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# An Infinitesimal Liouville–Arnold Theorem as a Criterion of Reducibility for Variational Hamiltonian Equations

**SERGEJ B. KUKSIN***

Institute for Information Transmission Problems, Moscow, Russia

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**Abstract**—We prove a criterion for the variational equation about a quasiperiodic solution of a Hamiltonian equation being reducible to a constant coefficient equation. We discuss applications of this criterion to the stability problem for lower dimensional invariant tori.

## 1. INTRODUCTION

The subject of investigation is a Hamiltonian vector-field $H_f$ on a $2N$-dimensional symplectic manifold $(M, \omega)$, which is integrable on some invariant symplectic submanifold $\mathcal{T} \subset M$, $\dim \mathcal{T} = 2n < 2N$. So $\mathcal{T}$ is foliated into invariant tori $T^n_p$ depending on an $n$-dimensional parameter $p \in \mathrm{P} \subset\subset \mathbb{R}^n$, and the flow on every torus $T^n_p$ is of the form $\dot{q} = \nabla_p f_0(p)$ ($f_0$ is a restriction of the Hamiltonian $f$ to $\mathcal{T}$). Let $(T\mathcal{T})^\perp \subset T_{\mathcal{T}}M \equiv \cup_{m \in \mathcal{T}} T_m M$ be the (skew-)normal bundle of $\mathcal{T}$. If $S_t$ is a flow of $H_f$, then the normal bundle $(T\mathcal{T})^\perp$ is invariant for the tangent flow $S_{t*}$. We call the restriction of $S_{t*}$ on $(T\mathcal{T})^\perp$ 'the flow of the normal variational equation (NVE) of $H_f$ along $\mathcal{T}$', and study the question: under what conditions is this flow reducible to the flow of a linear equation with coefficients independent of the point $q \in T^n_p$ (so-called reducibility problem; see e.g. [1]). If such reducibility occurs then in the 'nondegenerate case' $\mathcal{T}$ is 'KAM-stable'. That is most of the tori $T^n_p$, $p \in P$, survive after a small Hamiltonian perturbation of the system (this results from a perturbation theorem for lower-dimensional invariant tori of a linear system, see [2–4]).

It is known that if no additional conditions are imposed then the NVE may be non-reducible (see e.g. [5]). On the other hand, if in a neighborhood of $\mathcal{T}$ the conditions of the 'degenerate Liouville–Arnold theorem' are fulfilled, then the vector-field $H_f$ is integrable in the vicinity of $\mathcal{T}$ and NVE is trivially reducible (for the degenerate Liouville–Arnold theorem see [2] and its bibliography).

Our aim in this paper is to obtain some criterion of reducibility of the NVE, which is a rather straightforward infinitesimal version of the Liouville–Arnold theorem. In the important case of codimension 1 ($N = n + 1$) this criterion gives as a test for reducibility some zero-curvature equation.

We are most interested in elliptic invariant submanifolds $\mathcal{T}$. For such a $\mathcal{T}$ with reducible flow of the NVE we give a definition of a frequency spectrum of the flow and formulate the nondegeneracy condition sufficient for KAM-stability of $\mathcal{T}$ in terms of this spectrum.

When a previous version of this paper was done, I was informed that a result similar to

*Address for correspondence: Forschungsinstitut für Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland.

theorem 1 had been proved by Ju. Vorobjov. See [6] for the statement of the result and its applications to the quantization problem.

## 2. CRITERION OF REDUCIBILITY

We shall formulate the results in an analytic case. So all the manifolds and the mappings are supposed to be analytic. Let the symplectic manifold $(M, \omega)$ be provided with Riemann metric $dm$ (this is not a restriction due to the Grauert embedding theorem) and the submanifold $\mathcal{T}$ is symplectomorphic to $(\mathbb{T}^n \times P,\ dp \wedge dq)$, $\mathbb{T}^n = \{q\}$. That is, $\mathcal{T} = \Sigma_0(\mathbb{T}^n \times P)$ for an (analytic) map

$$\Sigma_0 : \mathbb{T}^n \times P \to M,\ \Sigma_0^* \omega = dp \wedge dq.$$

Below we identify $\mathcal{T}$ with $\mathbb{T}^n \times P$.

If $S_t$ is the flow of the Hamiltonian vector field $H_f$, then the subbundles $T_{\mathcal{T}} M = \cup_{m \in \mathcal{T}} T_m M \subset TM$, $T\mathcal{T} \subset T_{\mathcal{T}} M$ and $(T\mathcal{T})^{\perp} \subset T_{\mathcal{T}} M$ (the skew-normal bundle to $T\mathcal{T}$ in $T_{\mathcal{T}} M$) are invariant for the tangent flow $S_{t*}$.

*Definition* 1. The flow $S_{t*}$ of the NVE of the vector-field $H_f$ along $\mathcal{T}$ (together with the underlying normal bundle $(T\mathcal{T})^{\perp}$) is called reducible if

(1) there exist a symplectic trivialization of the bundle $(T\mathcal{T})^{\perp}$,

$$\begin{array}{ccc} \mathbb{T}^n \times P \times Y & \xrightarrow{\Phi} & (T\mathcal{T})^{\perp} \\ & \searrow \quad \swarrow & \\ & \mathcal{T} & \end{array} \tag{2.1}$$

where the fiber $Y = \mathbb{R}^{2m}_y = \mathbb{R}^m_{y+} \times \mathbb{R}^m_{y-}$, $m = N - n$, has the usual symplectic structure with the form $dy_+ \wedge dy_-$.

(2) There exists an analytic symmetric $2m \times 2m$-matrix $A(p)$ such that under this trivialisation the flow $S_{t*}$ on $(T\mathcal{T})^{\perp}$ corresponds on $\mathbb{T}^n \times P \times Y$ to the flow of the equation

$$\dot{q} = \nabla f_0(p),\ \dot{p} = 0,\ \dot{y} = JA(p)y \tag{2.2}$$

where $J(y_+, y_-) = (-y_-, y_+)$ (we use the same notation for operators and their matrices).

In the situation of the definition 1, we will say (with some abuse of language) that the NVE is reducible.

*Definition* 2. The flow $S_{t*}$ is called complex-reducible if its complexification in the bundle $(T\mathcal{T})^{\perp} \otimes_{\mathbb{R}} \mathbb{C}$ is reducible in the category of complex symplectic bundles, with some symmetric complex matrix $A(p)$.

*Proposition* 1. If the bundle $(T\mathcal{T})^{\perp}$ can be trivialized [i.e. if there exists an isomorphism $\Phi$ as in (2.1)], then some neighborhood of $\mathcal{T}$ in $M$ is symplectomorphic to a neighborhood $O$ of $\mathcal{T}_0 = \mathbb{T}^n \times P \times \{0\}$ in $\mathbb{T}^n \times P \times Y$ with 2-form $dp \wedge dq + dy_+ \wedge dy_-$.

*Proof.* Let us consider the restriction on $(T\mathcal{T})^{\perp}$ of the time-one shift along the geodesic flow on $TM$ and take its $M$-projection:

$$\Xi : (T\mathcal{T})^{\perp} \to M,\ (x, \xi) \to \exp_x \xi,$$

for $x \in M$, $\xi \in (T\mathcal{J})^{\perp}_{x}$. Let $(T\mathcal{J})^{\perp}_{0}$ be the zero-section of $(T\mathcal{J})^{\perp}$. Then for arbitrary $(x, 0) \in (T\mathcal{J})^{\perp}_{0}$ the tangent map

$$\Xi_{*}(x, 0) : T_{(x,0)}(T\mathcal{J})^{\perp} \simeq T_x\mathcal{J} \oplus (T_x\mathcal{J})^{\perp} \to T_x M \tag{2.3}$$

is a linear symplectomorphism. So by inverse function theorem the restriction of the map $\Xi \circ \Phi$ on some neighborhood $O^1$ of $\mathcal{J}_0$ in $\mathbb{T}^n \times P \times \mathbb{R}^n$ defines an isomorphism and

$$(\Xi \circ \Phi)^{*}\omega|_{\mathcal{J}_0} = dp \wedge dq + dy_{+} \wedge dy_{-}.$$

Now by the relative Darboux theorem (see [7–9]) in a smaller neighborhood $O$ of $\mathbb{T}^n \times P \times \{0\}$ there exists a change of coordinates $V$ such that

$$V_{*}|_{T\mathcal{J}(M)} = \mathrm{id} \tag{2.4}$$

and $(\Xi \circ \Phi \circ V)^{*}\alpha = dp \wedge dq + dy_{+} \wedge dy_{-}$. ■

*Proposition* 2. If the NVE for $H_f$ along $\mathcal{J}$ is reducible, then in the symplectic coordinates $(q, p, y)$ from proposition 1

$$f(q, p, y) = f_0(p) + \tfrac{1}{2} < A(p)y, y > + O(|y|^3). \tag{2.5}$$

*Proof.* Let us write $f(q, p, y)$ as a series in $y$:

$$f = f^0(q, p) + \langle f^1(q, p), y\rangle + \tfrac{1}{2}\langle f^2(q, p)y, y\rangle + O(|y|^3). \tag{2.6}$$

Here $f^1$ is a vector in $\mathbb{R}^{2m}$ and $f^2$ is a symmetric linear operator. As the manifold $\mathcal{J} = \{y = 0\}$ is invariant for the vector-field $H_f$, we have $f^1 \equiv 0$; as the restriction of $H_f$ on $\mathcal{J}$ is the Hamiltonian system with the Hamiltonian $f_0(p)$, we also have $f^0 = f_0(p)$. Then the flow of the NVE along $\mathcal{J}_0$ for the system with Hamiltonian (2.5) is the one of equations

$$\dot{q} = \nabla f_0(p),\ \dot{p} = 0,\ \dot{y} = Jf^2(q, p)y. \tag{2.7}$$

As $\Xi_{*}(x, 0)|_{(T_x\mathcal{J})^{\perp}}$ is identical map $\forall x \in \mathcal{J}$ and $V_{*}(m)$ is identical $\forall x \in \mathcal{J}$, then the map $\Phi$ transforms solutions of the system (2.7) into trajectories of the flow $S_{t*}|_{(T\mathcal{J})^{\perp}}$. So by the item (2) of definition 1 the set of solutions of equations (2.7) is equal to the one of the equation (2.2). Thus $f^2(q, p) \equiv A(p)$.■

In what follows for an analytic function $g$ on $M$ we write $g(m) = o(\mathrm{dist}(m, \mathcal{J}))^p$, $p \in \mathbb{Z}$, $p \geqslant 0$, if in every local chart $Q$ on $M$ with coordinates $(x_1, \ldots, x_{2N})$ we have:

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} g(x)\right| = o(\mathrm{dist}(x, \mathcal{J} \cap Q)^{p-|\alpha|})\ \forall \alpha \in \mathbb{Z}^{2N},\ |\alpha| \leqslant p.$$

*Theorem* 1. Let $f_1, \ldots, f_n$ be analytic functions in some neighborhood of $\mathcal{J}$ such that $f_1 = f$ and

(a) $$[f_j, f_k](m) = o(\mathrm{dist}(m, \mathcal{J}))^2\ \forall j, k, \tag{2.8}$$

(b) $\forall \tilde{q} \in \mathbb{T}^n$, $\tilde{p} \in P$ the vectors $H_{f_1}(\tilde{q}, \tilde{p}), \ldots, H_{f_n}(\tilde{q}, \tilde{p})$ are linearly independent and are tangent to $T^n_{\tilde{p}} = \{(q, p) \in \mathcal{J} | p = \tilde{p}\}$.

Then $\forall p_0 \in P$ there exists a neighborhood $P_0$ of $p_0$ such that the NVE for $H_f$ along $\mathcal{J}_0 = \mathbb{T}^n \times P_0$ is complex-reducible.

*Remark* 1. The assumption (b) of the theorem results from (a) and the following three assumptions:

(i) $f_j(m) = o(\text{dist}(m, \mathcal{T}))\ \forall j$,
(ii) Hess $f(p) \not\equiv 0$,
(iii) $\forall \tilde{q}, \tilde{p}$ the vectors $H_{f_1}(\tilde{q}, \tilde{p}), \ldots, H_{f_n}(\tilde{q}, \tilde{p})$ are linearly independent.

Indeed, by (i) the submanifold $\mathcal{T}$ is invariant for the flows $S_t^j$ of $H_{f_j}$ for all $j$ and these flows commute on $\mathcal{T}$ by (a) (see lemma 1 below). So a set $M_{\tilde{q},\tilde{p}} = \cup_{t_1,\ldots,t_n} S_{t_1}^1 \circ \cdots \circ S_{t_n}^n(\tilde{q}, \tilde{p})$ is invariant for $S_t$ for all $(\tilde{q}, \tilde{p})$. This set is $n$-dimensional by (iii) and contains a closure of the trajectory of $H_f$ starting from $(\tilde{q}, \tilde{p})$. By (ii) the last is equal to $T_{\tilde{p}}^n$ for almost all $\tilde{p}$. So $M_{\tilde{q},\tilde{p}} = T_{\tilde{p}}^n\ \forall \tilde{q}, \tilde{p}$ and the vector-fields $H_{f_1}, \ldots, H_{f_n}$ are tangent to $T_p^n$.

*Proof of the theorem.* Let $S_t^j (j = 1, \ldots, n)$ be the flows of $H_{f_j}$ and $S_{t*}^j$ be the tangent flows on $TM$. By the assumption (b) of the theorem the manifold $(T\mathcal{T})^\perp$ is invariant for $S_{t*}^j\ \forall j$.

*Lemma* 1. Restrictions of the flows $S_{t*}^j$ on $(T\mathcal{T})^\perp$, $j = 1, 2, \ldots, n$, commute. In particular, the flows $S_t^j|_{\mathcal{T}}$ commute.

*Proof.* We shall prove that the restrictions of the flows $(S_t^j)_*$ on $T_{\mathcal{T}}M$ commute. The statement is local and it is enough to prove it in a local chart $Q$ on $M$ with coordinates $(x_1, \ldots, x_{2N})$. Let in this chart

$$H_{f_j} = \mathbf{V} = (V^1, \ldots, V^{2N}),\ H_{f_k} = \mathbf{W} = (W^1, \ldots, W^{2N})$$

for some $1 \leq j, k \leq n$, and $\mathbf{TV}$: $TM \to T(TM)$ be a vector-field of a variational equation for $\mathbf{V}$. Let $(x_1, \ldots, x_{2N}, \xi_1, \ldots, \xi_{2N})$ be coordinates on $TQ$. Then $\mathbf{TV}(x,\xi) = (\mathbf{V}(x), \Sigma\, \partial/\partial_{x_e} \mathbf{V}(x)\xi_e)$ and the commutator $[\mathbf{TV},\mathbf{TW}]$ of the vector-fields $\mathbf{TV}$, $\mathbf{TW}$ is equal to

$$[\mathbf{TV}, \mathbf{TW}] = \left(\sum\left(\mathbf{W}^k \frac{\partial \mathbf{V}}{\partial x_k} - \mathbf{V}^k \frac{\partial \mathbf{W}}{\partial x_k}\right),\right.$$

$$\left.\sum\left(\frac{\partial^2 \mathbf{V}}{\partial x_j \partial x_k} \xi_j W^k + \frac{\partial \mathbf{V}}{\partial x_j} \frac{\partial W^j}{\partial x_k} \xi_k - \frac{\partial^2 \mathbf{W}}{\partial x_j \partial x_k} \xi_j \mathbf{V}^k - \frac{\partial \mathbf{W}}{\partial x_j} \frac{\partial \mathbf{V}^j}{\partial x_k} \xi_k\right)\right).$$

The right hand side of the last equality is equal to $T[\mathbf{V}, \mathbf{W}]$. So

$$[\mathbf{TV}, \mathbf{TW}] = T[\mathbf{V}, \mathbf{W}] = TH_{\{f_j, f_k\}}$$

and

$$[\mathbf{TV}|_{T_{\mathcal{T}}M}, \mathbf{TW}|_{T_{\mathcal{T}}M}] = TH_{\{f_j, f_k\}}|_{T_{\mathcal{T}}M}$$

because the commutation of vector-fields is a natural operation with respect to imbedding. By the assumption (2.8) $H_{\{f_j, f_k\}}(m) = o(\text{dist}(m, \mathcal{T}))$. So the right hand side in the last equality is equal to zero, the restrictions of vector-fields $\mathbf{TV}$, $\mathbf{TW}$ on $T_{\mathcal{T}}M$ commute and the lemma is proved. ■

Let us fix a point $q_0 \in \mathbb{T}^n$, $q_0 = 0 \bmod 2\pi\mathbb{Z}^n$, and fix some analytic trivialization of the restriction of $(T\mathcal{T})^\perp$ on $q_0 \times P$,

$$(T\mathcal{T})^\perp|_{q_0 \times \mathbf{P}} \simeq P \times E. \tag{2.9}$$

For $p \in P$, let $(T\mathcal{T}_p)^\perp$ be the restriction of $(T\mathcal{T})^\perp$ on the torus $T_p^n$. To prove the theorem, it is enough to trivialize the symplectic bundle $(T\mathcal{T}_p)^\perp$ by a map which depends on $p$ in an analytic way, and to check that the restriction of the flow $S_{t*}$ on $(T\mathcal{T}_p)^\perp$ is of the form (2.2).

Let $(e_1, \ldots, e_n)$ be the usual basis of $\mathbb{Z}^n$ and $\chi^j_t(q, p) = (q + te_j, p)$. By lemma 1 and assumption (b) of the theorem we can see that there exists a nondegenerate analytic matrix $D_{ij}(p)$ such that

$$\prod_{l=1}^{n} S^l_{tD_{jl}}|_{T^n_p} = \chi^j_t \; \forall j \tag{2.10}$$

(this is the first step from the classical proof of Liouville–Arnold theorem, see [7, 8, 10]). Let us denote by $\chi^j_{t_*}(p)$ the flow on $(T\mathcal{T}_p)^\perp$,

$$\chi^j_{t_*}(p) = \prod_{l=1}^{n} (S^l_{tD_{jl}})_*, \; j = 1, \ldots, n.$$

These flows are well-defined by lemma 1. By (2.9) the monodromy operators $\chi^j_{2\pi_*}$, $j = 1, \ldots, n$, define linear symplectomorphisms of $(T\mathcal{T})^\perp_{(q_0, p)} \simeq E$. By lemma A1 (see Appendix)

$$\chi^j_{2\pi_*}(p) = \mathrm{e}^{2\pi \mathrm{B}^j(p)} \tag{2.11}$$

with some analytic on $p$ linear Hamiltonian operators $B^j(p)$ in the complexification $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$ of $E$ (that is the matrix of $B^j$ in $E^c = \mathbb{C}^n_{y_+} \times \mathbb{C}^n_{y_-}$ is of the form $B^j = JB^{j(s)}$; here $J$ is the matrix of the operator $J(y_+, y_-) = (-y_-, y_+)$ and $B^{j(s)}$ is a symmetric matrix). As the operators $\chi^j_{2\pi_*}$, $j = 1, \ldots, n$, commute, their logarithms $B^j(p)$ commute as well (these results, for example, from the representation (A3) for $B^j(p)$). Now we can trivialize the bundle $(T\mathcal{T}_p)^\perp \otimes \mathbb{C}$ with the help of the map

$$\mathbb{T}^n \times \{p\} \times E^c \to (T\mathcal{T}_p)^\perp \otimes \mathbb{C},$$

$$(q, p, \xi) \to \prod_{j=1}^{n} \chi^j_{q j_*}(p)\left(0, p, \prod_{j=1}^{n} \mathrm{e}^{-q_j B^j(p)} \xi\right). \tag{2.12}$$

The definition of this map is correct because the image does not change if the vector $(q_1, \ldots, q_n)$ is replaced by $q_1, \ldots, q_j \pm 2\pi, \ldots, q_n)$. It is symplectic because every map $\exp \tau B^j(p) : E^c \to E^c$ and every flow $\chi^j_{t_*}$ are symplectic. The map (2.12) depends on $p$ in an analytic way because matrices $B^j(p)$ are analytic. Let us define a map $\Phi$ in (2.1) in such a way that $\Phi|_{\mathbb{T}^n \times \{p\} \times \mathrm{E}^c}$ is equal to the map (2.12).

Let us write for brevity

$$q \cdot \chi_*(p) = \prod \chi^j_{q_{j_*}}(p), \quad q \cdot \mathbf{B}(p) = \sum q_j B^j(p).$$

From (2.10) we see that

$$S^1_{t_*} = tD^-_1 \cdot \chi_* \tag{2.13}$$

here $D^-_1$ is the first row of the inverse matrix $D^{-1}_{ij}$. So if under the trivialization (2.12) $(T\mathcal{T}_p)^\perp \otimes \mathbb{C} \ni \eta \simeq (q, p, \xi)$ and $S^1_{t_*}\eta \simeq (q_1, p, \xi_1)$, i.e. if

$$\begin{array}{ccc} \eta & \xleftarrow{\Phi} & (q, p, \xi) \\ \downarrow & & \\ S^1_{t_*}\eta & \xleftarrow{\Phi} & (q_1, p, \xi_1), \end{array}$$

then $q_1 = q + tD^-_1$ and

$$\xi_1 = \mathrm{e}^{(q + tD_1^-) \cdot \mathbf{B}} \Pi_Y \circ ((-tD^-_1 - q) \cdot \chi_*) \circ (tD^-_1 \cdot \chi_*)$$
$$\circ (q \cdot \chi_*(0, p, \mathrm{e}^{-q \cdot \mathbf{B}} \xi)) = (0, p, \mathrm{e}^{tD_1^- \cdot \mathbf{B}} \xi)$$

(here $\Pi_Y$ is a projection of $(T\mathcal{T}_p)^\perp \otimes \mathbb{C})_{q_0} \simeq \mathbb{T}^n \times Y^c$ on $Y^c$).

So (2.2) holds with $A(p) = D_1^{-}(p)\cdot\mathbf{B}(p)$ and the theorem is proved. ■

An 'almost inverse' statement to theorem 1 easily results from proposition 2:

*Proposition* 3. If the NVE for $H_f$ along $\mathcal{T}$ is reducible, $p \in P$ and $P_0$ is a small enough neighborhood of $p$ in $P$, then in a neighborhood of $\mathcal{T}_0 = \mathbb{T}^n \times P_0$ there are $n$ analytic functions with the properties (a), (b).

To prove the statement it is enough to write the Hamiltonian $f$ in a form (2.5) and to choose $f_1 = f$, and for $j \geq 2$ $f_j(q, p, y) = f_j(p)$, where the vectors $\nabla f_1(p_0), \nabla f_2(p_0), \ldots, \nabla f_n(p_0)$ are linearly independent. ■

For the last proposition a natural question is whether the reduction of theorem 1 can be done in the category of real bundles. This is true if in (2.11) the logarithms $B_j(p)$ of the monodromy operators can be constructed as real matrices. For lemmas A1, A2 this is true if

$$\sigma(\chi^j_{2\pi*}) \cap (-\infty,0] = \phi \;\forall j \tag{2.14}$$

($\sigma$ = spectrum) or if $\chi^j_{2\pi_*}$, $j = 1, \ldots, n$ are replaced by their squares. The last takes place if the tori $T^n_p$ are replaced be their $2^n$-sheets covering 'doubled tori'

$$T^n_p \to T^n_p,\; q \longmapsto 2q.$$

This covering induces a bundle $(T\mathcal{T})^\perp_{\text{ind}}$ with the induced flow $(S_{t*})_{\text{ind}}$ in it.

*Corollary* 1. Under the assumptions of theorem 1, the bundle $(T\mathcal{T})^\perp_{\text{ind}}$ can be trivialized as a real bundle. For this trivialization the flow $(S_{t*})_{\text{ind}}$ is of the form (2.2).

To realize the first possibility let us mention that (2.14) holds if

$$\sigma(\chi^j_{2\pi_*}) \in i\mathbb{R}\;\forall_j. \tag{2.15}$$

*Definition* 3. An invariant manifold $\mathcal{T}$ is called linearly stable for a vector-field $H_f$, if all the Liapunov exponents of every solution of $H_f$ on $\mathcal{T}$ are equal to zero.

*Lemma* 2. Under the conditions of theorem 1 the assumption (2.15) holds if and only if the invariant manifold $\mathcal{T}$ is linearly stable for every vector-field $H_{f_j}$ $(j = 1, \ldots, n)$.

*Proof.* Let us suppose that $\mathcal{T}$ is linearly stable $\forall H_{f_j}$, $j = 1, \ldots, n$. Then by the definition of the flows $\chi^j_{t_*}(p)$ for every $\varepsilon > 0$ there exists $C_\varepsilon$ such that

$$\|\chi^j_{\pm 2\pi n}(p)\| \leq C_\varepsilon e^{\varepsilon n} \tag{2.16}$$

and so (2.15) is true.

Let us suppose that (2.15) holds. Then (2.16) is true $\forall\varepsilon > 0$ with some $C_\varepsilon$. By (2.13), (2.16) we see that $\|S^1_{n_*}\| \leq C^1_\varepsilon e^{\varepsilon n}$ and the same is true for all $S^j_{t_*}$. So $\mathcal{T}$ is linearly stable for all $H_{f_j}$. ■

*Theorem* 2. Suppose the invariant manifold $\mathcal{T}$ is linearly stable for $H_f$ and for some $p_0 \in P$ $H_f(p_0) \neq 0$. Then the NVE of $H_f$ along $\mathcal{T}_0 = \mathbb{T}^n \times P_0$ ($P_0$ is a small enough neighborhood of $p_0$ in $P$) is reducible if and only if there are analytic functions $f_1, \ldots, f_n$ such that $f_1 = f$ and the assumptions (a), (b) of theorem 1 are fulfilled for $\mathcal{T} = \mathcal{T}_0$, together with

(c) $\mathcal{T}_0$ is linearly stable for all $H_{f_j}$, $j = 1, \ldots, n$.

In such a case the spectrum of the operator $JA(p)$ is pure imaginary.

*Proof.* If the NVE is reducible then we can construct the functions $f_2, \ldots, f_n$ as in proposition 3. The manifold $\mathcal{T}_0$ is linearly stable for all $H_{f_j}$ trivially.

Suppose now that the assumptions (a)–(c) are fulfilled. Then by lemma 2 the assumption (2.15) holds and by lemma A1 the matrices $B_j(p)$ (and, so, the trivialization $\Phi$) can be chosen real. The last statement of the theorem is trivial because a system of the form (2.2) is linearly stable if and only if the spectrum of $JA(p)$ is pure imaginary. ■

*Remarks* 2. Propositions 1, 2 and theorems 1, 2 have direct smooth versions with the same proofs.

3. Our proof of theorems 1, 2 (but not of lemmas A1, A2) does not use the finite dimensionality of the fibers of the bundle $(T\mathcal{T})^{\perp}$. If in (2.9) $\dim Y = \infty$ and we have sufficient spectral information on the flows $S^j_{t*}$ and can construct 'regular' logarithms $B_j(p)$ of the monodromy operators $\chi^j_{2\pi*}$ [see (2.11)], then our proof is valid.

4. The reducibility of the NVE along $\mathcal{T}_0$ was proved via its reducibility along the tori $\{(q, p) \in \mathcal{T} | p = \text{const}\}$. So the proof can be used for proving the reducibility of a linear Hamiltonian equation

$$\dot{q} = \omega, \ \dot{y} = JA(q)y \quad (q \in \mathbb{T}^n, \ y \in Y)$$

to a constant-coefficient Hamiltonian equation $\dot{y} = J\bar{A}y$ by means of symplectic transformation $y = C(q)y$. This reduction is possible if in the phase space $\mathbb{T}^n \times \mathbb{R}^n \times Y$ there are functions $f_j(q, p, y)$ $(j = 1, 2, \ldots, n)$ of the form $f_j = \omega_j \cdot p + \frac{1}{2}\langle A_j(q)y, y\rangle$ such that $\omega_1 = \omega$, $A_1 = A$, $\omega_1 \wedge \ldots \wedge \omega_n \neq 0$ and $\forall j, k$

$$\tfrac{1}{2}\,(\omega_j \cdot \nabla)A_k(q) - \tfrac{1}{2}\,(\omega_k \cdot \nabla)A_j(q) + A_k(q)JA_j(q) - A_j(q)JA_k(q) \equiv 0.$$

5. In the special case $n = 1$ we need no 'infinitesimal integrals' other than $f_1 = f$, and the assumptions (a), (b) of theorem 1 are fulfilled in a trivial way. For $n = 1$ theorem 1 and corollary 1 coincide with the Floquet theorem (see [7]). For a less trivial example, see Section 4 below.

## 3. ELLIPTIC CASE

*Definition* 4. The invariant manifold $\mathcal{T}$ is called weakly elliptic if the NVE of $H_f$ along $\mathcal{T}$ is reducible and operator $JA(p)$ in (2.2) has pure imaginary spectrum $\{\pm i\lambda_j(p)\}$. $\mathcal{T}$ is called elliptic if it is weakly elliptic and operator $JA(p)$ is complex-diagonalizable $\forall p \in P$.

One can treat theorem 2 as a weak ellipticity criterion.

Clearly, submanifold $\mathcal{T}$ is elliptic if it is weakly elliptic and $\lambda_j(p) \neq \lambda_k(p)$ for $j \neq k$.

*Remark* 6. Finite-dimensional elliptic invariant submanifolds of infinite codimension appear in the study of nonlinear partial differential equations which are integrable in terms of theta-functions. See [3], section 4.

For an elliptic invariant submanifold $\mathcal{T}$ the spectrum $\{\pm i\lambda_j(p)\}$ is not defined in an unique way:

*Proposition* 4. Let the submanifold $\mathcal{T}$ be elliptic and the flow $S_t|_{\mathcal{T}}$ be nondegenerate:

$$\det \partial\omega/\partial p \neq 0, \ \omega(p) = \nabla f_0(p). \tag{3.1}$$

Let us consider some another trivialisation of $S_{t*}$ with $\Phi'$ and $A'$ in (2.1) and (2.2) instead of $\Phi$ and A. Let $\sigma(JA'(p)) = \{\pm i\mu_j(p)\}$. Then for every $j$ there exist $k = k(j)$, $s = s(j) \in \mathbb{Z}^n$ such that

$$\mu_j(p) = \pm \lambda_k(p) + s \cdot \omega(p)\ \forall p. \tag{3.2}$$

Moreover, every $n$ numbers of the form $\mu_j(p) = \lambda_j(p) + s_j \cdot \omega(p)$, $s_j \in \mathbb{Z}^n$, may be achieved as a spectrum of a Hamiltonian operator $JA'(p)$ for some trivialisation $\Phi'$.

*Proof.* Let $\{\varphi_j^{\pm}(p)\}$, $\varphi_j^{-}(p) = \overline{\varphi_j^{+}}(p)$, by symplectic basic of $Y^c$ such that $JA(p)\varphi_j^{\pm} = \pm i\lambda_j(p)\varphi_j^{\pm}$. Then the mapping $\Phi': \mathbb{T}^n \times P \times Y \to (T_{\mathcal{J}})^{\perp}$, which maps $(q, p, \varphi_j^{\pm})$ to $\exp(\mp s_j \cdot q)\varphi_j^{\pm}$, $j = 1, 2, \ldots, n$, transforms the flow $S_{t*}$ into a flow of an equation (2.2) with an operator $A'(p)$ such that

$$JA'(p)\varphi_j^{\pm} = \pm i(s_j \cdot \omega(p) + \lambda_j(p))\varphi_j^{\pm}\ \forall j.$$

Thus the second statement is proved.

To prove the first one let us mention that $\Phi^{-1} \circ \Phi'(q + \omega t, p, e^{i\mu_j t}\varphi_j'^{+})$ is a solution of (2.2) (here $JA'\varphi_j'^{\pm} = \pm i\mu_j\varphi'^{\pm}$). Let $\Phi^{-1} \circ \Phi'(q, p, \varphi_j'^{\pm}) = \Sigma x_k^{\pm}(q,p)\varphi_k^{\pm}$. Then the solution may be rewritten as $\exp(i\mu_j t)\Sigma x_k^{\pm}(q + \omega t, p)\varphi_k^{\pm}$. So

$$i\mu_j x_k^{\pm} + \frac{\partial}{\partial \omega} x_k^{\pm} = \pm i\lambda_k x_k^{\pm}\ \forall k. \tag{3.3}$$

Among the functions $x_k^{\pm}$ there are nonzero ones. Let us suppose that $x_{k_0}^{+}(q, p) \not\equiv 0$. By (3.1) the components of the vector $(\omega_1, \ldots, \omega_n)$ are rationally independent for almost all $p \in P$. Then by (3.3) $x_{k_0}^{+} = C(p) \exp is \cdot q$ for some $s \in \mathbb{Z}^n$ and $\mu_j = \pm\lambda_{k_0} - s \cdot \omega$. Thus the first assertion is proved, too. ■

Let us consider a family of subgroups of additive group $\mathbb{Z}$ of a form $\omega(p) \cdot \mathbb{Z}^n$, $p \in P$, and corresponding factor groups $G(p) = \mathbb{Z}/\omega(p) \cdot \mathbb{Z}^n$. For a weakly elliptic submanifold $\mathcal{J}$ let us define elements $\Lambda_1(p), \ldots, \Lambda_n(p)$ of $G(p)$ as follows:

$$\Lambda_j(p) = \lambda_j(p) + \omega(p) \cdot \mathbb{Z}^n \in G(p). \tag{3.4}$$

The following definition is motivated by proposition 4.

*Definition* 5. If $\mathcal{J}$ is a weakly elliptic invariant submanifold, then depending on the $p \in P$ set

$$\Lambda(p) = \{\pm \Lambda_1(p), \ldots, \pm \Lambda_n(p)\} \subset G(p)$$

is called the frequency spectrum of NVE.

The important reason to prove the reducibility of the NVE is proposition 2 which provide a Hamiltonian $f(q, p, y)$ with the useful normal form (2.5). For nondegenerate Hamiltonians of the form (2.5) (see the condition (3.5) below) one can prove that the family $\mathcal{J} = \Sigma_0(\mathbb{T}^n \times P)$ of invariant tori $\Sigma_0(\mathbb{T}^n \times \{p\})$, $p \in P$, is KAM-stable in the following sense:

*Definition* 6. A family of invariant tori $\mathcal{J} = \Sigma_0(\mathbb{T}^n \times P)$ of the Hamiltonian vector-field $H_f$ is called KAM-stable if for an arbitrary analytic function $\tilde{f}$ and for $\varepsilon$ small enough, the vector-field $H_{f+\varepsilon\tilde{f}}$ has an invariant set $\mathcal{J}_\varepsilon = \Sigma_\varepsilon(\mathbb{T}^n \times P_\varepsilon)$. Here

(1) $P_\varepsilon$ is a Cantor-set in $P$ and

$$\operatorname{mes}(P \backslash P_\varepsilon) \to 0 (\varepsilon \to 0),$$

(2) the map $\Sigma_\varepsilon: \mathbb{T}^n \times P_\varepsilon \to M$ is Lipschitz and

$$\operatorname{dist}\left(\Sigma_\varepsilon(\mathbb{T}^n \times \{p\}), \Sigma_0(\mathbb{T}^n \times \{p\})\right) \leqslant \mathrm{d}(\varepsilon) \to 0\ (\varepsilon \to 0)$$

for all $p \in P_\varepsilon$;

(3) the tori $\Sigma_\varepsilon(\mathbb{T}^n \times \{p\})$, $p \in P_\varepsilon$, are invariant for the vector-field $H_{f+\varepsilon f}$.

To prove the KAM-stability of $\mathcal{T}$ one has to apply a theorem on perturbation of a linear system (see [2–4]) to the vector-field $H_f$ with $f$ in the form (2.5) after a simple space-dilation (see [3], Section 1). In such a way the following result may be obtained.

*Theorem* 3. Suppose the invariant manifold $\mathcal{T}$ is weakly elliptic for the NVE of $H_f$ and for the frequency spectrum of NVE we have

$$\det \partial\omega/\partial p \neq 0,\ \Lambda_j(p) \neq 0\ \forall j,\ \pm\Lambda_j(p) \neq \Lambda_k(p)\ \forall j \neq \mathrm{k}. \tag{3.5}$$

Then $\mathcal{T}$ is KAM-stable.

*Remark* 7. There is a natural smooth version of theorem 3. In order to prove it one has to write down a smooth version of the perturbation theorem for lower-dimensional invariant tori using the usual smoothing techniques of J. Moser. Clearly it is possible but this work still has not been done.

*Remark* 8. In order to prove KAM-stability of $\mathcal{T}$ via a smooth version of the arguments (see remark 7) it is enough to prove 'KAM-reducibility' of NVE. That is, for every $\delta > 0$ we must be able to find a smooth trivialisation (2.1) such that in the equation (2.2) the matrix $\mathrm{A}(p)$ does not depend on $q$ if $p$ lies out of some Cantor set of measure $\delta$.

*Remark* 9. In [1] the hyperbolic situation was considered. It was proved that if the normal bundle $(T\mathcal{T})^\perp$ is trivial and the flow $S_{t*}$ has full Sacker–Sell spectrum then NVE is KAM-reducible and $\mathcal{T}$ as a family of 'doubled tori' (see corollary 1 above) is KAM-stable.

## 4. EXAMPLE

Let $N = n + 1$ and suppose the symplectic Riemann manifold $M$ is polarizable. Then the bundles $TM$ and $T\mathcal{T}$ are trivial and so the bundle $(T\mathcal{T})^\perp$ is also trivial. This results from the fact that the symplectic bundles $TM$, $T\mathcal{T}$, $(T\mathcal{T})^\perp$ can be given complex structures (see [7, 10]) and that a one-dimensional complex bundle which is a factor-bundle of a trivial complex bundle is trivial (see [11]). So by the proposition 1, in a neighborhood of $\mathcal{T}$ in $M$ there are symplectic coordinates $(q, p, y)(q \in \mathbb{T}^n,\ p \in P \subset \mathbb{R}^n,\ y = (y_+, y_-) \in \mathcal{O} \subset \mathbb{R}^2)$ and $\mathcal{T} = \{y = 0\}$. In this coordinates the Hamiltonians $f_1, \ldots, f_n$ we are looking for, can be written in the form

$$f_j(q, p, y) = f_j^0(p) + \tfrac{1}{2}\langle \mathrm{A}_j(q, p)y, y\rangle + O(|y|^3).$$

So $[f_j, f_k] = \langle \mathcal{A}_{jk}(q, p)y, y\rangle + O(|y|^3)$ with

$$\mathcal{A}_{jk} = \tfrac{1}{2}\,(\nabla_p f_j^0 \cdot \nabla_q A_k - \nabla_p f_k^0 \cdot \nabla_q A_j) + A_k J A_j - A_j J A_k.$$

Let us denote

$$\nabla_p f_j^0 = \omega_j(p) = (\omega_{j,1}, \ldots, \omega_{j,n}) \in \mathbb{R}^n,$$

$$\mathcal{A}_j(p) = JA_j(p) \in sl(2) = sl(2,\mathbb{R}).$$

The assumptions of theorem 1 are fulfilled if we can construct the functions $f_1, \ldots, f_n$ and the matrices $\mathcal{A}_1, \ldots, \mathcal{A}_n \in sl(2)$ in such a way that $f_1 = f_0$ and $\mathcal{A}_1 = JA_0$ ($f_0$, $A_0$ are given), for every $p \in P$ the vectors $\omega_1(p), \ldots, \omega_n(p)$ span $\mathbb{R}^n$ (i.e. $\omega_1 \wedge \ldots \wedge \omega_n \neq 0$), and

$$J\mathcal{A}_{jk} = (\omega_j \cdot \nabla_q)\mathcal{A}_k - (\omega_k \cdot \nabla_q)\mathcal{A}_j + [\mathcal{A}_k, \mathcal{A}_j] \equiv 0 \qquad (4.1)$$

$$\forall j,\ k = 1, \ldots, n.$$

Let us suppose that Hess $f_0(p_0) \neq 0$ and, so, near $p_0$ the map $p \mapsto \omega = \nabla f_0(p)$ is invertible. Then to prove KAM-reducibility of NVE (see remark 8), by remark 4 we have to construct smooth vectors $\omega_j = \omega_j(\omega)$ and smooth matrices $\mathcal{A}_j(q, \omega)$ $(j = 2, \ldots, n)$ which solve equations (4.1) with $\omega_1 = \omega$, $\mathcal{A}_1 = \mathcal{A}_1(q, p)$, $\nabla f_0(p) = \omega$, for $\omega$ out of a set of small measure $\delta$ in such a way that

$$\omega_1 \wedge \ldots \wedge \omega_n \neq 0. \qquad (4.2)$$

In particular, if $n = 2$, then we have to find a vector $\omega_2$ and a matrix $\mathcal{A}_2(q) \in sl(2)$ such that

$$(\omega_1 \cdot \nabla_q)\mathcal{A}_2 - (\omega_2 \cdot \nabla_q)\mathcal{A}_1 + [\mathcal{A}_2, \mathcal{A}_1] = 0,\ \omega_1 \wedge \omega_2 \neq 0 \qquad (4.3)$$

(the last relation excludes the trivial solution $\mathcal{A}_2 = \lambda\mathcal{A}_1$, $\omega_2 = \lambda\omega_1$).

The equation (4.3) is the equation of zero curvature (see [12]) with non-standard periodicity conditions (that is, the periodicity is not with respect to the directions $\omega_1, \omega_2$, but to some other directions). Well-known gauge transformations [12]

$$\mathcal{A}_2 \to (\omega_2 \cdot \nabla_q)GG^{-1} + G\mathcal{A}_2G^{-1}$$

$$\mathcal{A}_1 \to (\omega_1 \cdot \nabla_q)GG^{-1} + G\mathcal{A}_1G^{-1}$$

($G = G(q)$ is an analytic symplectic matrix) transform solutions of (4.3) into new ones. It provides a means to construct new solutions of (4.3) from the trivial ones.

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## APPENDIX: ON LOGARITHMS OF ANALYTIC SYMPLECTIC MATRICES.

Let $C_p$, $p \in P$, be a symplectic matrix of order $2m$ analytic on $p$. Let for some $p_0 \in P$ the matrix $C_{p_0}$ be invertible.

*Lemma A*1. There exist a neighborhood $P_0$ of $p_0$ and an analytic complex matrix $B_p$, $p \in P_0$, which is a branch of Ln $C_p$:

$$\exp B_p = C_p. \tag{A1}$$

The matrix is Hamiltonian

$$(JB_p) = (JB_p)^t \tag{A2}$$

and may be chosen real if the spectrum $\sigma(\mathbf{C}_{p_0})$ contains no real negative points.

*Proof.* As $\sigma(C_{p_0}) \not\ni 0$, there exists a contour $\Gamma \subset \mathbb{C}$ such that $\sigma(C_{p_0})$ lies inside $\Gamma$ and 0 lies outside $\Gamma$. The same is true for $\sigma(C_p)$, $p \in P_0$ if $P_0$ is small enough. For $\lambda \in \Gamma$ let us fix some branch ln $\lambda$ of Ln $\lambda$ and set

$$B_p = \frac{1}{2\pi i} \oint_{\Lambda} \frac{\ln \lambda}{C_p - \lambda}\, d\lambda. \tag{A3}$$

Then $\exp B_p = C_p$ (see [13], Ch. VII) and so (A1) is proved. To prove (A2) let us mention that:

(1) the operator $B_p$ in (A3) depends on $C_p$ in a continuous way;
(2) single-spectrum symplectic matrices are dense among symplectic matrices (e.g. because symplectic matrices form a connected analytical manifold [7, 10] and double-spectrum matrices form there a nontrivial analytical subset);
(3) a single-spectrum symplectic matrix is diagonal in some symplectic basis and for it (A2) is evident.

So it remains to prove the last statement. It is well-known [7, 10], that for an invertible symplectic matrix $C$, the spectrum $\sigma(C)$ consists of pairs of points $\lambda$, $\lambda^{-1} (\lambda \in \mathbb{R})$; pairs of points $\lambda$, $\bar{\lambda} (|\lambda| = 1)$ and quadruples $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1} (\lambda \in \mathbb{C} \setminus \mathbb{R}, |\lambda| \neq 1)$. So in the present situation $\sigma(C_{p_0}) = S_1 \cup (S_2 \cup \bar{S}_2)$, where $S_1 = \{\lambda_j\} \subset \mathbb{R}_+$, $S_2 = \{\mu_j\} \subset \{\lambda | \operatorname{Im} \lambda > 0\}$. Let us take a (nonconnected) contour $\Gamma_0$ of the form $\Gamma_0 = \cup_{\lambda_i} \Gamma(\lambda_j) \cup_{\mu_j} [\Gamma(\mu_j) \cup -\Gamma(\mu_j)]$. Here $\Gamma(\lambda_j)$, $\Gamma(\mu_j)$ are small circles centered at $\lambda_j$, $\mu_j$ (thus $\Gamma(\lambda_j) = -\bar{\Gamma}(\lambda_j)\ \forall \lambda_j$). We can do it in such a way that $\Gamma_0 \cap (-\infty, 0] = \phi$, and so we can take for ln $z$ a branch of Ln $z$ which is real for $\lambda \in \mathbb{R}_+$. With such a choice of $\Gamma$ in (A3) one can see in a trivial way that $B_p = \bar{B}_p$. ■

*Lemma A*2. Under the assumptions of lemma A1 there exists an analytic real Hamiltonian matrix $\bar{B}_p$, $p \in P_0$ such that $\exp \bar{B}_p = C_p^2$, $p \in P_0$.

*Proof.* Let $\tilde{\Gamma} \in \mathbb{C}$ be a contour containing all negative eigenvalues of $C_p$ and no other eigenvalues. Then for $p \in P_0$ ($P_0$ is small enough) there exists a smooth splitting $\mathbb{R}^{2m}$ into two invariant for $C_p$ symplectic subspaces, $\mathbb{R}^{2m} = E_1 \oplus E_2$, such that the spectrum of $C_p|_{E_1}$ is negative and lies inside $\tilde{\Gamma}$ and the spectrum of $C_p|_{E_2}$ lies outside $\tilde{\Gamma}$ and out of $(-\infty, 0]$. Then by lemma A1 $C_p|_{E_2} = \exp B_p^{(2)}$ for some real Hamiltonian operator $B_p^{(2)}$, and $(C_p|_{E_1})^2 = \exp B_p^{(1)}$. Now we can take $B_p = B_p^{(1)} \oplus 2B_p^{(2)}$. This operator has all the properties we need. ■