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PERTURBATION THEORY FOR QUASIPERIODIC SOLUTIONS OF INFINITE-DIMENSIONAL HAMILTONIAN SYSTEMS, AND ITS APPLICATION TO THE KORTEWEG-DE VRIES EQUATION

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ABSTRACT. A perturbation theory is constructed for quasiperiodic solutions of nonlinear conservative systems of large and of infinite dimension.

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Introduction

Let Z be a real Hilbert space with inner product $\langle\langle\cdot,\cdot\rangle\rangle$, let J^Z and A^Z be antiselfadjoint and selfadjoint operators on Z, and let H_0^Z and H_1^Z be analytic functions on Z. Give Z a symplectic structure using the 2-form $\omega_2^Z[\xi,\eta] = -\langle\langle (J^Z)^{-1}\xi,\eta\rangle\rangle$. Consider Hamilton's equation on Z with Hamiltonian

$$\mathscr{H}^Z = \langle\langle A^Z u, u\rangle\rangle/2 + H_0^Z(u),$$

namely,

$$\dot{u} = J^z(A^z u + \nabla H_0^z(u)), \qquad u \in Z; \tag{0.1}$$

and small Hamiltonian perturbations of it:

$$\dot{u} = J^z(A^z u + \nabla H_0^z(u) + \varepsilon\nabla H_1^z(u)), \qquad 0 < \varepsilon \ll 1. \tag{0.2}$$

Suppose that the operator $J^Z A^Z$ has purely imaginary spectrum $\{\pm i\lambda_j^Z \mid j \geq 1\}$, and that

$$\lambda_j^z = Cj^{d_1} + o(j^{d_1-1}), \qquad d_1 \geq 1. \tag{0.3}$$

Equations of the form (0.1) and (0.2) arise in the description of conservative physical systems. As a rule, (0.3) is satisfied if the system is spatially 1-dimensional (see [1] and [2]).

Up to now there have been rather a lot of examples of equations (0.1) and (0.3) that are integrable Hamiltonian systems (see [2]–[5]). For such systems it is typical for there to exist finite-dimensional invariant manifolds $\mathscr{T} = \mathscr{T}_n$ filled with quasiperiodic solutions with n basic frequencies ($n = 1, 2, \dots$). There are well-known averaging procedures that enable asymptotic solutions for perturbed equations to be constructed, starting from these quasiperiodic solutions (see the survey [4], and [6]). In this paper we study the following question: under what conditions does the perturbed equation (0.2) have exact quasiperiodic solutions close to those belonging to $\mathscr{T}$?

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The main result of the paper is Theorem 2 of §2. It is proved there that if $\dim \mathscr{T} = 2n$, if the variational equation for (0.1) along $\mathscr{T}$ is integrable, and if on $\mathscr{T}$ there is a nonresonant solution (a solution for which a certain finite set of resonance relations are not zero) then most nearby solutions on $\mathscr{T}$ are preserved under the perturbation (0.2).

In §4 we consider the Korteweg-de Vries (KdV) equation, for which $\mathscr{T}$ may be taken to be the manifold of n-gap potentials (see [3] and [5], for example). Analysis of the explicit formulas in [6] shows that the variational equation for the KdV equation along $\mathscr{T}$ is integrable in the sense of Theorem 2. Thus most nonresonant quasiperiodic solutions of the KdV equation are preserved under small Hamiltonian perturbation.

The proof of Theorem 2 is based on previous work [7], [8] of the author, in which a theory of perturbations of quasiperiodic solutions of linear equations (0.1), (0.3) that depend on a vector parameter is constructed using an infinite-dimensional Kolmogorov-Arnol′d-Moser (KAM) method (see [9] and [10], for example).

The author is indebted to B. A. Dubrovin and I. M. Krichever for numerous discussions on the properties of KdV-type equations.

We use the following notation. $C, C', C_1, \dots$ are various positive constants that do not depend on ε; $\delta, \delta', \delta_1 \dots$ are small positive constants independent of ε and representing radii of analyticity. For Hilbert spaces Z_1 and Z_2 the space of continuous linear operators from Z_1 to Z_2 is denoted by $L(Z_1, Z_2)$ and is given the operator norm $\|\cdot\|_{Z_1,Z_2}$. Analyticity of mappings defined on subdomains of Hilbert space is understood in the sense of Fréchet.

References to formulas from another section are given as (2.3) (formula (3) from §2), for example.

§1. Perturbation of quasiperiodic solutions of completely integrable systems

Let Y be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\{Y_t | t \in \mathbf{R}\}$ be a family of Hilbert spaces such that $Y_0 = Y$ and $Y_{t_1} \subset Y_{t_2}$ when $t_1 \geq t_2$, and such that the spaces Y_t and Y_{-t} are dual with respect to $\langle\cdot,\cdot\rangle$: $Y_{-t} = Y_t^*$. Denote the norm on Y_t by $\|\cdot\|_t$. Let $A^Y \colon Y_{j+d_A} \to Y_j$ and $J^Y \colon Y_{j+d_J} \to Y_j$, $j \in \mathbf{R}$, be linear isomorphisms, where $d_A, d_J \geq 0$, such that A^Y and J^Y define selfadjoint and antiselfadjoint (possibly unbounded) operators on Y. Suppose that A^Y and J^Y commute, and that Y admits an orthonormal basis $\{\varphi_j^\pm | j \geq 1\}$ with the following properties:

1) $A^Y \varphi_j^\pm = \lambda_{jA} \varphi_j^\pm$ and $J^Y \varphi_j^\pm = \mp \lambda_{jJ} \varphi_j^\mp$ for all j.

2) $\{j^{-t} \varphi_j^\pm | j \geq 1\}$ is an orthonormal basis in Y_t for all $t \in \mathbf{R}$.

Let $\mathscr{I}$ be a bounded domain in $\mathbf{R}^n$, and $\mathscr{I}_\delta^C$ ($\delta > 0$) its δ-neighborhood in $\mathbf{C}^n$.

Let $A(I)$ be a family of selfadjoint operators on Y, analytic with respect to $I \in \mathscr{I}$, such that for all j

$$A(I)\varphi_j^\pm = \lambda_j^A(I)\varphi_j^\pm, \qquad |\lambda_j^A(I_1) - \lambda_j^A(I_2)| \leq C|I_1 - I_2| \quad \forall I, I_1, I_2 \in \mathscr{I}_\delta^C, \tag{1}$$

where the inequality assumes that $\lambda_j^A(I)$ is analytically continued into $\mathscr{I}_\delta^C$. Fix a point $I_{00} \in \mathscr{I}$, and write $\lambda_{jA} = \lambda_j^A(I_{00})$ and $\lambda_j^0(I) = \lambda_j^A(I) - \lambda_{jA}$. Let

$$\begin{gathered} \mathscr{Y}_d = T^n \times \mathscr{I} \times Y_d, \quad T^n = \mathbf{R}^n / 2\pi \mathbf{Z}^n, \quad \mathscr{Y}_0 = \mathscr{Y}, \\ \mathscr{O}_d(\delta) = \mathscr{O}_d(\delta, \mathscr{I}) = T^n \times \mathscr{I} \times \{y \in Y_d \,|\, \|y\|_d < \delta\}. \end{gathered} \tag{2}$$

We identify the tangent space to $\mathscr{Y}_d$ at an arbitrary point with $E > Y_d$, $E = \mathbf{R}^{2n}$, and give $\mathscr{Y}_0$ a symplectic structure using the 2-form α_2, where

$$\alpha_2 = \langle \mathscr{J}_0 \cdot, \cdot \rangle_0, \qquad \mathscr{J}_0 = \Upsilon \times \mathscr{J}^Y : E + Y \to E \times Y, \tag{3}$$

in which $\langle \cdot, \cdot \rangle_0$ is the inner product on $E \times Y$, $\mathscr{J}^Y = -(J^Y)^{-1}$ and $\Upsilon(q, I) = (I, -q)$, $(q, I) \in E = \mathbf{R}^n \times \mathbf{R}^n$.

Consider the Hamiltonian

$$\begin{gathered} \mathscr{H}(q, I, y) = h(I) + \tfrac{1}{2}\langle A(I)y, y\rangle + H_2(q, I, y), \\ |H_2(q, I, y)| = O(\|y\|_{d_0}^3), \qquad d_0 \geq d_A/2, \end{gathered} \tag{4}$$

and perturbations of it having the form $\mathscr{H}_\Delta = \mathscr{H} + \varepsilon_0^3 H_3(q, I, y)$. To the Hamiltonian $\mathscr{H}_\Delta$ there corresponds the following Hamiltonian system (see [10] and [11]):

$$\dot{q} = \nabla_I \mathscr{H}_\Delta, \qquad \dot{I} = -\nabla_q \mathscr{H}_\Delta, \qquad \dot{y} = J^Y \nabla_y \mathscr{H}_\Delta = J^Y(A(I)y + \nabla_y H_2 + \varepsilon_0^3 \nabla_y H_3), \tag{5}$$

where ∇_y is the gradient with respect to y relative to $\langle \cdot, \cdot \rangle$. When $\varepsilon_0 = 0$ system (5) has invariant tori $T^n(I_0) = T^n \times \{I_0\} \times \{0\}$ filled with quasiperiodic orbits.

On making the following substitutions into (5) (see [9], p. 171)

$$q = \varphi, \quad I = I_0 + \varepsilon_0^2 p, \quad y = \varepsilon_0 y', \tag{6}$$

we find from (4) that in the new variables the Hamiltonian for system (5) becomes

$$\mathscr{H}_1 = \varepsilon_0^{-2} h(I_0) + \langle A(I_0)y', y'\rangle/2 + \nabla h(I_0) \cdot p + \varepsilon_0 H_4.$$

The analytic function $H_4 = H_4(\varphi, p, y'; I_0)$ and its gradients with respect to φ, p, and y' are uniformly bounded for $0 \leq \varepsilon_0 \leq 1$. We regard the vector I_0 as a parameter.

Suppose that

$$|\det(\partial^2 h/\partial I_i \partial I_j)(I_*)| \geq K_0 > 0. \tag{7}$$

Then on some neighborhood $\mathfrak{A} \subset \mathscr{T}$ of the point $I_0 = I_*$ the substitution $I_0 \to \omega = \nabla h(I_0)$ is analytically invertible. Subtracting the constant term $\varepsilon_0^{-2} h(I_0)$ from $\mathscr{H}$, we obtain that in terms of the variables (6) system (5) has the following form:

$$\begin{gathered} \dot{\varphi} = \nabla_p \mathscr{H}_2, \quad \dot{p} = -\nabla_\varphi \mathscr{H}_2, \quad y' = J^Y \nabla_{y'} \mathscr{H}_2, \\ \mathscr{H}_2 = \omega \cdot p + \langle A(\omega)y', y'\rangle/2 + \varepsilon_0 H_4(\varphi, p, y'; \omega). \end{gathered} \tag{8}$$

The vector ω is a parameter of the problem, and changes at the boundary of the region $\Omega = \nabla h(\mathfrak{A}) \subset \mathbf{R}^n$.

The torus $T_0^n = T^n \times \{0\} \times \{0\}$ is invariant for system (8) when $\varepsilon_0 = 0$ and for all ω. From the author's earlier work [7], [8] it follows that for most values of the parameter $\omega = \nabla h(I_0)$ and for sufficiently small $\varepsilon_0 > 0$ system (8) has an invariant torus close to T_0^n in $\mathscr{Y}_{d_0}$. In terms of (5) the results can be stated as follows:

THEOREM 1. *Suppose that the nondegeneracy condition* (7) *is satisfied as well as the following conditions* 1)–3):

1) (*Analyticity*). *For some $\delta > 0, d_0 \geq d_A/2$, and $d_2 \in \mathbf{R}$ with $d_2 + d_A \geq 2$ and $2d_2 + d_A > d_J$, and for all $d \geq d_0$ the maps*

$$H_j : \mathscr{O}_d(\delta) \to R, \qquad \nabla_y H_j : \mathscr{O}_d(\delta) \to Y_{d+d_2}, \qquad j = 2, 3, \tag{9}$$

are analytic.

2) (*Spectral asymptotics*).

$$|\lambda_j^0(I)| \le C_j \le C \qquad \forall j,\ \forall I \in \mathscr{I}_\delta^C; \tag{10}$$

$$\lambda_{jA} = K_1 j^{d_A} + o(j^{d_A}), \qquad \lambda_{jJ} = K_2 j^{d_J} + o(j^{d_J}),$$

$$\lambda_j \equiv \lambda_{jA}\lambda_{jJ} = K_3 j^{d_1} + K_4 + o(j^{d_1-1-\kappa}), \tag{11}$$

where $d_1 \equiv d_A + d_J \ge 1$ *and* $\kappa > 0$. *Here if* $d_1 = 1$ *then* $\kappa > 0$ *and* $C_j \le Cj^{-\kappa}$ *in* (10).

3) (*Local regularity*). *If* $d_2 < d_J$ *then for all* $0 \le \varepsilon_0 \le 1$ *solutions to* (5) *with initial conditions in* $\mathscr{O}_{d_3}(\delta)$ *with* $d_3 \equiv d_0 + d_2 + d_A$ *exist for all time up to some time* $T > 0$ *and remain uniformly bounded in* $T^n \times \mathscr{I} \times Y_{d_3}$.

Then with these assumptions there exist natural numbers j_1 *and* M_1 *such that if the following condition is satisfied*:

4) (*Nonresonance*).

$$|s \cdot \nabla h(I_*) \pm \lambda_j^A(I_*)\lambda_{jJ}| \ge K_5 > 0 \qquad \forall s \in \mathbf{Z}^n,\ |s| \le M_1,\ \forall j,\ 1 \le j \le j_1; \tag{12}$$

$$|s \cdot \nabla h(I_*) \pm \lambda_j^A(I_*)\lambda_{jJ} \pm \lambda_k^A(I_*)\lambda_{kJ}| \ge K_5 \quad \forall s \in \mathbf{Z}^n,\ |s| \le M_1,$$
$$\forall j,k,\ 1 \le j < k \le j_1,$$

then for sufficiently small $\varepsilon_0 > 0$ *there exist in a neighborhood*

$$\mathscr{K}(\delta_1) = \{I_0 \in \mathscr{I} \,|\, |I_{0j} - I_{*j}| \le \delta_1\ \forall j\}$$

of the point I_* (*where* $\delta_1 > 0$ *is sufficiently small and independent of* ε_0) *a measurable set* Θ_{ε_0} *and a smooth embedding* $\Sigma_{I_0}\colon T^n \to \mathscr{O}_{d_0}(\delta), I_0 \in \Theta_{\varepsilon_0}$, *with the following properties*:

a) $\operatorname{meas}(\mathscr{K}(\delta_1)\backslash\Theta_{\varepsilon_0}) \to 0$ $(\varepsilon_0 \to 0)$.

b) $\operatorname{dist}(\Sigma_{I_0}(T^n), T^n(I_0)) \le C\varepsilon_0^3$.

c) *The map*

$$\Sigma\colon T^n \times \Theta_{\varepsilon_0} \to \mathscr{O}_{d_0}(\delta), \qquad (q,I) \mapsto \Sigma_I(q)$$

is Lipschitz, and if $\mathfrak{I}(q,I) \equiv (q,I,0) \in \mathscr{Y}$ *then the norm of the difference* $\Sigma - \mathfrak{I}$ *and its Lipschitz constant are bounded above by* $C(\rho_1)\varepsilon_0^{\rho_1}$ *for all* $\rho_1 < 1/3$.

d) *The tori* $\Sigma_{I_0}(T^n)$ *are invariant with respect to the flow of* (5) *and are filled with quasiperiodic orbits.*

e) *The number* δ_1 *and the speed of convergence in* a) *are independent of* I_* (*but depend on the constants* K_0 *and* K_5 *in* (7) *and* (12)).

REMARK 1. In the case when $d_J = d_2 = 0$ (that is, when the unboundedness of the linear part of the system is due to the unboundedness of the Hamiltonian), the theorem is proved in [8]. When $d_A = 0$ (that is, when the unboundedness is due to the unboundedness of the operator J^Y of the Poisson structure), it is proved in [7]. The proof in the general case goes by repetition of the arguments of [7], taking account of two observations: 1) In Lemma 3 of [7] the fact that $B(q) \in \mathscr{L}^s_{d_0,d_0+d_2}$ implies that $G(q) \in \mathscr{L}^s_{d_0,d_0+d_2+d_1-1}$. 2) It is enough to prove the estimates of Lemma 1 of [7] with the a priori assumption $\|y(t)\|_{d_3} \le C$, with $T = C_1^{-1}$, and for initial conditions $(q(0),\xi(0),y(0))$ such that $|\xi(0)| \le \varepsilon_m^{2/3}/3$ and $\|y(0)\|_{d_0} \le \varepsilon_m^{1/3}/3$.

REMARK 2. Assertion c) of the theorem follows from intermediate lemmas of [7] and [8] (which were not included in the statement of the theorem). To obtain the estimates b), note that the first few open sets $\mathscr{O}_m$ that appear in the definition of the subsequent changes of variables S_m of [7] can be chosen independently of ε. For the corresponding changes S_m in the variables (q,I,y) we have $\|S_m - \mathrm{Id}\|_{d_0} \le C\varepsilon_0^3 = C\varepsilon$. For changes with high enough numbers in the list the estimate of Lemma 4 of [7] suffices.

COROLLARY 1. *Suppose conditions* 1)–3) *of Theorem* 1 *are satisfied, and* $K_0 > 0$. *Then there exist natural numbers* j_1 *and* M_1, *depending on* K_0, *such that if* $\mathcal{T}^0 \Subset \mathcal{T}$ *is an open set with smooth frontier, then, for all the points* I_* *that satisfy* (7) *and* (12), *for sufficiently small* ε_0 *there exist a measurable subset* $\Theta^0_{\varepsilon_0} \subset \mathcal{T}^0$ *and smooth embeddings* $\Sigma_{I_0}: T^n \to \mathcal{O}_{d_0}(\delta)$, $I_0 \in \Theta^0_{\varepsilon_0}$, *with the property that*

$$\operatorname{meas}(\mathcal{T}^0 \backslash \Theta^0_{\varepsilon_0}) \to 0 \qquad (\varepsilon_0 \to 0) \tag{13}$$

and assertions b)–d) *of Theorem* 1 *hold* (*with* Θ_{ε_0} *replaced by* $\Theta^0_{\varepsilon_0}$).

To deduce the corollary from Theorem 1 it is sufficient to approximate the region $\mathcal{T}^0$ from the inside by a union of nonintersecting cubes $\mathcal{K}(\delta_1)$. The convergence (13) follows from assertions a) and e) of the theorem.

EXAMPLE 1. Analytic sequences of equations of the form

$$\begin{gathered} \dot{x}_j^+ = \partial H/\partial x_j^-, \quad \dot{x}_j^- = -\partial H/\partial x_j^+, \qquad j = 1, 2, \ldots, \\ H = H(I_1, I_2, \ldots), \qquad I_j = (x_j^{+2} + x_j^{-2})/2 \end{gathered} \tag{14}$$

(where H is an analytic function of its arguments) reduce to systems of the form (5). The tori

$$T^n(I_0) = \{x | x_j^{+2} + x_j^{-2} = 2I_{0j},\ j = 1, \ldots, n;\ 0 = x_{n+1}^+ = x_{n+1}^- = x_{n+2}^+ = \cdots\}$$

are invariant for (14).

Close to the torus $T^n(I_0)$, system (14) has the form (5) with $\varepsilon_0 = 0$ after transforming to coordinates (q, I, y), where $I \in \mathcal{J} \Subset \mathbf{R}^n$, $q \in T^n$, $y = (y_1^\pm, y_2^\pm, \ldots)$, and

$$q_j = \operatorname{Arg}(x_j^+ + ix_j^-), \quad 1 \le j \le n; \qquad y_l^\pm = x_{n+l}^\pm, \quad l = 1, 2, \ldots. \tag{15}$$

Here $\lambda_{jJ} \equiv 1$ and $\lambda_j^A(I) = (\partial/\partial I_{n+j}) H(I_1, \ldots, I_n, 0, 0, \ldots)$.

Theorem 1 can thus be applied to Hamiltonian perturbations of (14) if condition (9) is satisfied (with $d_0 = d_A/2$ and $d_2 = 0$) and

$$(\partial/\partial I_{n+j}) H(I_1, \ldots, I_n, 0, 0, \ldots) = Cj^d + o(j^{d-1}), \qquad d \ge 2. \tag{16}$$

It is known that some equations of form (0.1) are completely integrable and by a canonical change of variables they reduce to chains of equations as in (14) (see [2], [5], and [12]). Roughly speaking, condition (16) indicates 1-dimensionality of the system and is easily verifiable. Condition (9) is pretty restrictive. It is satisfied if the change of variables taking (0.1) to (14) has the form

linear operator + smoothing.

For equations of Korteweg-de Vries type the integrating change of variables obviously does not have the necessary smoothing properties.

In §2 below we describe an approach that is applicable to a wider class of nonlinear systems.

§2. Integrable variational equations

Include the space Z in a family of Hilbert spaces $\{Z_t | t \in \mathbf{R}\}$ where $Z_0 = Z$, $Z_{t_1} \subset Z_{t_2}$ for $t_1 \ge t_2$, and $Z_t = Z_{-t}^*$, and suppose that the antiselfadjoint operator J^Z on Z defines an isomorphism $J^Z: Z_{t+d_J} \to Z_t$, $t \in \mathbf{R}$, while the selfadjoint operator A^Z on Z is a Fredholm mapping $A^Z: Z_{t+d_A} \to Z_t$, $t \in \mathbf{R}$. Write $\mathcal{J}^Z = -(J^Z)^{-1}$.

Assume the equation (0.1) has an invariant $2n$-dimensional surface $\mathcal{T}$ which is foliated into invariant n-tori, so that $\mathcal{T} = \Phi_0(T^n \times \mathcal{J})$ where $\mathcal{J}$ is a bounded open set in $\mathbf{R}^n$ and $\Phi_0\colon T^n \times \mathscr{I} \to Z_t$ is an analytic mapping for each $t \in \mathbf{R}$ and the tori $\Phi_0(T^n \times \{I\})$ are invariant for (0.1) for all $I \in \mathscr{I}$.

Suppose $\Phi_0^*(\omega_2^Z) = dI \wedge dq$, where $\omega_2^Z = \langle\langle \mathscr{J}^Z \cdot, \cdot \rangle\rangle$, and the system (0.1) induces a Hamiltonian vector field on $T^n \times \mathscr{I}$ with analytic Hamiltonian $h(I) = \mathscr{H}^Z(\Phi_0(q, I))$:

$$\dot q = \nabla h(I), \qquad \dot I = 0. \tag{1}$$

Then $u_0(t) = \Phi_0(q_0 + t\nabla h(I_0), I_0)$ is a solution to (0.1). Consider the variational equation for (0.1) along $u_0(t)$:

$$\dot v = J^Z(A^Z v + d\nabla H_0^Z(u_0(t))v). \tag{2}$$

We continue with the notation used in §1. Also, let

$$U(\delta) = \{q \in \mathbf{C}^n/2\pi\mathbf{Z}^n \,||\mathrm{Im}\, q| < \delta\},$$
$$\mathscr{O}_d^C(\delta) = \mathscr{O}_d^C(\delta, \mathscr{I}_\delta^C) = U(\delta) \times \mathscr{I}_\delta^C \times \{y \in Y_d^C \,|\, \|y\|_d < \delta\},$$

where $Y_d^C = Y_d \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of Y_d. Let $Z_k^C = Z_k \otimes_{\mathbf{R}} \mathbf{C}$ and $E^C = E \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}^{2n}$.

Definition 1. The variational equation for (0.1) on $\mathscr{T}$ is said to be *integrable* if there is a mapping

$$\Phi_1\colon T^n \times \mathscr{I} \times Y \to Z, \qquad (q, I, y) \mapsto \Phi_1(q, I)y,$$

linear in y such that the following conditions 1)–3) hold.

1) For each $(q, I) \in T^n \times \mathscr{I}$ the mapping $y \mapsto \Phi_1(q, I)y$ is a canonical isomorphism between Y and the skew-orthogonal completion (relative to ω_2^Z) of $T_{q,I}(\mathscr{T})$ in Z. The symplectic structure on Y comes from the form $\omega_2^Y = \langle \mathscr{J}^Y \cdot, \cdot \rangle$, $\mathscr{J}^Y = -(J^Y)^{-1}$.

2) For $t \geq 0$ the mapping

$$Y_t \to Z_t \cap (\Phi_1(q, I)Y), \qquad y \mapsto \Phi_1(q, I)y$$

is an isomorphism. For some $\delta_1 > 0$ the mapping Φ_1 defines a complex-analytic mapping $\Phi_1 : \mathscr{O}_t^C(\delta_1) \to Z_t^C$.

3) The mapping $y(t) \mapsto \Phi_1(q_0 + t\nabla h(I_0), I_0)y(t)$ takes solutions of the equation

$$\dot y(t) = J^Y A(I_0) y(t) \tag{3}$$

to solutions of the variational equation (2) (here the operators $A(I)$ are the same as in §1).

Write $\Phi(q, I, y) = \Phi_0(q, I) + \Phi_1(q, I)y$.

Example 2. Let $\dim Z = 2N_Z < \infty$, $n = 1$. Then equation (2) is linear with periodic coefficients. Since the surface $\mathscr{T}$ is 2-dimensional, the zero characteristic exponent of (2) has multiplicity no less than 2. Suppose its multiplicity is exactly 2 and the nonzero characteristic exponents are purely imaginary and each of multiplicity 1. Then, by the Floquet-Lyapunov theorem, equation (2) has complex solutions

$$\exp(\pm i\lambda_j(I_0)t) w_j^{\pm}(q_0 + th'(I_0); I_0), \qquad 0 \neq \lambda_j(I_0) \in \mathbf{R},\ j = 1, \dots, N_Z - 1,$$

where $w_j^{\pm}(\cdot; I_0)\colon T^1 \to Z^C$ is analytic and $w_j^- = \overline{w_j^+}$. Let $W_j^+ = \mu_j \,\mathrm{Re}\, w_j^+$ and $W_j^- = \mu_j \,\mathrm{Im}\, w_j^+$. By suitable choice of the real numbers μ_j it is possible to arrange that

$$\langle\langle \mathscr{J}^Z W_j^\alpha, W_l^\beta \rangle\rangle = \delta(j, l)\delta(\alpha, -\beta) \quad \forall j, l\ \forall \alpha, \beta = \pm.$$

Let $Y_{q,I}$ be the linear span of the vectors $\{W_j^{\pm}(q, I) | j \geq 1\}$, $Y = Y_{q_0, I_0}$, and let $\Phi_1(q, I)$ be the linear mapping taking $\Phi_j^{\pm} = W_j^{\pm}(q_0, I_0)$ to $W_j^{\pm}(q, I)$. It satisfies all the conditions of Definition 1.

EXAMPLE 3. Under the conditions of Example 1 the mapping $\Phi = \Phi_0 + \Phi_1$ can be expressed using (1.15) as $(q, I, y) \mapsto x$. The variational equation for the chain (1.14) along the surface $\{T^n(I_0) | I_0 \in \mathscr{I} \Subset \mathbf{R}^n\}$ is integrable in the sense of Definition 1.

LEMMA 1. *Under the conditions of Definition* 1, *for all* $(q, I) \in T^n \times \mathscr{I}$, *for some* $\delta_1 > 0$ *and* $u = \Phi_0(q, I)$ *we have*

$$\begin{aligned}\mathscr{F}^Z \Phi_1(q, I) J^Y A(I) + A^Z \Phi_1(q, I) \\ = \mathfrak{A}^0(q, I) \equiv -d\nabla H_0^Z(u)\Phi_1(q, I) - \mathscr{F}^Z(d_q\Phi_1(q, I)\nabla h(I));\end{aligned} \tag{4}$$

$$-A(I) + \Phi_1^*(q, I) A^Z \Phi_1(q, I) = \Phi_1^*(q, I)\mathfrak{A}^0(q, I); \tag{5}$$

$$\|\Phi_1(q, I)\|_{t,t} \le C(t) \quad \forall t \in \mathbf{R}\ \forall (q, I) \in U(\delta_1) \times \mathscr{I}_{\delta_1}^C. \tag{6}$$

PROOF. It follows from clause 3) of Definition 1 that the function $u_1(t) = u_0(t) + \varepsilon\Phi_1(q(t), I_0)y(t)$, where $q(t) = q_0 + t\nabla h(I_0)$ and $y(t)$ is a solution of (3), satisfies equation (0.1) up to terms of order ε^2. Therefore

$$\begin{aligned}\dot u_0 + \varepsilon(d_q\Phi_1(q(t), I_0)\nabla h(I))y + \varepsilon\Phi_1(q(t), I_0)\dot y = \dot u_1 \\ \overset{\mathrm{mod}\ \varepsilon^2}{=} J^Z(A^Z u_1 + \nabla H_0^Z(u_1)) \\ \overset{\mathrm{mod}\ \varepsilon^2}{=} J^Z(A^Z u_0 + \nabla H_0^Z(u_0)) \\ + \varepsilon J^Z(A^Z\Phi_1(q(t), I_0)y + d\nabla H_0^Z(u_0)(\Phi_1(q(t), I_0)y)).\end{aligned}$$

Hence for all q, I, and $u = \Phi_0(q, I)$ we obtain

$$d_q\Phi_1(q, I)\nabla h(I) + \Phi_1(q, I)J^Y A(I) = J^Z A^Z \Phi_1(q, I) + J^Z d\nabla H_0^Z(u)\Phi_1(q, I).$$

Thus (4) holds and

$$\Phi_1^*\mathscr{F}^Z\Phi_1 J^Y A + \Phi_1^* A^Z \Phi_1 = \Phi_1^*\mathfrak{A}^0.$$

But $\Phi_1^*\mathscr{J}^Z\Phi_1 = \mathscr{J}^Y$, and so (5) follows since Φ_1 is a canonical transformation.

The estimate (6) holds for $t \ge 0$ by clause 2) of Definition 1. Since $\Phi_1^* = -\mathscr{J}^Y\Phi_1^{-1}J^Z$ and Φ_1 is canonical, we have $\|\Phi_1^*\|_{l,l} \le C$ for $l \ge d_J$, and so (6) holds also for $t \le -d_J$. For $t \in (-d_J, 0)$ it now follows by an interpolation theorem (see [13], for example). ■

LEMMA 2. *Under the conditions for Definition* 1, *for arbitrary* $t \ge 0$ *and* $\mathscr{I}' \Subset \mathscr{I}$ *there is* $\delta > 0$ *such that* Φ *defines a complex-analytic isomorphism between* $\mathscr{O}_t^C(\delta, \mathscr{I}_\delta'^C)$ *and an open set in* Z_t^C *containing* $\Phi_0(T^n \times \mathscr{I}')$.

This lemma follows from 1), 2), and the inverse mapping theorem.

THEOREM 2. *Suppose equation* (0.1) *has an analytic invariant surface* $\mathscr{T} = \Phi_0(T^n \times \mathscr{I})$ *and the variational equation for* (0.1) *on* $\mathscr{T}$ *is integrable. Suppose that for* $d \ge d_0$ *the maps* $H_j^J : Z_d \to \mathbf{R}$ *and* $\nabla H_j^Z : Z_d \to Z_{d+d_2}$, $j = 1, 2$, *are analytic, and that* $C_j \le C_1 j^{-d_2}$ *in* (1.10); *suppose also condition* 2) *of Theorem* 1 *is satisfied, as well as the analog of condition* 3):

3′) *If* $d_2 < d_J$, *then for all* $0 \le \varepsilon_0 \le 1$ *there exist solutions to* (0.2) *with initial conditions in an arbitrary ball in* Z_{d_3} *defined for all time up to some* $T > 0$ *and remaining uniformly bounded in* Z_{d_3}.

Let $K_0 > 0$. *Then there are natural numbers* j_1 *and* M_1, *depending on* K_0, *such that if* $\mathscr{I}^0 \Subset \mathscr{I}$ *is an open set with smooth frontier then for all points* I_* *for which conditions* (1.7) *and* (1.12) *are satisfied, for all sufficiently small* ε *there exist a measurable subset* $\Theta_\varepsilon^0 \subset \mathscr{I}^0$ *and smooth embeddings* $\Sigma_I : T^n \to Z_{d_0}$, $I \in \Theta_\varepsilon^0$, *with the following properties*:

a) $\operatorname{meas}(\mathscr{I}^0 \setminus \Theta_\varepsilon^0) \to 0$ $(\varepsilon \to 0)$.

b) $\operatorname{dist}(\Sigma_I(T^n), \Phi_0(T^n \times \{I\})) \le C\varepsilon$.

c) *The map* $\Sigma\colon T^n \times \Theta_\varepsilon^0 \to Z_{d_0}$, $(q, I) \mapsto \Sigma_I(q)$ *is Lipschitz, and the norm of the difference* $(\Sigma - \Phi_0)(q, I)$ *and its Lipschitz constant in both variables is at most* $C(\rho_1)\varepsilon^{\rho_1}$ *for all* $\rho_1 < 1/9$.

d) *The tori* $\Sigma_I(T^n)$ *are invariant with respect to the flow of* (0.2) *and are filled with quasiperiodic orbits.*

PROOF. We restrict ourselves to the case $d_J = 0, d_2 = 0, d_A = 2, d_0 = 1$. The proof in the general case differs from this only in that the notation is more cumbersome.

Let $\omega_2 = \Phi^*(\omega_2^Z)$. By condition 1) of Definition 1, $\omega_2(q, I, 0) = \alpha_2$ for all $q \in T^n$ and $I \in \mathscr{I}'$ (see (1.3)). Let $w = (q, I) \in T^n \times \mathscr{I}'$ and $\Delta\omega_2(w, y) = \omega_2(w, y) = -\alpha_2$.

For operators from a family of spaces $\{Y_t^1\}$ to a family of spaces $\{Y_t^2\}$, write $\|\cdot\|_{t_1, t_2}$ for the norm of the operator regarded as a map from $Y_{t_1}^1$ to $Y_{t_2}^2$. In Lemma 3 below we have $Y_t^1 = Y_t^2 = E^C \times Y_t^C$.

LEMMA 3. *For all* $(w, y) \in \mathscr{O}_\theta(\delta)$ *and* $0 \le \theta \le 1$ *we have* $\Delta\omega_2(w, y) = \langle \mathscr{J}'_{wy}\cdot, \cdot\rangle$, *where* $\mathscr{J}'\colon \mathscr{O}_\theta(\delta) \to L(E \times Y_{-\theta}, E \times Y_\theta)$ *is an analytic map. If* δ' *is small enough, then for* $(w, y) \in \mathscr{O}_0^C(\delta')$ *the following estimates hold*:

$$\|\mathscr{F}'_{wy}\|_{-\theta, \theta} \le C\|y\|_\theta \quad \forall \theta \in [0, 1]; \tag{7}$$

$$\|d_y \mathscr{F}'_{wy}(\mathfrak{Z})\|_{-\theta, \theta} \le C\|\mathfrak{Z}\|_\theta (1 + \|y\|_{-\theta}) \quad \forall \mathfrak{Z} \in Y_\theta,\ \forall |\theta| \le 1. \tag{8}$$

The map $\mathscr{J}'_{wy}\colon E \times Y \to E \times Y$ *is given by the operator matrix*

$$\begin{bmatrix} j_{wy}^1 & j_{wy}^{12} \\ j_{wy}^{21} & 0 \end{bmatrix},$$

where

$$\begin{aligned} j^1 &\colon \mathscr{Y}_\theta \to L(E, E), \qquad \theta \ge 0, \\ (j_{wy}^{-1}(\delta w))_k &= \left\langle\!\!\left\langle \mathscr{F}^Z \frac{\partial \Phi_0}{\partial w}\delta w, \frac{\partial \Phi_1}{\partial w_k} y \right\rangle\!\!\right\rangle \\ &\quad + \left\langle\!\!\left\langle \mathscr{F}^Z \left(\frac{d\Phi_1}{\partial w}\delta w\right) y, \frac{\partial \Phi_0}{\partial w_k} \right\rangle\!\!\right\rangle \\ &\quad + \left\langle\!\!\left\langle \mathscr{F}^Z \left(\frac{\partial \Phi_1}{\partial w}\delta w\right) y, \frac{\partial \Phi_1}{\partial w_k} y \right\rangle\!\!\right\rangle; \end{aligned} \tag{9}$$

$$j^{12}\colon \mathscr{Y}_\theta \to L(Y_{-\theta}, E) \quad \forall\theta, \qquad (j_{wy}^{12}\delta y)_k = \left\langle\!\!\left\langle \mathscr{F}^Z \Phi_1 \delta y, \frac{\partial \Phi_1}{\partial w_k} y \right\rangle\!\!\right\rangle; \tag{10}$$

$$j^{21}\colon \mathscr{Y}_\theta \to L(E, Y_\theta) \quad \forall\theta, \qquad j_{wy}^{21}(\delta w) = \Phi_1^*(w)\mathscr{F}^Z \left(\frac{\partial \Phi_1}{\partial w}\delta w\right) y. \tag{11}$$

The maps (9)–(11) *are analytic.*

PROOF. It follows from the form of the map Φ that

$$\begin{aligned}
&\omega_2(w,y)[(\delta w_1,\delta y_1),(\delta w_2,\delta y_2)] \\
&\quad = \omega_2(w,0)[(\delta w_1,\delta y_1),(\delta w_2,\delta y_2)] \\
&\quad + \left\langle\left\langle \mathscr{F}^Z\left(\frac{\partial\Phi_1(w)}{\partial w}\delta w_1\right)y, \left(\frac{\partial\Phi_1(w)}{\partial w}\delta w_2\right)y\right\rangle\right\rangle \\
&\quad + \left\langle\left\langle \mathscr{F}^Z\left(\frac{\partial\Phi_0(w)}{\partial w}\delta w_1 + \Phi_1(w)\delta y_1\right), \left(\frac{\partial\Phi_1(w)}{\partial w}\delta w_2\right)y\right\rangle\right\rangle \\
&\quad + \left\langle\left\langle \mathscr{F}^Z\left(\frac{\partial\Phi_1(w)}{\partial w}\delta w_1\right)y, \frac{\partial\Phi_0(w)}{\partial w}\delta w_2 + \Phi_1(w)\delta y_2\right\rangle\right\rangle.
\end{aligned}$$

Since $\omega_2(w,0) = \alpha_2$, these equations give expressions for the elements of the operator matrix, and together with (6) they give estimates (7) and (8) and the analyticity of the maps (9)–(11). ■

Let $\langle\cdot,\cdot\rangle_E$ be the euclidean inner product on E, and denote the inner product on $E \times Y$ by $\langle\cdot,\cdot\rangle_0$.

LEMMA 4 (Relative Poincaré lemma; see [14]). *The form $\Delta\omega_2$ is exact*:

$$\Delta\omega_2 = d\omega_1, \qquad \omega_1(w,y)[\xi] = \langle\Omega(w,y),\xi\rangle_0,$$

where

$$\Omega\colon \mathscr{O}_0(\delta) \to E\times\{0\} \subset E\times Y, \qquad (w,y)\to \tfrac{1}{2}(j^{12}_{wy}y,0) \tag{12}$$

is an analytic map. Moreover,

$$\Omega(w,0) \equiv 0, \qquad d\Omega(w,0) \equiv 0. \tag{13}$$

PROOF. Since $\Delta\omega_2(w,0) = 0$, there exists a form ω_1 such that $d\omega_1 = \Delta\omega_2$ on the open set $\mathscr{O}_0(\delta)$, where $\delta > 0$ is sufficiently small, and ω_1 is obtained from $\Delta\omega_2$ by a cone retraction process ([10], §36; [15], Chapter 4):

$$\omega_1(w,y)[\xi] = \int_0^1 \Delta\omega_2(w,ty)[(0,y),\xi^t]\,dt,$$

where for $\xi = (\xi_w,\xi_y) \in E\times Y$ we have $\xi^t = (\xi_w, t\xi_y)$. Lemma 3 gives

$$\omega_1(w,y)[\xi] = \int_0^1 \langle J'_{wty}(0,y),\xi^t\rangle_0\,dt = \int_0^1 t\langle j^{12}_{wy}y,\xi_w\rangle_E\,dt,$$

which implies (12) and all the assertions of the lemma. ■

Let $J_0 = \Upsilon\times J^Y\colon E\times Y \to E\times Y$ (see (1.3)). Then $J_0 = -\mathscr{J}_0^{-1}$. Let $\mathscr{F}^t_{wy} = \mathscr{J}_0 + t\mathscr{J}'_{wy}$ for $0 \le t \le 1$. In view of estimate (7) the operator $\mathscr{F}^t_{wy}$ is invertible for small y (recall that $d_J = 0$).

LEMMA 5. *If δ is sufficiently small, $0 \le \theta \le 1$, and $(w,y) \in \mathscr{O}^C_\theta(\delta)$, then $(\mathscr{F}^t_{wy})^{-1}$: $Y^C_\theta \to Y^C_\theta$ is a bounded operator, depending analytically on $(w,y) \in \mathscr{O}^C_\theta(\delta)$. We have*

$$(\mathscr{F}^t_{wy})^{-1} = -(I - tJ_0\mathscr{F}'_{wy})^{-1}J_0 = -\sum_{n=0}^{\infty}(tJ_0\mathscr{F}'_{wy})^n J_0. \tag{14}$$

Consider the nonautonomous vector field on $\mathscr{O}^C_\theta(\delta)$, $0 \le \theta \le 1$:

$$V^t_{wy} = (\mathscr{F}^t_{wy})^{-1}\Omega(w,y) = -\frac{1}{2}\sum_{n=0}^{\infty}(tJ_0\mathscr{F}'_{wy})^n(\Upsilon j^{12}_{wy}y,0). \tag{15}$$

By Lemmas 4 and 5 the equation

$$\dot{\xi}(t) = V_\xi^t, \qquad \xi = (w, y),\ 0 \le t \le 1, \tag{16}$$

defines an analytic flow

$$S^t : \mathscr{O}_\theta^C(\delta') \to \mathscr{O}_\theta^C(\delta), \quad S^t(\mathscr{O}_\theta(\delta')) \subset \mathscr{O}_\theta(\delta); \qquad 0 \le \theta \le 1,\ 0 \le t \le 1,$$

for some $\delta' > 0$. By (13) we have

$$S^t(w, 0) \equiv (w, 0), \qquad dS^t(w, 0) = \mathrm{Id}. \tag{17}$$

LEMMA 6 (Relative Darboux lemma; see [14] and [15]). *The map S^1 takes ω_2 to α_2; that is, $S^{1*}\omega_2 = \alpha_2$.*

Let Π_y be the natural projection from $\mathscr{Y} = T^n \times \mathbf{R}^n \times Y$ to Y, and write

$$\tfrac{1}{2}\langle A(I)\Pi_Y S^t(w, y),\ \Pi_Y S^t(w, y)\rangle = \mathfrak{B}^t(w, y).$$

LEMMA 7. *There is $\delta_1 > 0$ such that the maps $\mathfrak{B}^1 - \mathfrak{B}^0 : \mathscr{O}_1(\delta_1) \to \mathbf{R}$ and $\nabla_y(\mathfrak{B}^1 - \mathfrak{B}^0) : \mathscr{O}_1(\delta_1) \to Y_1$ are analytic, and for $(w, y) \in \mathscr{O}_1^C(\delta_1)$*

$$\|(\mathfrak{B}^1 - \mathfrak{B}^0)(w, y)\| \le C\|y\|_0^4, \tag{18}$$

$$\|\nabla_y(\mathfrak{B}^1 - \mathfrak{B}^0)(w, y)\|_1 \le C\|y\|_1^3. \tag{19}$$

The lemma clearly holds in the finite-dimensional case. In the infinite-dimensional case it is a consequence of Lemma 1. We give the proof in §3.

By Lemma 6, the transformation $\Phi \circ S^1$ is a canonical diffeomorphism between $(\mathscr{O}_1(\delta), \alpha_2)$ and an open set in (Z_1, ω_2^Z). Under this transformation equation (0.1) is taken to an equation on $\mathscr{O}_1(\delta)$ with Hamiltonian $\mathscr{H} = \mathscr{H}^Z(\Phi \circ S^1(w, y))$. We shall find $\mathscr{H}$ in two stages.

LEMMA 8. *For $(w, y) \in \mathscr{O}_1(\delta)$*

$$\mathscr{H}_1(w, y) \equiv \mathscr{H}^Z(\Phi(w, y)) = \tfrac{1}{2}\langle A(I)y, y\rangle + H_5(w, y), \tag{20}$$

where the maps

$$H_5 : \mathscr{O}_1(\delta) \to \mathbf{R}, \qquad \nabla_y H_5 : \mathscr{O}_1(\delta) \to Y_1$$

are analytic.

PROOF. Since $\mathscr{H}^Z(\Phi_0(q, I)) = h(I)$, we have

$$\mathscr{H}_1(w, y) = h(I) + \tfrac{1}{2}\langle \Phi_1^* A^Z \Phi_1 y, y\rangle + \langle\langle \Phi_1 y, A^Z \Phi_0\rangle\rangle + H_0^Z(\Phi_0 + \Phi_1 y) - H_0^Z(\Phi_0).$$

By Lemma 1 the right-hand side of this equation is equal to the right-hand side of (20) if we put

$$H_5 = H_0^Z(\Phi_0 + \Phi_{1y}) - H_0^Z(\Phi_0) + \langle\langle \Phi_1 y, A^Z \Phi_0\rangle\rangle + \tfrac{1}{2}\langle \Phi_1^* \mathfrak{A}^0 y, y\rangle + h(I).$$

The analyticity of H_5 and $\nabla_y H_5$ follows from the form of H_5. ■

From (20) we obtain

$$\mathscr{H}(w, y) \equiv \mathscr{H}_1(S^1(w, y)) = \tfrac{1}{2}\langle A(I)y, y\rangle + H_6(w, y),$$

$$H_6(w, y) = \mathfrak{B}^1(w, y) - \mathfrak{B}^0(w, y) + H_5(S^1(w, y)).$$

By the analyticity of the flow S^t and Lemmas 7 and 8 the maps

$$H_6 : \mathscr{O}_1(\delta) \to \mathbf{R}, \qquad \nabla_y H_6 : \mathscr{O}_1(\delta) \to Y_1 \tag{21}$$

are analytic. Write

$$H_6(w,y) = h^0(w) + \langle h^1(w), y\rangle + \tfrac{1}{2}\langle A_h(w)y, y\rangle + H_7(w,y),$$
$$H_7(w,y) = O(\|y\|_1^3).$$

Since the surface $\mathscr{T}$ is invariant with respect to (0.1), it follows that the surface $T^n \times \mathscr{I}' \times \{0\}$ is invariant with respect to the equation with Hamiltonian $\mathscr{H}$. Thus $h^1(w) \equiv 0$. As the transformation S^1 is the identity when $y = 0$, the curves $q(t) = q_0 + t\nabla h(I_0)$, $I(t) = I_0$, are orbits of the equation with Hamiltonian $\mathscr{H}$. Therefore $h^0(w) = h(I)$. Hence

$$\mathscr{H}(w,y) = h(I) + \tfrac{1}{2}\langle (A(I) + A_h(w))y, y\rangle + H_7(w,y).$$

To the Hamiltonian $\mathscr{H}$ corresponds the following Hamiltonian system:

$$\dot{q} = \nabla_I h + \tfrac{1}{2}\langle \nabla_I (A(I) + A_h)y, y\rangle + \nabla_I H_7, \tag{22}$$
$$\dot{I} = -\tfrac{1}{2}\langle \nabla_q A_h y, y\rangle - \nabla_q H_7, \tag{23}$$
$$\dot{y}_1' = J^Y((A(I) + A_h)y + \nabla_y H_7). \tag{24}$$

Let $\mathfrak{z}_\varepsilon(t) = (q_0 + t\nabla h(I_0), I_0, \varepsilon y(t))$, where $\dot{y}(t) = J^Y A(I_0)y$. From Definition 1, the curve $\Phi(\mathfrak{z}_\varepsilon(t))$ satisfies (0.1) up to terms of order ε^2. By (17)

$$(S^1)^{-1}\mathfrak{z}_\varepsilon(t) = \mathfrak{z}_\varepsilon(t) \qquad \mathrm{mod}\ \varepsilon^2.$$

Thus $\mathfrak{z}_\varepsilon(t)$ satisfies (22)–(24) up to terms of order ε^2. From (24) we obtain

$$J^Y A(I_0)y = J^Y (A(I) + A_h(w))y \qquad \mathrm{mod}\ \varepsilon,$$

where $w = w(t) = (q_0 + t\nabla h(I_0), I_0)$.

Consequently, $A(I_0) + A_h(w) = A(I_0)$, and the canonical transformation $\Phi \circ S^1$ takes (0.1) to an equation with Hamiltonian

$$\mathscr{H}(q,I,y) = h(I) + \tfrac{1}{2}\langle A(I)y, y\rangle + H_7(q,I,y), \qquad |H_7(q,I,y)| = O(\|y\|_1^3),$$

that has the form (1.3), (1.4). In view of the analyticity of (21) and clause 3) of Definition 1 the Hamiltonian $\mathscr{H}$ satisfies conditions 1) and 2) of Theorem 1. Since $\Phi\circ S^1$ is analytic, it takes (0.2) to an equation with Hamiltonian $\mathscr{H} + \varepsilon H_8(q,I,y)$, and for H_8 the analogs of the maps (21) are analytic. Therefore with $\varepsilon_0 = \varepsilon^{1/3}$ Corollary 1 can be applied to the equation with Hamiltonian $\mathscr{H} + \varepsilon H_8$, and this gives Theorem 2.

§3. Proof of Lemma 7

Let $\Pi_E \colon E\times Y \to E$ and $\Pi_Y \colon E \times Y \to Y$ be the natural projections; let $\Pi_{E_j}(e,y) = e_j$, $1 \le j \le 2n$, and

$$\Pi_E^* \colon E \to E\times Y, \quad e \mapsto (e,0); \qquad \Pi_Y^* \colon Y \to E \times Y, \quad y \mapsto (0,y).$$

Then; if $\xi(t) = (w(t), y(t))$ is the solution to (2.16), we have

$$\dot{y}(t) = \Pi_Y V_\xi^t = -\frac{1}{2}\sum_{m=1}^{\infty} t^m \Pi_Y (J_0 \mathscr{F}_{wy}')^m \Pi_E^* \Upsilon j_{wy}^{12} y. \tag{1}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\mathfrak{B}^t(w_0,y_0) &= \langle A(I)y(t), \dot{y}(t)\rangle \\ &= -\frac{t}{2}\sum_{m=0}^{\infty} t^m \langle \mathscr{F}_{wy}^1 J_0 \Pi_Y^* A(I)y(t), (J_0\mathscr{F}_{wy}')^m \Pi_E^* \Upsilon j_{wy}^{12} y(t)\rangle_0.\end{aligned} \tag{2}$$

Let $t^m s_m$ denote the mth term under the summation sign. Then

$$\begin{aligned} s_m &= \langle \mathscr{F}'_{wy} J_0 \Pi_Y^* A(I) y(t), (J_0 \mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12} y(t) \rangle_0 \\ &= \langle \Pi_E^* j_{wy}^{12} J^Y A(I) y(t), (J_0 \mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12} y(t) \rangle_0 \\ &= \langle j_{wy}^{12} J^Y A(I) y(t), \Pi_E (\mathscr{F}_0 \mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12} y(t) \rangle_E . \end{aligned}$$

From (2.10) we find that

$$s_m = \sum_{j=1}^{2n} \left\langle\!\!\left\langle \mathscr{Y}^Z \Phi_1(w) J^Y A(I) y(t), \frac{\partial \Phi_1(w)}{\partial w_j} y(t) \right\rangle\!\!\right\rangle \Pi_{E_j} (J_0 \mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12} y(t).$$

By Lemma 1

$$\left\langle\!\!\left\langle \mathscr{F}^Z \Phi_1 J^Y A^Y y, \frac{\partial \Phi_1}{\partial w_j} y \right\rangle\!\!\right\rangle = - \left\langle\!\!\left\langle A^Z \Phi_1 y, \frac{\partial \Phi_1}{\partial w_j} y \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \mathfrak{A}^0(w) y, \frac{\partial \Phi_1}{\partial w_j} y \right\rangle\!\!\right\rangle .$$

From (2.5) of Lemma 1

$$\begin{aligned} \left\langle\!\!\left\langle A^Z \Phi_1(w) y, \frac{\partial \Phi_1(w)}{\partial w_j} y \right\rangle\!\!\right\rangle &= \frac{1}{2} \frac{\partial}{\partial w_j} \langle\langle A^Z \Phi_1(w) y, \Phi_1(w) y \rangle\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial w_j} [\langle\langle \mathfrak{A}^0(w) y, \Phi_1(w) y \rangle\rangle + \langle\langle (A(I) - A^Y) y, y \rangle\rangle] \end{aligned}$$

(using the fact that $(\partial/\partial w_j) A(I) = (\partial/\partial w_j)(A(I) - A^Y)$). Hence

$$s_m = \sum_{j=1}^{2n} s_m^{j1} s_m^{j2}, \qquad s_m^{j2} = \Pi_{E_j} (J_0 \mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12} y,$$

$$s_m^{j1} = \left\langle\!\!\left\langle \mathfrak{A}^0(w) y, \frac{\partial \Phi_1}{\partial \omega_j} y \right\rangle\!\!\right\rangle - \frac{1}{2} \frac{\partial}{\partial w_j} [\langle\langle \mathfrak{A}^0(w) y, \Phi_1(w) y \rangle\rangle + \langle\langle (A(I) - A^Y) y, y \rangle\rangle].$$

From clause 2) of Definition 1 and the Cauchy estimate for the derivatives of an analytic function, for $w \in U(\delta') \times \mathscr{J}_\delta^C$ and $0 \le \theta \le 1$ we have

$$\|(\partial/\partial w_j) \Phi_1(w)\|_{\theta,\theta} \le C, \qquad j = 1, \dots, 2n. \tag{3}$$

It follows from (3) and (2.7) that

$$|s_m^{j1}| \le \|y\|_0^2, \qquad |s_m^{j2}| \le \|y\|_0^{m+2} \tag{4}$$

for $(w, y) \in \mathscr{O}_0^C(\delta)$. The estimates (4) and equation (2) give us (2.18). Turning now to the estimate for the gradient of s_m:

$$\nabla_y s_m = \Sigma^1 + \Sigma^2, \quad \Sigma^1 = \sum_j (\nabla_y s_m^{j1}) s_m^{j2}, \quad \Sigma^2 = \sum_j s_m^{j1} \nabla_y s_m^{j2}$$

we find from (3) and Lemma 3 that

$$\|\Sigma^1\|_1 \le C \|y\|_1 \|y\|_0^{m+2}. \tag{5}$$

To estimate Σ^2, consider the map $s_m^{j2} : \mathscr{O}_0(\delta) \to \mathbf{R}$. Write its differential with respect to y in the following form:

$$d_y s_m^{j2}(w, y) \mathfrak{z} = D_1(\mathfrak{z}) + D_2(\mathfrak{z}) + \sum_{l=0}^{m-1} D_3^l(\mathfrak{z}),$$

where

$$D_1(\mathfrak{z}) = \Pi_{E_j}(J_0\mathscr{F}'_{wy})^m \Pi_E^* \Upsilon j_{wy}^{12}\mathfrak{z}, \qquad D_2(\mathfrak{z}) = \Pi_{E_j}(J_0\mathscr{F}'_{wy})^m \Pi_E^* \Upsilon (d_y j_{wy}^{12}(\mathfrak{z}))y,$$
$$D_3^l(\mathfrak{z}) = \Pi_{E_j}(J_0\mathscr{F}'_{wy})^l (J_0 d_y J'_{wy}(\mathfrak{z}))(J_0\mathscr{F}'_{wy})^{m-l-1} \circ \Pi_E^* \Upsilon j_{wy}^{12} y.$$

From (2.10) and (2.7) with $\theta = 1$ and the Cauchy estimate we have

$$\|D_1(\mathfrak{z})\|_1 \le C_1 \|y\|_1^{m+1} \|\mathfrak{z}\|_{-1}.$$

From (2.10) with $\theta = -1$, (2.7) with $\theta = 1$, and the Cauchy estimate we have

$$\|D_2(\mathfrak{z})\|_1 \le C_2 \|y\|_1^{m+1} \|\mathfrak{z}\|_{-1}.$$

From (2.10), (2.7) with $\theta = 1$, (2.8) with $\theta = -1$, and the Cauchy estimate we have

$$\|D_3^l(\mathfrak{z})\|_1 \le C_3 \|y\|_1^{m+1} \|\mathfrak{z}\|_{-1}.$$

Therefore $\|\nabla_y s_m^{j2}\|_1 \le C_4 m \|y\|_1^{m+1}$. This inequality together with (4) and (5) gives us $\|\nabla_y s_m\|_1 \le C_5 m \|y\|_1^{m+1}$. Hence (2) implies (2.19) if δ_1 is sufficiently small.

§4. Perturbation of quasiperiodic solutions of the Korteweg-de Vries equation

Equations of the form (0.1) which are integrable in terms of theta-functions (see [2] and [16]) possess a large supply of quasiperiodic solutions. The variational equations along manifolds filled with such solutions are integrable, and Theorem 2 can be applied to the study of perturbed equations. We illustrate this using the example of the Korteweg-de Vries equation in the space of 2π-periodic functions with zero mean:

$$\dot{u}(t,x) = -u_{xxx} + 6uu_x, \tag{1}$$

$$u(t,x) \equiv u(t,x+2\pi), \qquad \int_0^{2\pi} u(t,x)\,dx \equiv 0. \tag{2}$$

Problem (1), (2) is an equation of the form (0.1), where Z is the L_2-space of functions on $[0,2\pi]$ with zero mean; $J^Z = \partial/\partial x$; $-\mathscr{J}^Z = (J^Z)^{-1}\colon Z \to Z$ is the operator of integration with zero mean; $H_0^Z = \int u^3\,dx$; $A^Z = -\partial^2/\partial x^2$; $D(A^Z) = \{u \in H_p^2(0,2\pi) \cap Z\}$ (where $H_p^k(0,2\pi)$ is the Sobolev space of 2π-periodic functions).

We may take as the $2n$-dimensional invariant manifold $\mathscr{T}$ the family of n-gap potentials $u(x) \in Z$ the spectrum of whose Sturm-Liouville operator $L_u = -\partial^2/\partial x^2 + u$ in $L_2(\mathbf{R})$ has n prohibited gaps (lacunae) $[E_{2j}, E_{2j+1}]$, $j = 0, \dots, n$ $(E_0 = -\infty)$ and a double periodic spectrum $\{e_j | j \in \mathbf{N}\setminus\mathscr{N}\}$, $|\mathscr{N}| = n$ (see [3] and [5]). Let $\mu_1, \mu_2, \dots$ be the spectrum of the operator L_u on the space of functions with zero Dirichlet data at the ends of the interval $[0,2\pi]$ and $\mu_j \in [E_{2j}, E_{2j+1}]$, $1 \le j \le n$. The invariant tori $T^n(I)$ consist of n-gap potentials with fixed gap boundaries (depending on I) and with eigenvalues $\mu_1, \dots, \mu_n$ varying within the corresponding gaps.

By the It·s-Matveev formula (see [3]) the map $\Phi_0\colon T^n \times \mathscr{I} \to \mathscr{T}$, $\mathscr{I} \Subset \mathbf{R}^n$, has the following form:

$$\Phi_0(q,I)(x) = -2(\partial^2/\partial x^2)\theta(U_I x + iq + Z_I).$$

Here θ is the theta-function on $\mathbf{C}^n$ having periods $2\pi i f_1, \dots, 2\pi i f_n$ and quasiperiods $\xi_1, \dots, \xi_n$ ($f_1, \dots, f_n$ are unit vectors in $\mathbf{R}^n \subset \mathbf{C}^n$, while $\xi_1, \dots, \xi_n$ is a basis in $\mathbf{R}^n$ that depends on I (see [16]). The vectors U_I and Z_I depend on I and are purely imaginary, with $U_I \in i\mathbf{Z}^n$. Solutions to problem (1), (2) lying on $\mathscr{T}$ are

$$-2(\partial^2/\partial x^2)\theta(U_I x + \mathscr{W}_I t + Z_I), \qquad \mathscr{W}_I = i\nabla h(I) \in i\mathbf{R}^n, \tag{3}$$

where $h(I)$ is the Hamiltonian of the restriction of (1), (2) to $\mathscr{T}$, expressed in terms of the variables (q, I).

In [6] the following explicit formulas were given for solutions for the variational equation along the solution (3):

$$\begin{aligned} w_s^{\pm}(x,t) &= e^{\pm iV_s^0 t}\frac{1}{\pi s^2}\frac{\partial^2}{\partial x^2}\left[e^{\pm isx}\frac{\theta(U_I x + \mathscr{W}_I t + Z_I \pm 2A(e_s))}{\theta(U_I x + \mathscr{W}_I t + Z_I)}\right] \\ &= \frac{1}{\pi}e^{\pm iV_s^0 t}e^{\pm isx}(-1 + \Delta w_s^{\pm}(x; \mathscr{W}_I t + Z_I)), \\ \Delta w_s^{\pm}(x;z) &= e^{\mp isx}\frac{\partial^2}{\partial x^2}\left[e^{\pm isx}\frac{\theta(U_I x + z + 2A(e_s)) - \theta(U_I x + z)}{s^2\theta(U_I x + z)}\right]. \end{aligned} \tag{4}$$

Here $s \in \mathbf{N}\backslash\mathscr{N}$, $V_s^0 \in \mathbf{R}$, and $A(e_s) \in i\mathbf{R}^n$.

The restriction of θ to $i\mathbf{R}^n \subset \mathbf{C}^n$ defines an analytic function on the torus $iT^n = i\mathbf{R}^n/2\pi i\mathbf{Z}^n$, analytically depending on I and admitting an analytic extension to some open set $\mathscr{T}_\delta^C$, $\delta > 0$. Since $U_I \in i\mathbf{Z}^n$ we have $\Delta w_s^{\pm}(\cdot, q) \in H_p^k(0, 2\pi)$ for all k.

The vector $A(e_s)$ has the following form:

$$A_i(e_s) = \int_{E_1}^{e_s} \Omega_i,$$

where $\{\Omega_i\}$ is a normalized basis of the holomorphic differentials on the Riemann surface corresponding to the torus $T^n(I)$. Therefore $2A(\infty) = L \in \Gamma$ where Γ is the lattice in $\mathbf{C}^n$ generated by the period and quasiperiod vectors of the function θ. Since $e_s = s^2 + C_s$ where the C_s are uniformly bounded, we have

$$2A_i(e_s) - L_i \sim \int_{s^2}^{\infty} \Omega_i = \sum_{l=1}^{n} C_{il}\int_{s^2}^{\infty} E^{l-1}P_{2n+1}(E)^{-1/2}\,dE$$

(P_{2n+1} is a polynomial of degree $2n+1$). Therefore

$$|2A_i(e_s) - L_i| \le Cs^{-1}. \tag{5}$$

For $q \in T^n$ and $s \in \mathbf{N}\backslash\mathscr{N}$ let

$$\begin{aligned} W_s^+(x,q) &= \pi^{-1}\operatorname{Re} e^{isx}(-1 + \Delta w_s^+(x, iq)), \\ W_s^-(x,q) &= \pi^{-1}\operatorname{Im} e^{isx}(-1 + \Delta w_s^+(x, iq)). \end{aligned}$$

LEMMA 9. *For all q and s*

$$\langle \mathscr{F}^Z W_s^{\pm}(\cdot, q), W_r^{\pm}(\cdot, q)\rangle = 0 \quad \forall r \ne s, \tag{6}$$

$$\langle \mathscr{F}^Z W_s^+(\cdot, q), W_s^-(\cdot, q)\rangle = c_s \ne 0, \tag{7}$$

$$\langle \mathscr{F}^Z W_s^{\pm}(\cdot, q), \xi\rangle = 0 \quad \forall \xi \in T_q\mathscr{T}. \tag{8}$$

PROOF. Let $\mathscr{T}^{2n+4}$ be the manifold of finite-gap potentials which in addition have open gaps corresponding to eigenvalues e_r and e_s with $r, s \in \mathbf{N}\backslash\mathscr{N}$ and $r < s$. Let $(\varphi_1, I_1, \dots, \varphi_n, I_n, \varphi_r', I_r', \varphi_s', I_s')$, where $\varphi_j \in \mathbf{R}/2\pi\mathbf{Z}$ and $I_j \ge 0$, be action-angle variables for the system induced on $\mathscr{T}^{2n+4}$ (see [10] and [12]). Choose these in such a way that $\{(\varphi, I) | I_r' = I_s' = 0\} = \mathscr{T}$. Let $h^{2n+4}(I_1, \dots, I_n, I_r', I_s')$ be the Hamiltonian system in these variables. Then solutions of the variational equation along $\mathscr{T}$ that are obtained by varying the sth gap take the form

$$\begin{aligned} &\delta\varphi_\mu(t) = \text{const}, \quad \delta I_\mu(t) = \text{const}, \qquad \mu = 1, \dots, n, \\ &\delta I_r'(t) = 0, \quad \delta I_s'(t) = \text{const}, \quad \delta\varphi_s'(t) = \delta\varphi_s'(0) + t(\partial h^{2n+4}/\partial I_s')(I_1, \dots, I_n, 0, 0). \end{aligned} \tag{9}$$

However, the solutions (4) also correspond to varying the sth gap. Therefore their real and imaginary parts are as in (9). This means that the linear span of the vectors $W_s^{\pm}$ forms the tangent space at the point $(\varphi_1, I_1, \dots, \varphi_n, I_n) \in \mathscr{T}$ to the surface

$$\{(\varphi_1, I_1, \dots, \varphi_n, I_n, 0, 0, \varphi_s', I_s') | \varphi_s' \in \mathbf{R}/2\pi\mathbf{Z},\ I_s' \geq 0\}.$$

This implies (6) and (8), since in the action-angle variables the symplectic form is $dI \wedge d\varphi$.

Next we turn to (7). First we show that the left-hand side of (7) is independent of q. Translation along the orbits of the variational equation from q_0 to the point $q_0 + t\nabla h$ is a canonical transformation. By (4) it can be split into the composition of the transformation

$$W_s^{\pm}(\cdot, q_0) \mapsto W_s^{\pm}(\cdot, q_0 + t\nabla h) \tag{10}$$

with rotation through an angle of $\pm V_s^0 t$. The rotation transformation is canonical, and therefore (10) is also canonical. Hence the left-hand side of (7) does not depend on q. It is not equal to zero because of (6) and the nondegeneracy of $\mathscr{J}^Z$. ■

Let $\mathscr{L}_0(q)$ be the closure in Z of the linear span of the vectors $W_s^{\pm}(q) = W_s^{\pm}(\cdot, q)$, $s \in \mathbf{N}\backslash\mathscr{N}$, and let $\mathscr{L}_1(q)$ be the skew-orthogonal complement (relative to the form $\langle\mathscr{J}^Z\cdot, \cdot\rangle$) of $T_q(\mathscr{T})$ in Z.

LEMMA 10. *For all $(q, I) \in T^n \times \mathscr{T}$ we have $\mathscr{L}_0(q) = \mathscr{L}_1(q)$.*

PROOF. Lemma 9 gives $\mathscr{L}_0(q) \subset \mathscr{L}_1(q)$. Since the codimension of $\mathscr{L}_1(q)$ in Z is $2n$ it suffices to verify that $\operatorname{codim}\mathscr{L}_0(q) \leq 2n$.

Let the functional H be the highest KdV equation whose critical point set is $T^n(I)$. By Theorem 2 of [6] the splitting

$$L_2 \equiv L_2(0, 2\pi) = \operatorname{Ker} D^2H \oplus \overline{\operatorname{Im} D^2H}$$

has the form

$$L_2 = \operatorname{Ker} D^2H \oplus (\mathscr{L}_0 + \mathscr{L}_2),$$
$$\mathscr{L}_2 = \operatorname{span}\{\psi^2(\cdot, E_{2i-1}) | i = 1, \dots, n+1\}, \tag{11}$$

where $L_u\psi(x, E_l) = E_l\psi(x, E_l)$ and $\psi(x + 2\pi, E_l) \equiv \psi(x, E_l)$, $l = 1, \dots, 2n+1$.

Let $\Pi_Z : L_2 \to Z$ be orthogonal projection. Since $\Pi_Z\mathscr{L}_0 = \mathscr{L}_0$, it follows from (11) that $Z = \mathscr{L}_0 + \Pi_Z \operatorname{Ker} D^2H + \Pi_Z\mathscr{L}_2$. There are real numbers $\varepsilon_1, \dots, \varepsilon_{n+1}$ for which

$$\varepsilon_1\psi^2(x, E_1) + \varepsilon_2\psi^2(x, E_3) + \dots + \varepsilon_{n+1}\psi^2(x, E_{2n+1}) = 1$$

(see [5], §6, for example). Hence $\dim \Pi_Z\mathscr{L}_2 \leq n$.

From Theorem 1 of [6] it follows that $D^2H(\delta u) = 0$ only if the variation in the δu direction does not change the periodic spectrum of L_u. Thus $\operatorname{Ker} D^2H = T_q(T^n(I))$. Hence $\dim \operatorname{Ker} D^2H \leq n$. This implies $\operatorname{codim}\mathscr{L}_0 \leq 2n$, and the lemma is proved.

LEMMA 11. *For all $s \in \mathbf{N}\backslash\mathscr{N}$*

$$V_s^0 = s^3 + C(s)s, \qquad |C(s)| \leq C. \tag{12}$$

PROOF. The V_s^0 have the form

$$V_s^0 = \int_{E_1}^{e_s} \omega_3, \quad \omega_3 = K(z^{-4} + \varphi(z))\, dz, \qquad z = E^{-1/2}$$

(see [3]). Since $e_s = s^2 + c'(s)$, we have $V_s^0 = K's^3 + O(s)$. Substituting $w_s^{\pm}$ into the variational equation and considering only the terms of order s^3, we find that $K' = 1$. ■

LEMMA 12. *For all $q \in T^n$, $s \in \mathbf{N}\backslash\mathcal{N}$, and $m \in \mathbf{N}$,*

$$\|\Delta w_s^{\pm}(\cdot, q)\|_m \le C''(m)s^{m-1},$$

where $\|\cdot\|_m$ is the norm on the space $H_p^m(0, 2\pi)$.

The estimate follows from (5), the Cauchy inequality and the fact that the theta-function is bounded away from zero on the imaginary torus $i\mathbf{R}^n/2\pi i\mathbf{Z}^n$.

Let

$$Z_s = Z \cap H_p^s(0, 2\pi), \qquad Y^C = \bigoplus_{k\in\mathbf{N}\backslash\mathcal{N}} (\mathbf{C}e^{ikx} \oplus \mathbf{C}e^{-ikx}) \subset Z^C,$$

$$Y_s = Y^C \cap Z_s, \qquad J = J^Z|_Y.$$

For $s \ge 0$ define the following maps:

$$\Phi_1 : \mathscr{O}_s^C(\delta, \mathscr{T}_\delta^C) \to Z_s, \qquad (q, I, \cos lx) \to (\pi/\sqrt{c_l})W_l^+(x, q, I),$$
$$(q, I, \sin lx) \mapsto (\pi/\sqrt{c_l})W_l^-(x, q, I);$$
$$A(I) \colon Y_s^C \to Y_{s-2}^C, \qquad e^{\pm ilx} \mapsto (V_l^0/l)e^{\pm ilx}.$$

By Lemmas 9 and 10 the map Φ_1 satisfies condition 1) of Definition 1, by Lemma 12 it satisfies condition 2), and by Lemma 11 and (4) it satisfies condition 3). Therefore the variational equation for problem (1), (2) along the manifold of n-gap potentials is integrable in the sense of Definition 1.

We consider a perturbation of problem (1), (2), namely the problem (13), (2):

$$\dot{u}(t, x) = -u_{xxx} + \frac{\partial}{\partial x}(3u^2 + \varepsilon\varphi(u)), \tag{13}$$

where φ is a real-analytic function. This problem is locally regular in the sense of condition 3′) of Theorem 2 (see [17], for example).

As above, let $\mathscr{T}$ be the manifold of n-gap potentials (more precisely, its compact part, which is invariant under the flow of problem (1), (2)). As a corollary of Theorem 2 with $d_A = 2, d_0 = 1, d_J = 1$, and $d_2 = 0$ we obtain

THEOREM 3. *Let $K_0 > 0$. There are natural numbers j_1 and M_1, depending on K_0, such that if $\mathscr{T}^0 \Subset \mathscr{T}$ is an open set with smooth frontier, for all points for which*

$$|\det \partial^2 h/\partial I_i \partial I_j| \equiv |\det \partial \mathscr{W}_i/\partial I_j| \ge K_0 \tag{14}$$

and the nonresonance condition (1.12) holds, where $\nabla h(I_*) = \mathscr{W}(I_*)$ and $\lambda_j^A(I_*)\lambda_{jJ} = V_j^0$ (see (4)), there exist for sufficiently small $\varepsilon > 0$ a measurable subset $\Theta_\varepsilon^0 \subset \mathscr{T}^0$ and a smooth embedding $\Sigma_I \colon T^n \to H_p^1(0, 2\pi), I \in \Theta_\varepsilon^0$, such that the tori $\Sigma_I(T^n)$ are invariant under the flow of problem (13), (2), are filled with quasiperiodic orbits, and are such that assertions a)–c) of Theorem 2 hold (with $Z_{d_0} = H_p^1(0, 2\pi)$).

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