On the Inclusion of an Analytic Symplectomorphism Close to an Integrable One into a Hamiltonian Flow

SERGEI B. KUKSIN

Institute for Problems of Information Transmission, Russian Academy of Sciences, ul. Ermolovoj 19, Moscow, 101447, Russia

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It is proved that an analytic symplectomorphism close to an integrable one can be represented as an isoenergetic Poincaré succession map for a vector field close to an integrable one.

The paper is devoted to a rigorous proof of the following well-known claim: "In the real-analytic category, the study of perturbations of integrable diffeomorphisms is equivalent to the study of perturbations of integrable vector fields". The Poincaré succession map transforms vector fields into diffeomorphisms. The suspension construction provides the inverse transition and can be performed in the smooth category without essential difficulties [1–4]. The analytic case turns out to be much more complicated (see the discussion in [5, §26.3]). The only proof we know of the existence of the inverse transition in the analytic situation is contained in Raphael Douady's thesis [6], and thus is hardly available. Moreover, it has a more important shortcoming, since it is based on the Grauert analytic embedding theorem, and therefore is not constructive. That is why the proof [6] does not allow to estimate the magnitude of the perturbing vector field and its analyticity radius; thus, it is hardly suitable for a refined analysis of perturbed diffeomorphisms, e.g., for the proof of the analog of the Nekhoroshev theorem [8] for their iterations. In fact, the inclusion of the ε-perturbation of an integrable symplectomorphism into a Hamiltonian flow close to an integrable one permits to prove the following fact: let $\Delta p(N)$ be the variation of the action variable under N iterations of the symplectomorphism; then

$$| \Delta p(N) | \leqslant C_1 \varepsilon^b \quad \text{if} \quad N \leqslant C_2 \exp(C_3 \varepsilon^{-a}).$$

The exponents are determined by the unperturbed symplectomorphism, while the constants C_1, C_2, C_3 depend essentially on the analyticity radius for the Hamiltonian, whose flow includes the perturbed symplectomorphism. Therefore, if we are unable to control this radius, we can hardly claim that the "perturbed system is stable for a very large time interval if the perturbation is small", since *a priori* the stability interval may depend not only on the *magnitude* of the perturbation but on its other properties as well.

In the paper, in conformity with the author's taste, we pay attention mainly to symplectomorphisms (i.e., symplectic structure preserving diffeomorphisms). The proof below is valid also in the nonsymplectic case, which we discuss briefly at the end of the paper.

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§1. THE STATEMENT OF THE MAIN RESULT

For a subset $X \subset \mathbb{R}^m$ and $\delta > 0$, denote by $X(\delta)$ the complex δ-neighborhood of X in $\mathbb{C}^m$. For the torus $\mathbf{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, define $\mathbf{T}(\delta) = \{x + iy \in \mathbb{C}^m/\mathbb{Z}^m \mid |y| < \delta\}$ (here and henceforth $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \geqslant 1$). For $X \subset \mathbb{R}^m$ put

$$\mathbf{A}^m(X) = \mathbf{T}^m \times X, \quad \mathbf{A}^m_\delta(X) = \mathbf{T}^m(\delta) \times X(\delta), \quad \mathbf{A}^m = \mathbf{T}^m \times \mathbb{R}^m. \tag{1.1}$$

Fix $n \geqslant 2$. We denote points of $\mathbf{A}^{n-1}$ by (q,p), $q \in \mathbf{T}^{n-1}$, $p \in \mathbb{R}^{n-1}$, and points of $\mathbf{A}^n$ by $(\tilde q, \tilde p)$; then $\tilde q = (q, q_n)$, $\tilde p = (p, p_n)$, where $(q,p) \in \mathbf{A}^{n-1}$.

Let $\delta_0 > 0$, $\mathcal{P}$ be a convex neighborhood of the origin in $\mathbb{R}^{n-1}$, and

$$f \colon \mathbf{A}^{n-1}(\mathcal{P}) \times (-1,1) \to \mathbf{A}^{n-1}$$

be a smooth map such that

a) $\forall \varepsilon \in (-1,1)$ the map $f^\varepsilon \colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathbf{A}^{n-1}$, $(q,p) \mapsto f(q,p,\varepsilon)$ is globally canonical with respect to the symplectic structure defined by the form $\omega_2 = dp \wedge dq = d\omega_1$, $\omega_1 = p dq$ on $\mathbf{A}^{n-1}$. In other words,

$$f^{\varepsilon *}\omega_1 - \omega_1 = dg^\varepsilon(q,p) \tag{1.2}$$

for some smooth function g^ε (in particular, $f^{\varepsilon *}\omega_2 = df^{\varepsilon *}\omega_1 = \omega_2$);

b) the map f^0 is Liouville integrable, i.e., $f^0(q,p) = (q + \nabla h(p), p)$;

c) the maps f^ε, $\varepsilon \in (-1,1)$, and h can be extended to complex-analytic maps

$$f^\varepsilon \colon \mathbf{A}^{n-1}_{\delta_0}(\mathcal{P}) \to \mathbf{A}^{n-1}_1(\mathcal{P}), \quad h \colon \mathbf{A}^{n-1}_{\delta_0}(\mathcal{P}) \to \mathbb{C} \tag{1.3}$$

such that for some $K > 0$ we have

$$|h| \leqslant K, \quad |\nabla h| \leqslant K, \quad \|\partial^2 h/\partial p_i \partial p_j\| \leqslant K, \quad |\partial f/\partial \varepsilon| \leqslant K \tag{1.4}$$

in $\mathbf{A}^{n-1}_{\delta_0}(\mathcal{P}) \times (-1,1)$ (we denote by $\| \partial^2 h/\partial p_i \partial p_j \|$ the norm of the linear operator with the matrix $(\partial^2 h/\partial p_i \partial p_j)$).

Consider the symplectic manifold $(\mathbf{A}^n(Q), d\tilde p \wedge d\tilde q)$. Here the domain $Q \subset \mathbb{R}^n$ is the image of the domain $\mathcal{P} \times (-1,1)$ under the map $(p,v) \mapsto (p, v - h(p))$. Put $\mathcal{M}^0 = \{(\tilde q, \tilde p) \in \mathbf{A}^n(Q) \mid q_n = 0\}$ and consider the Hamiltonian $H^0(\tilde q, \tilde p) = p_n + h(p)$ on $\mathbf{A}^n(Q)$. It corresponds to the Hamiltonian vector field

$$V^0 = \frac{\partial}{\partial q_n} + \sum_{j=1}^{n-1} \frac{\partial h}{\partial p_j}(p) \frac{\partial}{\partial q_j} \tag{1.5}$$

and to the succession map S^0 with respect to the (transversal) plot $\mathcal{M}^0$,

$$S^0(q, 0, p, p_n) = (q + \nabla h(p), 0, p, p_n).$$

The isoenergetic succession map with respect to the plot ${\mathcal{M}_\theta}^0 = \mathcal{M}^0 \cap \{H^0 = \theta\}$, $\theta \in (-1,1)$, has the form

$$S^0_\theta(q,0,p,\theta - h(p)) = (q + \nabla h(p), 0, p, \theta - h(p)).$$

For any θ it is conjugate to the symplectomorphism f^0 by means of the symplectic embedding

$$j_\theta \colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathbf{A}^n, \quad (q,p) \mapsto ((q,0),(p,\theta - h(p))).$$

If $\mathcal{M}'$ is an analytic hypersurface in $\mathbf{A}^n$ close to $\mathcal{M}^0$, and H' is an analytic Hamiltonian close to H^0, then the succession map $S' : \mathcal{M}' \to \mathcal{M}'$ and the isoenergetic succession map

$$S'_\theta \colon \mathcal{M}'_\theta \equiv \mathcal{M}' \cap \{H' = \theta\} \to \mathcal{M}'_\theta$$

are defined for the Hamiltonian vector field $V_{H'}$. The latter notation is not quite correct, since the domains of the maps S' and S'_θ are, in general, a little bit smaller than $\mathcal{M}'$ and $\mathcal{M}'_\theta$. By virtue of analyticity, the hypersurface $\mathcal{M}'$ is a strict subdomain of its analytic extension $\mathcal{M}''$. Then, if the surface $\mathcal{M}''$ is sufficiently close to $\{(\tilde q, \tilde p) \mid q_n = 0\}$, and the Hamiltonian H' to H^0, the maps $S' \colon \mathcal{M}' \to \mathcal{M}''$ and $S'_\theta : \mathcal{M}'_\theta \to \mathcal{M}''_\theta$ are well-defined.

Fix an arbitrary δ' such that $\delta' < \delta_0(1+3K)^{-1}(1+2K)^{-1}$. For $\delta < \delta_0$ put

$$\mathcal{O}(\delta) = \mathbf{A}^n_\delta(Q), \quad Q = \{(p,p_n) \mid p \in \mathcal{P}, p_n \in \mathbb{R}, \mid p_n + h(p) \mid < 1\}.$$

Theorem 1. *For $|\varepsilon| < \varepsilon_0$, with sufficiently small $\varepsilon_0 > 0$, there exist a Hamiltonian vector field V_{H^ε} with an analytic Hamiltonian H^ε on the symplectic manifold $(\mathbf{A}^n(Q),\ d\tilde p \wedge d\tilde q)$ and a hypersurface $\mathcal{M}^\varepsilon \subset \mathbf{A}^n(Q)$ such that $\forall \theta \in (-1,1)$ the isoenergetic succession map*

$$S^\varepsilon_\theta \colon \mathcal{M}^\varepsilon_\theta \equiv \mathcal{M}^\varepsilon \cap \{H^\varepsilon = \theta\} \to \mathcal{M}^\varepsilon_\theta$$

is conjugate to f^ε. The conjugation is implemented by means of a symplectic embedding j^ε_θ,

$$j^\varepsilon_\theta \colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathbf{A}^n(Q), \quad j^\varepsilon_\theta(\mathbf{A}^{n-1}(\mathcal{P})) = \mathcal{M}^\varepsilon_\theta.$$

In other words,

$$S^\varepsilon_\theta \circ j^\varepsilon_\theta = j^\varepsilon_\theta \circ f^\varepsilon \tag{1.6}$$

in a subdomain of $\mathbf{A}^{n-1}(\mathcal{P})$ that is the common domain (of definition) for the left-and-right-hand sides. Moreover, the function H^ε admits the holomorphic extension to $\mathcal{O}(\delta')$, and the map j^ε_θ, to the domain $\mathbf{A}^{n-1}_{\delta'}(\mathcal{P})$. These analytic extensions satisfy the following estimates:

$$|H^\varepsilon - p_n - h(p)| \leqslant C|\varepsilon|, \quad |j^\varepsilon_\theta - j_\theta| \leqslant C|\varepsilon|. \tag{1.7}$$

The quantities ε_0 and C depend only on n, δ_0, δ', and K.

Remark 1. Due to bounds for the analyticity radii of the maps H^ε and j^ε_θ and estimate (1.7), both the left-and-right-hand sides of (1.6) admit an analytic extension to the entire domain $\mathbf{A}^{n-1}_{\delta'}(\mathcal{P})$.

Remark 2. By virtue of the second estimate (1.7), the hypersurface $\mathcal{M}^\varepsilon$ is $C|\varepsilon|$–close to the hypersurface $\mathcal{M}^0 = \{q_n = 0\}$. Therefore, the isoenergetic succession map S^ε_θ and, hence, the symplectomorphism f^ε are conjugate by means of analytic symplectomorphisms to the isoenergetic succession map with respect to the hypersurface $\mathcal{M}^0 \cap \{H^\varepsilon = \theta\}$. By the first of the estimates (1.7) and the Cauchy inequality for derivatives of holomorphic functions, for $|\varepsilon|$ sufficiently small and $|\theta| < 1$ the equation $H^\varepsilon(\tilde{q}, \tilde{p}) = \theta$ can be resolved with respect to p_n: $p_n = K(q, p, q_n; \theta)$. Moreover, the function K can be extended analytically to $\mathcal{O}(\delta')$ and $|K(q, p, q_n; \theta) + h(p) - \theta| \leqslant C|\varepsilon|$ there. Therefore, phase trajectories of the vector field V_{H^ε} on the surface $\{H^\varepsilon = \theta\}$ obey the following Hamiltonian system

$$\partial q_j / \partial q_n = \partial K / \partial p_j, \quad \partial p_j / \partial q_n = -\partial K / \partial q_j, \quad i = 1, ..., n-1 \tag{1.8}$$

(isoenergetic reduction, see [10, §45.B]). Thus, they are trajectories of a nonautonomous Hamiltonian system, which is 1-periodic with respect to the variable q_n playing the role of time. The map ${S_\theta}^\varepsilon$ is conjugate in the symplectic analytic category to the shift map along trajectories of system (1.8) during the time interval $[0, 1]$.

Theorem 1 allows to deduce theorems on invariant sets of analytic symplectomorphisms close to integrable ones from the corresponding statements for vector fields. It allows also to reduce the investigation of higher iterations of such symplectomorphisms to the asymptotic study of trajectories for Hamiltonian vector fields close to integrable ones.

We present the proof of the theorem in the following section. It can be thought of as a constructive version of a rather natural approach due to R. Douady [6]. It consists of two stages. First (step 1), we prove that the analytic manifold $\mathcal{B}^\varepsilon$ arising in the suspension construction [2, 3] for the diffeomorphism f^ε is analytically diffeomorphic to a subdomain of $\mathbf{A}^n$, and the diffeomorphism itself is conjugate to the succession map for an analytic perturbation V^ε of the vector field V^0. Then, (step 2) we prove that one can amend the constructed diffeomorphism by means of an automorphism of $\mathbf{A}^n$ such that the improved diffeomorphism $\mathcal{B}^\varepsilon \to \mathbf{A}^n$ becomes a symplectomorphism from $\mathcal{B}^\varepsilon$ to a subdomain of $\mathbf{A}^n$, and the field V^ε becomes Hamiltonian. The assertion of the first step is, essentially, a constructive version of the Grauert theorem for the manifold $\mathcal{B}^\varepsilon$. The proof of the assertion of the second step is based upon a rather traditional application of the Moser-Weinstein theorem.

§2. THE PROOF OF THEOREM 1

Henceforth we denote by C, C', C_1, $C_2, \dots$ constants in estimates, which can vary in different places and depend only on n, δ_0, δ', and K. All complex analytic maps below are real for real arguments. As a rule, we do not mention this and omit the trivial verification of this property. We write "diffeomorphism" to denote a diffeomorphic embedding. To derive estimates we make use of the condition "ε_0 is sufficiently small" in a systematic way.

Fix numbers $a < 0$, $b > 1$ such that $|b - a| < 3/2$. Denote

$$\mathcal{A} = \mathbf{A}^{n-1}(\mathcal{P}) \times (a, b) \times (-1, 1) = \{(q, p, u, v)\}, \quad \mathcal{A}_\delta = {\mathbf{A}_\delta}^{n-1}(\mathcal{P}) \times (a, b)_\delta \times (-1, 1)_\delta,$$

where $(c, d)_\delta$ with $c < d$ denotes the rectangle in the complex plane of the form

$$(c, d)_\delta = \{(x + iy) \in \mathbb{C} \mid c - \delta < x < d + \delta, |y| < \delta\}.$$

We equip $\mathcal{A}$ with an equivalence relation $\sim_\varepsilon$ such that $\xi \sim_\varepsilon \eta$ if the map

$$E^\varepsilon \colon (q, p, u, v) \mapsto (f^\varepsilon(q, p), u - 1, v).$$

either takes ξ to η, or, conversely, takes η to ξ (note that the domain of the map E^ε is a strict subdomain of $\mathcal{A}$). We denote the quotient space $\mathcal{A}/\sim_\varepsilon$ by $\mathcal{B}^\varepsilon$ and endow $\mathcal{B}^\varepsilon$ with a structure of $2n$-dimensional analytic manifold; the canonical map $\pi \colon \mathcal{A} \to \mathcal{B}^\varepsilon$ that assigns to a point of $\mathcal{A}$ its equivalence class is a local analytic diffeomorphism with respect to this structure. Then a function $h : \mathcal{B}^\varepsilon \to \mathbb{R}$ is analytic if and only if the function $\pi^* h \equiv h \circ \pi$ is analytic. Note that for any $\theta \in (-1, 1)$ the subset

$$\mathcal{B}^\varepsilon_\theta = \pi\{(q, p, u, v) \in \mathcal{A} \mid v = \theta\}$$

is an analytic submanifold of $\mathcal{B}^\varepsilon$.

Consider the 2-form $\alpha_2 = dp \wedge dq + dv \wedge du$ on the manifold $\mathcal{A}$. Since the map f^ε is a symplectomorphism, we have $E^{\varepsilon *}\alpha_2 = \alpha_2$, and α_2 defines a closed 2-form ω_2^ε on the manifold $\mathcal{B}^\varepsilon$. This form defines a symplectic structure on $\mathcal{B}^\varepsilon$ such that the map π is a local symplectomorphism with respect to this structure.

The map E^ε respects the vector field $\partial/\partial u$ on $\mathcal{A}$, thus defining a vector field $\tilde{V}$ on $\mathcal{B}^\varepsilon$. This field is tangent to every manifold $\mathcal{B}^\varepsilon_\theta$, $\theta \in (-1, 1)$, and is Hamiltonian with the Hamiltonian function $\tilde{H}$, where $\tilde{H}(\pi(q, p, u, v)) \equiv v$. Let $\tilde{S}_\theta$ be the isoenergetic succession map for the vector field $\tilde{V}$ on $\mathcal{B}^\varepsilon_\theta$ with respect to the submanifold

$$\mathcal{M}_\theta = \pi\{(q, p, u, v) \mid u = 0, v = \theta\} \subset \mathcal{B}^\varepsilon_\theta.$$

Lemma 1. *The map $\tilde{j}_\theta \colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathcal{B}^\varepsilon$, $(q, p) \mapsto \pi(q, p, 0, \theta)$, conjugates the maps $\tilde{S}_\theta$ and f^ε.*

Proof. Consider the point $\xi = \tilde{j}_\theta(q, p) = \pi(q, p, 0, \theta)$. The shift along trajectories of the field $\partial/\partial u$ in unit time takes $(q, p, 0, \theta)$ to $(q, p, 1, \theta)$. But $\pi(q, p, 1, \theta) = \pi(f^\varepsilon(q, p), 0, \theta) = \tilde{j}_\theta(f^\varepsilon(q, p))$. Therefore, the shift along trajectories of the field $\tilde{V}$ takes the point $\tilde{j}_\theta(q, p) \in \widetilde{\mathcal{M}}_\theta$ to the point $\tilde{j}_\theta(f^\varepsilon(q, p)) \in \widetilde{\mathcal{M}}_\theta$ as required.

Thus, the suspension construction conjugates the map f^ε with the succession map for a Hamiltonian vector field on the manifold $\mathcal{B}^\varepsilon$, depending on ε. To prove the theorem, a symplectomorphism φ^ε from the manifold $\mathcal{B}^\varepsilon$ to a subdomain of $\mathbf{A}^n$ is to be constructed. Thereafter we can put $H^\varepsilon = \tilde{H} \circ (\varphi^\varepsilon)^{-1}, {j_\theta}^\varepsilon = \varphi^\varepsilon \circ \tilde{j}_\theta$. We construct the map φ^ε as the superposition $\varphi^\varepsilon = K^\varepsilon \circ G^\varepsilon$, where G^ε is an analytic diffeomorphism from $\mathcal{B}^\varepsilon$ to a subdomain of $\mathbf{A}^n$ such that the form $(G^{\varepsilon *})^{-1}{\omega_2}^\varepsilon$ is equal to $dp \wedge dq + O(\varepsilon)$ and K^ε is a diffeomorphism of $\mathbf{A}^n$ compensating the discrepancy $O(\varepsilon)$ (and transforming the superposition $K^\varepsilon \circ G^\varepsilon$ into a symplectomorphism).

The map G^ε solves the problem of analytic conjugation of the map f^ε with the succession map for an analytic vector field on $\mathbf{A}^n$. The proof of its existence is the main step in the proof of the theorem. Below we state the corresponding result as a lemma, to be proved in the next section. Since $\delta' < \delta_0(1 + 3K)^{-1}(1 + 2K)^{-1}$, we can choose numbers $\delta^1, \delta^2, \dots,$ δ^6 such that

$$\delta_0 > \delta^1 > \delta^2 > \cdots > \delta^6 > \delta', \quad \delta_0 > (1 + 3K)\delta^1, \quad \delta^4 > (1 + 2K)\delta^5. \tag{2.1}$$

Lemma 2. *If $|\varepsilon| < \varepsilon_0$ with sufficiently small ε_0, then there exist a constant C and an analytic diffeomorphism $G_0^\varepsilon \colon \mathcal{B}^\varepsilon \to \mathbf{A}^n$ such that the map $G_0^\varepsilon \colon \mathcal{A} \to \mathbf{A}^n$ takes the point (q, p, u, v) to $((q', q_n), (p', v))$ (i.e., it is identical in the variable v) and can be extended analytically to the domain $\mathcal{A}_{\delta^3}$. In this domain it differs from G_0^0 by less than $C\varepsilon$ and*

$$G_0^0(q, p, u, v) = (q + u\nabla h(p), u, p, v).$$

Remark 3. The image of the map G_0^ε belongs to a polyannulus $\mathbf{A}^n(X)$, where X is a n-dimensional parallelepiped. Thus G_0^ε defines an analytic embedding of $\mathcal{B}^\varepsilon$ into $\mathbb{R}^{2n}$, with a known lower bound for its analyticity radius, and the lemma provides a constructive proof of the Grauert theorem for the manifold $\mathcal{B}^\varepsilon$. Conversely, a nonconstructive version of Lemma 2 follows easily from the Grauert theorem. For the proof, we introduce a natural structure of an analytic manifold in the set $\mathcal{B} = \{(\xi, \varepsilon) \mid \xi \in \mathcal{B}^\varepsilon, \varepsilon \in (-1,1)\}$ and construct an embedding $j\colon \mathcal{B} \to \mathbb{R}^N$ in accordance with the Grauert theorem. If $\mathcal{B}_c^\varepsilon$ is an arbitrary compact set in $\mathcal{B}^\varepsilon$ then, for $|\varepsilon|$ small enough, the analytic map $\pi_\varepsilon\colon j(\mathcal{B}_c^\varepsilon) \to j(\mathcal{B}^0)$ assigning to a point of $\mathcal{B}^\varepsilon$ the nearest point of $\mathcal{B}_0$ is defined. The map

$$\mathcal{A} \to \mathbf{A}^n, \quad (q,p,u,v) \mapsto (q + u\nabla h(p), u, p, v)$$

defines an analytic embedding $G_0^0\colon \mathcal{B}^0 \to \mathbf{A}^n$. We must still set

$$G_0^\varepsilon = G_0^0 \circ j^{-1} \circ \pi_\varepsilon \circ j\,|_{\mathcal{B}_c^\varepsilon}\ .$$

Consider the map $G_1\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $(q, q_n, p, p_n) \mapsto (q, q_n, p, p_n - h(p))$, and put $G^\varepsilon = G_1 \circ G_0^\varepsilon\colon \mathcal{B}^\varepsilon \to \mathbf{A}^n$. Then the map $\pi^* G^\varepsilon$ can be extended to $\mathcal{A}_{\delta^3}$ as well,

$$|\pi^* G^\varepsilon - \pi^* G^0| \leqslant C_1 \varepsilon \quad \forall (q,p,u,v) \in \mathcal{A}_{\delta^3}, \tag{2.2}$$

where

$$\pi^* G^0(q,p,u,v) = (q + u\nabla h(p), u, p, v - h(p)). \tag{2.3}$$

Now we proceed to the construction of the map K^ε.

Lemma 3. *The form ω_2^ε is exact : $\omega_2^\varepsilon = d\omega_1^\varepsilon$ for some smooth 1-form ω_1^ε on $\mathcal{B}^\varepsilon$.*

Proof. It suffices to find a smooth 1-form α_1^ε on $\mathcal{A}$ such that $d\alpha_1^\varepsilon = \alpha_2 = dp \wedge dq + dv \wedge du$ and

$$E^{\varepsilon *} \alpha_1^\varepsilon = \alpha_1^\varepsilon. \tag{2.4}$$

We put $\alpha_1^\varepsilon = pdq + vdu + dV(q,p,u,v)$. The differential of this form has the required expression and we must still verify equality (2.4). By (1.2), we see that

$$E^{\varepsilon *} \alpha_1^\varepsilon = pdq + vdu - dg^\varepsilon + d(V \circ E^\varepsilon).$$

Thus, equality (2.4) holds if the function V obeys the relation

$$V \circ E^\varepsilon = V + g \tag{2.5}$$

on the whole domain of the map E^ε.

A smooth function $g^\varepsilon(q,p)$ on $\mathbf{A}^{n-1}(\mathcal{P})$ can be recovered uniquely, up to a constant, from its differential by the formula

$$g^\varepsilon(q,p) = \int_\gamma dg^\varepsilon,$$

where γ is any smooth path from a fixed point (q_0, p_0) to (q,p). By virtue of (1.2), the form dg^ε can be extended analytically to $\mathbf{A}^{n-1}_{\delta_0/2}(\mathcal{P}) \cap \mathbf{A}^{n-1}$ if $|\varepsilon|$ is small enough. Therefore the function $g^\varepsilon(f^\varepsilon(q,p))$ can be extended analytically to $\mathbf{A}^{n-1}(\mathcal{P})$. To construct a smooth function V satisfying (2.5), we take for V a smooth extension on $\mathcal{A}$ of the function that is equal to zero for $a < u < b-1$ and equal to $g^\varepsilon(f^\varepsilon(q,p))$ for $a+1 < u < b$. We shall use in a systematic way the Cauchy estimate for derivatives of holomorphic maps, in particular, in the following form:

Lemma 4. *Let $g : \mathcal{A}_\delta \to \mathbb{C}^N$ be a holomorphic map whose modulus does not exceed one. Then for $\delta' < \delta$ and $j = 1, 2, ..., n$ we have*

$$|\partial g/\partial q_j| \leqslant (\delta - \delta')^{-1}, \quad |\partial g/\partial p_j| \leqslant (\delta - \delta')^{-1}$$

everywhere in $\mathcal{A}_{\delta'}$.

For $\delta \leqslant \delta^3$, denote

$$\mathcal{D}^\varepsilon = G^\varepsilon(\mathcal{B}^\varepsilon) = (\pi \circ G^\varepsilon)\mathcal{A} \subset \mathbf{A}^n, \quad \mathcal{D}^\varepsilon_\delta = (\pi^* G^\varepsilon)\mathcal{A}_\delta, \tag{2.6}$$

and define a 2-form on $\mathcal{D}^\varepsilon$ by putting

$$\gamma_2^\varepsilon = (G^{\varepsilon *})^{-1}\omega_2^\varepsilon = ((G^\varepsilon \circ \pi)^*)^{-1}\alpha_2.$$

The map $G^\varepsilon \circ \pi \colon \mathcal{A}_{\delta^3} \to \mathcal{D}^\varepsilon_{\delta^3}$ is a local biholomorphism. Since α_2 can be extended to a $(2,0)$-form (see [7]) with constant coefficients on $\mathcal{A}_{\delta^3}$, the form γ_2^ε can be extended to $\mathcal{D}^\varepsilon_{\delta^3}$ as a holomorphic form. By Lemmas 2 and 4, the maps $\pi^* G^\varepsilon$ and $\pi^* G^0$ are $C\varepsilon$-close on $\mathcal{A}_{\delta^4}$ with respect to the C^1-norm. One verifies directly that $(G^0 \circ \pi)^* d\tilde{p} \wedge d\tilde{q} = \alpha_2$. Therefore $\gamma_2^0 = d\tilde{p} \wedge d\tilde{q}$ and if $\Delta\gamma_2 = \gamma_2^\varepsilon - d\tilde{p} \wedge d\tilde{q}$, then

$$|\Delta\gamma_2| \leqslant C_2\varepsilon \quad \text{everywhere in} \quad \mathcal{D}^\varepsilon_{\delta^4}, \tag{2.7}$$

where $|\Delta\gamma_2|$ stands for the maximal modulus of the coefficients of the $(2,0)$-form in its expansion with respect to the basis $dx_k \wedge dx_l$, $x = (\tilde{p}, \tilde{q}), 1 \leqslant k, l \leqslant 2n$.

Lemma 5. *There exists an analytic differential form $\Delta\gamma_1$ in $\mathcal{D}^\varepsilon$ such that $d\Delta\gamma_1 = \Delta\gamma_2$. Moreover, $\Delta\gamma_1$ can be extended to $\mathcal{O}(\delta^5)$ as a holomorphic $(1,0)$-form, and the estimate $|\Delta\gamma_1| \leqslant C|\varepsilon|$ holds for the extended form.*

Proof. Observe that the map $G^\varepsilon \circ \pi$ is an affine mapping with respect to the variable v. Therefore, for all $C > 0$ the forms γ_2^ε and $\Delta\gamma_2$ admit an analytic extension to the domain $\mathcal{D}^\varepsilon_{\delta^4}(C)$ consisting of points of the form $(q, q_n, p, p_n + v)$, where $(q, q_n, p, p_n) \in \mathcal{D}^\varepsilon_{\delta^4}$ and $v \in \mathbb{C}, |v| \leqslant C$. Moreover, the estimate (2.7) is valid in $\mathcal{D}^\varepsilon{}_{\delta^4}(C)$, with a constant C_2 depending on C. For sufficiently large C, the map

$$W \colon \mathcal{D}^\varepsilon_{\delta^4} \times [0,1] \to \mathcal{D}^\varepsilon_{\delta^4}(C), \quad (\tilde{q}, \tilde{p}, t) \to (\tilde{q}, t\tilde{p})$$

determines a holomorphic homotopy between the identity map $W_1 \colon \mathcal{D}^\varepsilon_{\delta^4} \to \mathcal{D}^\varepsilon_{\delta^4}$ and the natural projection $W_0 \colon \mathcal{D}^\varepsilon_{\delta^4} \to \mathbf{T}^n(1) \times \{0\}$. Consider the form $\Omega = W^* \Delta\gamma_2$ on $\mathcal{D}^\varepsilon_{\delta^4} \times [0,1]$ and define a form $D\Omega$ on $\mathcal{D}^\varepsilon_{\delta^4}$ by

$$D\Omega = -\int_0^1 (\partial/\partial t \rfloor \Omega) dt$$

(the integral is to be understood as follows: we write out the form $\partial/\partial t \rfloor \Omega$ in the natural coordinate system for the manifold $\mathcal{D}^\varepsilon_{\delta^4} \times [0,1]$ and then integrate its components as functions). The following homotopy formula holds [11]:

$$\Delta\gamma_2 - \Delta_1\gamma_2 = dD\Omega, \quad \Delta_1\gamma_2 = W_0^* \Delta\gamma_2. \tag{2.8}$$

The forms $\Delta_1\gamma_2$ and $D\Omega$ are holomorphic in $\mathcal{D}^{\varepsilon}_{\delta^4}$. By virtue of (2.7) we have

$$|\Delta_1\gamma_2| + |D\Omega| \leqslant C'|\varepsilon| \tag{2.9}$$

everywhere in $\mathcal{D}^{\varepsilon}_{\delta^4}$.

It follows from the explicit form of the map $G^0 \circ \pi$(see (2.3)) that

$$(G^0 \circ \pi)^{-1}(q, q_n, p, p_n) = (q - q_n \nabla h(p), q_n, p, p_n + h(p)).$$

Set $\delta = \delta^4(1 + 2K)^{-1}$. If $(q, q_n, p, p_n) \in \mathcal{O}(\delta)$ then $0 \leqslant \Re q_n \leqslant 1$, and by (1.4) we have

$$\begin{aligned}|\Im(q - q_n \nabla h(p)| &\leqslant |\Im q| + |\Im q_n||\Re \nabla h(p)| + +|\Re q_n||\Im m\, \nabla h(p)|\\ &\leqslant \delta + \delta K + |\Im(\nabla h(p) - \nabla h(\Re p))| \leqslant \delta(1 + 2K)\end{aligned}$$

and

$$|\Im(p_n + h(p))| \leqslant \delta + \delta K.$$

Therefore $(G^0 \circ \pi)^{-1}\mathcal{O}(\delta) \subset \mathcal{A}_{\delta^4}$, and consequently

$$\mathcal{O}(\delta^4(1 + 2K)^{-1}) \subset (G^0 \circ \pi)\mathcal{A}_{\delta^4} = \mathcal{D}^0_{\delta^4}.$$

From estimate (2.2) and Lemma 4 we see that the set $\mathcal{D}^{\varepsilon}_{\delta^4} = (G^{\varepsilon} \circ \pi)\mathcal{A}_{\delta^4}$ contains the $C|\varepsilon|$-interior of the set $\mathcal{D}^0_{\delta^4} = (G^0 \circ \pi)\mathcal{A}_{\delta^4}$. Since $\delta^5 < \delta^4(1 + 2K)^{-1}$, the domain $\mathcal{D}^{\varepsilon}_{\delta^4}$ contains $\mathcal{O}(\delta^5)$ for sufficiently small $|\varepsilon|$. Now consider transforms S_l of the domain $\mathcal{O}(\delta^5)$:

$$S_l(\tilde{q}, \tilde{p}) = (\tilde{q} + l, \tilde{p}), \quad l \in \mathbb{R}^n, \quad |l| \leqslant \sqrt{n}.$$

The maps S_l are homotopic to the identity map. Therefore, in the same way as above, we have $\Delta_1\gamma_2 = S_l^*\Delta_1\gamma_2 + d\Omega_l$ for some 1-form Ω_l that can be extended to a holomorphic form in $\mathcal{O}(\delta^5)$ and does not exceed $C|\varepsilon|$ there. Integrating the latter equality with respect to $l \in [0, 1]^n$, we get

$$\Delta_1\gamma_2 = \Delta_2\gamma_2 + d\Omega^1, \quad \Delta_2\gamma_2 = \int S_l^*\Delta_1\gamma_2 dl. \tag{2.10}$$

Since the coefficients of the form $\Delta_1\gamma_2$ do not depend on p, the coefficients of the form $\Delta_2\gamma_2$ are constant, and $|\Delta_2\gamma_2| + |\Omega^1| \leqslant C|\varepsilon|$ everywhere in $\mathcal{O}(\delta^5)$.

From (2.8) and (2.10) we derive $\Delta\gamma_2 = \Delta_2\gamma_2 + d(\Omega^1 + D\Omega)$; by Lemma 3, the form $\Delta\gamma_2$ is exact. Therefore, the form $\Delta_2\gamma_2$ is exact as well, and $\Delta_2\gamma_2 = d\gamma_1$ for some smooth form γ_1 everywhere in $\mathcal{D}^{\varepsilon}$.

By averaging the latter equality with respect to $q \in \mathbf{T}^n$ we obtain

$$\Delta_2\gamma_2 = d\gamma_1^0, \quad \gamma_1^0 = \sum_{j=1}^{n} a_j(\tilde{p})d\tilde{p}_j + b_j(\tilde{p})d\tilde{q}_j$$

and

$$\Delta_2\gamma_2 = \sum(\frac{\partial a_j(\tilde{p})}{\partial \tilde{p}_k} d\tilde{p}_k \wedge d\tilde{p}_j + \frac{\partial b_j(\tilde{p})}{\partial \tilde{p}_k} d\tilde{p}_k \wedge d\tilde{q}_j).$$

This means that the derivatives

$$\partial a_j(\widetilde{p})/\partial \tilde{p}_k = C_j^k, \quad \partial b_j(\widetilde{p})/\partial \tilde{p}_k = {C'}_j^k, \quad |C_j^k| + |{C'}_j^k| \leqslant C|\varepsilon| \quad \forall j, k$$

are constants and $a_j(\tilde{p}), b_j(\tilde{p})$ are affine functions whose gradients do not exceed $C'|\varepsilon|$. Replace the functions $a_j(\tilde{p})$ and $b_j(\tilde{p})$ in the expression for the form γ_1^0 by $a_j(\tilde{p}) - a_j(0)$ and $b_j(\tilde{p}) - b_j(0)$, respectively, and denote the form thus obtained by γ_1^{00}.Then $d\gamma_1^{00} = d\gamma_1^0 = \Delta_2\gamma_2$ and $|\gamma_1^{00}| \leqslant C_1|\varepsilon|$.

Thus $\Delta\gamma_2 = d(\Omega^1 + D\Omega + \gamma_1^{00})$. Coefficients of the forms $\Omega^1, D\Omega$ and γ_1^{00} can be extended holomorphically into $\mathcal{O}(\delta^5)$ and do not exceed $C|\varepsilon|$ there.

Remark 4. Lemma 5 provides a constructive proof of the following fact: if a closed analytic form is exact in the smooth category then it is exact in the analytic category as well. This assertion is equivalent to the analytic version of the De Rham theorem: the kth cohomology group of a real analytic manifold is isomorphic to the quotient group of closed analytic k-forms by the subgroup of exact forms. For compact manifolds this theorem is an obvious consequence of the Hodge decomposition. In the noncompact case the latter is not at our disposal. Nevertheless, the assertion remains true. Its proof can be obtained by using the theory of coherent analytic sheaves [12]. The above proof of Lemma 5 (in fact, borrowed from [11]) leans heavily on the fact that the manifold $\mathbf{A}^n(Q)$ admits an analytic deformation retraction on the torus $\mathbf{T}^n \times \{0\}$. It seems that no constructive proofs of the analytic version of the De Rham theorem are known in the general case.

Lemma 6 (Moser-Weinstein). *There exists an analytic diffeomorphism $K^\varepsilon\colon \mathcal{D}^\varepsilon \to \mathbf{A}^n$ such that $K^{\varepsilon *}(d\tilde{p} \wedge d\tilde{q}) = \gamma_2^\varepsilon$. The map K^ε can be extended to a biholomorphic diffeomorphism $\mathcal{O}(\delta^6) \to \mathbf{T}^n(1) \times \mathbb{C}^n$ that is $C'|\varepsilon|$-close to the identity map.*

Proof. We put

$$\omega_t = \gamma_2^\varepsilon + t(d\tilde{p} \wedge d\tilde{q} - \gamma_2^\varepsilon) = d\tilde{p} \wedge d\tilde{q} + (1-t)\Delta\gamma_2$$

and construct a nonautonomous vector field $\xi_t(\tilde{p}, \tilde{q})$ on the manifold $\mathcal{D}^\varepsilon$ such that $\xi_t \rfloor \omega_t = \Delta\gamma_1$. By virtue of estimate (2.7) and Lemma 5, the vector field is well-defined and can be extended analytically to the domain $\mathcal{O}(\delta^5)$ so that

$$|\xi_t(\tilde{p}, \tilde{q})| \leqslant C|\varepsilon| \quad \forall (\tilde{p}, \tilde{q}) \in \mathcal{O}(\delta^5). \tag{2.11}$$

Denote by φ_t the shift operator along trajectories of the field ξ_t for the time interval $[0, t]$. By virtue of (2.11), for any $t \in [0, 1]$ the map φ_t can be extended analytically to the domain $\mathcal{O}(\delta^6)$ and differs there from the identity map by no more than $C_2 t|\varepsilon|$.

By the main formula of differential calculus [2, 13] we have

$$\frac{d}{dt}(\varphi_t^*(\omega_t)) = \varphi_t^*(\frac{d}{dt}\omega_t)) + d\varphi_t^*(\xi_t \rfloor \omega_t) = -\varphi_t{}^*(\Delta\gamma_2^\varepsilon) + d\varphi_t^*(\Delta\gamma_1) = d\varphi_t^*(-\Delta\gamma_1 + \Delta\gamma_1) = 0.$$

Hence $\varphi_1^*(d\tilde{p} \wedge d\tilde{q}) = \varphi_1^*(\omega_1) = \varphi_0^*(\omega_0) = \gamma_2^\varepsilon$, and the lemma is proved if we take $K^\varepsilon = \varphi_1$. $\square$

Now we complete the proof of Theorem 1. Denote by $\mathcal{M}_\theta^\varepsilon$ the image of the map j_θ^ε, where

$$j_\theta^\varepsilon = K^\varepsilon \circ G^\varepsilon \circ \tilde{j}_\theta\colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathbf{A}^n,$$

and put $\mathcal{M}^\varepsilon = \cup\{\mathcal{M}_\theta^\varepsilon | -1 < \theta < 1\}$. The map j_θ^ε is symplectic as the composition of the symplectic maps $\tilde{j}_\theta$ and $K^\varepsilon \circ G^\varepsilon$. By Lemmas 2 and 6 it can be extended analytically to $\mathbf{A}_{\delta'}^{n-1}(\mathcal{P})$. Since for $u = 0$ the map $K^\varepsilon \circ G^\varepsilon$ is $C|\varepsilon|$-close to the map $(q, p, 0, \theta) \mapsto (q, 0, p, \theta - h(p))$, we obtain estimate (1.6) for j_θ^ε.

By virtue of Lemma 1 the conjugation of f^ε by j_θ^ε is the succession map for the vector field $(K^\varepsilon \circ G^\varepsilon)_* \tilde{V}$. The field is Hamiltonian and corresponds to the Hamiltonian H^ε,

$$H^\varepsilon = \tilde{H} \circ (G^\varepsilon)^{-1} \circ (K^\varepsilon)^{-1} = (\pi^* \tilde{H}) \circ (\pi^* G^\varepsilon)^{-1} \circ (K^\varepsilon)^{-1}. \tag{2.12}$$

By Lemmas 2 and 6, the right-hand side of (2.12) can be extended to a complex analytic function on $\mathcal{O}(\delta')$, $C_2|\varepsilon|$-close to the function

$$(\pi^* \tilde{H}) \circ (\pi^* G^0)^{-1}(q, u, p, \theta) = \theta + h(p),$$

thus proving estimate (1.7) for the function H^ε.

The last assertion of the theorem follows from the definition of the constants C.

§3. PROOF OF LEMMA 2

We begin with the definition of a certain class of analytic manifolds containing the manifolds $\mathcal{B}^\varepsilon$. Let the domains $\mathcal{A}$ and $\mathcal{A}_\delta$ be as in §2, and g be an analytic map from $\mathbf{A}^{n-1}(\mathcal{P})\times(a,b)$ to $\mathbf{A}^{n-1}$. Introduce the following equivalence relation in $\mathcal{A}$: (q,p,u,v) is equivalent to (q',p',u',v') if either

$$(q',p')=g(q,p,u),\quad u'=u-1,\quad v'=v,$$

or

$$(q,p)=g(q',p',u),\quad u=u'-1, v=v'.$$

Denote by $\mathcal{B}_g$ the analytic manifold obtained by taking the quotient of $\mathcal{A}$ with respect to this equivalence relation.

Let F be a diffeomorphic analytic map of $\mathcal{A}$ into $\mathbf{A}^{n-1}\times(a,b)\times(-1,1)$ that is identical in the variables u and v. The assertion below follows directly from the definition. Here and further we denote by x the pair $(q,p)\in\mathbf{A}^{n-1}(\mathcal{P})$.

Lemma 7. *The map F defines an analytic diffeomorphism $\tilde F:\mathcal{B}_{g_1}\to\mathcal{B}_{g_2}$, provided the following diagram is commutative on the common domain:*

$$\begin{array}{ccc}(x,u,v) & \longrightarrow & (g_2(x,u),u-1,v)\\ {\scriptstyle F}\Big\downarrow & & \Big\downarrow{\scriptstyle F}\\ (x',u',v') & \longrightarrow & (g_1(x',u'),u'-1,v)\end{array}\tag{3.1}$$

In this case

$$\tilde F\circ\pi=\pi\circ F.\tag{3.2}$$

Note that if $g(x,u)\equiv x$, then the manifold $\mathcal{B}_g$ is analytically diffeomorphic to the product $\mathbf{T}^n\times\mathcal{P}\times(0,1)$. Therefore, to construct the map G_0^ε, it suffices to find a map F that makes the diagram (3.1) (with $g_2=f^\varepsilon$ and $g_1(x,u)\equiv x$) commutative.

All the functions arising in this section do not depend on the variable v, while the maps from $\mathcal{A}$ onto itself are identical with respect to v. That is why we omit, as a rule, the letter v below. By a certain abuse of language, we write $(x,u)\in\mathcal{A}_\delta$ instead of the more precise notation $(x,u)\in{\mathcal{A}_\delta}^{n-1}(\mathcal{P})\times(a,b)_\delta$.

The map G_0^ε will be constructed as the composition $G_0^\varepsilon=(\tilde G)^{-1}\circ\tilde F$. We begin with the map $\tilde F$. We set $g_2=f^\varepsilon=(f^{\varepsilon q},f^{\varepsilon p})$ and $F(q,p,u,v)=F_0=(q+u\nabla h(p),p,u,v)$ in the diagram (3.1). One can verify directly that the diagram (3.1) can be made commutative by means of a map $g_1(q,p,u)=(g_1^q,g_1^p)(q,p,u)$ that is defined as follows:

$$g_1^p(q,p,u)=f^{\varepsilon p}(q-u\nabla h(p),p),\tag{3.3}$$

$$g_1^q(q,p,u)=f^{\varepsilon q}(q-u\nabla h(p),p)+(u-1)\nabla h(f^{\varepsilon p}(q-u\nabla h(p),p)).\tag{3.4}$$

Observe that

$$(q-u\nabla h(p),p,u)\in\mathcal{A}_{\delta_0}\quad\forall(q,p,u)\in\mathcal{A}_{\delta_{0/(1+3K)}},\tag{3.5}$$

Since $\delta^1<\delta_0/(1+3K)$, for $(q,p,u)\in\mathcal{A}_{\delta^1}$ we obtain from (3.5) and condition (1.4) that

$$|f^{\varepsilon q}(q-u\nabla h(p),p)-(q-u\nabla h(p)+\nabla h(p))|\leqslant C|\varepsilon|,$$

$$|f^{\varepsilon p}(q-u\nabla h(p),p)-p|\leqslant C|\varepsilon|,\quad|\nabla h(f^{\varepsilon p})-\nabla h(p)|\leqslant C_1|\varepsilon|.$$

These inequalities and expressions (3.3), (3.4) for the map g_1 imply the proximity of g_1 and the identity map:

$$|g_1(x,u) - x| \leqslant \tilde{C}|\varepsilon| \quad \forall (x,u) \in \mathcal{A}_{\delta^1}.$$

Let $\tilde{F}\colon \mathcal{B}_{g_1} \to \mathcal{B}_{g_2}$ be the map assigned to F by Lemma 7.

Now we proceed to the construction of the map $\tilde{G}$. Take $g_2(x,u) \equiv x$ and $g_1(x,u) = (g_1^q, g_1^p)(x,u)$ in diagram (3.1). Let the map $F = G$ make the diagram commutative. Denote

$$g(x,u) = x - g_1(x,u), \quad G(x,u) = (x + f(x,u), u).$$

Then

$$|g| \leqslant \tilde{C}|\varepsilon| \quad \forall (x,u) \in \mathcal{A}_{\delta^1}, \tag{3.6}$$

and the maps g and f are related by

$$x + f(x, u-1) = x + f(x,u) - g(x + f(x,u), u),$$

or

$$f(x,u) - f(x,u-1) - g(x + f(x,u), u) = 0. \tag{3.7}$$

Lemma 8. *There exists a function $f(x,u)$ that satisfies* (3.7) *and can be extended to a holomorphic function on $\mathcal{A}_{\delta^2}$ not exceeding $C_1\varepsilon$ there.*

Take Lemma 8 for granted. Then, in view of Lemma 4, the map G is invertible and the inverse map takes the form

$$G^- : (x,u,v) \mapsto (x + f^-(x,u), u, v), \tag{3.8}$$

where the function $f^-(x,u)$ can be extended to a holomorphic function in $\mathcal{A}_{\delta^3}$, bounded by $C_2|\varepsilon|$. Hence, by Lemma 7, the map $\tilde{G}^- = (\tilde{G})^{-1}$ takes $\mathcal{B}_{g_1}$ to the manifold $\mathcal{B}_{g_2}$, $g_2(x,u) \equiv x$, i.e., into a subdomain of $\mathbf{A}^n$. Since

$$\widetilde{G}^- \circ \tilde{F} \circ \pi = \pi \circ g^- \circ F$$

by Lemma 7, Lemma 2 follows from (3.6), (3.8) and the bound for f^-.

Now, Lemma 8 is still to be proved. The rest of the section is devoted to this task. We carry out our proof by using the Newton accelerated convergence method and outline it without details. The reason for our brevity is explained in the concluding Remark 5.

Proof of Lemma 8. Denote by D the difference operator with respect to u:

$$Df(u) = f(u) - f(u-1).$$

We linearize equation (3.7) assuming that the map f is small. It is convenient to write out the linearized equation in the following form:

$$D\varphi(x,u) + G(x,u)\varphi(x,u) = g_1(x,u), \quad G(x,u) = -g'_x(x,u)|_{x = x + \varphi_0(x,u)}, \tag{3.9}$$

where $g_1(x,u) \equiv g(x,u)$ and $\varphi_0 \equiv 0$.

Lemma 9. *Suppose that the functions φ_0 and g_1 are holomorphic in a domain $\mathcal{A}_\delta$, $\delta < \delta^1$, and $|\varphi_0| \leqslant \varepsilon' < |\varepsilon|$, $|g_1| \leqslant 1$. If $\tilde{\delta} < \delta$ and*

$$\tilde{C}|\varepsilon| < \frac{1}{4}(\delta^1 - \delta), \quad \varepsilon' < \frac{1}{2}(\delta^1 - \delta) \tag{3.10}$$

(*the constant $\tilde{C}$ is the same as in* (3.6)), *then there exists a function $\varphi(x,u)$ in $\mathcal{A}_{\tilde{\delta}}$ such that*

$$|\varphi| < C_1 \exp\left\{ \frac{C_2|\varepsilon|}{(\delta - \tilde{\delta})(\delta' - \delta)} \right\} \tag{3.11}$$

and φ obeys relation (3.9) *for all the points $(x,u) \in \mathcal{A}_{\tilde{\delta}}$ such that $(x, u-1) \in \mathcal{A}_{\tilde{\delta}}$. The constants C_1 and C_2 in* (3.11) *do not depend on $\delta, \tilde{\delta}, \delta^1$.*

The proof of the lemma is based upon the study of the operator D in the space $\mathfrak{A}_\delta$ of holomorphic functions on $(a,b)_\delta$ that are real for a real argument, endowed with the sup-norm $\|\cdot\|_\delta$. The lemma stated below will be proved at the end of the section.

Lemma 10. *For any $0 < \delta' < \delta$ there exists a linear operator $L\colon \mathfrak{A}_\delta \to \mathfrak{A}_{\delta'}$ such that*

$$D \circ Lf(u) = f(u) \quad \forall u \in \{u \in \mathbb{C} \mid u, u-1 \in (a,b)_{\delta'}\} \tag{3.12}$$

and

$$\|Lf\|_{\delta'} \leqslant C\|f\|_\delta (\delta - \delta')^{-1}. \tag{3.13}$$

We deduce Lemma 9 from Lemma 10. To do this, denote $\bar{\delta} = (\delta + \tilde{\delta})/2$ and consider the function

$$\kappa(x,u) = L(\ln(1 + G(x,u))).$$

By (3.10) and the Cauchy estimate we have $|G| \leqslant 2\tilde{C}|\varepsilon|(\delta^1 - \delta)^{-1}$. Therefore, by virtue of Lemma 10, the function κ is holomorphic in the domain $\mathcal{A}_{\bar{\delta}}$ and

$$|\kappa(x,u)| \leqslant C|\varepsilon|(\delta - \tilde{\delta})^{-1}(\delta^1 - \delta)^{-1}. \tag{3.14}$$

The substitution

$$\varphi = \varphi_1 e^\kappa \tag{3.15}$$

in (3.9) gives an equation for φ_1:

$$D\varphi_1(x,u) = g_1(x,u)e^{-\kappa(x,u)}(1 + G(x,u))^{-1}. \tag{3.16}$$

We obtain the solution $\varphi_1(x,u)$ by applying the operator L to the right-hand side of (3.16). After that, we recover φ by means of expression (3.15). Then, by (3.15) and Lemma 10 with $\delta = \bar{\delta}$, $\delta' = \tilde{\delta}$, we see that φ is holomorphic in $\mathcal{A}_{\tilde{\delta}}$, and estimate (3.11) is valid.

On the base of Lemma 9, the statement of Lemma 8 can be established by applying a well-known version of the generalized implicit function theorem (see [4, 14]). However, the simplicity of the setup allows to proceed directly.

For $m = 1, 2, \ldots$, we put

$$\delta_m = \delta^1(1 - \gamma_*(1^{-2} + 2^{-2} + \cdots + m^{-2})), \quad \gamma_* = (1 - \delta^2/\delta^1)(1^{-2} + 2^{-2} + \ldots)^{-1}$$

(note that $\delta^1 > \delta_1 > \delta_2 > \cdots > \delta^2$). We search for the solution of (3.7) by the successive approximations method. Here, we look for the m-th approximation Φ_m in the form $\Phi_m =$

$\varphi_1(x,u)+\cdots+\varphi_m(x,u)$, where φ_m is a holomorphic function in $\mathcal{A}_{\delta_m}, m=1,2,\ldots$. Denote $\sup|\varphi_m|$ by ε_m and the discrepancy of the m-th approximation by Δ_m:

$$D\Phi_m = g(x+\Phi_m, u) + \Delta_m(x,u). \tag{3.17}$$

Then $D\Phi_{m+1} = g(x+\Phi_{m+1},u) - g(x+\Phi_m,u) + \Delta_{m+1} - \Delta_m$. Taking a solution of the equation

$$D\Phi_{m+1} - g'(x+\Phi_m,u)\Phi_{m+1} = -\Delta_m \tag{3.18}$$

for Φ_{m+1}, we see that

$$\Delta_{m+1} = -(g(x+\Phi_{m+1},u) - g(x+\Phi_m,u) - g'(x+\Phi_m,u)\Phi_{m+1}) \tag{3.19}$$

(we set $\Phi_0=0$, $\Delta_0 = g(x,u)$ for $m+1=1$). Under the *a priori* assumption

$$|\Phi_m(x,u)| \leqslant \frac{1}{2}(\delta^1-\delta_m) \quad \forall (x,u)\in\mathcal{A}_{\delta_m}, \tag{3.20}$$

by (3.19) and the Cauchy estimate, for $m\geqslant 1$ we obtain the inequality $|\Delta_{m+1}|\leqslant C|\varepsilon|\varepsilon_{m+1}^2$ in $\mathcal{A}_{\delta_{m+1}}$. In view of Lemma 9, equation (3.18) for $m\geqslant 1$ gives

$$\varepsilon_{m+1} = \sup\{|\varphi_{m+1}| \mid (x,u)\in\mathcal{A}_{\delta_{m+1}}\} \leqslant C_1|\varepsilon|\exp(C_2|\varepsilon|m^2)\varepsilon_m^2.$$

If $m+1=1$, then $|\Delta_0|\leqslant C|\varepsilon|$ and $\varepsilon_1\leqslant C'|\varepsilon|$. Hence, if $|\varepsilon|$ is sufficiently small, the sequence ε_m tends to zero superrapidly:

$$\varepsilon_m \leqslant C_1|\varepsilon|^{(3/2)^m}.$$

Therefore, the sequence Φ_m converges to a holomorphic function f in $\mathcal{A}_{\delta^2}$ and $|f|\leqslant C|\varepsilon|$. Thus, the *a priori* assumption (3.20) holds and, passing to the limit in (3.17), we see that the function $f(x,u)$ satisfies (3.7).

Proof of Lemma 10. Put $f^1(u) = (1+u^2)f(u)$ and rewrite $f^1(u)$ in the form of the Cauchy integral:

$$f^1(u) = \frac{1}{2\pi i}\oint_\Gamma \frac{f^1(\gamma)}{u-\gamma}d\gamma,$$

where Γ is the boundary of the domain $(a,b)_\delta$. The contour Γ is equal to $\Gamma^+\cup\Gamma^-$, where

$$\Gamma^+ = \Gamma\cap\{z \mid \Re z\leqslant 0\}, \quad \Gamma^- = \Gamma\cap\{z\mid \Re z\geqslant 0\}.$$

In accordance with this decomposition, the function $f^1(u)$ splits into the sum $f^{1+}+f^{1-}$, where $f^{1\pm}$ is the integral along $\Gamma^\pm$. The function $f^{1\pm}$ can be extended analytically to a band $\Pi^\pm$, where

$$\Pi^+ = \{x+iy \mid |y|<\delta', \quad x\geqslant a-\delta'\}, \quad \Pi^- = \{x+iy\mid |y|<\delta', \quad x\leqslant b+\delta'\}.$$

Then we have

$$|f^{1\pm}| \leqslant C|\delta-\delta'|^{-1}(1+|u|)^{-1}\|f\|_\delta \quad \forall u\in\Pi^\pm.$$

Thus

$$f(u) = (1+u^2)^{-1}f^1(u) = f^+(u)+f^-(u),$$

where $f^\pm = (1+u^2)^{-1}f^{1\pm}(u)$, the function $f^\pm$ is holomorphic in the band $\Pi^\pm$ and

$$|f^\pm| \leqslant C|\delta-\delta'|^{-1}(1+|u|)^{-3}\|f\|_\delta. \tag{3.21}$$

Put

$$L_+ f(u) = -\sum_{n\geqslant 1} f^+(u+n), \quad L_- f(u) = -\sum_{n\geqslant 0} f^-(u-n).$$

Now, by (3.21), the series defining the function $L_\pm f(u)$ converges absolutely and uniformly when $u \in (a,b)_{\delta'}$ and

$$\|L_\pm f\|_{\delta'} \leqslant C_2|\delta - \delta'|^{-1}\|f\|_\delta. \tag{3.22}$$

Moreover, $D(L_\pm f(u)) = f^\pm(u) \quad \forall u \in \mathfrak{A}_\delta$. Put $L = L_+ + L_-$. The operator L maps $\mathfrak{A}_\delta$ into $\mathfrak{A}_{\delta'}$; estimate (3.13) follows from (3.22), while (3.12) is a consequence of the relation $f^+ + f^- = f$.

Remark 5. It is proved by V. F. Lazutkin [9] that under the conditions of Lemma 10 there exists a *continuous* operator L in the space $\mathfrak{A}_\delta$. The assertion of Lemma 8 follows from this (fairly nontrivial) statement and the contraction mapping principle for Banach spaces. Indeed, the substitution $f(x,u) = L\varphi(x,u)$ in (3.7) leads to the equation

$$\varphi(x,u) = g(x + L(\varphi(x,u)), u)$$

for $\varphi(x,u)$, and its solution is a fixed point for the map

$$\varphi(x,u) \to g(x + L(\varphi(x,u)), u). \tag{3.23}$$

In virtue of estimate (3.6), for $|\varepsilon|$ sufficiently small, (3.23) is a self map of a ball of radius $\tilde{C}|\varepsilon|$ in the space of holomorphic functions on $\mathcal{A}_{\delta^2}$, endowed with the sup-norm. Moreover, (3.23) is a contraction in this space, and therefore it possesses a fixed point. Hence, Lemma 8 follows.

§4. THE CASE OF NONSYMPLECTIC PERTURBATIONS

It was mentioned above in §2 that the maps G_0^ε and G^ε constructed in Lemma 2 conjugate the diffeomorphism f^ε with a succession map for an analytic $C\varepsilon$-perturbation of the original vector field V^0 in a subdomain of $\mathbf{A}^n$. The proof of Lemma 2 does not use the symplectic structure of the manifold $\mathcal{B}^\varepsilon$ and, therefore, does not employ the fact that the map f^ε is symplectic. Since the vector field $\tilde{V}$ on $\mathcal{B}^\varepsilon$ is tangent to the manifold $\mathcal{B}_0^\varepsilon$ and the map G_0^ε takes $\mathcal{B}_0^\varepsilon$ to $\mathbf{A}_0^n = \mathbf{A}^n \cap \{V = 0\}$, the field $V^\varepsilon = (G_0^\varepsilon)_*\tilde{V} = (G_0^\varepsilon \circ \pi)_*(\partial/\partial u)$ is tangent to the manifold $\mathbf{A}_0^n$. Denote by V_0^ε the restriction of the field V^ε to the manifold $\mathbf{A}_0^n$ and consider the natural coordinate system $(q_1,\dots,q_n,p_1,\dots,p_n)$ on $\mathbf{A}_0^n$. By virtue of the estimates of Lemma 2 and the explicit form of the map $(G_0^\varepsilon \circ \pi)$, the field V_0^ε has the form

$$V_0^\varepsilon = Q_n^0(q,p)\frac{\partial}{\partial q_n} + \sum_{j=1}^{n-1}\left(Q_j^0(q,p)\frac{\partial}{\partial q_j} + P_j^0(q,p)\frac{\partial}{\partial p_j}\right),$$

where the functions Q_j^0, P_j^0 are analytic and

$$\begin{gathered} |Q_j^0 - \partial h/\partial p_j| \leqslant C|\varepsilon|, \quad |P_j^0| \leqslant C|\varepsilon| \quad \forall j = 1,\dots,n-1, \\ |Q_n^0 - 1| \leqslant C|\varepsilon| \end{gathered} \tag{4.1}$$

everywhere in $\mathbf{A}^{n-1}_{\delta_0/C_1}(\mathcal{P}) \times \mathbf{T}^1(\delta_0/C_1)$. The constant C_1 depends only on K. In view of Lemma 1, the map f^ε is conjugate to a succession map for the vector field V_0^ε by means of an analytic embedding of $\mathbf{A}^{n-1}(\mathcal{P})$ into $\mathbf{A}_0^n$, $C|\varepsilon|$-close to the embedding j, where

$$j\colon \mathbf{A}^{n-1}(\mathcal{P}) \to \mathbf{A}_0^n, \quad (q,p) \mapsto (q,p,0).$$

Therefore, the conjugation can be implemented by means of the embedding j itself. Denote by $W^\varepsilon(q,p)$ the vector field $(Q_n^0)^{-1}V_0^\varepsilon$. Then

$$W^\varepsilon = \frac{\partial}{\partial q_n} + \sum_{j=1}^{n-1}\left(Q_j(q,p)\frac{\partial}{\partial q_n} + P_j(q,p)\frac{\partial}{\partial p_n}\right),$$

and estimates (4.1) hold for the functions Q_j, P_j. The integral curves for the fields W^ε and V_0^ε coincide. Therefore, the corresponding succession maps are equal. We can identify the variable q_n with a time variable t and treat W^ε as a nonautonomous vector field with time-periodic coefficients. Thus we obtain the following assertion.

Theorem 2. *Suppose that a smooth map f is as in (1.1) and satisfies conditions b) and c). Then for $|\varepsilon| < \varepsilon_0$, with a sufficiently small positive ε_0, there exists a nonautonomous analytic 1–periodic in t vector field $W^\varepsilon(q,p,t)$ such that the shift map along trajectories of the field W^ε in time interval $[0,1]$ coincides with f^ε. Moreover, the field W^ε can be extended analytically to the domain $\mathbf{A}^{n-1}_{\delta_0/C_1}(\mathcal{P}) \times \{t \in \mathbb{C} \mid |\Im m\, t| < \delta_0/C_1\}$, and obeys estimates (4.1) there. Values of ε_0, C, and C_1 depend only on n, δ_0, and K.*

§5. PERTURBATIONS OF BIRKHOFF INTEGRABLE MAPS

Introduce a symplectic structure in the space

$$\mathbb{R}^{2n-2} = \{(x,y)\} = \{(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})\}$$

by means of the form $dx \wedge dy$. Fix numbers $a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}$ such that

$$a_j > b_j, a_j > 0, b_j \neq 0 \quad \forall j,$$

and consider the domain $\mathfrak{A} \subset \mathbb{R}^{2n-2}$,

$$\mathfrak{A} = \{(x,y) \mid 2a_j > x_j^2 + y_j^2 > 2b_j \quad \forall j\}.$$

In particular, when $b_j < 0\ \forall j$, the domain $\mathfrak{A}$ is a polydisc. Let $f\colon \mathfrak{A} \times (-1,1) \to \mathbb{R}^{2n-2}$ be a smooth map such that

a) $\forall \varepsilon \in (-1,1)$ the map $f^\varepsilon(x,y) \equiv f(x,y,\varepsilon)$ is globally canonical, i.e., $f^{\varepsilon *}xdy = xdy + dg^\varepsilon(x,y)$,

b) the map f^0 is Birkhoff integrable, i.e., $f^0(x,y) = (X,Y)$, where for $j = 1,\dots,n-1$

$$\begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \begin{pmatrix} \cos\omega_j & \sin\omega_j \\ \sin\omega_j & -\cos\omega_j \end{pmatrix}\begin{pmatrix} x_j \\ y_j \end{pmatrix},$$

$$(\omega_1,\dots,\omega_{n-1}) = \nabla h(p_1,\dots,p_{n-1}), p_j = \frac{1}{2}(x_j^2 + y_j^2) \quad \forall j,$$

c) the maps $f^\varepsilon, \varepsilon \in (-1,1)$, and h can be extended to complex–analytic maps

$$f^\varepsilon : \mathfrak{A}(\delta_0) \to \mathbf{C}^{2n-2}, \quad h\colon \mathfrak{A}(\delta_0) \to \mathbb{C}$$

with modulus bounded by a constant K.

Note that if $b_j > 0\ \forall j$, it is possible to substitute symplectic coordinates (q,p), where

$$q_j = Arg(x_j + iy_j), \quad p_j = \frac{1}{2}(x_j^2 + y_j^2),$$

for the coordinates (x,y). In these new coordinates the domain $\mathfrak{A}$ is equal to $\mathbf{A}^n(\mathcal{P}), \mathcal{P} = (b_1, a_1) \times \cdots \times (b_{n-1}, a_{n-1})$, and the map f obeys conditions a)–c) from §1 (estimate (1.4) follows from the Cauchy inequality, if δ_0 is replaced by $\delta_0' < \delta_0$). In particular, the coordinates (q,p) are action-angle variables for the map f^0, and the map takes the form (1.3) with respect to these variables.

If $b_j < 0 \quad \forall j$, the domain $\mathfrak{A}$ is a polydisc and the coordinate system (q,p) is singular at the points of $\mathfrak{A}$ where $x_j = y_j = 0$ for some j. The center of the polydisc is stable under all diffeomorphisms f^ε. In this case the global canonicity of the maps f^ε is equivalent to their canonicity. We need to study similar families of symplectomorphisms, for instance, when investigating a single fixed symplectomorphism in a small neighborhood of an elliptic fixed point. Then the small parameter ε is the radius of a neighborhood of the fixed point, while the map f^0 is the linearization of the diffeomorphism under consideration at the point.

In the intermediate case $b_1, \ldots, b_k > 0 > b_{k+1}, \ldots, b_{n-1}, 1 \leqslant k \leqslant n-2$, the Birkhoff integrable map f^0 is not simultaneously Liouville integrable (in the entire domain $\mathfrak{A}$). This case corresponds to perturbations of an integrable diffeomorphism in a neighborhood of a family of invariant low-dimensional tori (see [15] concerning similar question on perturbations of integrable vector fields).

The proof from §2–3 for perturbations of Liouville integrable symplectomorphisms remains essentially valid for the case of Birkhoff integrability. It allows to obtain the following assertion, where $\mathfrak{B} \subset \mathbb{R}^{2n-2} \times \mathbf{A}^1$ denotes the domain

$$\mathfrak{B} = \{(x, y, l, v - h(p_1, \ldots, p_{n-1})) \mid (x,y) \in \mathfrak{A}, \varphi \in S^1, |v| < 1\}$$

equipped with the symplectic form $dx \wedge dy + dI \wedge d\varphi$.

Theorem 3. *For $|\varepsilon| < \varepsilon_0$ with sufficiently small positive ε_0, there exist a Hamiltonian vector field V_{H^ε} with an analytic Hamiltonian H^ε on $\mathfrak{B}$ and a hypersurface $\mathcal{M}^\varepsilon \subset \mathfrak{B}$ such that for $\theta \in (-1,1)$ the isoenergetic succession map*

$$S_\theta^\varepsilon \colon \mathcal{M}_\theta^\varepsilon = \mathcal{M} \cap \{H^\varepsilon = \theta\} \to \mathcal{M}_\theta^\varepsilon$$

is conjugate to f^ε by means of an analytic embedding

$$j_\theta^\varepsilon \colon \mathfrak{A} \to \mathfrak{B}, \quad j_\theta^\varepsilon(\mathfrak{A}) = \mathcal{M}_\theta^\varepsilon.$$

Moreover, the function H^ε admits a holomorphic extension to the complex (δ_0/C_1)-neighborhood of the domain $\mathfrak{B}$, while the map j_θ^ε can be extended to $\mathfrak{A}(\delta_0/C_1)$. The following estimates hold for the analytic continuations:

$$|H^\varepsilon(x, y, I, \varphi) - I - h(p_1, \ldots, p_{n-1})| \leqslant C_2|\varepsilon|,$$
$$|j_\theta^\varepsilon(x,y) - (x, y, \theta - h(p_1, \ldots, p_{n-1}), 0)| \leqslant C_2|\varepsilon|.$$

Values of ε_0, C_1, and C_2 depend only on n, δ_0, and K.

REFERENCES

1. Douady, R., 'Une démonstration directe de l'équivalence des théorèmes de tores invariants pour difféomorphismes et champs de vecteurs', *C.R. Acad. Sci. Paris Ser. I Math.* vol. 295, no. 2 (1982), 201–204.
2. Lazutkin, V. F. (1992), KAM Theory and Semiclassical Approximation to Eigenvalues, Springer.
3. Palis, J., di Melu, V. (1982), Geometric Theory of Dynamical Systems, Springer-Verlag, New York-Heidelberg-Berlin.
4. Zehnder, E., 'Generalized implicit function theorem with applications to some small divizor problems, I and II', *Commun. Pure Appl. Math.* vol. 28 (1975), 91–140 vol. 29 (1976), 49–111.
5. Arnol'd, V. I. (1978), Supplementary Chapters of Ordinary Differential Equations, [in Russian], Nauka, Moscow.
6. Douady, R., 'Applications du théorème de tores invariants', *These* 3^{eme} *Cycle, Universite Paris VII* (1982).
7. Narasimhan, R. (1968), Analysis on Real and Complex Manifolds, North–Holland Publishing Company, Amsterdam.
8. Nekhoroshev, N. N., 'Exponential estimate of the stability interval for Hamiltonian systems close to integrable', *Russ. Math. Surveys* vol. 32 no. 6 (1977), 5–66.
9. Lazutkin, V. F., 'Exponential splitting of separatrices and an analytic integral for the semistandart map', *Preprint Universite Paris VII* (1991).
10. Arnol'd, V. I. (1974), Mathematical Methods of Classical Mechanics, [in Russian], Nauka, Moscow.
11. Dubrovin B. A., Novikov S. P., Fomenko A. T. (1984), Modern Geometry. Methods of Homology Theory, [in Russian], Nauka, Moscow.
12. Cartan, H., 'Varietes analitiques reelles et varietes analitiques complexes', *Bull. Soc. Math. France* vol. 85 (1957), 77–99.
13. Guillemin, V., Sternberg, Sh., 'Geometric asymptotics', *Mathematical Surveys* no. 14 (1977), AMS, Providence, Rhode Island.
14. Hamilton, R., 'The inverse function theorem of Nash and Mozer', *Bull. Amer. Math. Soc.* vol. 7 (1982), 65–222.
15. Kuksin, S. B., 'An infinitesimal Liouville–Arnold theorem as criterion of reducibility for variational Hamiltonian equations', *Preprint MPI/91–3 of Max–Plank–Institut für Mathematik* (1991), Bonn.

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