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Invariant Cantor Manifolds of Quasi-periodic Oscillations for a Nonlinear Schrödinger Equation

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1 Introduction and Results

This paper is concerned with the nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - mu - f(|u|^2)u, \tag{1}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t, 0) = 0 = u(t, \pi), \qquad -\infty < t < \infty.$$

The parameter m is real, and f is required to be real analytic in some neighbourhood of the origin in $\mathbb{C}$. Absorbing a constant into m we may assume that $f(0) = 0$. Furthermore, we require f to be *nondegenerate* in the sense that

$$f'(0) \neq 0.$$

As we will see later, the sign of the derivative of f is immaterial for our results and may be assumed to be positive for convenience. Then we have

$$\mathrm{i}u_t = u_{xx} - mu - |u|^2 u + O(u^5) \tag{2}$$

after rescaling u appropriately.

We study this equation as a hamiltonian system on some suitable phase space $\mathcal{P}$. We may take, for example, $\mathcal{P} = W_0^1([0,\pi])$, the Sobolev space of all complex valued L^2-functions on $[0,\pi]$ with an L^2-derivative and vanishing boundary values. With the inner product

$$\langle u, v\rangle = \operatorname{Re} \int_0^\pi u\bar{v}\,dx$$

and the hamiltonian

$$H = \tfrac{1}{2}\langle Au, u\rangle + \tfrac{1}{2}\int_0^\pi g(|u|^2)\,dx, \tag{3}$$

where $A = -d^2/dx^2 + m$ and $g = \int_0 f\,\mathrm{dz}$, equation (1) can be written in the hamiltonian form

$$\dot{u} = \mathrm{i}\nabla H(u),$$

where the gradient of H is defined with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$, and the dot indicates differentiation with respect to time.

We are going to study strong solutions of (2) in the phase space $\mathcal{P}$ given by continously differentiable (even analytic) curves $t \mapsto u(t)$ satisfying (1). We recall that a strong solution exists not for all initial data in $\mathcal{P}$. But if it exists, then it is unique.

Our aim is to construct plenty of small amplitude solutions that are *quasi-periodic in time*. Such quasi-periodic solutions can be written in the form

$$u(t,x) = U(\omega_1 t, \dots, \omega_n t, x),$$

where $\omega_1, \dots, \omega_n$ are rationally independent real numbers, the "basic frequencies" of u, and U is a continuous function of period 2π in all arguments except the last one, called the *hull* of u. Thus u admits a Fourier series expansion

$$u(t,x) \sim \sum_{k\in\mathbb{Z}^n} U_k(x)\, e^{\mathrm{i}k\cdot\omega t},$$

where $k\cdot\omega = \sum_{j=1}^n k_j\omega_j$. A special case are time-periodic solutions, which are quasi-periodic with exactly one basic frequency. Solutions of this kind may therefore be interpreted as quasi-periodic oscillations for the nonlinear Schrödinger equation.

We achieve our aim by constructing the hulls as *embeddings*

$$U\colon\ \mathbb{T}^n \to \mathcal{P}, \qquad \theta \mapsto U(\theta,\,\cdot\,)$$

of the n-torus $\mathbb{T}^n$ into the phase space $\mathcal{P}$ together with frequency vectors ω such that the straight windings $t \mapsto \omega t + \theta_o$ on the torus map into solutions of equation (1).

Thus those quasi-periodic solutions u arise from what we call *rotational tori*. These are embedded tori, which are invariant, and on which the flow is linear in suitable coordinates.

Now, all the rotational tori are not constructed individually, but they come in Cantor families forming finite dimensional *Cantor manifolds* through the stationary solution $u \equiv 0$. They are the remnants of invariant linear manifolds for the linear Schrödinger equation when the nonlinearity f is taken out.

To be more precise let

$$\phi_j(x) = \sqrt{\tfrac{2}{\pi}} \sin jx, \qquad \lambda_j = j^2 + m, \qquad j \geq 1$$

be the basic modes and their frequencies for the linear equation $\mathrm{i}u_t = u_{xx} - mu$ with Dirichlet boundary conditions. Then every solution is the superposition of oscillations of the basic modes, with the coefficients moving on circles:

$$u(t,x) = \sum_{j \geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = q_j^o e^{\mathrm{i}\lambda_j t}.$$

Together they move on a rotational torus of finite or infinite dimension, depending on how many modes are excited. In particular, for every choice

$$J = \{ j_1 < j_2 < \cdots < j_n \} \subset \mathbb{N}$$

of $n \geq 1$ basic modes there is an invariant linear space E_J of complex dimension n which is completely foliated into rotational tori:

$$E_J = \left\{ u = q_1\phi_{j_1} + \cdots + q_n\phi_{j_n} : q \in \mathbb{C}^n \right\} = \bigcup_{I \in \overline{\mathbb{P}^n}} \mathcal{T}_J(I),$$

where $\mathbb{P}^n = \{I : I_j > 0 \text{ for } 1 \leq j \leq n\}$ is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_J(I) = \left\{ u = q_1\phi_{j_1} + \cdots + q_n\phi_{j_n} : \left|q_j\right|^2 = 2I_j \text{ for } 1 \leq j \leq n \right\}.$$

In addition, each such torus is linearly stable, and all solutions have vanishing Lyapunov exponents. – This is the linear situation.

Upon restoring the nonlinearity f the invariant manifolds E_J will not persist in their entirety due to resonances among the modes and the strong perturbing effect of f for large amplitudes. We show, however, that in a sufficiently small neighbourhood of the origin a large Cantor subfamily of rotational n-tori persists and is only slightly deformed.

That is, there exists a Cantor set $\mathcal{C} \subset \mathbb{P}^n$, a family of n-tori

$$\mathcal{T}_J[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}_J(I) \subset E_J$$

over $\mathcal{C}$, and a Lipschitz continuous embedding

$$\Phi\colon\ \mathcal{T}_J[\mathcal{C}] \hookrightarrow \mathcal{P},$$

such that the restriction of Φ to each $\mathcal{T}_J(I)$ in the family is an embedding of a rotational n-torus for the nonlinear equation. The image $\mathcal{E}_J$ of $\mathcal{T}_J[\mathcal{C}]$ we call a *Cantor manifold of rotational n-tori* given by the embedding $\Phi\colon \mathcal{T}_J[\mathcal{C}] \to \mathcal{E}_J$.

The Cantor manifolds have a number of additional properties.

– The embedding Φ is a higher order perturbation of the inclusion mapping $\Phi_0\colon E_J \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{C}]$. Its restriction to each torus $\mathcal{T}_J(I)$ is real analytic, and it maps into the space of analytic functions on $[0,\pi]$, with uniform domains of analyticity.

– The Cantor set $\mathcal{C}$ has full density at the origin. That is,

$$\lim_{r\to 0} \frac{|\mathcal{C} \cap B_r|}{|\mathbb{P}^n \cap B_r|} = 1,$$

where $B_r = \{I : \|I\| < r\}$, and $|\cdot|$ denotes the n-dimensional Lebesgue measure for sets.

– In view of the previous remarks, $\mathcal{E}_J$ has a tangent space at the origin equal to E_J:

$$T_0\mathcal{E}_J = E_J.$$

– The frequencies ω of the rotational tori are diophantine, whence we also call them *diophantine tori*. That is, there exist positive α and τ such that

$$|k\cdot\omega| \geq \frac{\alpha}{|k|^\tau} \qquad \text{for } 0 \neq k \in \mathbb{Z}^n.$$

The exponent τ can be kept fixed, while α tends to zero as the tori approach the origin.

– All the tori are still linearly stable, and all their orbits have zero Lyapunov exponents.

Remarkably, the existence of such Cantor manifolds follows from the nondegeneracy of the nonlinearity without any further assumptions.

Theorem 1. *Suppose the nonlinearity f is real analytic and nondegenerate. Then for all $m \in \mathbb{R}$, all $n \in \mathbb{N}$ and all $J = \{j_1 < \cdots < j_n\} \subset \mathbb{N}$ there exists a Cantor manifold $\mathcal{E}_J$ of real analytic, linearly stable, diophantine n-tori for equation* (1) *given by a Lipschitz continuous embedding $\Phi\colon\ \mathcal{T}_J[\mathcal{C}] \to \mathcal{E}_J$, which is a higher order perturbation of the inclusion map $\Phi_0\colon E_J \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{C}]$. The Cantor set $\mathcal{C}$ has full density at the origin, whence $\mathcal{E}_J$ has a tangent space at the origin equal to E_J. Moreover, $\mathcal{E}_J$ is contained in the space of analytic functions on $[0,\pi]$.*

A more precise description of the asymptotic behavior of Φ and its analyticity properties is given later.

Remark 1. One could show that Φ is not only Lipschitz across the tori, but smooth in the sense of Whitney. We did not pursue this point.

Remark 2. The frequencies of the diophantine tori are also under control. They are

$$\omega(I) = \lambda_J + AI + O(\|I\|^2)$$

for $I \in \mathcal{C}$, where $\lambda_J = (\lambda_{j_1}, \ldots, \lambda_{j_n})$, and A is the $n \times n$-matrix with coefficients $A_{ij} = (4 - \delta_{ij})/4\pi$.

Remark 3. For $n = 1$ the theorem says that for each $j \geq 1$ there exists a "Cantor disc" $\mathcal{E}_j$ filled with time-periodic solutions of (1) which at the origin is tangent to the plane corresponding to the eigenfunction ϕ_j. However, it is not difficult to show directly that indeed there exists a "solid" disc of this kind, without any holes. A more detailed statement is given in Appendix C.

Remark 4. As we will explain below the Cantor manifolds also exist for nonlinearities f of the form

$$f(x,v) = av + \sum_{k\geq 2} f_k(x)v^k, \qquad a \neq 0,$$

where the coefficients f_k are real analytic in x, or in some Sobolev space $W^s([0,\pi])$, $s > \frac{1}{2}$, with norms growing at most exponentially to ensure analyticity in v. In the latter, non-analytic case the resulting quasi-periodic solutions are of class W^{s+2} in x.

The size of the Cantor manifolds $\mathcal{E}_J$ is not uniform, but depends on m, n and J, and in particular tends to zero as n tends to infinity. Thus, unlike the linear spaces E_J, they are *not* dense in some fixed neighbourhood of the origin. But they are *asymptotically dense* in the following sense.

Corollary 1. *The union of all Cantor manifolds $\mathcal{E}_J$ intersects every nonempty open cone in $W_0^1([0,\pi])$ with vertex at the origin.*

The result of Theorem 1 is not completely unexpected. Equation (2) may be viewed as a higher order perturbation of the Zakharov-Shabat equation

$$\mathrm{i}u_t = u_{xx} - mu - |u|^2 u$$

on the real line with periodic boundary condition, which is known to be integrable. In particular, the latter has plenty of x- and t-quasi-periodic solutions, given be exact formulas – the so called finite gap solutions. Some of them also satisfy the Dirichlet boundary condition [1]. The time-quasi-periodic solutions of our equation may thus be thought of as perturbations of small amplitude finite gap solutions of the Zakharov-Shabat equation.

This scheme may be converted into a proof, and exactly in this way a result similar to Theorem 1 was obtained in [3] for the nonlinear wave equation

$$u_{tt} = u_{xx} - mu + au^3 + O(u^5), \qquad m > 0, \quad a \neq 0$$

on $[0, \pi]$ with Dirichlet boundary conditions. Here the approximating integrable system is the Sine-Gordon equation or the Sinh-Gordon equation.

But in this paper a different approach is taken. Instead of approximating the problem by an integrable partial differential equation, we approximate it by an integrable infinite dimensional hamiltonian system, namely the *Birkhoff normal form* of the hamiltonian (3) in infinitely many coordinates up to order four.

To start, we use the complete set of eigenfunctions of the operator A with Dirichlet boundary conditions to write $u = \sum_j q_j \phi_j$ as in the linear case. We obtain a hamiltonian in infinitely many coordinates q_j which is real analytic near the origin in some suitable space of complex sequences $q = (q_1, q_2, \dots)$. Its equations of motion are

$$\dot{q}_j = 2\mathrm{i}\frac{\partial H}{\partial \bar{q}_j}, \qquad j \geq 1.$$

The linear equation in particular gives rise to the quadratic hamiltonian

$$\Lambda = \frac{1}{2}\sum_{j\geq 1} \lambda_j \left|q_j\right|^2,$$

while the nonlinearity f gives rise to terms of order four and more. Thus,

$$H = \Lambda + G, \qquad G = O(\|q\|^4),$$

which describes an elliptic fixed point in infinitely many degrees of freedom.

In classical hamiltonian theory, the standard tool to investigate such systems near the equilibrium is their Birkhoff normal form, or its generalizations. Remarkably, the Birkhoff normal form up to order four is available here without any further assumption, since the relevant nonresonance conditions among the λ_j hold uniformly. The upshot is that there is *one global* change of coordinates so that

$$H = \Lambda + \bar{G} + O(\|q\|^6), \qquad \bar{G} = \frac{1}{2} \sum_{i,j} \bar{G}_{ij} \, |q_i|^2 \left|q_j\right|^2 ,$$

for all m. Thus, the hamiltonian is *integrable* up to a perturbation of order 6 – a reflection of the integrability of the Zakharov-Shabat equation.

Now KAM theory comes into play. There is no genuine infinite dimensional KAM theory yet to establish the persistence of infinite dimensional rotational tori for hamiltonians of the type above. But there are also plenty of *finite dimensional* rotational tori, for which persistence results have been developed recently by the first author [9]. For example, the n-dimensional tori

$$\frac{1}{2} \left|q_j\right|^2 = \begin{cases} I_j > 0, & 1 \le j \le n \\ 0, & j \ge n+1 \end{cases}$$

are all invariant, if the O-term is omitted. Upon its inclusion, a large Cantor family of tori is shown to persist, forming a Cantor manifold of the kind described above.

Of course, KAM theory always requires some *nondegeneracy condition*, and this case makes no exception. Here, they involve the frequencies λ_j and the Birkhoff coefficients $\bar{G}_{ij}$, and it is remarkably easy and straightforward to verify the required conditions for all choices of m, n and J without making any further assumption or restriction. The only requirement is the *presence* of a third order term in the nonlinearity f, no matter what sign. In essence, the nondegeneracy of the nonlinearity f, as defined in the beginning, provides the nondegeneracy of the infinite dimensional integrable hamiltonian given by the fourth order Birkhoff normal form.

The technique described is not restricted to the nonlinear Schrödinger equation. It applies equally well to the nonlinear wave equation

$$u_{tt} = u_{xx} - mu - f(u)$$

on the x-interval $[0, \pi]$ with Dirichlet boundary conditions, a real parameter $m > 0$ and a real analytic nonlinearity

$$f(u) = au^3 + O(u^5), \qquad a \neq 0.$$

Theorem 1 holds *mutatis mutandis* also here, if for example the index set $J =$

$\{j_1 < \cdots < j_n\}$ satisfies $\min j_\nu - j_{\nu-1} = 1$. The solutions obtained are real analytic in t and x. Details will be given in [13].

One may also deal with more general nonlinearities of the form

$$f(x,u) = au^3 + \sum_{k\geq 5} f_k(x)u^k,$$

where the coefficients f_k are real analytic in x, or just in $W^s([0,\pi])$, $s > \frac{1}{2}$, with norms growing at most exponentially to ensure analyticity in u. In the latter, non-analytic case, however, the resulting quasi-periodic solutions are of class W^{s+2} in x only. The point is that the higher order terms have no bearing on the fourth order Birkhoff normal form and may therefore be of a more general form. On the other hand, any x-dependent coefficient of u^3 destroys the integrability of the Birkhoff normal form. – An analogous generalization applies to the nonlinear Schrödinger equation as mentioned above.

Finally, one may add a general perturbation term

$$\epsilon g(x,u) = \epsilon \sum_{k\geq 0} g_k(x)u^k$$

to the nonlinearity f, with coefficients g_k of the same type as the f_k. Then there still exist Cantor manifolds for all sufficiently small ϵ, the smallness depending on m, n and J. However, they have a *hole* at the origin instead of being dense there, since the perturbation no longer tends to zero as we approach the origin.

Hamiltonian perturbations of the KdV-equation, such as

$$u_t = u_{xxx} - uu_x - \frac{\mathrm{d}}{\mathrm{d}x} f(u), \qquad f(u) = O(u^3),$$

are also susceptible to our approach. Here the unperturbed hamiltonian is nonlinear and admits a nondegenerate normal form in the sense of section 4. So we obtain Cantor manifolds of time-quasi-periodic, space-periodic solutions of small amplitude. Of course, stronger and more global results are available based on the integrability of the KdV-equation [8]. But they also require a formidable amount of machinery, whereas by comparison the normal form technique looks elementary.

Our technique also has its limitations. First of all, to get started the linear operator involved needs to have a pure point spectrum with a complete set of eigenfunctions. Also, the point eigenvalues have to avoid certain lower order resonances. This is reminiscent of the Lyapunov center theorem in the finite dimensional theory: if, at an elliptic fixed point of a real analytic hamiltonian system with characteristic frequencies $\lambda_1, \ldots, \lambda_n$, one has

$$\frac{\lambda_j}{\lambda_{j_0}} \notin \mathbb{Z}, \qquad 1 \le j \ne j_0 \le n,$$

then there exists a disc through the origin filled with periodic solutions of the nonlinear system which is tangent to the plane of periodic solutions with frequency λ_0 for the linear part of the system. The nonresonance conditions are necessary: given a resonance there are nonlinearities so that such a family of periodic solutions does not persist [14].

For infinite dimensional systems there are similar results concerning the *non-persistence of breathers* under generic perturbations. See for example [5] and [15], respectively, for recent results concerning the sine-Gordon equation and certain classes of nonlinear wave equations in any space dimension, and the references therein. Very loosely speaking, the resonance occurs between the point eigenvalue of the unperturbed periodic solution – the breather – and the continuous spectrum of the unperturbed operator. These results indicate that the requirement of a pure point spectrum not in low order resonance is not a technical shortcoming, but essential for persistence results of the kind of Theorem 1.

Another limitation arises from requirements about the asymptotic nature of the point eigenvalues λ_j of the linear equation. They have to be simple and tend to infinity at least linearly. More precisely, $\lambda_j = j^d + \cdots + O(j^\delta)$ with $d \ge 1$ and $\delta < d - 1$. This restricts our approach essentially to one-dimensional problems with Sturm-Liouville type boundary conditions. Periodic boundary conditions are not admitted as they give rise to (asymptotically) double eigenvalues. Some of these restrictions, however, are probably of a technical nature. For example, Craig and Wayne [4] could allow for double eigenvalues in the construction of Cantor discs of periodic solutions for nonlinear wave equations. Their result may be viewed as an infinite dimensional extension of the Lyapunov center theorem.

Investigations into the existence of time-quasi-periodic solutions for nonlinear partial differential equations were started only rather recently, and independently, by Wayne [16] and the first author [7]. The monograph [9] gives an extensive list of references as well as some more historical background. All these results, however, are based on the assumption that the unperturbed equation is integrable and nondegenerate, or depends on sufficiently many parameters, which could be adjusted appropriately and eventually are restricted to some Cantor set. The use of a nondegenerate Birkhoff normal form, in this context introduced in [11], obviates the use of such parameters and leads to more “natural” results. – Incidentally it was shown that all known nonlinear integrable partial differential equations are nondegenerate [2,3].

The rest of the paper is organized as follows. In section 2 the hamiltonian of the nonlinear Schrödinger equation is written in infinitely many coordinates, and its regularity is established. In section 3 it is transformed into its Birkhoff normal form of order four. In section 4 another theorem about the existence of invariant Cantor manifolds for hamiltonians in such normal forms is formulated, which allows us to prove Theorem 1 in section 5. The latter theorem is subsequently reduced to a rather technical KAM-theorem about perturbations of families of linear hamiltonians. The statements and details of the reduction fill the last two sections, whereas the proof of the technical theorem is given in a separate paper.

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2 The Hamiltonian

The hamiltonian of the nonlinear Schrödinger equation is

$$H = \tfrac{1}{2}\,\langle Au, u\rangle + \tfrac{1}{2}\int_0^\pi g(|u|^2)\,\mathrm{d}x,$$

where $A = -\mathrm{d}^2/\mathrm{d}x^2 + m$ and $g = \int_0 f\,\mathrm{d}z$. We rewrite H as a hamiltonian in infinitely many coordinates by making the ansatz

$$u = \mathcal{S}q = \sum_{j\ge 1} q_j\phi_j, \qquad \phi_j = \sqrt{\tfrac{2}{\pi}}\sin jx, \qquad j \ge 1. \tag{4}$$

The coordinates are taken from the Hilbert space $\ell^{a,p}$ of all complex valued sequences $q = (q_1, q_2, \dots)$ with

$$\|q\|_{a,p}^2 = \sum_{j\ge 1} \left|q_j\right|^2 j^{2p} e^{2ja} < \infty.$$

We fix $a > 0$ and $p > \frac{1}{2}$ later. We then obtain the hamiltonian

$$\begin{aligned} H &= \Lambda + G \\ &= \tfrac{1}{2} \sum_{j \ge 1} \lambda_j \left| q_j \right|^2 + \tfrac{1}{2} \int_0^\pi g\left(|\mathcal{S} q|^2 \right) \,\mathrm{d}x \end{aligned} \tag{5}$$

on the phase space $\ell^{a,p}$ with symplectic structure $\frac{\mathrm{i}}{2} \sum_j \mathrm{d}q_j \wedge \mathrm{d}\bar{q}_j$. Its equations of motion are

$$\dot{q}_j = 2\mathrm{i} \frac{\partial H}{\partial \bar{q}_j}, \qquad j \ge 1. \tag{6}$$

They are the classical hamiltonian equations of motion for the real and imaginary parts of $q_j = x_j + \mathrm{i} y_j$ written in complex notation.

At this point we do not discuss the validity of this transformation or its symplectic nature. Rather we take the latter hamiltonian as our new starting point and make the following simple observation.

Lemma 1. *Let $a > 0$ and $p \ge 0$. If a curve $I \to \ell^{a,p}$, $t \mapsto q(t)$ is an analytic solution of* (6)*, then*

$$u(t,x) = \sum_{j \ge 1} q_j(t) \phi_j(x)$$

is a solution of (1) *that is analytic on $I \times [0, \pi]$.*

Proof. For $a > 0$ and $p \ge 0$, the sum is absolutely convergent in some complex neighbourhood of the x-interval $[0, \pi]$ and some complex disc around a given t in I. The same is true for its termwise t-derivative. Therefore u is analytic in t and x, and we can differentiate under the summation sign. We find that

$$\begin{aligned} \mathrm{i}u_t &= \sum_{j \ge 1} \mathrm{i} \dot{q}_j \phi_j \\ &= -\sum_{j \ge 1} \left(\lambda_j q_j + \int_0^\pi f(|u|^2) u \phi_j \,\mathrm{d}x \right) \phi_j \\ &= -\sum_{j \ge 1} A\phi_j q_j - \sum_{j \ge 1} \left(\int_0^\pi f(|u|^2) u \phi_j \,\mathrm{d}x \right) \phi_j \\ &= -Au - f(|u|^2) u \end{aligned}$$

by the orthonormality and completeness of the ϕ_j. ∎

To continue our investigation of the hamiltonian in (5) we need to establish the regularity of the nonlinear hamiltonian vectorfield X_G associated with G.

To this end, let ℓ_b^2 and L^2, respectively, be the Hilbert spaces of all *bi*-infinite, square summable sequences with complex coefficients and all square-integrable complex valued functions on $[-\pi,\pi]$. Let

$$\mathcal{F}\colon\ \ell_b^2 \to L^2, \quad q \mapsto \mathcal{F}q = \frac{1}{\sqrt{2\pi}} \sum_j q_j e^{\mathrm{i}jx}$$

be the inverse discrete Fourier transform, which defines an isometry between the two spaces.

Let $a \ge 0$ and $p \ge 0$. The subspaces $\ell_b^{a,p} \subset \ell_b^2$ consist, by definition, of all bi-infinite sequences with finite norm

$$\|q\|_{a,p}^2 = |q_0|^2 + \sum_j \left|q_j\right|^2 |j|^{2p}\, e^{2|j|a}.$$

Through $\mathcal{F}$ they define subspaces $W^{a,p} \subset L^2$ that are normed by setting $\|\mathcal{F}q\|_{a,p} = \|q\|_{a,p}$. For $a > 0$, the space $W^{a,p}$ may be identified with the space of all 2π-periodic functions which are analytic and bounded in the complex strip $|\mathrm{Im}\, z| < a$ with trace functions on $|\mathrm{Im}\, z| = a$ belonging to the usual Sobolev space W^p.

Lemma 2. *For $a \ge 0$ and $p > \frac{1}{2}$, the space $\ell_b^{a,p}$ is a Hilbert algebra with respect to convolution of sequences, and*

$$\|q * r\|_{a,p} \le c\, \|q\|_{a,p}\, \|r\|_{a,p}$$

with a constant c depending only on p. Consequently, $W^{a,p}$ is a Hilbert algebra with respect to multiplication of functions.

The proof is given in Appendix A.

Lemma 3. *For $a \ge 0$ and $p > \frac{1}{2}$, the hamiltonian vectorfield X_G is real analytic as a map from some neighbourhood of the origin in $\ell^{a,p}$ into $\ell^{a,p}$, with*

$$\|X_G\|_{a,p} = O\big(\|q\|_{a,p}^3\big).$$

Thus, X_G is a genuine vectorfield on $\ell^{a,p}$. On the other hand the linear vectorfield X_Λ is unbounded on $\ell^{a,p}$, since it takes values in $\ell^{a,p-2}$.

Proof. We have

$$\frac{\partial G}{\partial \bar{q}_j} = \int_0^\pi f(|u|^2) u \phi_j \,\mathrm{d}x, \qquad u = \mathcal{S} q.$$

Let q be in $\ell^{a,p}$. Considered as a function on $[-\pi, \pi]$, $u = \mathcal{S} q$ is in $W^{a,p}$, with $\|u\|_{a,p} = \|q\|_{a,p}$. By the algebra property and the analyticity of f, the function $f(|u|^2)u$ also belongs to $W^{a,p}$ with

$$\left\| f(|u|^2) u \right\|_{a,p} \le c \, \|u\|_{a,p}^3$$

in a sufficiently small neighbourhood of the origin. The components of the gradient $G_{\bar{q}}$ are its Fourier sine coefficients, so $G_{\bar{q}}$ belongs to $\ell^{a,p}$, with

$$\left\| G_{\bar{q}} \right\|_{a,p} \le \left\| f(|u|^2) u \right\|_{a,p} \le c \, \|u\|_{a,p}^3 \le c \, \|q\|_{a,p}^3 .$$

The regularity of X_G follows from the regularity of its components. ∎

For the nonlinearity $|u|^2 u$ we find

$$G = \tfrac{1}{4} \int_0^\pi |u(x)|^4 \,\mathrm{d}x = \tfrac{1}{4} \sum_{i,j,k,l} G_{ijkl} \, q_i q_j \bar{q}_k \bar{q}_l \tag{7}$$

with

$$G_{ijkl} = \int_0^\pi \phi_i \phi_j \phi_k \phi_l \,\mathrm{d}x.$$

It is easy to verify that $G_{ijkl} = 0$ *unless* $i \pm j \pm k \pm l = 0$, for *some* combination of plus and minus signs. Thus, only a codimension one set of coefficients is actually different from zero, and the sum extends only over $i \pm j \pm k \pm l = 0$. In particular, we have

$$\begin{aligned} G_{ijij} &= \frac{4}{\pi^2} \int_0^\pi \sin^2 ix \sin^2 jx \,\mathrm{d}x \\ &= \frac{1}{\pi^2} \int_0^\pi (1 - \cos 2ix)(1 - \cos 2jx) \,\mathrm{d}x \\ &= \frac{1}{\pi^2} \int_0^\pi \bigl(1 + \tfrac{1}{2} \cos 2(i-j)x\bigr) \,\mathrm{d}x \\ &= \frac{1}{2\pi} (2 + \delta_{ij}), \end{aligned}$$

using $2\sin^2 u = 1 - \cos 2u$ and $2 \cos u \cos v = \cos(u+v) + \cos(u-v)$. These coefficients determine the Birkhoff coefficients of the next section.

From now on we focus our attention on the nonlinearity $|u|^2 u$, since higher order terms do not matter.

3 The Birkhoff Normal Form

Lemma 4. *For the hamiltonian $H = \Lambda + G$ with nonlinearity* (7) *there exists a real analytic, symplectic change of coordinates Γ in a neighbourhood of the origin in $\ell^{a,p}$ that for all real values of m takes it into its Birkhoff normal form up to order four. That is,*

$$H \circ \Gamma = \Lambda + \bar{G} + K,$$

where $X_{\bar{G}}$ and X_K are real analytic vectorfields in a neighbourhood of the origin in $\ell^{a,p}$,

$$\bar{G} = \frac{1}{2} \sum_{i,j \geq 1} \bar{G}_{ij} \, |q_i|^2 \left|q_j\right|^2, \qquad |K| = O\big(\|q\|_{a,p}^6\big),$$

with uniquely determined coefficients $\bar{G}_{ij} = (4 - \delta_{ij})/4\pi$.

The key ingredient of the proof is the observation that all the relevant divisors in the normalizing transformation are independent of m and uniformly bounded away from zero, since they are nonvanishing integers.

Lemma 5. *If $i \pm j \pm k \pm l = 0$ and $\{i, j\} \neq \{k, l\}$, then*

$$\lambda_i + \lambda_j - \lambda_k - \lambda_l = i^2 + j^2 - k^2 - l^2 \neq 0.$$

Proof. If one pair of indices in the two sets is equal, say $j = l$, then $i \neq k$ by hypotheses, and so $i^2 + j^2 - k^2 - l^2 = i^2 - k^2 \neq 0$. So suppose that there is no pair of equal indices in the two sets, but that to the contrary

$$i^2 + j^2 = k^2 + l^2.$$

We may also assume that $i < k \leq l < j$. From $i \pm j \pm k \pm l = 0$ we conclude that indeed $i + j = k + l$. Squaring and subtracting the first equation yields $ij = kl$, and hence we find $(k - i)(k - j) = 0$. This is a contradiction. ∎

Proof of Lemma 4. Let $\Gamma = X_F^t\big|_{t=1}$ be the time-1-map of the flow of the hamiltonian vectorfield X_F given by the hamiltonian

$$F = \tfrac{1}{4} \sum_{i,j,k,l} F_{ijkl}\, q_i q_j \bar{q}_k \bar{q}_l$$

with coefficients

$$\mathrm{i}F_{ijkl} = \begin{cases} \dfrac{G_{ijkl}}{\lambda_i + \lambda_j - \lambda_k - \lambda_l}, & \text{if } i \pm j \pm k \pm l = 0 \\ & \text{and } \{i, j\} \neq \{k, l\} \\ 0 & \text{otherwise.} \end{cases},$$

As we will show in a moment, X_F is a real analytic vectorfield on $\ell^{a,p}$ of order three at the origin. Hence Γ is a real analytic, symplectic change of coordinates defined at least in a neighbourhood of the origin in $\ell^{a,p}$.

Expanding at $t = 0$ and using Taylor's formula we have

$$\begin{aligned} H \circ \Gamma &= H \circ X_F^t\big|_{t=1} \\ &= H + \{H, F\} + \int_0^1 (1-t)\, \{\{H, F\}, F\} \circ X_F^t \, dt \\ &= \Lambda + G + \{\Lambda, F\} \\ &\qquad + \{G, F\} + \int_0^1 (1-t)\, \{\{H, F\}, F\} \circ X_F^t \, dt, \end{aligned}$$

in a neighbourhood of the origin, where $\{H, F\}$ denotes the Poisson bracket of H and F. The last line consists of terms of order six or more in q and constitutes the higher order term K. In the second to last line,

$$\begin{aligned} G + \{\Lambda, F\} &= \tfrac{1}{4} \sum_{i \pm j \pm k \pm l = 0} \big(G_{ijkl} - \mathrm{i}(\lambda_i + \lambda_j - \lambda_k - \lambda_l) F_{ijkl} \big)\, q_i q_j \bar{q}_k \bar{q}_l \\ &= \tfrac{1}{4} \sum_{\{i,j\} = \{k,l\}} G_{ijkl}\, q_i q_j \bar{q}_k \bar{q}_l \\ &= \tfrac{1}{2} \sum_{i,j \geq 1} \bar{G}_{ij}\, |q_i|^2 \left|q_j\right|^2 = \bar{G} \end{aligned}$$

with $2\bar{G}_{ii} = G_{iiii}$ and $\bar{G}_{ij} = G_{ijij}$ for $i \neq j$. Hence $H \circ \Gamma = \Lambda + \bar{G} + K$ as claimed.

It remains to prove the analyticity. In view of Lemma 5 we have

$$\left| \frac{\partial F}{\partial \bar{q}_l} \right| \leq \tfrac{1}{2} \sum_{\pm i \pm j \pm k = l} \left| G_{ijkl} \right| \left| q_i q_j \bar{q}_k \right| \leq \sum_{\pm i \pm j \pm k = l} \left| q_i q_j q_k \right|.$$

Introducing the doubly infinite sequence p by setting $p_j = \left| q_{|j|} \right|$ for $j \neq 0$ and $p_0 = 0$ we can write

$$\sum_{\pm i \pm j \pm k = l} \left| q_i q_j q_k \right| = \sum_{i+j+k=l} p_i p_j p_k = (p * p * p)_l ,$$

where the second sum extends over *all* integers i, j, k. Hence, by Lemma A.1,

$$\left\| F_{\bar{q}} \right\|_{a,p} \leq \| p * p * p \|_{a,p} \leq c \, \| p \|_{a,p}^3 \leq c \, \| q \|_{a,p}^3 .$$

Finally, uniqueness is proven in the classical way [14]. ∎

Thus, the hamiltonian is brought into the infinite dimensional analogue of the classical fourth order Birkhoff normal form,

$$H \circ \Gamma = \langle \lambda, I \rangle + \tfrac{1}{2} \langle AI, I \rangle + K ,$$

where $I = (I_1, I_2, \dots)$ with $I_j = \frac{1}{2} \left| q_j \right|^2$ and $A_{ij} = \bar{G}_{ij}$. This result is in fact stronger than what we can make use of, since we have no KAM theory yet for infinite dimensional tori of such hamiltonians. We rather focus our attention now on certain families of finite dimensional tori, for which a partial Birkhoff normal form would suffice.

With the normal form at hand, Theorem 1 could be deduced by applying an abstract infinite-dimensional KAM-theorem from [9] in the same way as it was done in [2] for the nonlinear string equation. There a similar normal form was obtained by an analysis of the time-quasi-periodic solutions of the sine-Gordon-equation. This way one obtains Cantor manifolds of solutions which are t-analytic and x-smooth. The x-analyticity then results from the uniqueness assertion of a version of the Cauchy Kowalewski theorem due to Ovsjannikov, Nirenberg and Nishida [10], which is applied to equation (1) with x as time and $u|_{x=x_0}$ as analytic Cauchy data. The point is that $u|_{x=x_0}$ is already known to be analytic in t for every x_0 in $[0, \pi]$.

Below we proceed in a similar way but deduce Theorem 1 from a somewhat different KAM-like result, which is proven in [12].

4 The Cantor Manifold Theorem

In a neighbourhood of the origin in $\ell^{a,p}$ we now consider a hamiltonian $H = \Lambda + Q + R$, where R represents some higher order perturbation of an integrable normal form $\Lambda + Q$. The latter describes a family of linearly stable invariant tori of dimension n with quasi-periodic motions. The dimension n is fixed, $1 \leq n < \infty$.

In complex coordinates $q = (\tilde{q}, \hat{q})$ on $\ell^{a,p}$, where $\tilde{q} = (q_1, \dots, q_n)$ and $\hat{q} = (q_{n+1}, q_{n+2}, \dots)$, and with

$$I = \tfrac{1}{2}(|q_1|^2, \dots, |q_n|^2), \quad Z = \tfrac{1}{2}(|q_{n+1}|^2, |q_{n+2}|^2, \dots),$$

the normal form consists of the terms

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \qquad Q = \tfrac{1}{2} \langle AI, I \rangle + \langle BI, Z \rangle,$$

where α, β and A, B denote vectors and matrices with constant coefficients, respectively. Its equations of motion are

$$\dot{\tilde{q}} = \mathrm{i}\,\mathrm{diag}(\alpha + AI + B^T Z)\tilde{q}, \qquad \dot{\hat{q}} = \mathrm{i}\,\mathrm{diag}(\beta + BI)\hat{q}.$$

They admit a complex n-dimensional invariant manifold $E = \{\hat{q} = 0\}$, which is completely filled, up to the origin, by the invariant tori

$$\mathcal{T}(I) = \left\{\tilde{q} : \left|\tilde{q}_j\right|^2 = 2I_j \text{ for } 1 \le j \le n\right\}, \qquad I \in \overline{\mathbb{P}^n}.$$

On $\mathcal{T}(I)$ and in its normal space, respectively, the flows are given by

$$\begin{aligned} \dot{\tilde{q}} &= \mathrm{i}\,\mathrm{diag}(\omega(I))\tilde{q}, \qquad \omega(I) = \alpha + AI, \\ \dot{\hat{q}} &= \mathrm{i}\,\mathrm{diag}(\Omega(I))\hat{q}, \qquad \Omega(I) = \beta + BI. \end{aligned}$$

They are linear and in diagonal form. In particular, since $\Omega(I)$ is real, $\hat{q} = 0$ is an elliptic fixed point, all the tori are linearly stable, and all their orbits have zero Lyapunov exponents. We call $\mathcal{T}(I)$ an *elliptic rotational torus with frequencies* $\omega(I)$.

Note that although looking very much alike there is an important difference between an n-dimensional space E_J for the linear Schrödinger equation and the space E for the nonlinear normal form hamiltonian. In the latter the frequencies in general vary from torus to torus, while in the former they do not. This so called amplitude-frequency modulation is essential for obtaining the stability result below.

Due to resonances the manifold E does in general not persist in its entirety under the inclusion of the higher order terms R. Instead, our aim is to prove the persistence of a *large portion* of E forming an invariant Cantor manifold $\mathcal{E}$ for the hamiltonian $H = \Lambda + Q + R$.

That is, there exists a family of n-tori

$$\mathcal{T}[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}(I) \subset E$$

over a Cantor set $\mathcal{C} \subset \mathbb{P}^n$ and a Lipschitz continuous embedding

$$\Psi:\ \mathcal{T}[\mathcal{C}] \hookrightarrow \ell^{a,p},$$

such that the restriction of Ψ to each torus $\mathcal{T}(I)$ in the family is an embedding of an elliptic rotational n-torus for the hamiltonian H. The image $\mathcal{E}$ of $\mathcal{T}[\mathcal{C}]$ we call a *Cantor manifold of elliptic rotational n-tori* given by the embedding $\Psi\colon \mathcal{T}[\mathcal{C}] \to \mathcal{E}$.

In addition, the Cantor set $\mathcal{C}$ has full density at the origin, the embedding Ψ is close to the inclusion map $\Psi_0\colon E \hookrightarrow \ell^{a,p}$, and the Cantor manifold $\mathcal{E}$ is tangent to E at the origin.

For the existence of $\mathcal{E}$ the following assumptions are made.

A. Nondegeneracy. The normal form $\Lambda + Q$ is *nondegenerate* in the sense that

$$\det A \neq 0,$$
$$\langle l, \beta\rangle \neq 0,$$
$$\langle k, \omega(I)\rangle + \langle l, \Omega(I)\rangle \not\equiv 0,$$

for all $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \le |l| \le 2$, where $\omega = \alpha + AI$ and $\Omega = \beta + BI$.

B. Spectral Asymptotics. There exists $d \ge 1$ and $\delta < d-1$ such that

$$\beta_j = j^d + \cdots + O(j^\delta),$$

where the dots stand for terms of order less than d in j.

C. Regularity.

$$X_Q, X_R \in \mathcal{A}(\ell^{a,p}, \ell^{a,\bar{p}}), \qquad \begin{cases} \bar{p} \ge p & \text{for } d > 1, \\ \bar{p} > p & \text{for } d = 1, \end{cases}$$

where $\mathcal{A}(\ell^{a,p}, \ell^{a,\bar{p}})$ denotes the class of all maps from some neighbourhood of the origin in $\ell^{a,p}$ into $\ell^{a,\bar{p}}$, which are real analytic in the real and imaginary parts of the complex coordinate q.

By the regularity assumption the coefficients of $B = (B_{ij})_{1\le j\le n<i}$ satisfy the estimate $B_{ij} = O(i^{p-\bar{p}})$ uniformly in $1 \le j \le n$. Consequently, for $d = 1$ there exists a maximal positive constant κ such that

$$\frac{\Omega_i - \Omega_j}{i-j} = 1 + O(j^{-\kappa}), \qquad i > j,$$

uniformly for bounded I. For $d > 1$, we set $\kappa = \infty$.

Theorem 2 (The Cantor Manifold Theorem). *Suppose the hamiltonian $H = \Lambda + Q + R$ satisfies assumptions A, B and C, and*

$$|R| = O\big(\|\hat{q}\|_{a,p}^4\big) + O\big(\|q\|_{a,p}^g\big)$$

with

$$g > 4 + \frac{4-\Delta}{\kappa}, \qquad \Delta = \min(\bar{p} - p, 1).$$

Then there exists a Cantor manifold $\mathcal{E}$ of real analytic, elliptic diophantine n-tori given by a Lipschitz continuous embedding $\Psi\colon\ \mathcal{T}[\mathcal{C}] \to \mathcal{E}$, where $\mathcal{C}$ has full density at the origin, and Ψ is close to the inclusion map Ψ_0:

$$\|\Psi - \Psi_0\|_{a,\bar{p},B_r\cap\mathcal{T}[\mathcal{C}]} = O(r^\sigma),$$

with $\sigma > 1$ specified at the end of section 7. Consequently, $\mathcal{E}$ is tangent to E at the origin.

Remark 1. The embedding Ψ can be chosen to include a parametrization of each torus in which the flow is linear (although the estimate is then worse, see Section 7). Then, for each $I \in \mathcal{C}$ and $v_0 \in \mathcal{T}(I)$,

$$\psi_{I,v_0}\colon\ t \mapsto \Psi\big(e^{\mathrm{i}\omega(I)t}v_0\big)$$

is a real analytic solution curve in $\ell^{a,p}$ for the hamiltonian $H = \Lambda + Q + R$. The frequencies $\omega(I)$ are diophantine for all $I \in \mathcal{C}$, so each such orbit is quasi-periodic with n basic frequencies.

Remark 2. The map Ψ is not only Lipschitz but could be shown to be smooth on $\mathcal{T}[\mathcal{C}]$ in the sense of Whitney. But we did not pursue this technical question. Moreover, Ψ may be extended to a global Lipschitz map $\bar{\Psi}\colon E \to \ell^{a,p}$ satisfying the same estimates as Ψ – see Appendix B. So $\mathcal{E}$ may be viewed as part of a global Lipschitz manifold. The latter, however, has no invariant meaning for the hamiltonian system outside the Cantor set.

5 Proof of Theorem 1

We prove Theorem 1 by deducing it from Theorem 2. By Lemma 3 our hamiltonian is $H = \Lambda + G$ with Λ as in (5) and X_G in $\mathcal{A}(\ell^{a,p}, \ell^{a,p})$, where we can fix $a > 0$ and $p > \frac{1}{2}$ arbitrarily. The domain of analyticity is then, of course, determined with respect to the norm $\|\cdot\|_{a,p}$. With the help of Lemma 4 we put H into its Birkhoff normal form up to order four by a real analytic symplectic map Γ, such that $H \circ \Gamma = \Lambda + \bar{G} + K$.

Now we choose any finite number n of normal modes $\phi_{j_1}, \phi_{j_2}, \dots, \phi_{j_n}$ and renumber them in such a way that they become the first n modes. With the notation of the previous section we then write

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \qquad \bar{G} = \tfrac{1}{2} \langle AI, I \rangle + \langle BI, Z \rangle + \tilde{G},$$

where, after renumbering, we have $\alpha = (\lambda_1, \dots, \lambda_n)$, $\beta = (\lambda_{n+1}, \lambda_{n+2}, \dots)$, and $A = (\bar{G}_{ij})_{1 \le i, j \le n}$, $B = (\bar{G}_{ij})_{1 \le j \le n < i}$. The term $\tilde{G}$ comprises “the rest” of $\bar{G}$, which is quadratic in Z and therefore satisfies

$$|\tilde{G}| = O\big(\|\hat{q}\|_{a,p}^4\big).$$

Thus $H \circ \Gamma = \Lambda + Q + R$ with $Q = \frac{1}{2} \langle AI, I \rangle + \langle BI, Z \rangle$ and $R = \tilde{G} + K$, for which we verify the hypotheses of Theorem 2.

Lemma 6. *For any choice of finitely many normal modes the normal form $\Lambda + Q$ is nondegenerate. So condition A is satisfied.*

Proof. We have $\bar{G}_{ij} = (4 - \delta_{ij})/4\pi$, independently of any renumbering of coordinates. Hence $4\pi A = 4X - I$, where I is the identity matrix and all components of X are 1. So X has rank 1, its spectrum is $\sigma(X) = \{0, n\}$, and

$$\sigma(4\pi A) = 4\sigma(X) - 1 = \{-1, 4n - 1\} \not\ni 0,$$

whence $\det A \ne 0$.

Clearly, $\langle l, \beta \rangle \ne 0$ for $1 \le |l| \le 2$. For the last nondegeneracy condition we have to check that

$$\langle \alpha, k \rangle + \langle \beta, l \rangle \ne 0 \quad \text{or} \quad Ak + B^T l \ne 0$$

for all (k, l) with $1 \le |l| \le 2$. Suppose $Ak + B^T l = 0$. Multiplying by 4π we have

$$(4X - I)k + 4\pi B^T l = 0,$$

and all coefficients of $4\pi B^T$ are 4. Thus all components of k are equal, say p, and $(4n - 1)p + 4q = 0$, where q is the sum of the at most two nonzero components of l. The only integer solution to this equation is $p = 0$, $q = 0$, so $k = 0$, and l has one ‘1’ and one ‘-1’. But then $\langle \alpha, k \rangle + \langle \beta, l \rangle = \lambda_i - \lambda_j$ for some $i \ne j$, and this expression does not vanish. ∎

Since $\lambda_j = j^2 + m$, the spectral sequence β satisfies condition B with $d = 2$. Hence it is sufficient to have $X_Q, X_R \in \mathcal{A}(\ell^{a,p}, \ell^{a,p})$, which follows from Lemmata 3 and 4. Finally,

$$|R| \le |\tilde{G}| + |K| = O\big(\|\hat{q}\|_{a,p}^4\big) + O\big(\|q\|_{a,p}^6\big),$$

so also condition C is satisfied with $g = 6 > 4$.

So Theorem 2 applies, and we obtain in particular

$$\|\Psi - \Psi_0\|_{a,p,B_r \cap \mathcal{T}[\mathcal{C}]} = O(r^2).$$

Composing with Γ we obtain a Cantor manifold $\mathcal{E}$ of real analytic diophantine n-tori in $\ell^{a,p}$ carrying quasi-periodic solutions

$$\gamma_{I,v_0}\colon\ t \mapsto q(t) = \Gamma \circ \Psi\big(e^{\mathrm{i}\omega(I)t} v_0\big)$$

for the hamiltonian $H = \Lambda + G$. Going back to $W^{a,p}$ by the isometry

$$\ell^{a,p} \to W^{a,p}, \qquad q \mapsto \mathcal{S}q = \sum_{j \ge 1} q_j \phi_j,$$

$\mathcal{E}$ is mapped into another Cantor manifold of real analytic diophantine tori in $W^{a,p}$, which by Lemma 3 carry analytic time quasi-periodic solutions u of the given nonlinear Schrödinger equation. This proves Theorem 1.

6 The Basic KAM Theorem

The Cantor Manifold Theorem is obtained from a KAM theorem that is concerned with perturbations of a family of linear integrable hamiltonians

$$N = \sum_{j=1}^{n} \omega_j(\xi) y_j + \tfrac{1}{2} \sum_{j=n+1}^{\infty} \Omega_j(\xi)(u_j^2 + v_j^2),$$

given in n-dimensional angle action coordinates (x, y) and infinite dimensional cartesian coordinates (u, v) with symplectic structure

$$\sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j=n+1}^{\infty} \mathrm{d}u_j \wedge \mathrm{d}v_j.$$

The frequencies $\omega = (\omega_1, \dots, \omega_n)$ and $\Omega = (\Omega_{n+1}, \Omega_{n+2}, \dots)$ depend on n parameters

$$\xi \in \Pi \subset \mathbb{R}^n,$$

Π a closed bounded set of positive Lebesgue measure, in a way described below.

For each ξ there is an invariant n-torus $\mathcal{T}_0 = \mathbb{T}^n \times \{0, 0, 0\}$ with frequencies $\omega(\xi)$. In its normal space described by the uv-coordinates the origin is an elliptic fixed point with characteristic frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large portion of this family of linearly stable rotational tori under small perturbations $H = N + P$ of N. To this end the following assumptions are made.

A. Nondegeneracy.* The map $\xi \mapsto \omega(\xi)$ between Π and its image is a homeomorphism which is Lipschitz continuous in both directions. Moreover,

$$\left| \{\xi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\} \right| = 0$$

and

$$\langle l, \Omega(\xi)\rangle \neq 0 \quad \text{on } \Pi$$

for all integer vectors $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \le |l| \le 2$.

B. Spectral Asymptotics and Lipschitz Property.* There exists $d \ge 1$ and $\delta < d - 1$ such that

$$\Omega_j(\xi) = j^d + \cdots + O(j^\delta),$$

where the dots stand for fixed lower order terms in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\bar{\Omega}$ with $\bar{\Omega}_j = j^d + \cdots$ such that the tails $\hat{\Omega}_j = \Omega_j - \bar{\Omega}_j$ give rise to a Lipschitz map

$$\hat{\Omega}\colon \; \Pi \to \ell_\infty^{-\delta},$$

where ℓ_∞^p denotes the space of all complex sequences with finite norm $|w|_p = \sup_j |w_j|\, j^p$.

C. Regularity.* The perturbation P is real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi \in \Pi$ its gradients with respect to u and v satisfy

$$P_u, P_v \in \mathcal{A}(\ell^{a,p}, \ell^{a,\bar{p}}), \qquad \begin{cases} \bar{p} \ge p & \text{for } d > 1, \\ \bar{p} > p & \text{for } d = 1. \end{cases}$$

To make this more precise we introduce complex neighbourhoods

$$D(s, r)\colon \; |\operatorname{Im} x| < s, \quad |y| < r^2, \quad \|u\|_{a,p} + \|v\|_{a,p} < r$$

of $\mathcal{T}_0$ and weighted phase space norms

$$\left|(x, y, u, v)\right|_r = |x| + \frac{1}{r^2}\,|y| + \frac{1}{r}\,\|u\|_{a,\bar{p}} + \frac{1}{r}\,\|v\|_{a,\bar{p}}\,,$$

where we omit a, p and $\bar{p}$ from the notation for brevity, and where $|\cdot|$ is the max-norm for complex vectors. Then we assume that the hamiltonian vectorfield X_P is real analytic on $D(s,r)$ for some s and r uniformly in ξ with finite norm $\left|X_P\right|_r$, and that the same holds for its Lipschitz semi-norm

$$\left|X_P\right|_r^{\mathcal{L}} = \sup_{\xi \neq \zeta} \frac{\left|\Delta_{\xi\zeta} X_P\right|_r}{|\xi - \zeta|}$$

over the parameter domain Π, where $\Delta_{\xi\zeta} X_P = X_P(\,\cdot\,,\xi) - X_P(\,\cdot\,,\zeta)$.

To state the following theorem we also assume that

$$|\omega|_{\Pi}^{\mathcal{L}} + \left|\hat{\Omega}\right|_{-\delta,\Pi}^{\mathcal{L}} \le M < \infty, \qquad \left|\omega^{-1}\right|_{\omega(\Pi)}^{\mathcal{L}} \le L < \infty,$$

with Lipschitz semi-norms defined analogously to $\left|X_P\right|_r^{\mathcal{L}}$. In the case $d = 1$, let $\kappa > 0$ be the largest exponent such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}), \qquad i > j,$$

uniformly on Π. The following theorem is proven in [12].

Theorem 3 (The Basic KAM-Theorem). *Suppose $H = N + P$ satisfies assumptions A*, B* and C*, and*

$$\epsilon = \left|X_P\right|_{r,D(s,r)} + \frac{\alpha}{M}\left|X_P\right|_{r,D(s,r)}^{\mathcal{L}} \le \gamma\alpha,$$

where $0 < \alpha \le 1$ is a parameter, and γ depends on the parameters described below. Then there exists a Cantor set $\Pi_\alpha \subset \Pi$ with $|\Pi \backslash \Pi_\alpha| \to 0$ as $\alpha \to 0$, a Lipschitz continuous family of torus embeddings

$$\Phi\colon\ \mathbb{T}^n \times \Pi_\alpha \to \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a,\bar{p}} \times \ell^{a,\bar{p}},$$

and a Lipschitz continuous map $\tilde{\omega}\colon \Pi_\alpha \to \mathbb{R}^n$, such that for each ξ in Π_α the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\tilde{\omega}(\xi)$ for the hamiltonian H at ξ.

Each embedding is real analytic on $|\mathrm{Im}\, x| < s/2$*, and*

$$\|\Phi - \Phi_0\|_r + \frac{\alpha}{M} \|\Phi - \Phi_0\|_r^{\mathcal{L}} \le \frac{c\epsilon}{\alpha},$$
$$|\tilde{\omega} - \omega| + \frac{\alpha}{M} |\tilde{\omega} - \omega|^{\mathcal{L}} \le c\epsilon,$$

uniformly on that domain and Π_α*, where* $\Phi_0 \colon \mathbb{T}^n \times \Pi \to \mathcal{T}_0$ *is the trivial embedding.*

If the unperturbed frequencies are affine functions of the parameter ξ*, then*

$$|\Pi \backslash \Pi_\alpha| \le \tilde{c}\rho^{n-1}\alpha^\mu, \qquad \mu = \begin{cases} 1 & \text{for } d > 1, \\ \dfrac{\kappa}{\kappa + 1 - \pi/4} & \text{for } d = 1, \end{cases}$$

where $\rho = \mathrm{diam}\, \Pi$ *and* π *is any number in* $0 \le \pi < \min(\bar{p} - p, 1)$*.*

The constants γ *and* c*,* $\tilde{c}$ *depend on the parameters* $n, d, \delta, \kappa, \bar{p} - p, s$*, the product* LM *and the frequencies* ω *and* Ω *in a 'monotone' way. That is,* γ^{-1} *and* $c, \tilde{c}$ *do not increase for closed subsets of* Π*. In addition, for* $d = 1$*,* $\tilde{c}$ *also depends on* π*.*

Remark 1. There is not only a family of embeddings but indeed a Lipschitz continuous family of real analytic coordinate changes $\bar{\Phi}$ in a neighbourhood of $\mathcal{T}_0$ such that

$$H \circ \bar{\Phi} = \sum_{j=1}^{n} \tilde{\omega}_j(\xi) y_j + \tfrac{1}{2} \sum_{j=n+1}^{\infty} \tilde{\Omega}_j(\xi)(u_j^2 + v_j^2) + \dots,$$

where the dots stand for higher order terms in y, u, v, and the new frequencies are strongly nonresonant. See [12] for more details.

Remark 2. Actually, Π_α may contain isolated points, so it may not be a Cantor set in the strict sense. But it will be a Cantor set up to a set of measure zero.

Remark 3. The rôle of the parameter α is the following. In applications the size of the perturbation usually depends on a small parameter, for example the size of the neighbourhood around an elliptic fixed point. One then wants to choose α as another function of this parameter in order to obtain useful estimates for $|\Pi \backslash \Pi_\alpha|$. The next section provides an example.

7 Proof of the Cantor Manifold Theorem

We finally prove the Cantor Manifold Theorem with the help of the Basic KAM-Theorem. Recall that we are given a hamiltonian $H = \Lambda + Q + R$ in complex coordinates $q = (\tilde{q}, \hat{q})$, where R is some perturbation of the normal form

$$\Lambda + Q = \langle \alpha, I \rangle + \langle \beta, Z \rangle + \tfrac{1}{2} \langle AI, I \rangle + \langle BI, Z \rangle \,,$$

with $I = \frac{1}{2}(|q_1|^2, \ldots, |q_n|^2)$ and $Z = \frac{1}{2}(|q_{n+1}|^2, |q_{n+2}|^2, \ldots)$. Assumptions A, B and C are supposed to hold.

Step 1. New coordinates. We introduce symplectic polar and real coordinates by setting

$$q_j = \begin{cases} \sqrt{2(\xi_j + y_j)}\, e^{-\mathrm{i}x_j}, & 1 \le j \le n, \\ u_j + \mathrm{i}v_j, & j \ge n+1, \end{cases}$$

depending on parameters $\xi \in \Pi = [0,1]^n$. The precise domains will be specified later when they become important. Then we have

$$\frac{\mathrm{i}}{2} \sum_{j \ge 1} \mathrm{d}q_j \wedge \mathrm{d}\bar{q}_j = \sum_{1 \le j \le n} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j \ge n+1} \mathrm{d}u_j \wedge \mathrm{d}v_j \,,$$

$I = \xi + y$ and $Z = \frac{1}{2}(u^2 + v^2)$, with the obvious componentwise interpretation. The normal form becomes

$$\Lambda + Q = \langle \omega(\xi), y \rangle + \tfrac{1}{2} \langle \Omega(\xi), u^2 + v^2 \rangle + \tilde{Q}$$

with frequencies $\omega(\xi) = \alpha + A\xi$, $\Omega(\xi) = \beta + B\xi$ and remainder $\tilde{Q} = O(|y|^2) + O(\|u^2 + v^2\|^2)$. The total hamiltonian is $H = N + P$ with $P = \tilde{Q} + R$.

Step 2. Checking assumptions A, B* and C*.* The map $\xi \mapsto \omega(\xi)$ is a lipeomorphism of $\mathbb{R}^n$ onto itself, since A is invertible by assumption. The measure condition is satisfied, since $\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle$ is a nontrivial affine function of ξ by assumption, which vanishes on a codimension 1 subspace. Finally, $\langle l, \Omega(\xi) \rangle$ does not vanish for small $|\xi|$ because of the asymptotic behavior of the frequencies and the assumption $\langle l, \beta \rangle \neq 0$. So condition A* is satisfied.

As to condition B* we have $\Omega(\xi) = \beta + B\xi$ with $\beta_j = j^d + \cdots + O(j^\delta)$ by assumption. We already noticed that the regularity assumption implies that $B_{ij} = O(i^{p-\bar{p}})$ uniformly in $1 \le j \le n$. Hence $\hat{\Omega} = \Omega - \beta$ is Lipschitz as a map $\Pi \to \ell_\infty^{\bar{p}-p}$ with finite Lipschitz constant $|\hat{\Omega}|^{\mathcal{L}}_{\bar{p}-p}$ on Π.

The above statements hold *a fortiori*, if δ is replaced by

$$\delta_* = \max(\delta, p - \bar{p}) < d - 1.$$

So condition B* is satisfies with this choice for δ. Finally, the regularity condition C* follows immediately from assumption C.

The Lipschitz constants $L = |\omega^{-1}|^{\mathcal{L}}$ and $M = |\omega|^{\mathcal{L}} + |\hat{\Omega}|^{\mathcal{L}}_{-\delta}$ are fixed and finite on Π. For convenience we may therefore ignore them in the following.

Step 3. Domains and estimates. Now let $r > 0$ and consider the phase space domain

$$D(2, r)\colon\ |\operatorname{Im} x| < 2, \quad |y| < r^2, \quad \|u\|_{a,p} + \|v\|_{a,p} < r,$$

and the parameter domain

$$\Xi_r^- = U_{-4r^2}\,\Xi_r, \qquad \Xi_r = \left\{\xi : 0 < \xi < r^{2\lambda}\right\}, \qquad \lambda = \frac{4}{g} < 1,$$

where $U_{-\rho}\,\Xi$ is the subset of all points in Ξ with boundary distance greater than ρ.

The total hamiltonian H is well defined on these domains, and $|\tilde{Q}| = O(r^4)$ as well as $|R| = O(r^{g\lambda} + r^4) = O(r^4)$ on $D(2, 2r)$. Using Cauchy estimates for R_x, R_y and the hypotheses for R_u, R_v we then obtain

$$|X_{\tilde{Q}}|_{r,D(1,r)} + |X_R|_{r,D(1,r)} = O(r^2).$$

Using again Cauchy with respect to ξ, we have $|X_{\tilde{Q}}|^{\mathcal{L}}_r + |X_R|^{\mathcal{L}}_r = O(r^2/\alpha)$ on $U_{-\alpha/2}\,\Xi_r^-$, $\alpha > 0$. Altogether we obtain

$$|X_P|_{r,D(1,r)} + \alpha\,|X_P|^{\mathcal{L}}_{r,D(1,r)} = O(r^2)$$

with respect to the parameter domain

$$\Pi_r = U_{-\alpha}\,\Xi_r, \qquad \alpha \geq 8r^2,$$

where α will be chosen as a function of r later.

Step 4. Application of Theorem 3. To apply Theorem 3 it now suffices to require that as a function of r,

$$\alpha(r) \geq c_1 r^2 \tag{8}$$

for all small r with a sufficiently large constant c_1 which depends on the parameters indicated in that theorem, but *not* on the parameter sub-domain Π_r and hence *not* on r.

As a result we obtain a Cantor set $\Pi_{r,\alpha} \subset \Pi_r$ of parameters, a Lipschitz continuous family of real analytic torus embeddings

$$\Phi_r\colon\ \mathbb{T}^n \times \Pi_{r,\alpha} \to D(1,r),$$

and a Lipschitz continuous frequency map $\tilde{\omega}_r\colon \Pi_{r,\alpha} \to \mathbb{R}^n$, such that for each ξ in $\Pi_{r,\alpha}$ the map Φ_r restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of an elliptic, rotational torus with freqnecies $\tilde{\omega}_r(\xi)$ for the hamiltonian H at ξ. Also the estimates

$$\begin{gathered} |\!|\Phi_r - \Phi_0|\!|_r + \alpha\, |\!|\Phi_r - \Phi_0|\!|_r^{\mathcal{L}} \leq cr^2/\alpha, \\ |\tilde{\omega}_r - \omega| + \alpha\, |\tilde{\omega}_r - \omega|^{\mathcal{L}} \leq cr^2, \end{gathered} \tag{9}$$

hold on $|\mathrm{Im}\, x| < \frac{1}{2}$ and $\Pi_{r,\alpha}$, where the generic constant c depends on the same parameters as c_1. Moreover, we have the measure estimate

$$\left| \Xi_r \backslash \Pi_{r,\alpha} \right| \leq \frac{c\alpha^\mu}{r^{2\lambda}}\, |\Xi_r| \tag{10}$$

with a constant c independent of Π_r and μ depending on κ and η. Hence, to obtain a non-empty Cantor set we also need

$$\alpha^\mu(r) \leq c_1^{-1} r^{2\lambda}.$$

The embedding Φ_r describes the invariant tori in terms of the “relative” actions y. The translated embedding $\Phi_r^t = \Phi_r + T_\xi$, where $T_\xi(\phi,\xi) = (0,\xi,0,0)$, gives the same tori in terms of the “absolute” actions $I = \xi + y$. It is a small perturbation of the trivial embedding $\Phi_0^t\colon (\phi, I) \mapsto (\phi, I, 0, 0)$, and we have $\Phi_r^t - \Phi_0^t = \Phi_r - \Phi_0$. So the above estimates are preserved, and henceforth we will write again Φ_r for brevity.

Step 5. Patching up domains. The Cantor sets $\Pi_{r,\alpha}$ by themselves are not dense at the origin. To obtain such a set we have to take the union of a suitable sequence of subsets of $\Pi_{r,\alpha}$. Fix some $0 < \eta \le \frac{1}{2}$ and set $R_r = \Xi_r \setminus \Xi_{\eta r}$, $R_r^- = U_{-\alpha} R_r$, and

$$\mathcal{C}_r = \Pi_{r,\alpha} \cap R_r^- .$$

In view of (10) we then have $|R_r \setminus \mathcal{C}_r| \le \dfrac{c\alpha^\mu}{r^{2\lambda}}\,|R_r|$. Now choose the sequence $r_k = \eta^k r_0$, $k \ge 0$, and set

$$\mathcal{C} = \bigcup_{k\ge 0} \mathcal{C}_{r_k},$$

choosing r_0 small enough so that all the $\mathcal{C}_{r_k}$ are not empty. Define the embedding $\Phi\colon\ \mathbb{T}^n \times \mathcal{C} \to \mathcal{P} = \mathbb{T}^n \times \Pi^n \times \ell^{a,\bar p} \times \ell^{a,\bar p}$ by piecing together the corresponding definitions on each component. That is, $\Phi\,|_{\mathbb{T}^n\times\mathcal{C}_{r_k}} = \Phi_{r_k}$. Similarly for $\tilde\omega$. These definitions are correct, since the $\mathcal{C}_{r_k}$ are disjoint and even have a pairwise positive distance to each other. So Φ is certainly continuous in I and real analytic in each fiber $\mathbb{T}^n \times \{I\}$, $I \in \mathcal{C}$.

Step 6. Estimates. We now show that if $r^2/\alpha(r)$ is a nondecreasing function of r, then on $|\mathrm{Im}\,\phi| < \frac{1}{2}$ and $\mathcal{C} \cap \Xi_{r_k}$ one has

$$|\Phi - \Phi_0|_{r_k}, \quad \alpha(r_k)\,\frac{|\Delta_{IJ}(\Phi-\Phi_0)|_{r_k}}{|I-J|} \le \frac{c r_k^2}{\alpha(r_k)}, \tag{11}$$

provided $I \in \mathcal{C}_{r_k}$. This holds for all $k \ge 0$. Analogous estimates hold for $\tilde\omega$, which we forego. Moreover, if also $\alpha^\mu(r)/r^{2\lambda}$ is a nondecreasing function of r, then

$$\frac{\left|\mathcal{C} \cap \Xi_{r_k}\right|}{\left|\Xi_{r_k}\right|} \ge 1 - \frac{c\alpha^\mu(r_k)}{r_k^{2\lambda}}.$$

To prove the first estimate we observe that $|\Phi - \Phi_0|_r$ increases as r decreases, so that

$$\begin{aligned}
|\Phi - \Phi_0|_{r_k,\mathcal{C}\cap\Xi_{r_k}} &= \sup_{l\ge k} |\Phi - \Phi_0|_{r_k,\mathcal{C}_{r_l}} \\
&\le \sup_{l\ge k} |\Phi - \Phi_0|_{r_l,\mathcal{C}_{r_l}} \le \sup_{l\ge k} \frac{c r_l^2}{\alpha(r_l)} \le \frac{c r_k^2}{\alpha(r_k)}.
\end{aligned}$$

The second estimate we immediately obtain from (9), if I and J are in the same patch $\mathcal{C}_{r_k}$. Otherwise, $J \in \mathcal{C}_{r_l}$ with $l > k$. Then $|I - J| \ge \alpha(r_k)$ and

$$|\Delta_{IJ}(\Phi-\Phi_0)|_{r_k} \le |\Phi-\Phi_0|_{r_k,\mathcal{C}_{r_k}} + |\Phi-\Phi_0|_{r_l,\mathcal{C}_{r_l}} \le \frac{c r_k^2}{\alpha(r_k)},$$

from which the statement follows. As to the measure estimate, let $\mathcal{E}$ be the complement of $\mathcal{C}$. Then

$$\left|\mathcal{E} \cap \Xi_{r_k}\right| = \sum_{l \geq k} \left|\mathcal{E} \cap R_{r_l}\right| = \sum_{l \geq k} \left|R_{r_l} \setminus \mathcal{C}_{r_l}\right|$$
$$\leq \sum_{l \geq k} \frac{c\alpha^{\mu}(r_l)}{r_l^{2\lambda}} \left|R_{r_l}\right| \leq \frac{c\alpha^{\mu}(r_k)}{r_k^{2\lambda}} \left|\Xi_{r_k}\right|,$$

which gives the claim.

Step 7. The embedding Ψ. The map Φ describes the invariant tori in terms of symplectic polar coordinates, and it also includes a parametrization of each torus in which the flow is linear. To describe and estimate the same tori in symplectic cartesian coordinates, however, it is advantageous to undo this parametrization in order to obtain a “radial map”.

Let $X\colon \mathbb{T}^n \times \mathcal{C} \to \mathbb{T}^n$ be the angular component of the embedding Φ. By (11) X is close to the identity on $|\mathrm{Im}\, x| < \frac{1}{2}$ uniformly in I, by choosing the constant c_1 in (8) sufficiently large. Hence there exists an inverse on each torus, $X^{-1}\colon \mathbb{T}^n \times \mathcal{C} \to \mathbb{T}^n$, which is real analytic on $|\mathrm{Im}\, x| < \frac{1}{4}$ and satisfies the same estimate as X. So we can define the embedding

$$\Psi = \Phi \circ X^{-1}\colon\ \mathbb{T}^n \times \mathcal{C} \to \mathcal{P},$$

which is of the form

$$\phi' = \phi, \quad I' = I + y(\phi, I), \quad u' = u(\phi, I), \quad v' = v(\phi, I)$$

and satisfies the same estimates as Φ.

It remains to estimate Ψ in terms of the cartesian coordinates. Indeed, we show that

$$\|\Psi - \Psi_0\|_{a,\bar{p}} \leq \frac{c r_k^2}{\alpha(r_k)} \cdot r_k$$

uniformly on $\mathcal{T}[\mathcal{C} \cap \Xi_{r_k}]$ for $k \geq 0$.

For the proof, consider $\tilde{q} = \sqrt{2I}\, e^{\mathrm{i}\phi}$ and $\hat{q} = u + \mathrm{i}v$, understood componentwise. On $\mathcal{T}[\mathcal{C} \cap R_{r_k}]$ we have

$$\begin{aligned}\frac{1}{\sqrt{2}}\left|\tilde{q}'-\tilde{q}\right| &= \left|\sqrt{I+y}-\sqrt{I}\right| \\ &\le |y|\left(\min_{I\in R_{r_k}}\left|\sqrt{I_j}\right|\right)^{-1} \\ &\le r_k^2\,|\Psi-\Psi_0|_{r_k}\,\frac{1}{\sqrt{\alpha(r_k)}} \\ &\le \frac{cr_k^2}{\alpha(r_k)}\,r_k,\end{aligned}$$

using $\alpha(r)\ge c_1 r^2$ and (11). By the same token,

$$\left\|\hat{q}'\right\|_{a,\bar{p}} \le \|u\|_{a,\bar{p}} + \|v\|_{a,\bar{p}} \le r_k\,|\Psi-\Psi_0|_{r_k} \le \frac{cr_k^2}{\alpha(r_k)}\,r_k.$$

The right hand sides decrease as k increases, so this bound holds also on $\mathcal{T}[\mathcal{C}\cap R_{r_l}]$ with $l>k$ and thus on all of $\mathcal{T}[\mathcal{C}\cap\Xi_{r_k}]$.

Step 8. Choice of $\alpha(r)$. Eventually we have to choose α as a function of r satisfying

$$c_1 r^{2\mu} \le \alpha^{\mu}(r) \le c_1^{-1} r^{2\lambda}$$

for all small r. Thus we need $\mu>\lambda$. For $d>1$ with $\mu=1$ this clearly holds. For $d=1$, this can also be arranged, since $g>4+(4-\Delta)/\kappa$ by assumption, hence

$$\frac{4}{g}=\lambda<\bar{\mu}=\frac{\kappa}{\kappa+1-\Delta/4},$$

and $\mu<\bar{\mu}$ can be chosen arbitrarily close to μ.

The choice $\alpha=c_1r^2$ would lead to a large Cantor set $\mathcal{C}$, but rather weak estimates for $\Psi-\Psi_0$. The other extreme $\alpha^{\mu}=c_1^{-1}r^{2\lambda}$ would lead to good estimates for $\Psi-\Psi_0$, but the density of $\mathcal{C}$ at 0 would not be 1. Unfortunately, we have no uniqueness results for the lower dimensional tori, so different choices of α may lead to quite different and maybe disjoint Cantor manifolds $\mathcal{E}$.

So some choice has to be made. For example, taking the geometric mean

$$\alpha^{\bar{\mu}}(r)=r^{\lambda+\bar{\mu}}$$

and choosing μ properly, the above condition is met for all small r, the Cantor set $\mathcal{C}$ has full density at the origin, and

$$\|\Psi-\Psi_0\|_{a,\bar{p}} \le cr^{2-\lambda/\bar{\mu}} = cr^{\sigma\lambda}, \qquad \sigma=\frac{g}{2}-\frac{1}{\bar{\mu}},$$

on $\mathcal{T}[\mathcal{C} \cap \Xi_r]$. The latter contains the set $\mathcal{T}[\mathcal{C}] \cap B_{r^\lambda}$, and so the estimate of Theorem 2 is obtained. The proof of Theorem 2 is now complete.

A The Banach Algebra Property

Consider the Hilbert space $\ell^{a,p}$ of all doubly infinite complex sequences $q = (\ldots, q_{-1}, q_0, q_1, \ldots)$ with

$$\|q\|_{a,p}^2 = \sum_j |q_j|^2 [j]^{2p} e^{2a|j|} < \infty, \qquad [j] = \max(|j|, 1).$$

The convolution $q * p$ of two such sequences is defined by $(q * p)_j = \sum_k q_{j-k} p_k$.

Lemma. *If $a \geq 0$ and $p > \frac{1}{2}$, then $\|q * p\|_{a,p} \leq c\,\|q\|_{a,p}\,\|p\|_{a,p}$ for $q, p \in \ell^{a,p}$ with a finite constant c depending only on p.*

Proof. Let $\gamma_{jk} = \dfrac{[j-k]\,[k]}{[j]}$. By the Schwarz inequality,

$$\Big|\sum_k x_k\Big|^2 = \Big|\sum_k \frac{\gamma_{jk}^p x_k}{\gamma_{jk}^p}\Big|^2 \leq c_j^2 \sum_k \gamma_{jk}^{2p} |x_k|^2, \qquad c_j^2 = \sum_k \frac{1}{\gamma_{jk}^{2p}},$$

for all j. We have

$$\frac{1}{\gamma_{jk}} \leq \frac{[j-k] + [k]}{[j-k]\,[k]} = \frac{1}{[j-k]} + \frac{1}{[k]},$$

so that

$$c_j^2 \leq \sum_k \left(\frac{1}{[j-k]} + \frac{1}{[k]}\right)^{2p} \leq 4^p \sum_k \frac{1}{[k]^{2p}} \stackrel{\text{def}}{=} c^2 < \infty$$

for all j. It follows that for $a = 0$,

$$\begin{aligned}
\|q * p\|_{a,p}^2 &= \sum_j [j]^{2p} \Big|\sum_k q_{j-k} p_k\Big|^2 \\
&\leq c^2 \sum_j [j]^{2p} \sum_k \gamma_{jk}^{2p} \left|q_{j-k} p_k\right|^2 \\
&= c^2 \sum_{j,k} [j-k]^{2p} \left|q_{j-k}\right|^2 [k]^{2p} |p_k|^2 \\
&= c^2 \, \|q\|_{a,p}^2 \, \|p\|_{a,p}^2 .
\end{aligned}$$

The case $a > 0$ is a simple variation of the last estimate. ∎

B The Kirszbraun Theorem

Let E and F be a finite dimensional and a separable Hilbert space, respectively, with norms $\|\cdot\|_E$ and $\|\cdot\|_F$. The Lipschitz seminorm of a map $f: E \supset S \to F$ is

$$[f]_S = \sup_{\substack{x,y\in S\\ x\neq y}} \frac{\|f(x)-f(y)\|_F}{\|x-y\|_E}.$$

Theorem (Kirszbraun). *Let $S \subset E$, and let $f: S \to F$ be a map with finite Lipschitz seminorm. Then there exists an extension $\phi: E \to F$ of f with the same Lipschitz seminorm: $[\phi]_E = [f]_S$.*

Proof. Let $F_1 \subset F_2 \subset \ldots$ be an increasing sequence of closed subspaces of F with $\bigcup_n F_n = F$ and orthogonal projections P_n onto them. Let

$$f_n = P_n f : S \to F_n.$$

By the finite dimensional Kirszbraun Theorem [6] there is an extension $\phi_n: E \to F_n$ of f_n with

$$[\phi_n]_E = [f_n]_S \leq [f]_S.$$

Thus, the ϕ_n are uniformly Lipschitz with $\phi_n|_S = f_n$.

Now choose a dense countable set $X \subset E - S$. Since they are uniformly Lipschitz and equal to f_n on S, the ϕ_n are uniformly bounded on bounded subsets of E. Hence, by the usual diagonal trick, we can extract a subsequence, again denoted by ϕ_n, that converges *weakly* in every point of X to a map $\varphi: X \cup S \to F$. They also converge pointwise in S so that $\varphi|_S = f$. Moreover,

$$[\varphi]_{X\cup S} \leq \liminf_{n\to\infty} [\phi_n]_E \leq [f]_S,$$

since for weak limits,

$$\|\varphi(x)-\varphi(y)\|_F \leq \liminf_{n\to\infty} \|\phi_n(x)-\phi_n(y)\|_F.$$

Hence we can uniquely extend φ to a Lipschitz continuous map $\phi: E \to F$ with $[\phi]_E = [\varphi]_{X\cup S} \leq [f]_S$. But indeed $[\phi]_E = [f]_S$, since $\phi|_S = f$. ∎

C Time Periodic Solutions

We discuss the nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - mu - f(|u|^2)u \tag{1}$$

with Dirichlet boundary conditions and a real analytic nonlinearity f with $f(0) = 0$, but not necessarily $f'(0) \neq 0$. We show that there exist "solid" discs through $u \equiv 0$ which are filled with time periodic solutions.

Recall that ϕ_j and λ_j are the basic modes and their frequencies for the linear equation $\mathrm{i}u_t = u_{xx} - mu$ with Dirichlet boundary conditions.

Theorem. *Suppose f is real analytic near* 0 *with $f(r) = O(r^s)$ for some $s \geq 1$. Then for every $j \geq 1$ there exists an embedded disc*

$$\mathcal{E}_j = \left\{ u(t,x) = r v_j(x,r) e^{\mathrm{i}\mu_j(r)t},\ \ 0 \leq r < r_j \right\}$$

of real analytic, time periodic solutions of (1)*, where*

$$v_j = \phi_j + O(r^{2s}), \qquad \mu_j = \lambda_j + O(r^{2s})$$

are real analytic in r and x and in r, respectively.

Proof. Let $u(t,x) = r v(x) e^{\mathrm{i}\mu t}$, where the dependence of μ and v on j and r is not indicated. Then u is a solution of (1) with Dirichlet boundary conditions if and only if v vanishes at 0 and π and satisfies

$$\mu v = -v_{xx} + mv + f(r^2\,|v|^2)v. \tag{2}$$

This equation is independent of t, which is crucial for our argument.

Fix some $a > 0$ and $p > \frac{1}{2}$, and let $W_{\mathrm{o}}^{a,p}$ be the space of all *odd*, 2π-periodic functions in the space $W^{a,p}$ introduced in section 2. We solve equation (2) by applying the implicit function theorem to the map

$$\begin{aligned} \Phi\colon\ W_{\mathrm{o}}^{a,p} \times \mathbb{R} \times \mathbb{R} &\to W_{\mathrm{o}}^{a,p-2} \times \mathbb{R} \\ (v, \mu, r) &\mapsto \left(-v_{xx} + mv + f(r^2\,|v|^2) - \mu v,\ \|v\|^2 - 1\right) \end{aligned}$$

where $\|\cdot\|$ denotes the L^2-norm. The aim is to find solutions of

$$\Phi(v, \mu, r) = 0.$$

We normalize $\|v\| = 1$ to make solutions locally unique.

We have

$$\Phi(\phi_j, \lambda_j, 0) = 0$$

for every $j \geq 1$. The map Φ is real analytic in some neighbourhood of each of these points (which depends on a, p and j) by the same arguments we used to prove Lemma 3 in section 2. Its Jacobian with respect to v and μ at $(\phi_j, \lambda_j, 0)$ is the linear map

$$\begin{aligned} W_o^{a,p} \times \mathbb{R} &\to W_o^{a,p-2} \times \mathbb{R} \\ (w, \nu) &\mapsto \big(-w_{xx} + mw - \lambda_j w - \nu\phi_j, 2\langle w, \phi_j\rangle\big) . \end{aligned}$$

This is an *isomorphism*, as one verifies by writing $w = \sum_{k\geq 1} \hat{w}_k \phi_k$ and comparing coefficients. Thus the implicit function theorem applies, and for every $j \geq 1$ we obtain a unique real analytic arc of solutions

$$(-r_j, r_j) \to W_o^{a,p} \times \mathbb{R}, \qquad r \mapsto \big(v_j(r), \mu_j(r)\big)$$

through $(v_j(0), \mu_j(0) = (\phi_j, \lambda_j)$. By uniqueness, this arc is even even in r. Moreover, we have the standard estimate

$$\|v_j(r) - v_j(0)\| + |\mu_j(r) - \mu_j(0)| = O(\|\Phi(\phi_j, \lambda_j, r)\|) = O(r^{2s}),$$

which yields the asymptotic behavior of v_j and μ_j. ∎

Remark 1. The discs $\mathcal{E}_j$ are real analytic outside the origin. At the origin they are at least C^2, since they have a third order tangency to the plane $\{q\phi_j : q \in \mathbb{C}\}$. It would be interesting to know whether they are analytic at the origin.

Remark 2. For the discs $\mathcal{E}_j$ to exist the nonlinearity f need not be analytic. It suffices that f is smooth or sufficiently often differentiable. Of course, the regularity of the periodic solutions with respect to x changes accordingly.

Remark 3. The argument applies equally well to nonlinear Schrödinger operators on higher dimensional bounded domains Ω. For the linearized equation to define an isomorphism, the boundary of Ω has to be regular enough so that standard elliptic regularity theory applies.

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