s| \leq M_1$, $1 \leq j < k \leq j_1$, γ_1, $\gamma_2 = \pm 1$;

4)
$$K_0 \leqslant \min(K_3/(M_1 + 2K),\ K^{-2}/4), \tag{16}$$

then for $\varepsilon_0 \in (0, \varepsilon_*]$, where $\varepsilon_* > 0$ is sufficiently small, there exist a measurable subset $\Theta_{\varepsilon_0} \subset \Theta_0 = \Omega_0 \times \mathcal{J}$ and smooth imbeddings

$$\Sigma = \Sigma_\theta : T^n \to Q, \Sigma(T^n) \subset \mathcal{Y}^1, \quad \theta = (\omega, I) \in \Theta_{\varepsilon_0}, \tag{17}$$

with the following properties:

a)
$$\text{mes}\,\{\omega \in \Omega_0\,|\,(\omega, I) \notin \Theta_{\varepsilon_0}\} \to 0$$

as $\varepsilon_0 \to 0$ uniformly with respect to $I \in \mathcal{J}$;

b)
$$\text{dist}\,(\Sigma_\theta(T^n), T^n(I)) = o(\varepsilon_0^\rho)\ \forall\theta \in \Theta_{\varepsilon_0}, \forall\rho < 1/3;$$

c) for $\omega \in \Omega_0$ and for every I such that $\theta = (\omega, I) \in \Theta_{\varepsilon_0}$, the tori $\Sigma(T^m)$ are invariant with respect to the flow of Eq. (5) and are filled with its quasi-periodic solutions. Under the imbedding Σ the trajectories of (5) are carried on T^m into trajectories of the equation $\dot{q} = \mathfrak{S}(\theta)$, where $\mathfrak{S} : \Theta_{\varepsilon_0} \to \mathbf{R}^n$ is a Lipschitzian map close to the projection $(\omega, I) \mapsto \omega$.

Remark. Assume that the domain $Q_1 \subset Q_C$ contains $\bigcup\{U_\omega(O)\,|\,\omega \in \Omega_0\}$. By Eq. (4), condition 1) of the Theorem is fulfilled (for some large K), if H_Δ^0: $O_1 \to \mathbf{C}$, ∇H_Δ^0: $O_1 \to Q_C$ are Frechet complex-analytic maps; for all $\omega \in O_1$ the estimates (10) hold with replacement of H_Δ by H_Δ^0 and moreover

$$|U_\cdot^{-1}|^{\omega}_{\mathcal{Y}^2, \mathcal{Y}^2} + |W_\cdot^{-1}|^{\omega}_{\mathcal{Y}^2, \mathcal{Y}^2} + |U_\cdot|^{\omega}_{Q, Q} \leqslant K. \tag{18}$$

The proof of the Theorem is carried out by the following scheme. We construct a sequence of domains $O \supset O_0^R \supset O_1^R \supset \ldots$, $\bigcap O_m^R = T^n(I)$ and canonic maps S_m: $O_{m+1}^R \to O_m^R$ taking (5) into an equation of the same form but with a smaller ε_0. After this we set the map Σ equal to the restriction of the composition $S_0 \circ S_1 \circ \ldots$ to $T^n(I)$. The proof is given in Sec. 2. It is based on lemmas from Secs. 3 and 4. In Sec. 5 we give an application of Theorem 1 to the nonlinear string equation.

We use the following notation: C, C_1, ... are positive constants, independent of ε_0 and m (m is the iteration index); C(m), C_1(m), ... are functions of m of the form $C_1 m^{C_2}$; $C_*(m)$, $C_{*1}(m)$, ... are fixed functions of the form C(m); $e(m) = (1^{-2} + \ldots + m^{-2})/\gamma_0$, $\gamma_0 = 2(1^{-2} + 2^{-2} + \ldots)$. The space of continuous linear maps $Y^a \to Y^b$, $a, b \in Z$ is denoted by $\mathcal{L}_{a,b}$, and the norm in $\mathcal{L}_{a,b}$ is denoted by $|\cdot|_{a,b}$; we denote by $\mathcal{L}^s_{0,a}$, $a \geqslant 0$, the symmetric maps with respect to the pairing <•, •>. If P_1 and P_2 are complex Banach spaces with real elements P_{1R}, P_{2R}, if O is a domain in P_1, $O \cap P_{1R} \neq \varnothing$, and $M = \{\mu\} \subset R^N$, then we denote by A_R^μ (O, P_2) the set of maps F: O × M → P_2 that are analytic in the first variable, map $(O \cap P_{1R}) \times M$ into P_2R and are such that $|F(p, \cdot)|^{\mu}_{P_2} \leqslant C = C(F)$ for every $p \in O$. By the real elements of $\mathcal{L}_{a,b}$ we mean the maps taking Y_R^a into Y_R^b.

The references to formulas from a different paragraph are made as follows: (2.3) means Sec. 2, formula (3).

2. Proof of Theorem 1

Let us write the Hamiltonian of Eq. (1.5) in the variables q, ξ, y [see (1.8)]:

$$H_0(q, \xi, y; I, \omega) = \|w\|_0^2/2 + H_\Delta(w; \omega),$$
$$w = w(q, \xi, y; I) = \sum_{j=1}^{n} (2(\xi_j + I_j))^{1/2}(\cos q_j w^+_{j-n} + \sin q_j w^-_{j-n}) + y. \tag{1}$$

Let us set

$$U(\delta)=\{q\in \mathbf{C}^n/(2\pi \mathbf{Z}^n) \mid |\operatorname{Im} q|<\delta\},$$
$$O(\varepsilon,\delta)=U(\delta)\times\{\xi\in \mathbf{C}^n \mid |\xi|<\varepsilon^{2/3}\}\times\{y\in Y^0 \mid \|y\|_0<\varepsilon^{1/3}\}. \tag{2}$$

By (1.9) for sufficiently small ε_0 and δ_0 the function (1) maps analytically $O(\varepsilon_0, \delta_0)$ onto a subdomain of O and has a Lipschitzian dependence on $I\in \mathcal{J}$.

Let $0 < \rho < 1/3$. Let us define sequences $\{\varepsilon_m\}$ and $\{\delta_m\}$ and numbers δ_m^j, $1 \le j \le 4$:

$$\varepsilon_m=\varepsilon_0^{(1+\rho)^m}.\quad \delta_m=\delta_0(1-e(m)),\quad m\geqslant 1;\quad \delta_m^j=\frac{5-j}{5}\delta_m+\frac{j}{5}\delta_{m+1}. \tag{3}$$

Let us denote

$$U_m=U(\delta_m),\quad O_m=O(\varepsilon_m,\delta_m),\quad U_m^j=U(\delta_m^j),\quad O_m^j=O(2^{-j}\varepsilon_m,\delta_m^j). \tag{4}$$

Let us assume that $\Theta_m\subset\Theta_0$ is a Lebesgue measurable set such that for every $I\in\mathcal{J}$

$$\operatorname{mes}\Theta_m[I]\geqslant K_4(1-\gamma_* e(m)),\ \gamma_*\in(0,1],\ K_4=\operatorname{mes}\Omega_0, \tag{5}$$
$$\Theta_m[I]=\{\omega\in\Omega_0 \mid (\omega,I)\in\Theta_m\}. \tag{5'}$$

Let us consider in the domain O_m the Hamiltonian

$$H_m=H_{0m}(\xi,y;\theta)+\varepsilon_m H_{\Delta m}(q,\xi,y;\theta),$$
$$H_{0m}=\xi\cdot(1_n+\mathfrak{S}_m(\theta))+\langle A_m(\theta)y,y\rangle/2, \tag{6}$$

where $1_n=(1,\ldots,1)\in\mathbf{R}^n$, $A_m=I+A_{1m}(\theta)$ $\mathfrak{S}_m:\Theta_m\to\mathbf{R}^n$, $|\mathfrak{S}_m(\cdot)|^\theta\leqslant\varepsilon_0^\rho e(m)$,

$$A_{1m}w_j^\pm=\beta_{jm}(\theta)w_j^\pm,\quad |\beta_{jm}(\cdot)|^\theta\leqslant\varepsilon_0^\rho e(m)\lambda_{j*}^{-2}\quad \forall j\geqslant 1. \tag{7}$$

Assume that $(H_{\Delta m},\nabla_y H_{\Delta m})\in A_R^\theta(O_m,\mathbf{C}\times Y^2)$ $(\theta\in\Theta_m)$, and that for every $\mathfrak{h}\in O_m$

$$|H_{\Delta m}(\mathfrak{h};\cdot)|^\theta+\varepsilon_m^{1/3}\|\nabla_y H_{\Delta m}(\mathfrak{h};\cdot)\|_2^\theta\leqslant C_*(m)\equiv K_5^{m+1}. \tag{8}$$

For m = 0 and sufficiently small ε_0 and δ_0 the Hamiltonian H_0 has the form (6) with $\mathfrak{S}_0\equiv 0$ and $A_{10}\equiv 0$. Condition (8) holds by condition 1) of Theorem 1 and the analyticity of the map (1).

Let us denote

$$Q_\pi^j=T_q^n\times\mathbf{R}_\xi^n\times Y_R^j,\quad j\in\mathbf{Z};\quad O_m^R=O_m\cap Q_\pi^0. \tag{9}$$

We identify the tangent space to Q_π^0 with Q and introduce in Q_π^0 the metric of Q. To the Hamiltonian H_m there corresponds the system of equations in O_m^R (we omit the parameter θ)

$$\dot{q}_j=\omega_j(1+\mathfrak{S}_{mj}+\varepsilon_m\partial/\partial\xi_j H_{\Delta m}),\quad \dot{\xi}_j=-\omega_j\varepsilon_m\partial/\partial q_j H_{\Delta m}, \tag{10}$$
$$\dot{y}=J(A_m y+\varepsilon_m\nabla_y H_{\Delta m}). \tag{11}$$

This system is a Lipschitzian perturbation of the equation $(\dot{p}, \xi, \dot{y}) = (0, 0, Jy)$, whose operator has domain Q_π^1 and defines a group of isometric transformations of the space Q_π^0. Hence for $\mathfrak{h}(0)\in O_m^2\cap Q_\pi^1$ and sufficiently small T Eqs. (10) and (11) have a unique solution:

$$\mathfrak{h}(t)=(q(t),\xi(t),y(t))\in O_m^1\cap Q_\pi^1,\quad 0\leqslant t\leqslant T;\quad \dot{\mathfrak{h}}\in L_\infty(0,T;Q) \tag{12}$$

(see [14] and [8, pp. 105, 106]). Taking the inner product in Z_R of Eq. (11) by y(t) and using estimate (8), we obtain

$$d/dt\,\|y(t)\|_0\leqslant C(m)\varepsilon^{2/3}. \tag{13}$$

Let us assume that $\|y(0)\|_0<\varepsilon_m^{1/2}/2$ and $TC(m)\varepsilon_m^{1/6}<1/2$. Then $\|y(t)\|_0<\varepsilon_m^{1/2}$ and by (8) we obtain from Eqs. (10):

$$|d/dt\,\xi(t)|+\varepsilon_m^{2/3}|d/dt(q(t)-q(0)-\omega'' t)|\leqslant C(m)\varepsilon_m,\quad \omega_j''=\omega_j(1+\mathfrak{S}_{mj}). \tag{14}$$

From estimates (13) and (14) we have the following assertion.

LEMMA 1. If $\varepsilon_0 \ll 1$ (i.e., if $\varepsilon_0 > 0$ is sufficiently small), $\mathfrak{h}(0)\in Q_\pi^1$ and $|\xi(0)|<\varepsilon_m^{2/3}/3$, $\|y(0)\|_0<\varepsilon_m^{1/2}/2$, then for T = 1 there exists a unique solution of the system of Eqs. (10), (11) of the form (12). It satisfies the following estimates:

$$|q(t)-q(0)-\omega''t|<C(m)\varepsilon_m^{1/3},\ |\xi(t)|<\varepsilon_m^{2/3}/2,\ \|y(t)\|_0<\varepsilon_m^{1/2}. \tag{15}$$

Let us single out from $H_{\Delta m}$ the linear terms in ξ, as well as the linear and quadratic terms in y:

$$H_{\Delta m}=g(q;\theta)+\xi\cdot h_1(q;\theta)+\langle y,\mathfrak{z}(q;\theta)\rangle+\langle B_1(q;\theta)y,y\rangle+H_{3m}(q,\xi,y;\theta);$$
$$B_1\in A_R^\theta(O_m,\mathcal{L}_{0,0}^s).$$

Changing if necessary $H_{\Delta m}$ by a constant we may assume that $\bar{g}(\theta)=0$ (the bar above the symbol means averaging with respect to $q\in T^n$).

Let us define a linear operator B_0: $Y^0\to Y^0$:

$$B_0(\theta)w_j^{\pm}=\bar{b}_j(\theta)w_j^{\pm}/2,\quad b_j(q,\theta)=\langle B_1w_j^+,w_j^+\rangle+\langle B_1w_j^-,w_j^-\rangle\quad\forall j. \tag{16}$$

We set

$$B=B_1-B_0,\quad h=h_1-\bar{h}_1,\quad \mathfrak{S}_{m+1}=\mathfrak{S}_m+\varepsilon_m\bar{h}_1,\quad A_{m+1}=A_m+2\varepsilon_mB_0 \tag{17}$$

and rewrite H_m as follows:

$$H_m=H_{0\,m+1}(\xi,y;\theta)+\varepsilon_mH_{1m}(q,\xi,y;\theta);\quad H_{1m}=H_{2m}+H_{3m},$$
$$H_{2m}=g(q;\theta)+\xi\cdot h(q;\theta)+\langle y,\mathfrak{z}(q;\theta)\rangle+\langle B(q;\theta)y,y\rangle \tag{18}$$

[the functional H_{0m+1} is the same as in (6)].

LEMMA 2. If $\varepsilon_0\ll 1$, then: a) for $q\in U_m$

$$|g(q;\cdot)|^\theta\leqslant C_*(m),\ \|\mathfrak{z}(q;\cdot)\|_2^\theta\leqslant C_*(m)\varepsilon_m^{-1/3},$$
$$|h(q;\cdot)|^\theta\leqslant 2C_*(m)\varepsilon_m^{-2/3},\ |\bar{h}_1(\cdot)|^\theta\leqslant C_*(m)\varepsilon_m^{-2/3},$$
$$|B(q;\cdot)|_{0,2}^\theta\leqslant 2C_*(m)\varepsilon_m^{-2/3},\ |\bar{b}_j(\cdot)|^\theta\leqslant\lambda_{j*}^{-2}\varepsilon_m^{-2/3}C(m)\ \forall j;$$

b) if $\mathfrak{h}\in O_{m+1}$ and K_5 in (8) is sufficiently large, then

$$|H_{3m}(\mathfrak{h};\cdot)|^\theta+\varepsilon_{m+1}^{1/3}\|\nabla_yH_{3m}(\mathfrak{h};\cdot)\|_2^\theta\leqslant C_*(m+1)\varepsilon_m^\rho/2; \tag{19}$$

c) for the map $A_{1m+1}=A_{1m}+2\varepsilon_mB_0$ conditions (7) hold with the replacement of m by m + 1;

d)
$$|\mathfrak{S}_{m+1}(\cdot)|^\theta\leqslant\varepsilon_0^\rho e(m+1),\quad\theta\in\Theta_m. \tag{20}$$

Proof. The assertions of part a) follow from inequalities (8) and Cauchy estimates applied to the following functions of the argument $t\in\mathbf{C}$, $|t|\leq 1$: $H_{\Delta m}(q,t\eta;\theta)$, $\nabla_yH_{\Delta m}(q,t\eta;\theta)$, where $(q,\eta)\in O_m$ and $\eta=(\xi,0)$ or $\eta=(0,y)$.

Let $\mathfrak{h}=(q,\xi,y)\in O_{m+1}$ and $\nu=\varepsilon_m^{\rho/3}$. Then $(q,(t/\nu)^2\xi,(t/\nu)y)\in O_m$ for $|t|\leq 1$. Let us consider the function

$$H_{\Delta m}(q,(t/\nu)^2\xi,(t/\nu)y;\theta)=f_0+f_1t+f_2t^2+\ldots.$$

By (8) we have $f_k<C_*(m)$ for all k. Since $H_{3m}(\mathfrak{h};\theta)=f_3\nu^3+f_4\nu^4+\ldots$, then $|H_{3m}(\mathfrak{h};\theta)|\leqslant C_*(m)\varepsilon_m^\rho/(1-\nu)\leqslant C_*(m+1)\varepsilon_m^\rho/8$. We estimate similarly the Lipschitz constant for the function H_{3m}. The estimate for ∇H_{3m} is obtained analogously. Assertions c) and d) of the lemma follow from a). ■

Let us define the auxiliary Hamiltonian $\varepsilon_m\Xi$:

$$\Xi=\xi\cdot\chi(q;\theta)+\varkappa(q;\theta)+\langle y,\mathfrak{Y}(q;\theta)\rangle+\langle G(q;\theta)y,y\rangle. \tag{21}$$

To (21) there corresponds a canonical transformation S_m, which is a time one shift along the trajectories of the system of equations

$$\dot{q}_j=\varepsilon_mF_j^q(q;\theta),\quad\dot{\xi}_j=\varepsilon_mF_j^\xi(q,\xi,y;\theta),\quad j=1,\ldots,n; \tag{22}$$
$$\dot{y}=\varepsilon_mF^y(q,\xi,y;\theta); \tag{23}$$
$$F_j^q=\omega_j\chi_j,$$
$$F_j^\xi=-\omega_j(\xi\cdot\partial\chi/\partial q_j+\partial\varkappa/\partial q_j+\langle y,\partial\mathfrak{Y}/\partial q_j\rangle+\langle(\partial G/\partial q_j)y,y\rangle),$$
$$F^y=J(\mathfrak{Y}(q)+2G(q)y).$$

Let us write S_m as

$$q \mapsto q + \varepsilon_m q^1, \quad \xi \mapsto \xi + \varepsilon_m \xi^1, \quad y \mapsto y + \varepsilon_m y^1. \tag{24}$$

Then $q^1 = F^q + \varepsilon_m \ldots$, $\xi^1 = F + \xi_m \ldots$, $y^1 = F^y + \varepsilon_m \ldots$. Omitting the parameter θ and denoting $(q, \xi, y) = \mathfrak{h}$, $(q^1, \xi^1, y^1) = \mathfrak{h}^1$, we write the transformed Hamiltonian as follows:

$$\begin{aligned} H_m(S_m(\mathfrak{h}; 0); \theta) = H_{0\,m+1}(\xi, y) + \varepsilon_m \xi^1 \cdot (1_n + \mathfrak{S}_{m+1}) + \\ + \varepsilon_m \langle A_{m+1} y, y^1 \rangle + \varepsilon_m^2 \langle A_{m+1} y^1, y^1 \rangle / 2 + \varepsilon_m (g(q) + \xi \cdot h(q) + \\ + \langle y, \mathfrak{z}(q) \rangle + \langle B(q) y, y \rangle) + \varepsilon_m (H_{3m}(\mathfrak{h}) + H_{1m}(\mathfrak{h} + \varepsilon \mathfrak{h}^1) - H_{1m}(\mathfrak{h})). \end{aligned}$$

We set $\omega_j' = \omega_j (1 + \mathfrak{S}_{m+1\,j}(\omega, I))$ and denote $\omega' \cdot \nabla_q = \partial/\partial\omega'$. Then the transformed Hamiltonian equals

$$\begin{aligned} H_{0\,m+1}(\xi, y) + \varepsilon_m [-\xi \cdot \partial\chi/\partial\omega' - \partial\varkappa/\partial\omega' - \langle y, \partial\mathfrak{Y}/\partial\omega' \rangle - \langle \partial G/\partial\omega' y, y \rangle + \\ + \langle A_{m+1} y, J\mathfrak{Y} \rangle + 2 \langle A_{m+1} y, JGy \rangle + g(q) + \xi \cdot h(q) + \langle y, \mathfrak{z}(q) \rangle + \langle B(q) y, y \rangle] + \varepsilon_m^{1+\rho} \ldots . \end{aligned}$$

Equating to zero the contents of the square brackets, we obtain the following homological equations for $\varkappa$, χ, G, and $\mathfrak{Y}$:

$$\partial\varkappa/\partial\omega' = g(q; \theta), \quad \partial\chi/\partial\omega' = h(q; \theta); \tag{25}$$

$$\partial G/\partial\omega' + GJ(\omega) A_{m+1}(\theta) - A_{m+1}(\theta) J(\omega) G = B(q; \theta) + \varepsilon_m \Delta B(q; \theta); \tag{26}$$

$$\partial\mathfrak{Y}/\partial\omega' - A_{m+1}(\theta) J(\omega) \mathfrak{Y} = \mathfrak{z}(q; \theta) + \varepsilon_m \Delta\mathfrak{z}(q; \theta), \tag{27}$$

where we denoted by $\varepsilon_m \Delta B$ and $\varepsilon_m \Delta\xi$ admissible small discrepancies.

LEMMA 3. If $\varepsilon_0 \ll 1$, then in Θ_m there exists a measurable subset Θ_{m+1} such that mes $(\Theta_m \setminus \Theta_{m+1})[I] \leq \gamma K_4 (m+1)^{-2}/\gamma_0$ for every $I \in \mathcal{I}$ [see (5')]. For $\theta \in \Theta_{m+1}$:

a) equation (26) has a solution $\varkappa \in A_R^\theta(U_m, \mathbf{C})$, $\chi \in A_R^\theta(U_m^1, \mathbf{C}^n)$; everywhere in U_m^1 the following estimate is valid:

$$|\varkappa(q; \cdot)|^\theta + \varepsilon_m^{2/3} |\chi(q; \cdot)|^\theta \leqslant C(m); \tag{28}$$

b) equation (26) has a solution $G, \Delta B \in A_R^\theta(U_m^1, \mathcal{L}_{0,2}^s)$. For some c > 0 independent of ε_0, δ_0 and m, for every $q \in U_m^1$ the following estimates hold:

$$|\Delta B(q; \cdot)|_{0,2}^\theta \leqslant C(m) \varepsilon_m^{-2/3}, \tag{29}$$

$$|G(q; \cdot)|_{0,2}^\theta \leqslant C(m) \varepsilon_m^{-2/3} \ln^c \varepsilon_m^{-1}, \tag{30}$$

$$|(A_{m+1} JG - GJA_{m+1})(q; \cdot)|_{0,2}^\theta \leqslant C(m) \varepsilon_m^{-2/3} \ln^c \varepsilon_m^{-1}; \tag{31}$$

c) equation (27) has a solution $\mathfrak{Y}$, $\Delta\mathfrak{z} \in A_R^\theta(U_m^1, Y^2)$. For $q \in U_m^1$ the following estimates hold:

$$\|\Delta\mathfrak{z}(q; \cdot)\|_2^\theta \leqslant C(m) \varepsilon_m^{-1/3}, \tag{32}$$

$$\|\mathfrak{Y}(q; \cdot)\|_3^\theta \leqslant C(m) \varepsilon_m^{-1/3} \ln^2 \varepsilon_m^{-1}. \tag{33}$$

Lemma 3 is proven below in Sec. 3. Assume, as before, that S_m is the time one shift operator along the trajectories of Eqs. (22) and (23).

LEMMA 4. If $\theta \in \Theta_{m+1}$ and $\varepsilon_0 \ll 1$, then:

a) $S_m \in A_R^\theta(O_{m+1}, Q_\pi^0)$ and the restriction of S_m to O_{m+1}^R [see (9)] is the canonical map S_m: $O_{m+1}^R \to O_m^R$ satisfying

$$|d^r(S_m - I)|_{Q \times \ldots \times Q, Q}^\theta \leqslant C_r(m) \varepsilon_m^{1/3} \ln^c \varepsilon_m^{-1}, \quad r \geqslant 0, \tag{34}$$

$$|q^1| + \varepsilon_m^{-2/3} |\xi^1| + \varepsilon_m^{-1/3} \|y^1\|_1 \leqslant C(m) \varepsilon_m^{-2/3} \ln^c \varepsilon_m^{-1}; \tag{35}$$

b) under the map S_m the equation on O_m^R with Hamiltonian H_m is carried into an equation on O_{m+1}^R with a Hamiltonian H_{m+1} satisfying conditions (5)-(8) with the replacement of m by m + 1.

Lemma 4 is proved in Sec. 4.

Let us set $\Theta_{\varepsilon_0} = \bigcap \Theta_m$. Then Θ_{ε_0} is a measurable subset of Θ_0 and

$$\operatorname{mes} \Theta_{\varepsilon_0} [I] \geqslant (1 - \gamma_*/2) \operatorname{mes} \Omega_0 \quad \forall I \in \mathcal{Y}. \tag{36}$$

For $\theta \in \Theta_{\varepsilon_0}$ and $m \geq 0$, $N \geq 0$ we consider the maps

$$\Sigma^m_{m+N+1}(\theta) = S_m(\theta) \circ \ldots \circ S_{m+N}(\theta) : O^R_{m+N+1} \to O^R_m.$$

LEMMA 5. If $\varepsilon_0 \ll 1$, then as $N \to \infty$ the maps

$$\Sigma^m_{m+N+1}(\theta) : T^n_0 = T^n \times \{0\} \times \{0\} \to O^R_m$$

converge to a smooth limit map $\Sigma^m_\infty(\theta) : T^n_0 \to O^R_m$ such that:

a) $\Sigma^m_\infty(\theta)(T^n_0) \subset Q^1_\pi$;

b) for every $0 \leq m \leq m'$

$$\Sigma^m_{m'} \circ \Sigma^{m'}_\infty = \Sigma^m_\infty;$$

c) the norm in $Q_\pi{}^0$ of the difference $\Sigma_\infty{}^m(\theta) - I$ and its Lipschitz constant do not exceed $\varepsilon_m{}^\rho$. The same is true for $(\Sigma_\infty{}^m(\theta))^{-1} - I$.

The lemma follows from estimates (34) and (35).

By the recurrent formula for $\mathfrak{S}_{m+1}$ [see (17)] as $m \to \infty$, the restrictions of the maps $\mathfrak{S}_m$ to Θ_{ε_0} converge to the Lipschitzian map

$$\mathfrak{S}_{\varepsilon_0} : \Theta_{\varepsilon_0} \to \mathbf{R}^n, \quad |\mathfrak{S}_{\varepsilon_0}(\cdot) - I|^\theta \leqslant \varepsilon_0^\rho. \tag{37}$$

We fix $\theta_0 \in \Theta_{\varepsilon_0}$, and denote $\omega_m = \mathfrak{S}_m(\theta_0)$, $\omega = \mathfrak{S}_{\varepsilon_0}(\theta_0)$. By (17), we have $|\omega_m - \omega_{m+j}| \leqslant C(m)\varepsilon_m^{1/3}$, $j \geq 1$. Let us consider on the torus $T_0{}^n$ the curve $t \mapsto \mathfrak{h}_*(t) = (q + t\omega, 0, 0)$, $0 \leqslant t \leqslant 1$. Under the map $\Sigma_\infty{}^m(\theta_0)$ it is carried into the curve $\mathfrak{h}_m(t) = (q_m(t), \xi_m(t), y_m(t))$, and by part c) of Lemma 5 and (35) we have

$$|q_m(t) - q_m(0) - t\omega| \leqslant 3\varepsilon_m^\rho, \tag{38}$$

$$|\xi_m(t)| + \varepsilon_m^{1/3} \|y_m(t)\|_1 \leqslant C(m)\,\varepsilon_m \ln^c \varepsilon_m^{-1}. \tag{39}$$

Hence if $\varepsilon_0 \ll 1$, then by Lemma 1 the system of Eqs. (10) and (11) has a unique solution $\mathfrak{h}(t)$, $0 \leq t \leq 1$, with the initial condition $\mathfrak{h}_m(0)$. From (15), (38), and (39) it follows that $|\mathfrak{h}_m(t) - \mathfrak{h}(t)|_Q \leqslant C\varepsilon_m^\rho$, $0 \leqslant t \leqslant 1$. Carrying out the transformation $\Sigma_m{}^0$, we find that the distance from the curve $\mathfrak{h}_0(t)$, $0 \leq t \leq 1$ to the solution of the initial system with initial condition $\Sigma_\infty{}^0(q, 0, 0)$ is no larger than $C_1\varepsilon_m{}^\rho$. Letting m tend to ∞, we obtain that $\mathfrak{h}_0(t) = \Sigma^0_\infty(\mathfrak{h}_*(t))$ is a solution of the original Hamiltonian system of equations. Assertions b) and c) of the theorem are proved by setting $\Sigma_\theta(q) = \Sigma_\infty{}^0(\theta)(q, 0, 0)$.

In order to prove assertion a), we set in (5) $\gamma_* = \gamma_*(M) \searrow 0$, where M is a natural parameter tending to infinity. Assertions b) and c) of the theorem are valid for $\varepsilon_* = \varepsilon_*(M) > 0$, and we may assume that $\varepsilon_*(M) \searrow 0$. Then by (36) for $\varepsilon_* \in (\varepsilon_*(M+1), \varepsilon_*(M)]$ $\operatorname{mes}(\Omega_0 \setminus \Theta_{\varepsilon_0}[I]) \leqslant \gamma_*(M)/2 \searrow 0$. The theorem is proved.

3. Proof of Lemma 3

Everywhere in Secs. 3 and 4 we write ε, δ instead of ε_m, δ_m and sometimes we omit the argument θ for functions and maps. In the deduction of estimates, we use systematically the condition $\varepsilon_0 \ll 1$.

The assertions of the lemma will be proven for $\Theta_{m+1} = \Theta_m \setminus (\Theta^1 \cup \ldots \cup \Theta^4)$, where Θ^r are measurable sets, and for r = 1, ..., 4

$$\operatorname{mes} \Theta^r(I) \leqslant \gamma_* K_4 (m+1)^{-2}/(4\gamma_0) \quad \forall I \in \mathcal{Y}. \tag{1}$$

By Lemma 2 the map

$$\omega \mapsto \omega', \ \omega'_j = \omega_j (1 + \mathfrak{S}_{m+1\, j}(\omega, I)) \tag{2}$$

is a Lipschitzian homeomorphism for all I, changing the volume by a factor no greater than two. Therefore, if

$$\Theta^1 = \{\theta \in \Theta_m \mid |k - \omega'| \leqslant (m+1)^{-2} |k|^{-n-1} C^{-1} \quad \forall k \in \mathbf{Z}^n, \quad k \neq 0\}, \tag{3}$$

then Θ^1 is a measurable set satisfying condition (1) (see [2, Sec. 4.1]). For $\theta \in \Theta_m \setminus \Theta^1$, $q \in U_m^1$, the solutions of Eqs. (2.25) are given by convergent trigonometric series and satisfy the estimate (2.28) (see [2, Sec. 4.2]).

We turn to Eq. (2.26). We set $W_j^{\pm} = (w_j^+ \pm i w_j^-)/\sqrt{2}$. Then $JA_{m+1}W_j^{\pm} = \pm i\lambda_j(\omega)(1 + \beta_{j\, m+1}(\theta))W_j^{\pm}$, $\| W_j^{\pm} \|_0 = 1$.

Since the operator $B \in A_R^{\theta}(O_m, \mathcal{L}_{0,2}^s)$ satisfies the estimates of Lemma 2, then

$$B(q) = \sum e^{iq\cdot s}\widehat{B}(s), \quad |\widehat{B}(s)|_{0,2}^{\theta} \leqslant C(m)\varepsilon^{-2/3}e^{-\delta|s|} \tag{4}$$

(see [2, Sec. 4.2]). Therefore, if $M = C(m) \ln \varepsilon^{-1}$ and

$$\varepsilon\Delta B(q) = \sum_{|s| \geqslant M} e^{iq\cdot s}\widehat{B}(s),$$

then for $q \in U_m^1$, ΔB satisfies (2.29). Let us set $B_{k\,j}^{\gamma_1\gamma_2}(q) = \langle BW_j^{\gamma_2}, W_k^{\gamma_1}\rangle$, $k, j = 1, 2, \ldots$; $\gamma_1, \gamma_2 = \pm 1$ (if $\gamma = \pm 1$ figures as an upper index, then instead of γ we take its sign). We define $\widehat{B}_{k\,j}^{\gamma_1\gamma_2}(s)$ and $G_{k\,j}^{\gamma_1\gamma_2}(q)$ analogously. Be definition of the operator B_0 [see (2.16)], we have $\widehat{B}_{k\,k}^{\gamma-\gamma}(0) \equiv 0$.

We look for $G_{k\,j}^{\gamma_1\gamma_2}$ in the form

$$G_{k\,j}^{\gamma_1\gamma_2}(q) = \sum_{|s| \leqslant M} e^{iq\cdot s}\widehat{G}_{k\,j}^{\gamma_1\gamma_2}(s), \quad \widehat{G}_{k\,k}^{\gamma-\gamma}(0) = 0. \tag{5}$$

Then (2.26) is equivalent to:

$$\widehat{G}_{k\,j}^{\gamma_1\gamma_2}(s) = \widehat{B}_{k\,j}^{\gamma_1\gamma_2}(s)/D(k, j), \quad D(k, j) = i(\omega'\cdot s + \gamma_1\lambda_k' + \gamma_2\lambda_j'), \tag{6}$$

where $\lambda_p' = \lambda_p(\omega)(1 + \beta_{p\,m+1}(\theta))$ and $\gamma_1 = \gamma_2$, if $k = j$ and $s = 0$.

LEMMA 6. There exist measurable sets $\Theta^2, \Theta^3 \subset \Theta_m$ satisfying (1) and a constant $c > 0$ such that for $\theta \in \Theta_m \setminus (\Theta^2 \cup \Theta^3)$ and for every k, j, s ($|s| \leq M$), γ_1, γ_2

$$|D^{-1}(k, j; \cdot)|^{\theta} \leqslant C(m)(1 + |s|)^{2n+3}(\ln^c \varepsilon^{-1})/(1 + |\lambda_{k*} - \lambda_{j*}|). \tag{7}$$

Proof. If $s = 0$ then $\gamma_1 = \gamma_2$ for $k = j$, and by conditions (1.11) and (1.12)

$$|D| \geqslant |\lambda_{k*} - \lambda_{j*}|/2 + K^{-1}/2 - 2KK_0 - |\lambda_k\beta_{k\,m+1}| - |\lambda_j\beta_{j\,m+1}|.$$

Consequently, if $s = 0$ then from conditions (1.16) and (2.7) for $\varepsilon_0 \ll 1$, the inequality $|D| \geq (1 + |\lambda_{k*} - \lambda_{j*}|)/C$ follows, which implies (7). If $2|\omega'\cdot s| \leq |\gamma_1\lambda_k' + \gamma_2\lambda_j'|$), then for small ε_0 inequality (7) clearly holds. Hence we assume below that $s \neq 0$ and

$$2|\omega'\cdot s| \geqslant |\gamma_1\lambda_k' + \gamma_2\lambda_j'|. \tag{8}$$

If $d > 1$, then from (8) it follows that either $k = j$ or $k, j \leq 1 + (CM)^{1/(d-1)}$, which simplifies further reasonings. Therefore we restrict ourselves to the consideration of the more complicated case $d = 1$. From (1.6), (1.11) and (1.13) we obtain that for every $l \geq 1$

$$|\lambda_l' - \lambda_{l*}| \leqslant Kl^{-d_1}(K_0 + \varepsilon_0^{\rho}), \quad |\lambda_l'(\cdot) - \lambda_{l*}|^{\theta} \leqslant (1 + K)l^{-d_1}. \tag{9}$$

From (8) and (9) it follows that

$$|\lambda_{k*} - \lambda_{j*}| \leqslant CM. \tag{10}$$

To fix the ideas, assume that $k \leq j$. Depending on the relation between k and s, we consider three cases.

1) Let $|s| < 4(K + 1)k^{-d_1} + 1/2$. Then $k \leq (8(K + 1))^{1/d_1}$, $|s| \leq 4K + 5$. From (8) it follows that $|\lambda_j'| \leq |\lambda_k'| + 2|\omega'\cdot s|$. Hence

$$|\lambda_{j*}| \leqslant K(8(K+1))^{1/d_1} + 2(1+K) + 2(4K+5)(|\omega_*| + 1).$$

Let us choose j_1 in the assumptions of Theorem 1 such that from (1.11) and this inequality it follows that $j \leq j_1$. Then by (9), (1.14) and (1.16),

$$|D| \geqslant |\omega_* \cdot s + \gamma_1\lambda_{k*} + \gamma_2\lambda_{j*}| - (|s| + 2K)(K_0 + \varepsilon_0^0) \geqslant K_3,$$

from here and from (10), estimate (7) follows.

2) Let $|s| \geq 4\ (K + 1)k^{-d_1} + 1/2$, $k \leq C_{*1}(m)\ \ln^{n'} \varepsilon^{-1}$, $n' = (n + 1)/d_1$ (we choose below the function C_{*1}). By (10) there are no more than $CC_{*1}(m)\ \ln^{1+n'} \varepsilon^{-1}$ such pairs k, j, and by (9)

$$|s| \geqslant 2\,|\gamma_1\lambda_k'(\cdot) + \gamma_2\lambda_j'(\cdot) - \gamma_1\lambda_{k*} - \gamma_2\lambda_{j*}|^{\theta} + 1/2. \tag{11}$$

Let us set $T = (C_{*2}(m)\,|s|^n \ln^{1+n'} \varepsilon^{-1})^{-1}$ and $\Theta' = \Theta'(k, j, s, \gamma_1, \gamma_2) = \{\theta \in \Theta_m \mid |D| \leqslant T\}$, $\Omega'[I] = \Omega'(k, \ldots, \gamma_2)[I] = \{\omega'(\omega, I) \mid \omega \in \Theta'(k, \ldots, \gamma_2)[I]\}$. Let $\Theta^2 = \bigcup\Theta'$, where the union is taken over all k, j, s, γ_1, γ_2 such that $1/2 + 4(K+1)k^{-d_1} \leqslant |s| \leqslant M$, $k \leqslant j$, $k \leqslant C_{*1}(m) \ln^{n'} \varepsilon^{-1}$, $|\lambda_{k*} - \lambda_{j*}| \leqslant CM$.

The map (2) is invertible, and for all I the Lipschitz constant of the map $\omega' \mapsto \omega$ does not exceed 4/3. The set of values $t \in \mathbf{R}$, such that $\omega'(t) \equiv \omega_0 + ts/|s| \in \Omega'[I]$, is contained in the set

$$\{t \mid |t|s| + (\gamma_1\lambda_k' + \gamma_2\lambda_j')(\omega'(t)) + \omega_0' \cdot s| \leqslant T\}. \tag{12}$$

By (11) the Lipschitz constant with respect to t of the function $(\gamma_1\lambda_k' + \gamma_2\lambda_j')(\omega'(t))$ does not exceed $(2|s| - 1)/3$. Consequently, the set (12) does not contain points t_1, t_2 such that $|t_1 - t_2| \geq 6T/(|s| + 1)$. Since the vector ω_0' may be chosen arbitrarily, we have $\Omega'[I] \leq CT/(|s| + 1)$. Therefore, under a suitable choice of the function $C_{*2}(m)$ we have $\operatorname{mes} \bigcup \Omega'[I] \leqslant \gamma_* K_4 (m+1)^{-2}/(8\gamma_0)$. Since the map (2) changes the measure of the sets by a factor no greater than two, the set Θ^2 satisfies (1). If $\theta \in \Theta_m \setminus \Theta^2$, then $|D| \geqslant T$, and from here and (10), estimate (7) follows.

3) Let $k \geq C_{*1}(m)\ \ln^{n'} \varepsilon^{-1}$. Then by (8) we have $\gamma_1 = -\gamma_2$. From (1.13), (9), and (10) it follows that

$$|\lambda_k' - \lambda_j' - K_2(k-j)| \leqslant 2Kk^{-d_1} + |\zeta(k) - \zeta(j)| \leqslant 2Kk^{-d_1} + Kk^{-d_1}C_1M \leqslant CC_{*1}^{-d_1} \ln^{-n}\varepsilon^{-1}. \tag{13}$$

Let us set

$$\Theta^3 = \{\theta \in \Theta_m \mid \omega'(\theta) \in \bigcup \Omega''(s, N)\ s \in \mathbf{Z}^n,\ 1 \leqslant |s| \leqslant M;\ N \in \mathbf{Z},\ |N| \leqslant CM\},$$

$$\Omega''(s, N) = \{\omega' \mid |\omega' - \omega_*| \leqslant 1,\ |\omega' \cdot s - NK_2| \leqslant C_{*3}^{-1}(m) \ln^{-n}\varepsilon^{-1}\}.$$

Since $\operatorname{mes} \Omega'' < C/(C_{*3}(m)\ |s|\ \ln^n \varepsilon^{-1}$, then $\operatorname{mes} \Theta^3[I] \leq CC_{*3}^{-1}(m)M^n \ln^{-n}\varepsilon^{-1}$. Therefore condition (1) for Θ^3 holds if $C_{*3}(m)$ is sufficiently large. If $\theta \notin \Theta^3$, then by (13)

$$|D| \geqslant C_{*3(m)}^{-1} \ln^{-n} \varepsilon^{-1} - CC_{*1}^{-d_1}(m) \ln^{-n} \varepsilon^{-1} \geqslant C^{-1}(m) \ln^{-n}\varepsilon^{-1},$$

if $C_{*1}(m)$ is sufficiently large. Hence from (10) we obtain (7).

Since the collection of vectors $\{\lambda_{k*}^2 W_k^{\pm}\}$ forms an orthonormal basis in Y_{-2} [see (1.7)], then by (4)

$$|\lambda_k^2 \widehat{B}_{k\ j}^{\gamma_1\gamma_2}(s)|_{l^2(k,\gamma_1)}^{\theta} \leqslant C(m)\,\varepsilon^{-2/3} e^{-|s|\delta} \quad \forall j, \nu. \tag{14}$$

[we denote by $l^2(k, \gamma_1)$ the space of l^2-sequences in k, γ_1]. By (6), (14) and Lemma 6, for $\theta \in \Theta_m \setminus (\Theta^2 \cup \Theta^3)$ we have

$$|\lambda_k^2 \widehat{G}_{k\ j}^{\gamma_1\gamma_2}(s)|_{l^2(k,\gamma_1)}^{\theta} \leqslant T_1 \Big(\sum_k (1 + |\lambda_{k*} - \lambda_{j*}|)^{-2}\Big)^{1/2} \leqslant T_1 C, \tag{15}$$

where $T_1 = C(m)e^{-|s|\delta}\varepsilon^{-2/3}(\ln^c \varepsilon^{-1})(1 + |s|)^{2n+3}$ and in the deduction of the last inequality we used condition (1.12).

Since $\hat{B}(s) \in \mathscr{L}_{0,2}^s$, then by (4) $|\hat{B}(s)|_{-2,0}^{\theta} \leqslant C(m)\varepsilon^{-2/3}e^{-|s|\delta}$. Therefore, by the interpolation theorem (see, for instance, [9]), $|\hat{B}(s)|_{-1,1}^{\theta} \leqslant C_1(m)\varepsilon^{-2/3}e^{-|s|\delta}$. Hence

$$|\lambda_k\lambda_j \widehat{B}_{k\ j}^{\gamma_1\gamma_2}(s)|_{l^2(j,\gamma_2)}^{\theta} \leqslant C_1(m)\,\varepsilon^{-2/3}e^{-\delta|s|} \quad \forall k, \gamma_1$$

and consequently

$$|\lambda_k^2 \widehat{G}_{k\ j}^{\gamma_1\gamma_2}(s)|_{l^2(j,\gamma_2)}^{\theta} \leqslant CT_1 \left(\sum_j \frac{\lambda_k^2\lambda_j^{-2}}{(1 + |\lambda_{k*} - \lambda_{j*}|)^2}\right)^{1/2} \leqslant C_1 T_1. \tag{16}$$

From (15) and (16) by Schur's criterion (see [10, p. 33]) we obtain that $|\hat{G}(s)|^{\theta}_{0,2} \leqslant CT_1$, whence for $q \in U_m^1$ follow estimate (2.30) and an analogous estimate for $\partial G/\partial\omega'$. Estimate (2.31) follows from the form of Eq. (2.26). The inclusion $G \in A^{\theta}_{R}(U^1_m, \mathscr{L}^s_{0,2})$ follows from an analogous fact for the operator B. Part b) of the lemma is proved.

The existence of a measurable set Θ^4 satisfying (1) and the assertions of part c) are proved analogously.

4. Proof of Lemma 4

Let us denote by E_s the space $C^{2n}_{q,\xi} \times Y^s$ with the norm $|(q,\xi,y)|^2_{(s)} = |q|^2 + \varepsilon^{-4/9}|\xi|^2 + \varepsilon^{-1/9}\|y\|^2_s$, and by E_s^* the space conjugate to E_s with respect to the pairing $\langle\langle\cdot,\cdot\rangle\rangle$ with the norm $|(q,\xi,y)|^2_{(*,s)} = |q|^2 + \varepsilon^{4/9}|\xi|^2 + \varepsilon^{1/9}\|y\|^2_{-s}$. If $\varepsilon_0 \ll 1$, then for j = 1, 2, 3,

$$\operatorname{dist}_{E_0}(O^{j+1}_m, O_m \setminus O^j_m) \geqslant C^{-1}(m) \tag{1}$$

[see (2.4)]. By (1), Lemma 3 and the Cauchy estimates

$$F \in A^{\theta}_{R}(O^2_m, E_0), \quad \varepsilon|F(\mathfrak{h})|^{\theta}_{(0)} \leqslant C(m)\varepsilon^{1/3}\ln^c\varepsilon^{-1}, \tag{2}$$

$$\varepsilon|F(\mathfrak{h}_1) - F(\mathfrak{h}_2)|^{\theta}_{(0)} \leqslant C(m)|\mathfrak{h}_1 - \mathfrak{h}_2|_{(0)}\varepsilon^{1/3}\ln^c\varepsilon^{-1} \tag{3}$$

for every $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2 \in O^3_m$. We denote $(q(t), \xi(t), y(t)) = \mathfrak{h}(t)$ and rewrite Eqs. (2.22), (2.23) as follows:

$$\dot{\mathfrak{h}} = \varepsilon F(\mathfrak{h}(t)). \tag{4}$$

By (1) and (2), for $0 \leq t \leq 1$ and $\varepsilon_0 \ll 1$ the solution of (4) depends analytically on $\mathfrak{h}(0) \in O^3_m$ and does not leave O^2_m. Let $S^t\colon O^3_m \to O^2_m$, $0 \leq t \leq 1$ be the shift by time t along the trajectories of (4). From (2) and (3) it follows that

$$|\mathfrak{h} - S^t(\mathfrak{h})|^{\theta}_{(0)} \leqslant C(m)t\varepsilon^{1/3}\ln^c\varepsilon^{-1}, \tag{5}$$

$$|S^t(\mathfrak{h}) - \mathfrak{h} - t\varepsilon F(\mathfrak{h})|^{\theta}_{(0)} \leqslant C(m)t^2\varepsilon^{2/3}\ln^{2c}\varepsilon^{-1}. \tag{6}$$

Since the map F is linear in ξ and quadratic in y, then by Lemma 3 and Cauchy inequality for $\mathfrak{h} \in O^2_m$ and r = 0, 1, 2, ...

$$|d^2F(\mathfrak{h})|_{Q\times\ldots\times Q,Q} \leqslant C_r(m)\varepsilon^{-2/3}\ln^c\varepsilon^{-1},$$

whence estimate (2.34) follows.

Since $G(q) \in \mathscr{L}^s_{0,2}$, then $|G(q)|_{-2,0} = G|(q)|_{0,2}$. Therefore, by the estimates of Lemma 3 for $\mathfrak{h} \in O^2_m$ $|\varepsilon dF(\mathfrak{h})|^{\theta}_{(-2),(-2)} \leqslant tC(m)\varepsilon^{1/3}\ln^c\varepsilon^{-1}$; and for $\mathfrak{h} \in O^3_m$

$$|I - dS^t(\mathfrak{h})|^{\theta}_{(-2),(-2)} \leqslant tC(m)\varepsilon^{1/3}\ln^c\varepsilon^{-1}, \tag{7}$$

$$|I - dS^t(\mathfrak{h}) + t\varepsilon\, dF(\mathfrak{h})|^{\theta}_{(-2),(-2)} \leqslant t^2C(m)\varepsilon^{2/3}\ln^{2c}\varepsilon^{-1}. \tag{8}$$

Let $\mathfrak{h}(t)$ be a solution of (4) and $\mathfrak{h}(0) = (q_0, \xi_0, y_0) \in O^3_m$. Then y(t) is expanded in a convergent series in powers of ε: $y(t) = y_0 + \varepsilon y^1(t)$, $\varepsilon y^1(t) = \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots$, where

$$y_1(t) = J\int_0^t \mathfrak{Y}(q(\tau)) + 2G(q(\tau))y_0\,d\tau,$$

$$y_{k+1}(t) = 2J\int_0^t G(q(\tau))y_k(\tau)\,d\tau, \quad k \geqslant 1.$$

Using Lemma 3 and estimates (5)-(8), we conclude from these formulas that for $y^1 = y^1(1)$, $y_j = y_j(1)$, j = 1, 2, ...

$$\varepsilon^{1/3}\|y^1\|^{\theta}_1 + \varepsilon^{2/3}(|dy^1|^{\theta}_{0,1} + |dy^1|^{\theta}_{-2,-1}) \leqslant C(m)\ln^c\varepsilon^{-1}, \tag{9}$$

$$\|y^1 - y_1\|^{\theta}_2 + \varepsilon^{1/3}(|d(y^1 - y_1)|^{\theta}_{0,2} + |d(y^1 - y_1)|^{\theta}_{-2,0}) \leqslant C(m)\ln^{2c}\varepsilon^{-1}, \tag{10}$$

$$\| \nabla_y \langle A_{m+1}y, \varepsilon(y_1 - F^y)\rangle \|_2^\theta \leqslant C(m)\varepsilon \ln^{2c}\varepsilon^{-1}. \tag{11}$$

From (5) and (9), the estimate (2.35) of Lemma 4 follows.

By (2), $S^t(O_{m+1}^R) \subset O_m^R$; the transformation S^t is canonical as a shift along the trajectories of a Hamiltonian flow [3, 14]. Assertion a) of the lemma is proved.

Let $\mathfrak{h} \in O_{m+1}$, $S_m(\mathfrak{h}) = \mathfrak{h} + \varepsilon\mathfrak{h}^1$. We write the transformed Hamiltonian as follows:

$$\begin{aligned} H_m(S_m(\mathfrak{h};\theta);\theta) = H_{0\,m+1}(\xi, y; \theta) + \varepsilon[(\xi^1 - F^\xi)\cdot(1_n + \mathfrak{S}_{m+1})]_1 + \\ + \varepsilon[\langle A_{m+1}y, y^1 - F^y\rangle]_2 + (\varepsilon^2/2)[\langle A_{m+1}y^1, y^1\rangle]_3 + \varepsilon[-\partial\varkappa/\partial\omega' + g]_4 + \\ + \varepsilon[(-\partial\chi/\partial\omega' + h)\cdot\xi]_5 - \varepsilon[\langle\partial\mathfrak{Y}/\partial\omega' - A_{m+1}J\mathfrak{Y} - \mathfrak{z}, y\rangle]_6 - \\ - \varepsilon[\langle(\partial G/\partial\omega' - 2A_{m+1}JG - B)y, y]_7 + \varepsilon[H_{1m}(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - H_{1m}(\mathfrak{h})]_8 + \varepsilon[H_{3m}(\mathfrak{h})]_9. \end{aligned} \tag{12}$$

We denote by $\Delta_j H$ the functional in the brackets $[\cdot]_j$ (together with the preceding factor).

<u>LEMMA 7.</u> For $\mathfrak{h} \in O_{m+1}$ and j = 1, ..., 8 the following estimates hold:

$$| \Delta_j H |^\theta \leqslant C(m)\varepsilon^{4/3}\ln^{2c}\varepsilon^{-1}, \tag{13}$$

$$\| \nabla_y \Delta_j H \|_2^\theta \leqslant C(m)\varepsilon \ln^{2c}\varepsilon^{-1}. \tag{14}$$

<u>Proof.</u> Estimates (13) for j = 1, ..., 7 follow easily from inequalities (5), (6), and for j = 8 they follow from (1), (5), (2.8) and the Cauchy estimate.

Let us prove inequalities (14). Let Π_y: $E_{-2*} \to Y^2$ be the natural projection. Then $| \Pi_y |_{(-2,*),2} \leqslant \varepsilon^{-1/3}$. Since

$$\nabla_y(\varepsilon(\xi^1 - F^\xi)\cdot(1_n + \mathfrak{S}_{m+1})) = \Pi_y \circ d(S_m - I - \varepsilon F)^*(\mathfrak{h})(0, 1_n + \mathfrak{S}_{m+1}, 0),$$

then for j = 1 estimate (14) follows from (8). Since

$$\nabla_y \Delta_2 H = \nabla_y \langle A_{m+1}y, \varepsilon(y_1 - F^y)\rangle + \varepsilon A_{m+1}(y^1 - y_1) + \varepsilon d(y^1 - y_1)^* A_{m+1}y,$$

then for j = 2 estimate (14) follows from (10) and (11). By the equality $\nabla\Delta_3 H = \varepsilon^2 d(\varepsilon y^1)^* \cdot A_{m+1}\varepsilon y^1$, estimate (14) for j = 3 follows from (9).

By Lemma 3, $\Delta_4 H \equiv \Delta_5 H = 0$ and

$$\| \nabla\Delta_6 H \|_2^\theta = \varepsilon^2 \| \Delta\mathfrak{z} \|_2^\theta \leqslant C(m)\varepsilon^{5/3}, \quad \| \nabla\Delta_7 H \|_2^\theta = 2\varepsilon^2 \| \Delta By \|_2^\theta \leqslant C(m)\varepsilon^{5/3},$$

which proves (14) for j = 4-7.

It remains to verify (14) for j = 8. Let us write $\nabla y \Delta_8 H$ as follows:

$$\nabla_y \Delta_8 H = \varepsilon[\nabla_y H_{1m}(\mathfrak{h} + \varepsilon\mathfrak{h}^1) - \nabla_y H_{1m}(\mathfrak{h})] + \varepsilon\Pi_y(d\varepsilon\mathfrak{h}^1)^* \nabla_y H_{1m}(\mathfrak{h})|_{\mathfrak{h}=\mathfrak{h}+\varepsilon\mathfrak{h}^1}.$$

From (1), (5), (2.8) and the Cauchy estimate it follows that the first term in the right hand side of the inequality satisfies (14). Since $| \nabla_{\mathfrak{h}} H_{1m}(\mathfrak{h}) |_{(-2,*)} \leqslant C(m)$, the estimate (14) for the second term follows from (7).

Assertion b) of Lemma 4 follows from inequality (12), Lemmas 2 and 7, and the fact that $\varepsilon_0 \ll 1$.

<u>5. Equation of Oscillations of a Nonlinear String</u>

Let $V(a, x) \in C^\infty(\mathbf{R}^n \times [0, \pi])$, $-1/2 \leqslant V(a, x) \leqslant K_6$. The Sturm–Liouville operator $-\partial^2/\partial x^2 + V(a, x)$ defines a self-adjoint positive operator $\mathcal{A}_a$ in L_2, $D(\mathcal{A}_a) = \overset{\circ}{H}{}^1 \cap {}_{\circ}H^2 \equiv {}_{\circ}\mathcal{H}$ (here and everywhere further $L_2 = L_2(0, \pi)$, $\overset{\circ}{H}{}^1 = \overset{\circ}{H}{}^1(0, \pi)$, ...). Let us set $Q = \overset{\circ}{H}{}^1 \times \overset{\circ}{H}{}^1$, $M_a = \mathcal{A}_a \times \mathcal{A}_a$, $| w |_Q = | M_0^{1/4} w |_{L_2 \times L_2}$. Let us define in Q the following families of operators: $W_a = M_0^{-1} \circ M_a$, $J_a^0(w^1, w^2) = (\mathcal{A}_a^{1/2}w^2, -\mathcal{A}_a^{1/2}w^1)$. Let $\{\varphi_j(a, \cdot) \mid j \geqslant -n+1\}$ be an L_2-orthonormalized system of eigenfunctions of $\mathcal{A}_a$ with eigenvalues $\mu_j(a)$, $j \geq -n + 1$ depending smoothly on a such that

$$\mu_{l+1}(a) > \mu_l(a) \quad \forall a, \quad \forall l \geqslant l_1. \tag{1}$$

Since $V \geq -1/2$, then $\mu_j(a) \geq 1/2$ for all j. Let us set $w_j^+ = (\varphi_j, 0)/\lambda_j$, $w_j^- = (0, \varphi_j)/\lambda_j$, $\lambda_j = \mu_j^{1/2}$. The family $\{w_j^\pm \mid j \geq -n + 1\}$ forms an orthonormal basis in Q_a satisfying (1.1).

Let $f(u, x) \in C^1 (\mathbf{C}^1 \times [0, \pi])$, $f(0, x) \equiv 0$ be a function holomorphic in u that is real for real u. We denote $f_t = \sup \{|f(u,x)| \mid |u| \leqslant t, x \in [0,\pi]\}$ and define the functional $H^0_\Delta (w^1(x), w^2(x)) = \int f(w^1(x), x)dx$. Then

$$\nabla^a H^0_\Delta (w^1, w^2) = (\mathcal{A}_a^{-1} \varphi'_w (w^1(x), x), 0) \tag{2}$$

and in the case under consideration Eq. (1.3) becomes

$$\dot{w}^1 = \mathcal{A}_a^{1/2} w^2, \quad \dot{w}^2 = - \mathcal{A}_a^{1/2} (w^1 + \varepsilon \mathcal{A}_a^{-1} \varphi'_w (w^1(x), x)). \tag{3}$$

Hence

$$\ddot{w}^1 = (\partial^2/\partial x^2 - V(a, x)) w^1 - \varepsilon \varphi'_w (w^1(x), x). \tag{4}$$

Equation (4) is the equation of oscillations of a string in a nonlinear-elastic medium.

By (1) and the asymptotics of the spectrum of the Sturm–Liouville operator, we have

$$|\mu_j(a) - (j+n)^2| \leqslant C \quad \forall j, \quad \forall a. \tag{5}$$

Moreover, $\mu_j \neq \mu_l$ for $j \neq l$. Therefore, applying the perturbation theory to the simple eigenvalue μ_j (see [11]) we obtain that φ_j and μ_j are differentiable functions of the parameter a and that for all j

$$\partial \mu_j / \partial a_k = \int (\partial V / \partial a_k) \varphi_j^2 dx, \tag{6}$$

$$|\varphi_j(a_1, x) - \varphi_j(a_2, x)|_{L_2} \leqslant C |a_1 - a_2| / j; \quad |a_1|, |a_2| \leqslant 2. \tag{7}$$

From (5) and (7) for every $j \geq 1$ and $l \geq 0$ by induction on l we obtain for $|a_1|$, $|a_2| \leq 2$ the inequalities

$$|\varphi_j(a_1, x) - \varphi_j(a_2, x)|_{H^l} \leqslant C_l |a_1 - a_2| j^{l-1}. \tag{8}$$

Assume that the following general position condition is fulfilled for the family $V(a, x)$:

$$K_7 \equiv |\det(\Lambda_{jk})| \neq 0, \quad \Lambda_{jk} = (4\mu_j)^{-1/2} \int (\partial V(0, x)/\partial a_k) \varphi_j^2 (0, x) \, dx. \tag{9}$$

Then by (6) the map

$$a \mapsto \omega = (\lambda_{1-n}, \lambda_{2-n}, \ldots, \lambda_0) \tag{10}$$

defines a diffeomorphism of a closed neighborhood of zero onto a ball Ω_0 [see (1.2)], $\omega_* = \omega(0)$. The radius K_0 of the ball may be chosen depending only on K_7, K_6, n, and l_1.

Equations (3) for $\varepsilon = 0$ and $\omega \in \Omega_0$ have invariant tori $U_a T^n(I)$, $I \in \mathcal{J} \Subset \mathbf{R}^n$, determined as in Sec. 1. Let us verify that for Eqs. (3) and the families $T^n(I)$ conditions 1) and 2) of Theorem 1 are fulfilled. Let $O_1 \subset Q_C$ be bounded domain containing $\bigcup \{U_a T^n(I) \mid I \in \mathcal{J}, |a| \leqslant 2\}$. Since $f(u, x)$ is analytic in u, the map $O \to H^1$, $(w^1, w^2) \mapsto \varphi'_w (w^1(x), x)$ is bounded and analytic. Therefore, by (2) and the remark following Theorem 1, in order to verify condition 1) it suffices to show that (1.18) is valid.

Let us consider the linear operator U_a^0 in $\mathring{H}^1$ carrying $\varphi_j(0, x)$ into $\varphi_j(a, x)$. Let us introduce in $\mathcal{H}$ the inner product $\langle u_1, u_2 \rangle_2 = (\mathcal{A}_0 u_1, \mathcal{A}_0 u_2)_{L_2}$. The family of functions $\{\varphi_{j2}(0, x) \mid j \geqslant -n+1\}$, $\varphi_{j2} = \varphi_j / \mu_j(0)$, is an orthonormal basis in $\mathcal{H}$. By (8), for $l = 2$ we have $|\varphi_{j2}(a_1, x) - \varphi_{j2}(a_2, x)|_{\mathcal{H}} \leqslant C |a_1 - a_2| / j$. Hence

$$|U_\cdot|^a_{\mathcal{Y}^2, \mathcal{Y}^2} \leqslant C |U^0_\cdot|^a_{\mathcal{H}, \mathcal{H}} \leqslant C_1. \tag{11}$$

Since $U_0 = I$, then for sufficiently small K_0 the estimate for the first term in (1.18) follows from (11). The estimate for the third term follows from (7).

For $|a| \leq 2$, we have $|\mathcal{A}_\cdot|^a_{\mathcal{H}, L_2} + |\mathcal{A}_0^{-1}|_{L_2, \mathcal{H}} \leqslant C$. Therefore $|\mathcal{A}_0^{-1} \circ \mathcal{A}_\cdot|^a_{\mathcal{H}, \mathcal{H}} \leqslant C_1$, whence the estimate for the second term in (1.18) follows.

Condition 2) of Theorem 1 for $d = d_1 = 1, k_2 = 1, \zeta(j) \equiv n$ follows from (5) and (6). We observe that the constant K in (1.10)-(1.14) and the constant K_0 in (1.2) depend only on $\mathcal{J}$, n, K_6, K_7, l_1 and the function f_t that characterizes the growth of $f(u, x)$ as $|u| \to \infty$.

Let

$$\mathfrak{A}_1 = \{V_* (a, x) \in C^\infty \mid | V_* (a, x) - V (a, x) |_{C^1} \leqslant K_8\} \tag{12}$$

be a neighborhood of the family $V(a, x)$. If K_8 is sufficiently small, then for every $V_*(a, x)$ from $\mathfrak{A}_1$ we have $|\det(\Lambda_{jk})| \geqslant K_7/2$. Therefore, for $V = V_* \in \mathfrak{A}_1$ the numbers M_1 and j_1 in part 3) of Theorem 1 may be chosen to be independent of V_*. Let us set $\Upsilon(V) = \Pi(s\cdot\omega_* + \gamma_1\lambda_{j*} + \gamma_2\lambda_{k*})$, where the product is taken over all s, γ_1, γ_2, k, j as in part 3) of the theorem. The functional T(V) is analytic and not identically zero.

THEOREM 2. Assume that conditions (1.9) and (9) hold and let the function f(u, x) and the set $\mathfrak{A}_1$ be the same as above. Let $K_8 > 0$ be sufficiently small, $V_* \in \mathfrak{A}_1$ and $\Upsilon(V_*) \neq 0$. Then there exist an $\varepsilon_* > 0$ and a sufficiently small neighborhood $\mathfrak{A}_2$ of $a = 0$ in $\mathbf{R}^n$ such that for $V = V_*$ and $\varepsilon \in [0, \varepsilon_*]$ there exist a measurable subset $\Theta_\varepsilon \subset \Theta_0 = \mathfrak{A}_2 \times \mathcal{J}$ smooth imbedding $\Sigma = \Sigma_\theta\colon T^n \to Q$, $\theta \in \Theta_\varepsilon$ and linear maps $U_a\colon Q \to Q$, $a \in \mathfrak{A}_2$, with the following properties:

a) $\Sigma_\theta(T^n) \subset \mathcal{H} \times \mathcal{H}$, $U_a(\mathcal{H} \times \mathcal{H}) = \mathcal{H} \times \mathcal{H}$;

b) for $a \in \mathfrak{A}_2$ and all I such that $\theta = (a, I) \in \Theta_\varepsilon$, the tori $U_a\Sigma_\theta(T^n)$ are invariant with respect to the flow of Eqs. (3). Under the imbedding $U_a\Sigma_\theta$ the trajectories of (3) are carried on T^m into the curves of $\dot{q} = \mathfrak{S}_\varepsilon(\theta)$, where $\mathfrak{S}_\varepsilon\colon \Theta_\varepsilon \to \mathbf{R}^n$ is a Lipschitzian map;

c) $\operatorname{mes}\{a \in \mathfrak{A}_2 \mid (a, I) \notin \Theta_\varepsilon\} \to 0$ as $\varepsilon \to 0$ uniformly with respect to $I \in \mathcal{J}$.

6. Change of the Symplectic Structure in Equation (0.1)

In the notations of Sec. 1 let us assume that $W_a \equiv I$, $Q_a \equiv Q$ and that $J_a{}^0 = J_s A_a{}^0$, where $A_a{}^0 = |J_a{}^0|$ and the operator J_s does not depend on a. Then

$$J_s w_j^\pm(a) = \mp w_j^\mp(a), \quad A_a^0 w_j^\pm(a) = \lambda_j(a) w_j^\pm(a). \tag{1}$$

Let us consider a perturbation of (0.1) that is Hamiltonian with respect to the symplectic structure given by the 2-form $\langle\langle J_s\cdot, \cdot\rangle\rangle$ (see [3, 14]):

$$\dot{w} = J_s(A_a^0 w + \varepsilon\nabla H_\Delta^0(w, a)). \tag{2}$$

As in Sec. 1, let us pass from the parameter a to the parameter $\omega \in \Omega_0$, $\omega_j = \lambda_{j-n}(a)$. By conditions (1), the map U_ω: $Q \to Q$ is canonical. It takes Eq. (2) into

$$\dot{w} = J_s(A_\omega w + \varepsilon\nabla H_\Delta(w, \omega)),$$
$$A_\omega w_j^\pm = \lambda_j(\omega) w_j^\pm, \quad w_j^\pm = w_j^\pm(a)|_{a=0}, \quad j \geqslant -n+1; \quad H_\Delta = H_\Delta^0(U_\omega(w), \omega). \tag{3}$$

Let us define the tori $T^n(I)$ $(I \in \mathcal{J})$, the domain O and the spaces $\mathcal{Y}^r$ as in Sec. 1. Let us denote $O_Y = O \cap \mathcal{Y}^{1/2}$.

THEOREM 3. Assume that for $d \geq 2$ conditions (1.6), (1.9)-(1.11) of Theorem 1 are fulfilled and the maps H_Δ and ∇H_Δ may be extended to Frechet complex-analytic maps $H_\Delta\colon O_Y \to \mathbf{C}$, $\nabla H_\Delta\colon O_Y \to \mathcal{Y}^{1/2}$, and

$$|H_\Delta(w; \cdot)|^\omega + |\nabla_w H_\Delta(w; \cdot)|^\omega_{\mathcal{Y}^{1/2}} \leqslant K \quad \forall w \in O_Y.$$

Then there exist natural numbers j_1, M_1 and a real number $K_3 > 0$ depending on K, K_1, d, and n such that if for every $|s| \leq M_1$, $1 \leq j < k \leq j_1$ and $\gamma_1, \gamma_2 = \pm 1$ conditions (1.14) and (1.15) hold and $K_0 \leq K_0'$, where $K_0' > 0$ depends only on K, K_1, d, n, and K_3, then for $0 < \varepsilon_0 \leq \varepsilon_*$, where $\varepsilon_* > 0$ is sufficiently small, there exist a measurable subset $\Theta_{\varepsilon_0} \subset \Theta_0 = \Omega_0 \times \mathcal{J}$ and smooth imbeddings

$$\Sigma = \Sigma_\theta : T^n \to Q, \ \Sigma(T^n) \subset \mathcal{Y}^1, \quad \theta = (\omega, I) \in \Theta_{\varepsilon_0},$$

satisfying assertions a)-c) of Theorem 1 [with the obvious replacement of Eq. (1.15) by (3)].

The proof of Theorem 3 is based on the same ideas as the proof of Theorem 1. It will be published in an author's paper in the journal "Izv. Akad. Nauk SSSR (Ser. Mat.)."

As an example of application of Theorem 3 let us consider the boundary-value problem for the one-dimensional nonlinear Schrodinger equation with real potential $V(x, a)$ ($a \in \mathbf{R}^n$ is a parameter):

$$\dot{u} = i(-u'' + V(x, a)u + 2\varepsilon\varphi'_w(|u|^2, x)u),$$
$$u = u(t, x), \ x \in (0, \pi), \quad u(t, 0) = u(t, \pi) = 0. \tag{4}$$

The potential V(x, a) and the function φ(w, x) are as in Sec. 5. Problem (4) is a particular case of Eq. (2) for Q = L_2 (0, π; **C**) with the inner product $\langle\langle u, v\rangle\rangle = \mathrm{Re}\int u(x)\,\bar{v}(x)\,dx$ and

$$J_s u = iu,\ A_a^0 = -\partial^2/\partial x^2 + V(x, a),\ H_\Delta^0 = \int \varphi(\,|\,u\,|^2(x), x)\,dx.$$

Assume that conditions (1.9) and (5.9) hold, and the set $\mathfrak{A}_1$ is as in (5.12).

THEOREM 4. Assume that the number K_8 defining the set $\mathfrak{A}_1$ is sufficiently small. Then there is a nontrivial analytic function $\Upsilon\colon \mathfrak{A}_1 \to \mathbf{R}$ such that if $\Upsilon(V_*) \neq 0$, then there exist $\varepsilon_* > 0$ and a neighborhood $\mathfrak{A}_2'$ of a = 0 with the following property. For V = V_* and $0 < \varepsilon \le \varepsilon_*$ there are a measurable subset $\Theta_\varepsilon \subset \Theta_0 = \mathfrak{A}_2' \times \mathcal{J}$, satisfying assertion c) of Theorem 2, and smooth imbeddings

$$\Sigma_\theta : T^n \to Q,\quad \Sigma_0(T^n) \subset (\mathring{H}^1 \cap H^2)(0, \pi; \mathbf{C}),\quad \theta = (a, I) \in \Theta_\varepsilon,$$

such that the tori $\Sigma_\theta(T^n)$ are invariant for the problem (4).

Theorem 4 is deduced from Theorem 3 in the same way as Theorem 2 is deduced from Theorem 1.

LITERATURE CITED

1. V. E. Zakharov, "Hamiltonian formalism for waves in nonlinear media with dispersion," Radiofizika, 17, No. 4, 431-453 (1974).
2. V. I. Arnol'd, "Proof of a theorem of A. N. Kolmogorov on the conservation of quasi-periodic motions under a small change of the Hamilton function," Usp. Mat. Nauk, 18, No. 1, 13-39 (1963).
3. V. I. Arnol'd, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York (1978).
4. V. K. Mel'nikov, "A family of quasiperiodic solutions of a Hamilton system," Dokl. Akad. Nauk SSSR, 181, No. 3, 546-549 (1968).
5. J. Moser, "Convergent series expansions for quasiperiodic motions," Math. Ann., 169, 136-176 (1967).
6. N. V. Nikolenko, "Poincare's method of normal forms in the problems of intergrability of evolution-type equations," Usp. Mat. Nauk, 41, No. 5, 109-152 (1986).
7. N. V. Nikolenko, "Invariant asymptotically stable tori of the perturbed Korteweg—de-Vries equation," Usp. Mat. Nauk, 35, No. 5, 121-180 (1980).
8. H. Brezis, Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam (1973).
9. J.-L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Springer-Verlag, New York (1972).
10. P. Halmos and V. Sunder, Bounded Integral Operators on L^2-Spaces, Springer-Verlag, New York (1978).
11. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Academic Press, New York (1972-1977).
12. V. K. Mel'nikov, "Some cases of conservation of quasiperiodic motions under a small change of the Hamilton function," Dokl. Akad. Nauk SSSR, 165, No. 6, 1245-1248 (1965).
13. S. M. Graff, "On the conservation of hyperbolic invariant tori for Hamiltonian systems," J. Diff. Equations, 15, No. 1, 1-69 (1974).
14. P. R. Chernoff and J. E. Marsden, "Properties of infinite dimensional Hamiltonian systems," Lect. Notes Math., No. 425, Springer-Verlag, Berlin (1974).