DOUBLE-PERIODIC SOLUTIONS OF QUASILINEAR CAUCHY–RIEMANN EQUATIONS

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Abstract. For a compact Kählerian manifold $\mathfrak{M}$ we study the equation

$$u_{\bar z}(z)+f(z,u)=0, \quad z\in\mathbb{T}^2, u\in\mathfrak{M},$$

where $f(z,\cdot)$ is a vectorfield on $\mathfrak{M}$, sufficiently smooth in z,u. We define V as the infimum of areas of nontrivial holomorphic spheroids $S^2\subset\mathfrak{M}$. So $V=\infty$ if $\mathfrak{M}$ is aspherical and, for example, $V=1$ if $\mathfrak{M}=S^2$ (more general, if $\mathfrak{M}=\mathbb{C}P^n$). Let $\|f(z,u)\|_u<\sqrt{V}$ where $\|\cdot\|_u$ stands for the normalized Kählerian metric. Then, first, the equation has at least $|\chi(\mathfrak{M})|$ contractible solutions ($\chi(\mathfrak{M})$ is the Euler characteristic) if f is taken from an open dense subset of the set of all admissible f and, second, for each admissible f it has at least one solution if $\chi(\mathfrak{M})\neq 0$. These estimates are sharp.

Introduction

We study double-periodic solutions of quasilinear Cauchy-Riemann equations for maps valued in a compact complex manifold $\mathfrak{M}=\mathfrak{M}^n$:

$$u_{\bar z}(z)+f(z,u)=0, \quad z=x+iy\in\mathbb{T}^2=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z}),^1 \;\; u\in\mathfrak{M}, \tag{1}$$

where $f(z,\cdot)$ is a vectorfield on $\mathfrak{M}$, sufficiently smooth in z, u and

$$u_{\bar z}=(\partial/\partial\bar z)u=\frac{1}{2}(u_x+iu_y)\in T_u\mathfrak{M}.$$

We are mostly concerned with contractible solutions,

$$u:\mathbb{T}^2\to\mathfrak{M} \quad \text{is contractible.} \tag{2}$$

Equation (1) linearized about a contractible solution defines a zero-index Fredholm operator. Essentially for this reason for a typical f the equation has a finite number of solutions. Our goal is to estimate this number from below by investigating the manifold formed by all pairs (u,f) where u solves (1), (2) with $f=f$.

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[1] all the results remain true if one replace $\mathbb{T}^2$ by any complex torus $\mathbb{C}/(\mathbb{Z}+\nu\,\mathbb{Z})$ of the unit area.

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The problem (1) is a two-dimensional-time analogy of the popular problem of estimating the number of periodic solutions of a nonautonomous vectorfield R on a compact manifold $\mathcal{N}$:

$$\dot{v}(t) + R(t, v(t)) = 0, \quad t \in S^1 = \mathbb{R}/\mathbb{Z}, \quad v \in \mathcal{N}. \tag{3}$$

To study the problem (3) one can use the succession map S which sends $v_0 \in \mathcal{N}$ to $v(1) \in \mathcal{N}$ where $v(t)$, $0 \le t \le 1$, solves the equation $\dot{v} + R(t, v) = 0$, $v(0) = v_0$. Indeed, solutions of (3) correspond to fixed points of S. Since S is homotopic to identity then due to the Lefschetz theorem [5] in the nondegenerate case the number of fixed points counted with their orientations equals the Euler characteristic $\chi(\mathcal{N})$. In particular, if $\chi(\mathcal{N}) \neq 0$ then (3) has at least one solution.[2]

For the equation (1) there exists no analogy of the succession map. Still a direct analogy of the stated result holds for (1) if

a) we take into account contractible solutions only,

b) the manifold $\mathfrak{M}$ is Kählerian,

c) $\|f(z, u)\|_u < \sqrt{V}$ for each $z \in \mathbb{T}^2$, $u \in \mathfrak{M}$, where $\|\cdot\|_u$ is the normalized Kählerian metric and

$$V = \inf\{\text{Area } g(S^2) \mid g : S^2 \to \mathfrak{M} \text{ is a nontrivial holomorphic map}\}. \tag{4}$$

To state the result we denote by $\mathcal{F}_V$ the set of all sufficiently smooth $f(z, u)$ as in c). The following statement reformulates Theorem 4.1 from Part 4:

Theorem 1. *If $\mathfrak{M}$ is a compact Kählerian manifold, then:*

1) for f from an open dense subset of $\mathcal{F}_V$ all solutions of (1), (2) are nondegenerate[3]*, their number is finite and is $\ge |\chi(\mathfrak{M})|$ ($\chi(\mathfrak{M})$ is the Euler characteristic of $\mathfrak{M}$);*

2) if $\chi(\mathfrak{M}) \neq 0$, then for each $f \in \mathcal{F}_V$ the problem (1), (2) has a solution;

3) the estimates in 1), 2) are sharp: the equation can have no solutions if $\chi(\mathfrak{M}) = 0$, for each $\mathfrak{M}$ under an appropriate choice of $f \in \mathcal{F}_V$ it can have only one solution and can have only $|\chi(\mathfrak{M})|$ nondegenerate solutions.

In other words, the *hyperbolic* dynamical system $u_x = -iu_y - 2f$ (where x is "time") in the space of contractible loops $\{S^1 = \mathbb{R}/\mathbb{Z} \to \mathfrak{M}\}$, $S^1 = \{y\}$, has as many contractible periodic trajectories as (3) has due to Lefschetz. *Possibly, only one!*

To study the problem (1) we have imposed the restrictions a)-c). Are they essential? The first one certainly is since if the tangent bundle $T\mathfrak{M}$ is not trivial then for a typical f the set of solutions of (1) could be empty or be a manifold of a positive dimension, depending on the homotopy type of the solutions[4]. See [4, Part 2] and [10].

Assumption b) can be relaxed to the assumption that the complex structure of $\mathfrak{M}$ is tamed by a symplectic structure. In this case the constant V should be modified (see Remark 2.1).

Concerning the assumption c) we do not know if it is necessary or not for the theorem's assertions to be true (but see Proposition 2.1 and its discussion below).

Next we give some examples of theorem's applications and discuss the proof.

[2]Below in Part 6.3) we apply our techniques to prove that the algebraic number of all *contractible* solutions of (3) also equals $\chi(\mathcal{N})$

[3]a solution u_0 of (1), (2) with $f = f_0$ is called nondegenerate if for f close to f_0 the equation (1), (2) has the only solution u close to u_0 and u smoothly depends on f.

[4]but the dimension is always zero if the bundle $T\mathfrak{M}$ is trivial, e.g. if $\mathfrak{M}$ is a torus - see in [6]

Example 1. (see Part 5.1). Let $\mathfrak{M}$ be the sphere S^2 of unit area. Then $V = 1$, $\chi(\mathfrak{M}) = 2$ and due to the theorem the equation (1), (2) with $\|f\| < 1$ is solvable and for typical f has at least two nondegenerate solutions. If $f = 0$, then the solutions are elliptic functions. So the theorem provides us with two contractible "nonlinear elliptic functions".

We can identify S^2 with $\mathbb{C} \cup \infty$ and write (1) using the chart $\mathbb{C} \subset S^2$:

$$u_{\bar{z}}(z) + f(z,u) = 0,\ u \in \mathbb{C} \cup \infty,\ z \in \mathbb{T}^2. \tag{5}$$

Solutions of (5) with $f(z,u)$ linear in u, $\bar{u}$ have been intensively studied as "generalized analytic functions" (see in [15]).

Example 2. (see Part 5.2, cf. [11, 6]). Let us choose $\mathfrak{M} = \mathbb{T}^2$. Now $\mathfrak{M}$ is aspherical, so $V = \infty$. Each solution of (1), (2) can be lifted to a solution $U(z)$ of the equation

$$U_{\bar{z}} + f(z,U) = 0, \quad z \in \mathbb{T}^2, \quad U \in \mathbb{C}, \tag{6}$$

with double-periodic in z and U function f. Since $\chi(\mathbb{T}^2) = 0$, then the theorem does not guarantee solvability of (6). To see that an equation without solutions exists we take $f \equiv -1$. Then (6) takes the form $U_{\bar{z}} = 1$. This equation can not be solved since the mean value of its r.h.s. is nonzero. Still Theorem 1 can be used to find a nontrivial solution given a nondegenerate trivial one, say $u \equiv 0$ – see the main text.

Example 3. If $\mathfrak{M}$ is a Riemann surface of genus $g \geq 2$, then $\chi(\mathfrak{M}) = 2 - 2g$ and $\mathfrak{M}$ can be holomorphically covered by an open complex disc. So $\mathfrak{M}$ is aspherical, for each f equation (1), (2) has a solution and for typical f it has at least $2g - 2$ nondegenerate solutions.

Example 4. (see Part 5.3) Let us take $\mathfrak{M} = \mathbb{C}P^n$ with the normalized Fubini-Studi metric [2, Appendix 3]. Now $V = 1$ and $\chi(\mathfrak{M}) = n + 1$. So if $f(z,u)$ is a nonautonomous vectorfield on $\mathbb{C}P^n$ such that $\|f(z,u)\|_u < 1$ for all z, u, then the equation $u_{\bar{z}} + f(z,u) = 0,\ z \in \mathbb{T}^2,\ u \in \mathbb{C}P^n$, has a solution. For a typical f all solutions are different and there are at least $n+1$ of them. This statement admits a "deprojectivization". Take any one-homogeneous (in ξ) nonautonomous vectorfield $F(z,\xi)$ in $\mathbb{C}^{n+1}$ such that

$$|F(z,\xi)|^2|\xi|^2 - |\langle F(z,\xi), \xi\rangle|^2 < \pi|\xi|^4$$

for all z, ξ. Then the "nonlinear eigenvalue problem"

$$U_{\bar{z}} + F(z,U) = \hat{\lambda}U, \quad U(z) \in \mathbb{C}^{n+1}, \quad \hat{\lambda} = \mathrm{const} \in \mathbb{C},$$

has a family of solutions $U = Ce^{2\pi i(mx+ny)}U_0(x+iy)$, $\hat{\lambda} = \hat{\lambda}_0 + \pi(im - n)$, where C is a nonzero complex constant and $m, n \in \mathbb{N}$. For a typical F the number of different families is finite and is at least $n + 1$.

Proof of Theorem 1 essentially is based on continuation of solutions of an equation (1) with f equal to some distinguished f_0 : we deform f_0 to given f_1 and use Gromov's compactness to prove that the solutions can not go to infinity during the deformation.

To realize this scheme we study the set $\mathfrak{A}$ formed by all pairs $(u(z),\, f(z,u))$ which satisfy (1), (2) with $f \in \mathcal{F}$, where $\mathcal{F}$ is a Hilbert space of "sufficiently smooth" vectorfields f. We prove that $\mathfrak{A}$ is a Hilbert manifold modeled by the space $\mathcal{F}$ and that the projection-map π,

$$\pi : \mathfrak{A} \ni (u, f) \mapsto f \in \mathcal{F},$$

is zero-index Fredholm. Moreover, following [16] we show that the manifold $\mathfrak{A}$ is quasi finite-dimensional: it admits an atlas formed by charts with transition maps of the form

$$\langle \text{finite-dimensional map} \rangle \;+\; \langle \text{shift} \rangle, \tag{7}$$

where a map of $\mathcal{F}$ to itself is called finite-dimensional if in decomposition in some *fixed* basis of $\mathcal{F}$ only finite number of its component does not vanish. In each chart of $\mathfrak{A}$ as above the projection map π also has the form (7).

Besides, as we show in Part 3, the manifold $\mathfrak{A}$ carries a natural orientation. The task to orient $\mathfrak{A}$ is simplified by the quasi finite-dimensional structure.

We wish to define a degree of the map π. Since $\mathfrak{A}$ is not compact, we should first check that the map π is proper. An example constructed in Proposition 2.1 shows that it is not the case since a break of compactness for the multi-valued inverse map $\pi^{-1} : \mathcal{F} \to \mathfrak{A}$ can happen due to the bubbling-of phenomenon. However in Lemma 2.1 we show (following [4, 9]) that the map π becomes proper if we replace $\mathcal{F}$ by the subset $\mathcal{F}_V$ of bounded vectorfields,

$$\mathcal{F}_V = \{ f(z,u) \in \mathcal{F} \mid \|f(z,u)\|_u < \sqrt{V} \quad \forall z, u \}$$

(V was defined in (4)) and replace $\mathfrak{A}$ by $\mathfrak{A}_V = \pi^{-1}\mathcal{F}_V \subset \mathfrak{A}$.

The degree $\deg(\pi)$ of $\pi\,|_{\mathfrak{A}_V}$ is well-defined. To calculate it we in Part 4 choose in (1) $f = \varepsilon f_0(u)$, where f_0 is an autonomous vectorfield on $\mathfrak{M}$ with nondegenerate singular points $u_1, \dots, u_m$, $m = |\chi(\mathfrak{M})|$. We prove that if positive ε is sufficiently small, then the only solutions of (1), (2) are the constants $u(z) \equiv u_j,\ j = 1, \dots, m$. Besides, the index of the linearization π_* at the point $(u_j, \varepsilon f_0) \in \mathfrak{A}_V$ agrees with the index of u_j as a singular point of the vectorfield f_0 (equal to sgn $\chi(\mathfrak{M})$ since we have exactly $|\chi(\mathfrak{M})|$ nondegenerate singular points). Therefore

$$\deg\,(\pi) = \chi(\mathfrak{M}). \tag{8}$$

As the map $\pi\,|_{\mathfrak{A}_V}$ is proper and regular values of π are dense (due to Sard-Smale), then (8) implies the assertions of Theorem 1.

Study of nonlinear Cauchy-Riemann equations for maps valued in a manifold was initiated by M.Gromov in his pioneer work [4], which is mostly concerned with pseudoholomorphic spheres in an almost complex manifold (M, J) (J is an almost complex structure on M). Among other important results it is proven in [4] that if $M = S^2 \times N$, where the manifold N possesses some special properties, then for each point $m_0 \in M$ and each almost complex structure J which is tamed by a symplectic form on M (for definitions see Part 1), one can find a pseudoholomorphic sphere $u : S^2 \to (M, J)$ which is homotopic to $S^2 \times \{pt\}$ and contains m_0. In the language we use in this work, the proof of [4] can be described as follows. All pairs

$$\big(\langle \text{pseudoholomorphic sphere } u(S^2) \text{ in } (M,J) \text{ which contains } m_0 \rangle,\\ \langle \text{tamed almost complex structure } J \rangle\big)$$

form a Hilbert manifold $\mathfrak{A}^{\rm sph}$ such that the projection $\pi^{\rm sph} : \mathfrak{A}^{\rm sph} \ni (u, J) \mapsto J$ is a zero-index Fredholm map. Under the restrictions imposed on the manifold M, one can find a distinguished structure J_0 such that $(\pi^{\rm sph})^{-1} J_0$ contains the only point and J_0 is a regular value of $\pi^{\rm sph}$. Therefore the degree modulo 2 of $\pi^{\rm sph}$ at J_0 equals 1. The map $\pi^{\rm sph}$ is proper; so the global degree modulo 2, $\deg_2 \pi^{\rm sph}$, is well-defined (see [14]) and also equals 1. Hence, the map $\pi^{\rm sph}$ is onto.

For the problem we study in this paper, $\deg_2$ vanishes for many important manifolds $\mathfrak{M}$ (e.g., for all compact Riemann surfaces). Therefore we orient $\mathfrak{A}$ and use sharper arguments based on the "absolute" degree.

The approach to study a manifold formed by pairs

$$\big(\langle \text{solution of a nonlinear elliptic equation} \rangle, \langle \text{parameters of the equation} \rangle\big)$$

and its projection π to the parameter-space we develop in this paper is rather general. We hope it can be used to study other quasilinear elliptic equations for maps valued in a compact manifold (cf. Part 6.3 below).

When a preprint of this paper has appeared we were informed by L. Polterovich about the paper [3] where – among other results – pseudoholomorphic tori in tamed almost complex manifolds are studied and the number of different nondegenerate tori is estimated from below via the Euler characteristic. Most likely our results are related to the results of [3], although quite different goals and techniques of the two works rather complicate their comparison.

1. Geometric preliminaries

We recall that a complex n-dimensional manifold $\mathfrak{M}$ is called Kählerian if it admits an Hermitian structure $\langle \cdot, \cdot \rangle$ such that the 2-form $\omega_2(\xi, \eta) = -\mathrm{Im}\langle \xi, \eta \rangle, \xi, \eta \in T_m\mathfrak{M}, m \in \mathfrak{M}$ is closed nondegenerate.[5] The form ω_2 provides $\mathfrak{M}$ with a structure of a symplectic manifold. We denote $\omega_2^n = \omega_2 \wedge \cdots \wedge \omega_2$ and say that Kählerian $\mathfrak{M}$ is *normalized* if it is compact connected and $\int_{\mathfrak{M}} \omega_2^n = 1$. Simple calculations show that if D is a complex domain and $u : D \to \mathfrak{M}$ is a holomorphic map, then

$$\langle \omega_2, u(D) \rangle \overset{\text{def}}{=} \int_D u^* \omega_2 = \mathrm{Area}\, u(D), \tag{1.1}$$

where the area is calculated with respect to the Riemann metric $(\cdot, \cdot) = \mathrm{Re}\, \langle \cdot, \cdot \rangle$ on $\mathfrak{M}$, called the Kählerian metric.

Example 1.1. Any Riemann surface is Kählerian since for $n = 1$ the form ω_2 is closed as a two-form on a two-dimensional manifold.

Next we take any even-dimensional manifold M. It is called *almost complex* if each tangent space T_mM is given a complex structure by means of a linear operator J_m, $J_m^2 = -I$, which smoothly depends on m. Clearly each complex manifold is almost complex. Any symplectic manifold (M, ω_2) also can be given an almost complex structure:

Example 1.2. Let (M, ω_2) be a symplectic manifold. We fix any smooth Riemann metric $(\cdot, \cdot)$ on M and write the symplectic form ω_2 as $\omega_2 = (\Upsilon \cdot, \cdot)$, where $\Upsilon = \Upsilon_m$ is a linear operator in T_mM, antisymmetric with respect to the scalar product $(\cdot, \cdot) = (\cdot, \cdot)_m$ in T_mM. We define J_m as $J_m = \Upsilon_m / \sqrt{\Upsilon_m \Upsilon_m^*} = \Upsilon_m / \sqrt{-\Upsilon_m^2}$. This operator is well-defined and $J_m^2 = -I$.

[5] this form is always nondegenerate. So only its closedness is a restriction.

Example 1.3. Take the complex 2-torus $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, a Kählerian manifold $\mathfrak{M}$ and define M as $M = \mathbb{T}^2 \times \mathfrak{M}$. Denote by Π the natural projection $\Pi : M \to \mathfrak{M}$ and take any smooth section f of the bundle $\Pi^* T\mathfrak{M}$. It can be treated as a "nonautonomous" (z-dependent) vectorfield on $\mathfrak{M}$. Now for $(z, u) \in M$ we have $T_{(z,u)}M \simeq \mathbb{C} \times T_u\mathfrak{M}$. Following [4], we define the operator $J^f_{(z,u)}$ in $T_{(z,u)}M$ as follows:

$$J^f_{(z,u)}(\xi, \eta) = (i\xi,\, i\eta - 2if(z, u)\bar{\xi}).$$

Clearly $\left(J^f\right)^2 = -I$. So (M, J^f) is an almost complex manifold.

Take a Riemann surface Σ. A map $\tilde{u} : \Sigma \to (M, J)$ is called pseudoholomorphic (and $\tilde{u}(\Sigma) \subset (M, J)$ is called a pseudoholomorphic curve) if its linearization agrees the complex structures of $T\Sigma$ and TM:

$$\tilde{u}(z)_*(i\xi) = J(\tilde{u}(z))\tilde{u}(z)_*\xi \quad \forall\, (z, \xi) \in T\Sigma. \tag{1.2}$$

We recall that each complex 2-torus is biholomorphically equivalent to a torus $T^2_\nu = \mathbb{C}/(\mathbb{Z} + \nu\mathbb{Z})$ where ν is the uniquely defined element of the fundamental domain for the action of the modular group on the complex upper half-plane [13]. If $z = x + iy$ is the natural complex coordinate on T^2_ν, then the relation (1.2) takes the form of the nonlinear Cauchy-Riemann equation

$$\frac{\partial \tilde{u}}{\partial x} + J(\tilde{u})\frac{\partial \tilde{u}}{\partial y} = 0, \quad \tilde{u} \in M, \tag{1.3}$$

since for $\xi = i$ one has $\tilde{u}_*(i) = \partial\tilde{u}/\partial y$, $\tilde{u}_*(i^2) = -\tilde{u}_*(1) = \partial\tilde{u}/\partial x$.

In particular, if (M, J^f) is the almost complex manifold from Example 1.3, then a map $z \mapsto \tilde{u}(z) = (v(z), u(z)) \in M = \mathbb{T}^2 \times \mathfrak{M}$ is pseudoholomorphic if

$$\frac{\partial v}{\partial \bar{z}} = 0, \qquad \frac{\partial u}{\partial \bar{z}} = if(z, u)v_y. \tag{1.4}$$

Now suppose that

$$\tilde{u} \quad \text{is homotopic to a map} \quad \mathbb{T}^2 \to M,\ z \mapsto (z, \zeta(z)), \tag{1.5}$$

where $\zeta : \mathbb{T}^2 \to \mathfrak{M}$ is fixed. Then due to the first equation in (1.4) the map v is a biholomorphic isomorphism $T^2_\nu \underset{\sim}{\longrightarrow} \mathbb{T}^2$. Therefore $T^2_\nu = \mathbb{T}^2$, v is a translation, $v(z) = z + z_0$, and the pseudoholomorphic torus $\tilde{u}(T^2_\nu) \subset M$ equals the graph of the map $u^0(z) = u(z - z_0)$. Finally using the second equation in (1.4) we get

Proposition 1.1. *Each pseudoholomorphic torus $\tilde{u}(T^2_\nu)$ in (M, J^f) satisfying (1.5) is the graph of a solution $u(z)$ of the quasilinear Cauchy-Riemann equation*

$$\frac{\partial u}{\partial \bar{z}} + f(z, u) = 0. \tag{1.6}$$

The map $u : \mathbb{T}^2 \to \mathfrak{M}$ is homotopic to ζ.

This result forms a bridge between the nonlinear and quasilinear Cauchy-Riemann equations (1.3) and (1.6).

Definition 1.1. An almost complex structure J on M is tamed by a symplectic structure ω_2 if $\omega_2(\xi, J\xi) > 0$ for each nonzero $\xi \in TM$.

Example 1.4. For a Kählerian manifold $\mathfrak{M}$, $\omega_2(\xi, J\eta)$ is the Kählerian metric $\operatorname{Re}\langle\xi,\eta\rangle$. So the complex structure of $\mathfrak{M}$ is tamed by its symplectic structure.

Now let $\mathbb{T}^2$ and $\mathfrak{M}$ be normalized Kählerian manifolds and $(M = \mathbb{T}^2 \times \mathfrak{M},\, J^f)$ be the almost complex manifold from Example 1.3. Following [4] we supply M with the symplectic structure $\omega_2 = \mu^2\omega_2^{(1)} \oplus \omega_2^{(2)}$ where $\mu^2 > 0$ and $\omega_2^{(1)}$, $\omega_2^{(2)}$ are the normalized Kählerian forms on $\mathbb{T}^2$ and $\mathfrak{M}$. For $(\xi,\eta) \in T_{(z,u)}M \backsimeq \mathbb{C} \times T_u\mathfrak{M}$ we have

$$\omega_2\left((\xi,\eta),\, J^f(\xi,\eta)\right) \geq \mu^2|\xi|^2 + \|\eta\|^2 - 2|\xi|\,\|\eta\|\sup\|f\| \geq \varepsilon\left(|\xi|^2 + \|\eta\|^2\right)$$

with some $\varepsilon > 0$ if $\sup\|f\| < \mu$. Therefore we have proved

Lemma 1.1. *The almost complex structure J^f is tamed by the symplectic form $\omega_2 = \mu^2\omega_2^{(1)} \oplus \omega_2^{(2)}$ if $\|f(z,u)\|_u < \mu$ for each z and u.*

If $\mathcal{O}$ is a complex domain, (M,J) is an almost complex manifold with J tamed by ω_2 and $v : \mathcal{O} \to M$ is a pseudoholomorphic map then

$$\int_{v(\mathcal{O})} \omega_2 = \int_{\mathcal{O}} \omega_2(v_x, v_y)\,dxdy = \int_{\mathcal{O}} \omega_2(v_x, Jv_x)\,dxdy \geq C^{-1}\int_{\mathcal{O}} |v_x|^2\,dxdy.$$

Similar $\int \omega_2 \geq C^{-1}\int |v_y|^2$ and

$$\int_{v(\mathcal{O})} \omega_2 \geq C_1^{-1}\int_{\mathcal{O}} |\nabla v|^2\,dxdy \geq C_2^{-1}\operatorname{Area} v(\mathcal{O}).$$

In particular, symplectic area of any pseudoholomorphic curve in a tamed almost complex manifold is nonnegative[6] and symplectic area of a piece of a pseudoholomorphic curve is no larger than the area of the whole curve (cf. (1.1)).

2. The infinite dimensional moduli space $\mathfrak{A}_V$

Let $\mathfrak{M}$ be any normalized Kählerian manifold. We denote by V the minimal area (=minimal symplectic area) of a nontrivial holomorphic spheroid in $\mathfrak{M}$,

$$V := \inf\{\langle\omega_2, u(S^2)\rangle \mid u : S^2 \to \mathfrak{M} \text{ is holomorphic nontrivial}\}. \tag{2.1}$$

As usual, infimum over the empty set equals infinity. So $V = \infty$ if $\mathfrak{M}$ is aspherical.

The number V is always positive. Indeed, $\mathfrak{M}$ can be covered by a finite system of charts in such a way that each point in $\mathfrak{M}$ lies in some chart with its γ-neighborhood (measured in the chart's metric). Here $\gamma > 0$ does not depend on the point. Now let us take any nontrivial holomorphic spheroid, any point on it and the chart as above. Due to the well-known isoperimetric inequality (see [4, p.335, 9, p.333]) the area of the holomorphic disc given by the intersection of the spheroid with the γ-ball is at least $\pi\gamma^2$ (again the area is measured in the chart's metric). So $V \geq c\gamma^2$ with some $c > 0$.

[6]in fact - positive

Examples. 2.1. If $\mathfrak{M}$ is a compact Riemann surface of a genus $g \geq 0$, then $V = 1$ if $g = 0$ (i.e., if $\mathfrak{M} = S^2$) and $V = \infty$ if $g > 0$.

2.2 If $\mathfrak{M} = \mathbb{C}P^n$ with the normalized Fubini–Studi metric (see in [2]), then $V = 1$. Indeed, the embedding $\mathbb{C}P^1 \to \mathbb{C}P^n$, $(z_0 : z_1) \mapsto (z_0 : z_1 : 0 : \cdots : 0)$ defines an isomorphism of $\pi_2(\mathbb{C}P^1) = \mathbb{Z}$ and $\pi_2(\mathbb{C}P^n)$. As this map is symplectic, then the result for $n \geq 2$ follows from the previous example with $\mathfrak{M} = S^2 = \mathbb{C}P^1$.

In this part we start to study the manifold formed by all pairs $(u(z), f(z,u))$ such that

$$u_{\bar{z}} + f(z,u) = 0, \quad u : \mathbb{T}^2 \to \mathfrak{M} \text{ is contractible,} \tag{2.2}$$

and the projection π which sends a (u, f) from the manifold to f.

Remark 2.1. Below we suppose that the complex manifold $\mathfrak{M}$ is Kählerian and compact. Nevertheless our proofs run without changes if we suppose that $\mathfrak{M}$ is a compact complex manifold with the complex structure tamed by some symplectic form $\omega_2^{(2)}$. In this case we can find $K > 0$ and some Riemann structure $(\cdot,\cdot)$ on $\mathfrak{M}$ such that

$$\omega_2^{(2)}(\xi, i\xi) \geq \|\xi\|^2, \;\; |\omega_2^{(2)}(\xi, z\eta)| \leq K|z|\|\xi\|\,\|\eta\|$$

for all $z \in \mathbb{C}$, $\xi, \eta \in T_m\mathfrak{M}$, $m \in \mathfrak{M}$. Now the almost complex structure J^f on $M = \mathbb{T}^2 \times \mathfrak{M}$ is tamed by the form $\mu^2\omega_2^{(1)} \oplus \omega_2^{(2)}$ if $\|f\| < K^{-1}\mu$, and all arguments in our proofs are valid if we replace V as in (2.1) by

$$V = K^{-1}\inf\{\langle\omega_2^{(2)}, u(S^2)\rangle \mid u : S^2 \to \mathfrak{M} \text{ is holomorphic nontrivial}\}. \quad \square$$

Let us denote

$$W = \{u \in H^3(\mathbb{T}^2; \mathfrak{M}) \mid u \text{ is contractible to the map } \mathbb{T}^2 \to m_0\},$$

where m_0 is a fixed point in $\mathfrak{M}$ and – as usual – H^m is the Sobolev space $W^{m,2}$ (see Appendix 1.1). The tangent space T_uW equals to the space $H^3(u^*T\mathfrak{M})$ of H^3-smooth sections of the bundle $u^*T\mathfrak{M}$ (see [7, Part 1.2], [10] and below). Since u is contractible, then $u^*T\mathfrak{M}$ is diffeomorphic to a trivial linear bundle and $T_uW \simeq H^3(\mathbb{T}^2, \mathbb{C}^n)$. Therefore W is a Hilbert manifold, modeled by the space

$$H^3(\mathbb{T}^2, \mathbb{C}^n).$$

We supply W with a metric, compatible with its smooth structure (see Appendix 1.1). We normalize the metric in such a way that for any u, v in W we have

$$\operatorname{dist}(u,v) \geq \sup_z \operatorname{dist}_K(u(z), v(z)),$$

where dist_K stands for the Kählerian distance in $\mathfrak{M}$.

Take any integer $\tilde{m}$ such that

$$\tilde{m} \geq n + 6,$$

and denote by $\mathcal{F}$ the Hilbert space formed by $H^{\tilde{m}}$-smooth nonautonomous vector-fields $f(z,u)$, i.e., by $H^{\tilde{m}}$-smooth sections of the bundle $\Pi^*T\mathfrak{M}$ where $\Pi : \mathbb{T}^2 \times \mathfrak{M} \to$

$\mathfrak{M}$ is the natural projection. We fix any Hilbert basis $\{\psi_j\}$ of $\mathcal{F} = H^{\tilde{m}}(\Pi^* T\mathfrak{M})$ and denote

$$\mathcal{F}^N = \operatorname{span}(\psi_1, \dots, \psi_N), \quad {}^N\mathcal{F} = \overline{\operatorname{span}}(\psi_{N+1}, \psi_{N+2}, \dots)$$

(bar means closure). Let $\mathcal{O}$ be an open domain in $\mathcal{F}$. We call a map $G : \mathcal{O} \to \mathcal{F}$ *finite-dimensional* (with respect to the basis $\{\psi_j\}$) if

$$G(\mathcal{O}) \subset \mathcal{F}^N \quad \text{for some } N = N(G). \tag{2.3}$$

Maps of the form

$$\mathcal{O} \ni f \mapsto G(f) + f_0, \quad G \text{ is finite-dimensional, } f_0 \in \mathcal{F} \text{ is a fixed vector,} \tag{2.4}$$

are called *quasi finite-dimensional.*

Let us define $\mathcal{F}_V \subset \mathcal{F}$ as

$$\mathcal{F}_V = \{f \in \mathcal{F} \mid \|f(z,u)\|_u < \sqrt{V} \quad \forall z, u\}$$

(here $\|\cdot\|_u$ is the Kählerian norm in $T_u\mathfrak{M}$). In particular, $\mathcal{F}_V = \mathcal{F}$ if $\mathfrak{M}$ is aspherical.

Example 2.3. Let $\mathfrak{M}$ be the normalized sphere S^2 (i.e., $\langle \omega_2, S^2 \rangle = 1$). We treat S^2 as the Riemann sphere, $S^2 = \mathbb{C} \cup \infty$, and write the equation (2.2) using the chart $\mathbb{C} \subset S^2$:

$$u_{\bar{z}}(z) + f(z, u(z)) = 0, \quad u \in \mathbb{C} \cup \infty.$$

Now $V = 1$. As the normalized Kählerian metric in $T_u S^2$, $u \in \mathbb{C}$, is the Euclidean metric divided by $\pi(1+|u|^2)^2$ (see [2, Appendix 3]), then

$$\mathcal{F}_V = \left\{ f(z,u) \in H^{\tilde{m}}(\mathbb{T}^2 \times \mathbb{C}, \mathbb{C}) \mid \sup_{z,u} \frac{|f(z,u)|}{1+|u|^2} < \sqrt{\pi} \right\}. \tag{2.5}$$

Now we consider the map Φ defined by the l.h.s. of (2.2) with f in $\mathcal{F}_V$:

$$\Phi : W \times \mathcal{F}_V \ni (u, f) \mapsto u_{\bar{z}} + f(z,u) \in H^2(u^* T\mathfrak{M}),$$

denote by $\mathfrak{A}_V$ its zero-set, $\mathfrak{A}_V = \Phi^{-1}(0)$ (where 0 is the zero section of $u^*T\mathfrak{M}$) and denote by π the natural projection

$$\pi : \mathfrak{A}_V \to \mathcal{F}_V, \quad (u, f) \to f.$$

As $\tilde{m}$ is at least $n+6$, then $\mathcal{F} \subset C^4(\Pi^* T\mathfrak{M})$ by the Sobolev theorem (see Appendix 1.1) and

$$\mathfrak{A}_V \subset H^5(\mathbb{T}^2, \mathfrak{M}) \times C^4(\Pi^* T\mathfrak{M}) \tag{2.6}$$

by the regularity theory for first-order elliptic equations (Appendix 1.3).

Lemma 2.1. *The map π is proper.*

Proof (following [4,9]). We should check that if $(u_j, f_j) \in \mathfrak{A}_V$ and $f_j \to f_0 \in \mathcal{F}_V$, then for a subsequence one has $u_{j_m} \to u_0 \in W$, where $(u_0, f_0) \in \mathfrak{A}_V$.

Suppose we can prove that the maps u_j are uniformly bounded in the C^1-topology. As they satisfy (2.2) with $f = f_j$ converging to $f_0 \in \mathcal{F}$ and since $\mathcal{F} \subset C^4$, then the maps are uniformly bounded in H^5 by the elliptic regularity (see in Appendix 1.3). So a subsequence $\{u_{j_m}\}$ converges to a solution of the limiting equation with $f = f_0$ and the assertion follows.

We prove the C^1-boundedness by a contradiction and suppose that C^1-norms of some subsequence tend to infinity. We re-denote the subsequence as $\{u_j\}$ and write this assumption as

$$\|u_j(z_j)_*\| = \sup_z \|u_j(z)_*\| = C_j \to \infty. \qquad (*)$$

We can suppose that $z_j \to z_0 \in \mathbb{T}^2$. The shifted maps $u_j(z + z_j)$ satisfy (2.2) with $f_j(z, u)$ replaced by $f_j(z+z_j, u)$. The shifted vectorfields $f_j(z+z_j, u)$ also converge in $\mathcal{F}_V$. So we can replace u_j, f_j by the shifted objects and suppose that in (*) $z_j \equiv 0$.

Now we rescale z-variable near $0 \in \mathbb{T}^2$ by means of the substitution $z = \psi_j(\mathfrak{z})$,

$$\psi_j : D_j := \{\mathfrak{z} \in \mathbb{C} \mid |\mathfrak{z}| < \sqrt{C_j}\} \to \mathbb{T}^2, \;\; \mathfrak{z} \mapsto \mathfrak{z}/C_j,$$

and consider the maps v_j,

$$v_j = u_j \circ \psi_j : D_j \to \mathfrak{M}.$$

We have $\|v_j(0)_*\| = 1, \; \|v_j(\mathfrak{z})_*\| \le 1$ for each $\mathfrak{z} \in D_j$ and

$$\frac{\partial}{\partial \bar{\mathfrak{z}}} v_j = C_j^{-1} \frac{\partial}{\partial \bar z} u_j(\psi_j(\mathfrak{z})) = C_j^{-1} f_j\left(\frac{\mathfrak{z}}{C_j}, v_j\right) =: g_j(\mathfrak{z}, v_j).$$

Take any $m \in \mathbb{N}$. For $j = m, m+1, \dots$ the maps $v_j : D_m \to \mathfrak{M}$ are uniformly bounded in the C^1-norm. Since g_j's converge to zero in $C^1(\Pi^* T\mathfrak{M})$ as j grows, then using again the elliptic regularity theory we find a subsequence $\{v_{j_n}\}$ converging to some $v_0 \in C^1(D_m, \mathfrak{M})$ in the C^1-topology, where $(\partial/\partial\bar z)v_0 = 0$ in D_m. We can repeat these arguments for $m = m+1, m+2, \dots$ to obtain a sparse subsequence $\{v_{j_r}\}$ converging to a holomorphic map $v : \mathbb{C} \to \mathfrak{M}$ uniformly on compacts in $\mathbb{C}$.

For a map $g(z)$ we denote by $\tilde g(z)$ its graph, $\tilde g(z) = (z, g(z))$, take any $K \subset D_n \subset \mathbb{C}$ and for $j_m \ge n$ consider the curves

$$\begin{aligned}\Gamma_{j_m}(K) := \tilde u_{j_m}(K/C_{j_m}) = \tilde u_{j_m}(\psi_{j_m}(K)) = \\ \{(\mathfrak{z}/C_{j_m}, v_{j_m}(\mathfrak{z})) \mid \mathfrak{z} \in K\} \subset M = \mathbb{T}^2 \times \mathfrak{M}.\end{aligned}$$

For any compact K these curves C^1-converge to $\Gamma(K) := \{0\} \times v(K) \subset M$.

As $f_j \to f \in \mathcal{F}_V$ then there exists $\mu^2 < V$ such that $\|f_j(z,u)\|_u \le \mu$ for all j and all z, u. By Lemma 1.1 the form $\omega_2 = \mu^2 \omega_2^{(1)} \oplus \omega_2^{(2)}$ tames the structures $J^0, J^1, \dots$. Therefore symplectic area of each piece of $\tilde u_n(\mathbb{T}^2)$ is nonnegative (see (1.7)) and

$$\langle \omega_2, \Gamma_{j_m}(K) \rangle \le \langle \omega_2, \tilde u_{j_m}(\mathbb{T}^2) \rangle = \mu^2,$$

where the equality follows from a homotopy of the torus $\tilde{u}_{j_m}(\mathbb{T}^2)$ and $\mathbb{T}^2 \times \{pt\}$.

Thus, for each $K \Subset \mathbb{C}$,

$$0 < \langle \omega_2, \Gamma(K) \rangle = \langle \omega_2^{(2)}, v(K) \rangle \leq \mu^2$$

(the area is strictly positive since the map v is not trivial: $\|v_*(0)\| = 1$). As the holomorphic map $v : \mathbb{C} = S^2 \backslash \infty \to \mathfrak{M}$ has a finite area, then it can be extended to a continuous map $\hat{v} : S^2 \to \mathfrak{M}$, $\hat{v} = v$ in $\mathbb{C} \subset S^2$ (see e.g., [9, p.330]). Take any chart $\mathcal{O} \ni \hat{v}(\infty)$ in $\mathfrak{M}$. The map $\hat{v}$ sends a small disc in S^2 centered at ∞ to $\mathcal{O} \Subset \mathbb{C}^n$ and is holomorphic outside ∞. Therefore it is holomorphic in the whole disc and $\hat{v}$ defines a nontrivial holomorphic spheroid in $\mathfrak{M}$ such that $0 < \langle \omega_2^{(2)}, \hat{v}(S^2) \rangle \leq \mu^2 < V$. This contradicts the choice (2.1) of V and proves that (*) is impossible. $\square$

The set $\mathfrak{A}_V$ is a subset of $\mathfrak{A}$, where $\mathfrak{A}$ is formed by all pairs $(u, f) \subset W \times \mathcal{F}$ which satisfy (2.2). Accordingly π is the restriction to $\mathfrak{A}_V$ of the projection $\tilde{\pi} : \mathfrak{A} \to \mathcal{F}$. It is interesting that the statement of Lemma 2.1 becomes false if we replace $\mathfrak{A}_V$ by the larger set $\mathfrak{A}$. Indeed, in Appendix 3 we prove

Proposition 2.1. *If $\mathfrak{M} = S^2$ and $\mathfrak{A}$ is defined as above, then the map $\tilde{\pi} : \mathfrak{A} \to \mathcal{F}$ is not proper.*

Our next goal is to supply the set $\mathfrak{A}_V$ with a structure of a quasi finite-dimensional Hilbert manifold, where

Definition 2.1 ([16]). A C^r-smooth Hilbert manifold modeled by the space $\mathcal{F}$ is called *quasi finite-dimensional* (with respect to the basis $\{\psi_j\}$) if it admits an atlas formed by charts with C^r-smooth quasi finite-dimensional transition maps (see (2.4)).

The main difficulty which complicates construction of the smooth structure on $\mathfrak{A}_V$ is that $\Phi(u, f)$ is a section of the u-dependent bundle $u^* T\mathfrak{M}$. To handle with this obstacle we shall construct locally near arbitrary point $(\hat{u}, \hat{f}) \in \mathfrak{A}_V$ an isomorphism L_u of $u^* T\mathfrak{M}$ and the trivial bundle $\mathbb{T}^2 \times \mathbb{C}^n$. After this we shall replace the equation $\Phi = 0$ by $\tilde{\Phi} := L_u \circ \Phi(u, f) = 0$ and shall study $\mathfrak{A}_V$ as a zero-set of the map $\tilde{\Phi}$ (locally near $(\hat{u}, \hat{f})$).

Fix any complex connection on $T\mathfrak{M}$ [8, Chapter IX]. Then given any piecewise smooth path $\gamma : [0, 1] \to \mathfrak{M}$ we get a complex linear isomorphism of $T_{\gamma(1)}\mathfrak{M}$ and $T_{\gamma(0)}\mathfrak{M}$, smoothly depending on γ. If we take $u \in W$ and fix a homotopy $u_{(t)}(z)$ of $u_{(0)}(z) = u(z)$ and $u_{(1)}(z) \equiv m_0$, then we get a complex-linear isomorphism

$$\mathcal{U} : T_{u(z)}\mathfrak{M} \underset{\sim}{\longrightarrow} T_{m_0}\mathfrak{M}.$$

Now we shall show that the homotopy $u_{(t)}(z)$ can be chosen in such a way that the operator $\mathcal{U}$ is a smooth function of z and $u = u(z)$. Let us take any smooth $\hat{u} \in W$, and fix a smooth homotopy of $\hat{u}$ and the map $u_{(1)}(z) \equiv m_0$. We fix $\delta > 0$ so small that each point from the δ-neighborhood of $\hat{u}(z)$ in $\mathfrak{M}$ is connected with $\hat{u}(z)$ by the unique minimal geodesic (corresponding to the Kählerian metric). So for all $u \in W$, δ-close to $\hat{u}$, we get a homotopy of u and $u_{(1)}$, connecting $u(z)$ with $\hat{u}(z)$ by the minimal geodesic and then – with m_0 by the fixed homotopy. Now

$$\mathcal{U} = \mathcal{U}(\hat{u}(z), u(z)) = \mathcal{U}(\hat{u}(z), u) \mid_{u=u(z)}, \tag{2.7}$$

where

$$\text{the map } \mathcal{U} = \mathcal{U}(\hat{u}(z), u) \text{ is smooth in } z \text{ and in } u,\ \delta\text{-close to } \hat{u}(z). \tag{2.8}$$

We choose $L_u = \mathcal{U}$ and replace the map Φ by $\tilde{\Phi}$, where

$$\begin{aligned}\tilde{\Phi}(u,f)(z) = \mathcal{U}(\hat{u}(z),u(z)) \circ \Phi(u,f)(z) = &\mathcal{U}(\hat{u}(z),u(z)) \circ u_{\bar{z}} + \\ &+ \mathcal{U}(\hat{u}(z),u(z)) \circ f(z,u) \in H^2(u_1^* T\mathfrak{M}) \simeq H^2(\mathbb{T}^2;\mathbb{C}^n),\end{aligned}$$

and $u_1(z) \equiv m_0$. In Appendix 1.4 we prove that

$$\text{the map } \ \tilde{\Phi} : W \times \mathcal{F}_V \to H^2(\mathbb{T}^2;\mathbb{C}^n) \ \text{ is } \ C^2\text{-smooth}$$

(we abuse the notation: the map $\tilde{\Phi}$ is defined in a neighborhood of $(\hat{u}, \hat{f})$ in $W \times \mathcal{F}_V$, not everywhere).

Consider the linearization $\tilde{\Phi}'_u(\xi)$ of $\tilde{\Phi}$ in u at a point $\xi = (u,f) \in \mathfrak{A}_V$ and define $\Psi = \Psi_{(\xi)}$ as $\Psi = \tilde{\Phi}'_u(\xi) \circ \mathcal{U}(\hat{u}(z),u(z))^{-1}$. We have the commutative diagram

$$\begin{array}{ccc} H^3(u^* T\mathfrak{M}) \ni v & \xrightarrow{\tilde{\Phi}'_u} & H^2(u_1^* T\mathfrak{M}) \simeq H^2(\mathbb{T}^2;\mathbb{C}^n)\,, \\ \Big\downarrow{\scriptstyle \mathcal{U}(\hat{u}(z),u(z))} & & \Psi \\ H^3(\mathbb{T}^2,\mathbb{C}^n) \ni w & & \end{array}$$

Lemma 2.2. *For each $\xi = (u,f) \in \mathfrak{A}_V$ we have $\Psi(w) = w_{\bar{z}} + \Psi^{\triangle} w$, where*

$$\Psi^{\triangle} : H^3(\mathbb{T}^2;\mathbb{C}^n) \to H^2(\mathbb{T}^2;\mathbb{C}^n)$$

is a compact linear operator.

Proof. The operator Ψ is local in $w(z)$. So to evaluate $\Psi(w(z))$ for z in a small disc $D^2 \subset \mathbb{T}^2$ we can use local coordinates in $\mathfrak{M}$, can suppose that

$$u(z) \in B \Subset \mathbb{C}^n,\ v(z) \in \mathbb{C}^n \quad \forall z \in D^2$$

and that f is a map $D^2 \times B \to \mathbb{C}^n$. Then

$$\begin{aligned}\tilde{\Phi}'_u(\xi)v = &\mathcal{U}(\hat{u}(z),u(z))(v_{\bar{z}}(z) + f'_u(z,u)v) + \left(\frac{\partial \mathcal{U}(\hat{u}(z),u(z)))}{\partial u(z)} v(z)\right)\Phi = \\ &= \mathcal{U}((\hat{u}(z),u(z))(v_{\bar{z}} + f'_u(z,u)v),\ \ v \in T_u W = u^* T\mathfrak{M}.\end{aligned}$$

So

$$\begin{aligned}\Psi w(z) = \tilde{\Phi}'_u(\xi)\mathcal{U}^{-1}(\hat{u}(z),u(z))w(z) = \mathcal{U}(\hat{u}(z),u(z)) \frac{\partial}{\partial \bar{z}} \big(\mathcal{U}^{-1}(\hat{u}(z),u(z))w(z)\big) + \\ + \mathcal{U}(\hat{u}(z),u(z)) f'_u(z,u)\mathcal{U}^{-1}(\hat{u}(z),u(z))w(z) = w_{\bar{z}}(z) + g(z)w(z),\end{aligned}$$

where due to (2.8) $g(z)$ is a linear over reals operator in $\mathbb{C}^n$, H^4-smooth in z. Now the lemma's assertion follows since the embedding $H^3 \subset H^2$ is compact and the multiplication by g is continuous in H^2. $\square$

Remark 2.2. The map $\tilde\Phi$ is defined up to a choice of the homotopy $u_{(t)}$ of u and $u_{(1)} \equiv m_0$, which gives rise to the operator $\mathcal{U}$ as in (2.7), (2.8). If we replace $u_{(t)}$ by another homotopy then the operator $\tilde\Phi$ will be multiplied from the left by the holonomy operator $U(\gamma^z)$, corresponding to a contractible loop $\gamma^z(t)$, $\gamma^z(0) = \gamma^z(1) = m_0$, which is H^3-smooth in z. Accordingly the new operator Ψ will be obtained from the old one by the conjugation by $U(\gamma^z)$. □

Remark 2.3. If we replace the point m_0 by $m_1 \in \mathfrak{M}$, then $\tilde\Phi$ will be multiplied from the left by a z-independent complex-linear operator $\mathcal{U}_{m_0,m_1} : T_{m_0}\mathfrak{M} \simeq \mathbb{C}^n \to T_{m_1}\mathfrak{M} \simeq \mathbb{C}^n$, which is a parallel transport along some path from m_0 to m_1. □

The operator

$$\Psi : H^3(\mathbb{T}^2;\mathbb{C}^n) \to H^2(\mathbb{T}^2;\mathbb{C}^n)$$

is zero-index Fredholm since by By Lemma 2.2 it is a compact perturbation of the Cauchy-Riemann operator $\partial/\partial\bar z$ and $\mathrm{ind}\,(\partial/\partial\bar z) = 0$. Therefore the image of Ψ is a closed subspace of a finite codimension and there exists a finite-dimensional subspace $Z_1 \subset C^2(\mathbb{T}^2;\mathbb{C}^n) \subset H^2$ such that

$$Z_1 \cap \text{ range } \Psi = \{0\}, \quad Z_1 + \text{ range } \Psi = H^2 \tag{2.9}$$

(for example, one can take for Z_1 a kernel of the adjoint operator). Also let us fix some subspace $H_2 \subset H^3$ such that $\mathrm{codim}\, H_2 = \dim Z_1$ and

$$\Psi\,|_{H_2} : H_2 \underset{\sim}{\longrightarrow} \text{range } \Psi \quad \text{is an isomorphism.}$$

For $\xi_0 = (u_0, f_0) \in \mathfrak{A}_V$ we consider the map

$$\tilde\Phi'_f(\xi_0) : \mathcal{F} \ni g \mapsto \mathcal{U}(\hat u(z), u_0(z)) \circ g(z, u_0(z)) \in H^2(u_1^* T\mathfrak{M}) \simeq H^2(\mathbb{T}^2;\mathbb{C}^n).$$

Lemma 2.3. *The map $\tilde\Phi'_f(\xi_0)$ has a right inverse $(\tilde\Phi'_f)^{-1}$ which is a bounded linear map*

$$C^2(\mathbb{T}^2,\mathbb{C}^n) \to C^2(\Pi^* T\mathfrak{M}).$$

Proof. There exists a smooth map

$$U : T\mathfrak{M} \to C^2(T\mathfrak{M}) \in H^2(u_1^* T\mathfrak{M}) \simeq H^2(\mathbb{T}^2;\mathbb{C}^n),$$

linear in fibers of $T\mathfrak{M}$, such that $U(u,\xi)(u) = \xi$ for each $(u,\xi) \in T\mathfrak{M}$ and the vectorfield $U(u,\xi)$ is supported in the vicinity of u. Indeed, given $(u,\xi) \in T\mathfrak{M}$ we can, first, use any smooth connection in $T\mathfrak{M}$ to spread the vector ξ to the tangent spaces $T_v\mathfrak{M}$ with v close to u and, second, multiply this local vectorfield by $\phi(u,v)$ where smooth $\phi : \mathfrak{M}\times\mathfrak{M} \to \mathbb{R}$ is supported in a small neighborhood of the diagonal and equals one on the diagonal. We set $U(u,\xi)$ be equal to the vectorfield we have got.

Now for $w \in C^2(\mathbb{T}^2;\mathbb{C}^n)$ we set $(\tilde\Phi'_f)^{-1} w = h$, where

$$h(z,u) = U\big(u_0(z), \mathcal{U}^{-1}(\hat u(z), u_0(z)) w(z)\big)(u).$$

This operator is a right inverse with the desired properties. □

Let us take $\tilde{\mathcal{F}}_1 = (\tilde{\Phi}'_f)^{-1} Z_1$, approximate $\tilde{\mathcal{F}}_1$ by a subspace of the same dimension from $H^{\tilde{m}}$ and approximate the latter by a subspace $\mathcal{F}_1 \subset \mathcal{F}^N$ with sufficiently large N in such a way that

$$(\tilde{\Phi}'_f)\mathcal{F}_1 \cap (\tilde{\Phi}'_u)H_2 = \{0\}, \quad (\tilde{\Phi}'_f)\mathcal{F}_1 + (\tilde{\Phi}'_u)H_2 = H^2 \tag{2.9$'$}$$

(this follows from (2.9) since $(\tilde{\Phi}'_u)H_2 =$ range Ψ). Then $\dim \mathcal{F}_1 = \dim Z_1$ and the map

$$\tilde{\Phi}_*(\xi)\mid_{H_2\times\mathcal{F}_1} : H_2 \times \mathcal{F}_1 \underset{\sim}{\longrightarrow} H^2(\mathbb{T}^2; \mathbb{C}^n) \tag{2.10}$$

is an isomorphism. Finally we define

$$\begin{aligned} &H_1 = H^3 \ominus H_2 \overset{\text{def}}{=} \left\{u \in H^3(\mathbb{T}^2, \mathbb{C}^n) \mid \langle u, v\rangle_{H^3} = 0 \quad \forall v \in H_2\right\}, \\ &\mathcal{F}_2 = \mathcal{F} \ominus \mathcal{F}_1. \end{aligned}$$

Then $\dim H_1 = \dim Z_1$ and

$$H^3 = H_1 \oplus H_2, \quad \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2, \quad \mathcal{F}_2 \supset {}^N\mathcal{F}.$$

We write

$$W \times \mathcal{F} = (H_2 \times \mathcal{F}_1) \oplus (H_1 \times \mathcal{F}_2),$$

denote points in $H^3 = H_1 \oplus H_2$ (in $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$) as (u_1, u_2) (as (f_1, f_2)) and accordingly identify a small neighborhood $\mathcal{O}$ of $\xi = (u, f) \in W \times \mathcal{F}$ with

$$\mathcal{O} = (\langle \text{ball in } H_2\rangle \times \langle \text{ball in } \mathcal{F}_1\rangle) \oplus (\langle \text{ball in } H_1\rangle \times \langle \text{ball in } \mathcal{F}_2\rangle). \tag{2.11}$$

The map $\tilde{\Phi}$ is C^2-smooth (see in Appendix 1.4). So due to (2.10) and the implicit function theorem, in $\mathcal{O}$ one can use equation $\tilde{\Phi} = 0$ to find the first component in the decomposition (2.11) as a C^2-smooth implicit function of the second. That is,

$$\mathcal{O} \cap \mathfrak{A}_V = \{(u_1, u_2 = u_2(u_1, f_1)),\ (f_1 = f_1(u_1, f_2), f_2)\}. \tag{2.12}$$

Thus for each $(u, f) = \xi \in \mathfrak{A}_V$ we have constructed a chart $\mathcal{O} \cap \mathfrak{A}_V \ni \xi$ which is homeomorphic to a ball in $H_1 \times \mathcal{F}_2$. As $\dim H_1 = \dim \mathcal{F}_1$, then we can use any linear isomorphism

$$M : H_1 \underset{\sim}{\longrightarrow} \mathcal{F}_1 \tag{2.13}$$

to construct a smaller chart, homeomorphic to a ball in $\mathcal{F}$. With some abuse of language we call both the charts – homeomorphic to balls in $H_1 \times \mathcal{F}_2$ and in $\mathcal{F}$ – "the charts centered at ξ".

Take any two charts $\mathcal{O}_1, \mathcal{O}_2$ with a nontrivial intersection and denote by N maximum of the two N's. Then the transition map from one chart to another changes the ${}^N\mathcal{F}$-component of f_2 by a shift. The map and its inverse both are C^2-smooth since the maps u_2, f_1 in (2.12) are smooth. Jointly the charts form an atlas and we see that $\mathfrak{A}_V$ is a quasi finite-dimensional Hilbert manifold, modeled by $\mathcal{F}$.

Let us also observe that in any chart as above the projection map π has the form

$$\pi(u_1, f_2) = (f_1(u_1, f_2), f_2), \tag{2.14}$$

i.e., is quasi finite-dimensional (see (2.4)). Its linearization is a zero-index Fredholm map and we have proved

Theorem 2.1. *The set $\mathfrak{A}_V$ is a C^2-smooth quasi finite-dimensional Hilbert manifold modeled by $\mathcal{F}$. The projection $\pi : \mathfrak{A}_V \to \mathcal{F}_V$ is a proper zero-index Fredholm map.*

The notion of quasi finite-dimensional manifold depends on the basis $\{\psi_j\}$ of $\mathcal{F}$. The manifold $\mathfrak{A}$ is quasi finite-dimensional for each choice of this basis.

We recall that a point $(u, f) \in \mathfrak{A}_V$ is regular for the map π if the linearization $\pi_*(u, f)$ is surjective and a point $f \in \mathcal{F}$ is a regular value of π if all points in $\pi^{-1}f$ are regular (e.g., if $\pi^{-1}f = \emptyset$). Since $ind\,\pi_* = 0$, then for a regular point $(u, f) \in \mathfrak{A}_V$ the map $\pi_*(u, f)$ is an isomorphism and u is a nondegenerate solution of (2.2) (see footnote 3).

As the map π is proper, then the set of its singular values is closed in $\mathcal{F}_V$. As π is Fredholm, then due to the Sard-Smale theorem [14] regular values of π are dense. So we get

Corollary 2.1. *Regular values of π form an open dense subset of $\mathcal{F}_V$.*

3. The orientation of $\mathfrak{A}_V$

To orient the charts we need a notion of relative signum of determinant of the map $\tilde{\Phi}'_u$, which we denote $Sgn\,det_{\rm rel}\tilde{\Phi}'_u$ (capital S stresses generalized nature of this object).

To start we suppose that Ker $\tilde{\Phi}'_u = \{0\}$. Then also Ker $\Psi = \{0\}$. We shall define the $Sgn\,det$ in terms of Ψ – this is natural since the real object we study is not the map $\tilde{\Phi}$ but Φ and Φ'_u is conjugated with Ψ in points of $\mathfrak{A}_V$ since (at least formally) $\Psi = \mathcal{U} \circ \Phi'_u \circ \mathcal{U}^{-1}$. The operator $A_0 = (\partial/\partial\bar{z}) + \gamma$, where $\gamma \in \mathbb{C} \setminus \frac{1}{2}(\mathbb{Z} + i\mathbb{Z})$, defines a complex-linear isomorphism $H^3 \underset{\sim}{\longrightarrow} H^2$. Each *finite-dimensional* complex-linear isomorphism has a positive determinant. Thus motivated we set by definition

$$\text{Sgn det } \tilde{\Phi}'_u = \text{sgn det } A_0^{-1}\Psi.$$

The operator $A_0^{-1}\Psi$ is of the form $I + compact$. So its Leray-Shauder (L-S) index is well-defined (see [12]). For an invertible *finite-dimensional* linear map the L-S index equals signum of determinant, and we set

Definition 3.1. For $(u, f) \in \mathfrak{A}_V$ with Ker $\tilde{\Phi}'_u = \{0\}$,

$$\text{Sgn det } \tilde{\Phi}'_u(u, f) := \text{the L-S index of } A_0^{-1}\Psi.$$

Remark 3.1. If in definition of the operator A_0 we replace γ by γ_1, then the operator $A_0^{-1}\Psi$ will be multiplied from the left by $B = ((\partial/\partial\bar{z}) + \gamma_1)^{-1}(\partial/\partial\bar{z}) + \gamma)$ and the *Sgn det* will not change since the L-S index is multiplicative and the index of B equals one because this operator is complex-linear (and has the form $I +$ compact), see in [12]. Therefore the *Sgn det* is γ-independent. $\square$

Now we pass to a definition of relative $Sgn\,det$. Given an operator $\Psi = \Psi_\xi$: $H^3 \to H^2$, $\xi \in \mathfrak{A}_V$, possibly with a nontrivial kernel, we take any decompositions

$$H^3(\mathbb{T}^2; \mathbb{C}^n) = H_1 \oplus H_2, \quad H^2(\mathbb{T}^2; \mathbb{C}^n) = Z_1 \oplus Z_2, \tag{3.1}$$

such that

1) $Z_2 \subset \text{range } \Psi,\ Z_1 \subset C^2(\mathbb{T}^2; \mathbb{C}^n) \subset H^2$,
2) $\infty > \dim H_1 = \dim Z_1 \geq \dim \text{Ker } \Psi$,
3) Ψ restricted to H_2 defines an isomorphism $H_2 \underset{\sim}{\longrightarrow} Z_2$.

Given a decomposition (3.1) take any linear isomorphism

$$L : H_1 \underset{\sim}{\longrightarrow} Z_1. \tag{3.2}$$

We call the operator

$$\Psi_{\text{reg}} := L \oplus \left(\Psi\, |_{H_2}\right) : H_1 \oplus H_2 = H^3 \to Z_1 \oplus Z_2 = H^2$$

the regularization of Ψ, related to the decomposition (3.1) and the operator L. Finally, we give

Definition 3.2. Given a decomposition (3.1) and an isomorphism (3.2) we set

$$\text{Sgn } \det_{\text{rel}} \tilde{\Phi}'_u := \text{the L-S index of } A_0^{-1}\Psi_{\text{reg}}.$$

Now take any chart centered at $(u_0, f_0) \in \mathfrak{A}_V$. If Ker $\Phi'_u(u_0, f_0) = 0$, then in (2.10) $\mathcal{F}_1 = \{0\}$, $H_2 = H^3$ and we take for an oriented chart centered at (u_0, f_0) a ball in the oriented space

$$\text{Sgn } \det\, \Phi'_u \cdot \mathcal{F}, \tag{3.3}$$

where the space $\mathcal{F}$ is given some fixed orientation. We shall call the oriented charts of this kind *charts of the main type*.

Next take any chart as in (2.12) with $H_1 \neq \{0\}$. The chart-map uses a linear isomorphism M (see (2.13)). Besides, we have the linear isomorphism $\tilde{\Phi}'_f : \mathcal{F}_1 \underset{\sim}{\longrightarrow} Z_1$. Denote by L the composition map with the sign "minus":

$$L = -\tilde{\Phi}'_f \circ M : H_1 \underset{\sim}{\longrightarrow} Z_1, \tag{3.4}$$

and take for decompositions (3.1) the decomposition (2.9′). Then

$$Z_1 = \tilde{\Phi}'_f \mathcal{F}_1, \quad Z_2 = \Psi H_2 = \text{range } \Psi, \quad H_1 \oplus H_2 = H^3(\mathbb{T}^2; \mathbb{C}^n).$$

Now for the oriented chart we take a ball in the space

$$\text{Sgn } \det_{\text{rel}} \tilde{\Phi}'_u \cdot \mathcal{F}. \tag{3.3′}$$

Each transition map from one chart to another has the form (2.4), where the range of the finite-dimensional map lies in some $\mathcal{F}^N$. Therefore the L-S index of the transition map is well-defined and equals the signum of determinant of the $N \times N$-matrix, equal to Jacobian of the map

$$\mathcal{F}^N \to \mathcal{F} \xrightarrow{\text{transition map}} \mathcal{F} \to \mathcal{F}^N \tag{3.5}$$

(the first map is the natural embedding, the last one is the projection along the space ${}^N\mathcal{F}$). If we increase N, the Jacobian of (3.5) will not change. We call this well-defined number *Jacobian of the transition map*.

We say that orientations of two intersecting charts *agree* if the L-S index of the corresponding quasi finite-dimensional transition map equals +1 (i.e., if Jacobian of the transition map is positive).

Definition 3.3. A quasi finite-dimensional manifold modeled by $\mathcal{F}$ is orientable if it admits an oriented atlas such that orientations of any two its intersecting charts agree.

Below we show that the orientations of the charts of $\mathfrak{A}$ we have constructed agree. The proof is based on a kind of stabilization procedure (we shall "stabilize" – enlarge – the spaces H_1). To carry it out we extend the oriented atlas we have constructed. To do so we take any decomposition (3.1) and define the spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as above – so the maps (2.10), (3.4) are isomorphisms and $\mathcal{F}_1 \subset \mathcal{F}^N$, ${}^N\mathcal{F} \subset \mathcal{F}_2$. After this we use the same arguments as above to construct a neighborhood of (u_0, f_0), homeomorphic to a ball in the space $H_1 \times \mathcal{F}_2$ and use a map $M \times E$ to construct a chart in $\mathcal{F}$.

We call the quadruple formed by the decomposition (3.1), the spaces $\mathcal{F}_1, \mathcal{F}_2$ and the map M *the type of a chart* .

Clearly for a chart centered at a point $(u_0, f_0) \in \mathfrak{A}_V$ each sufficiently close point is the center of a chart of the same type.

Remark 3.2. Our definition of the chart's orientation uses a contraction $u_{(t)}(z)$ of u to $u_1(z) \equiv m_0$. If we change the contraction, then the new operator Ψ^{new} will be conjugated to the old one, $\Psi^{\mathrm{new}} w(z) = U(\gamma^z)^{-1}\Psi U(\gamma^z) w(z)$, where U is the complex-linear holonomy map, corresponding to a contractible loop $t \mapsto \gamma_t^z$ (see Remark 2.2). The operator Ψ^{new} also has the form $(\partial/\partial\bar{z}) + perturbation$. Therefore if Ker $\tilde{\Phi}'_u = \{0\}$, then

$$\text{Sgn det } \tilde{\Phi}'_u = \text{Sgn det } \left(\tilde{\Phi}^{\mathrm{new}}\right)'_u,$$

since the L-S index is homotopy invariant. Thus the orientation of a chart of the main type does not depend on the contraction $u_{(t)}$. In general the chart's orientation is based on a decomposition (3.1) (which meets 1)-3)). The spaces $UH_1, \ldots, UZ_2$ form decompositions (3.1) for the operator Ψ^{new} and – jointly with $L^{\mathrm{new}} = U^{-1}LU$ – define an oriented chart centered at the same point. For the same homotopy-invariance arguments the $Sgn\, det_{\mathrm{rel}}\Phi'_u$ is the same for both charts and the Jacobian of the transition map equals one. So the orientation does not depend on the contraction $u_{(t)}(z)$. $\square$

Remark 3.3. The choice of the trivialization of $T_{m_0}\mathfrak{M}$ we have used is inessential for similar reasons. A change of the point m_0 affects the map Ψ as a change of trivialization (Remark 2.3). So it also does not change the orientation. $\square$

Theorem 3.1. *The orientations of the charts forming the constructed atlas of* $\mathfrak{A}_V$ *agree, so the manifold* $\mathfrak{A}_V$ *is orientable.*

Proof. The problem is local since we should only examine intersecting charts and can suppose that the charts are sufficiently small. So we consider $u(z)$ close to some fixed $\hat{u}(z)$, fix the contraction of $\hat{u}(z)$ to m_0 and the trivialization of $T_{m_0}\mathfrak{M}$.

Let us make two trivial observations. First, if orientations of two charts agree in a point of their intersection, then they agree in the corresponding connected component of the intersection (since Jacobian of the transition-map is continuous and does not vanish). Second, if in an intersection of three charts the first agrees with the second and the second agrees with the third, then all the three agree. We shall refer to this property as *the transitivity.*

We need two lemmas.

Lemma 3.1. *Orientations of any two charts, centered at the same point, agree.*

For a proof see Appendix 2.

Lemma 3.2. *Let $\xi_\lambda = (u_\lambda, f_\lambda)$, $\lambda \in [0,1]$, be a curve in $\mathfrak{A}_V$ such that for each λ there exists an admissible chart $\mathcal{O}_\lambda$, centered at ξ_λ, of the same λ-independent type. Suppose that an intersection of any two $\mathcal{O}_\lambda$'s is connected (or empty). Then orientations of the charts $\mathcal{O}_\lambda$ agree.*

Proof. Take any two charts $\mathcal{O}_{\lambda_1}, \mathcal{O}_{\lambda_2}$, centered at $\xi_{\lambda_1}, \xi_{\lambda_2}$. Take any $\xi_\lambda \in \mathcal{O}_{\lambda_1} \cap \mathcal{O}_{\lambda_2}$, $\lambda \in [\lambda_1, \lambda_2]$, and consider the Jacobian of the transition map from $\mathcal{O}_{\lambda_1}$ to $\mathcal{O}_{\lambda_2}$, calculated at ξ_λ. This is a continuous function of $(\lambda, \lambda_1, \lambda_2)$ which never vanishes and equals one for $\lambda = \lambda_1 = \lambda_2$. Therefore it is positive everywhere and the assertion follows. □

Now we complete the proof of the theorem. Take any two charts $\mathcal{O}_0$ and $\mathcal{O}_1$ centered at $\xi_0 = (u_0, f_0)$ and $\xi_1 = (u_1, f_1)$ with a nonempty intersection. Let $\xi_\lambda = (u_\lambda, f_\lambda)$, $\lambda \in \Lambda = [0,1]$, be a path in $\mathcal{O}_0$ from its center ξ_0 to a point $\xi_* \in \mathcal{O}_0 \cap \mathcal{O}_1$. We cover the segment Λ by open in Λ intervals Λ_j such that for each j each point $\xi \in \Lambda_j$ has a chart centered at this point such that its type does not depend on ξ (but depends on j). We choose these charts to be so small that they lie in $\mathcal{O}_1$ and choose them to be convex in the geometry of $\mathcal{O}_1$. For the type of the interval which contains ξ_0 we choose the type of the chart $\mathcal{O}_0$. We extract a finite sub-covering $\{\Lambda_k \mid 1 \le k \le N\}$ which covers Λ "from left to right". By Lemma 3.2 each chart centered at $\xi \in \Lambda_1 = [a_0 = 0, a_1]$ has the same orientation as $\mathcal{O}_0$. By Lemma 3.3 the chart of the type of Λ_1, centered at $\xi(a_1)$, has the same orientation as the chart of the type of Λ_2, centered at the same point. So each chart centered at a point from Λ_2 agrees with $\mathcal{O}_0$, etc. Finally we get that the chart centered at ξ_* agrees with $\mathcal{O}_0$. For the same reason it agrees with $\mathcal{O}_1$. By the transitivity orientations of $\mathcal{O}_0$ and $\mathcal{O}_1$ agree and the theorem is proven. □

We say that a map $G : \mathfrak{A}_V \to \mathcal{F}_V$ is quasi finite-dimensional if in the charts as in Definition 2.1 it is quasi finite-dimensional (see (2.4)). If a quasi finite-dimensional map G is C^1-smooth, then it is zero-index Fredholm.

Definition 3.4. If G is proper and quasi finite-dimensional, then for its regular value $f \in \mathcal{F}_V$ the degree $deg\, G(f)$ of G at the point f is

$$\deg\, G(f) = \sum_{\xi : G(\xi) = f} \operatorname{sgn}\, \text{Jacobian}\; G(\xi)$$

(the sum in the r.h.s. is finite since G is proper and is well-defined since Jacobians of the transition-maps between different charts are positive).

Since $\mathcal{F}_V$ is connected then – as for the finite-dimensional situation [5,12] – the degree is f-independent:

Proposition 3.1. *If G as above is C^2-smooth and f_1, f_2 are its regular values, then deg $G(f_1) =$ deg $G(f_2)$.*

Proof. Since the function $d(f) = \deg\, G(f)$ is locally constant, then we can find sufficiently large N_1 and regular values $g_1, g_2 \in \mathcal{F}^{N_1}$ such that

$$d(g_1) = d(f_1), \quad d(g_2) = d(f_2).$$

Take any path μ in $\mathcal{F}_V$ from g_1 to g_2 such that G is transversal to μ (it exists due to Sard-Smale [14]) and approximate it by some path $\gamma \subset \mathcal{F}^{N_2}$, $N_2 < \infty$, which still is transversal to G.

The set $G^{-1}(\gamma)$ is a compact one-dimensional submanifold of $\mathfrak{A}_V$, covered by a finite system of charts from the oriented atlas. Let $N \geq N_1, N_2$ be an integer which is larger than:

1) all the N's used to obtain the quasi finite-dimensional form (2.4) for the transition maps for intersecting charts as above,

2) all the N's used in the quasi finite-dimensional form (2.4) for the map G.

Denote by $\mathfrak{A}_V^N$ a neighborhood of $G^{-1}(\gamma)$ in $(W \times \mathcal{F}^N) \cap \mathfrak{A}_V$ which is contained in the union of the charts as above. Then $\mathfrak{A}_V^N$ is a C^2-smooth N-manifold and locally $\mathfrak{A}_V$ equals $\mathfrak{A}_V^N$ times a ball in ${}^N\mathcal{F}$. This presentation supplies $\mathfrak{A}_V^N$ with an orientation induced from $\mathfrak{A}_V$. Let us set $G^N = G\mid_{\mathfrak{A}_V^N} : \mathfrak{A}_V^N \to \mathcal{F}^N$. For $\xi \in G^{-1}(\gamma)$ we have

$$\text{sgn Jacobian } G(\xi) = \text{sgn Jacobian } G^N(\xi).$$

So $d(f) = \deg G^N(f)$ for regular $f \in \gamma$ and the assertion follows since $d(g_1) = d(g_2)$ by the finite-dimensional theory [5,12]. $\square$

Now for a map G as above we define its *degree* $\deg G$ as

$$\deg G = \deg G(f),$$

where f is any regular value of G.

By Theorems 2.1, 3.1 and Proposition 3.1 the degree $\deg \pi$ of the map $\pi : \mathfrak{A}_V \to \mathcal{F}_V$ is well-defined. In the next part we shall calculate it.

4. Degree of the projection map

We continue to study the manifold $\mathfrak{A}_V$ and the projection π.

Lemma 4.1. *If $(u, 0) \in \mathfrak{A}_V$, then $u = const$.*

Proof. Now u is a holomorphic map $u : \mathbb{T}^2 \to \mathfrak{M}$. Therefore the area of $u(\mathbb{T}^2)$ equals its symplectic area (see (1.1)) and equals zero since u is contractible. As locally u is a holomorphic vector-function of zero area, then locally u is a constant. So u is a global constant and the assertion follows. $\square$

Lemma 4.2. *Let $f = \varepsilon g(u)$. Then (2.2) has only constant solutions, provided that ε is sufficiently small.*

Proof. Let $(\partial/\partial\bar{z})u_\varepsilon + \varepsilon g(u) = 0$. As $\varepsilon g \to 0$ when $\varepsilon \to 0$, then by Lemma 2.1 $u_{\varepsilon_j} \to u_0$ and u_0 solves (2.2) with $f = 0$. By Lemma 4.1 $u_0 = \text{const}$. Take any chart in $\mathfrak{M}$ which contains u_0 and identify it with a domain $\mathcal{O} \subset \mathbb{C}^n$ in such a way that u_0 corresponds to the origin. Then we can treat u_ε ($\varepsilon = \varepsilon_j$) with ε sufficiently small as a map $\mathbb{T}^2 \to \mathcal{O} \subset \mathbb{C}^n$. We extract from u_ε its mean value ρ_ε, write u_ε as

$$u_\varepsilon(z) = \rho_\varepsilon + v_\varepsilon(z), \quad \int v_\varepsilon \, d^2 z = 0,$$

and rewrite the equation for u_ε as

$$\frac{\partial v_\varepsilon}{\partial \bar{z}} + \varepsilon g(\rho_\varepsilon) + \varepsilon(g(\rho_\varepsilon + v_\varepsilon) - g(\rho_\varepsilon)) = 0.$$

We denote by P the operator which subtracts from a function its mean value and get

$$|v_\varepsilon|_{L_\infty} \le C\varepsilon |P(g(\rho_\varepsilon + v_\varepsilon) - g(\rho_\varepsilon))|_{L_\infty} \le C_1\varepsilon |v_\varepsilon|_{L_\infty}$$

(see Appendix 1.2). Now the assertion follows if ε is chosen to be smaller than C_1^{-1}. $\square$

Lemma 4.3. *A constant solution $u(z) \equiv \tilde u$ of (2.2) with $f = \tilde f$ is nondegenerate (and the point $(\tilde u, \tilde f)$ is regular for the map π) if the linearized equation*

$$\frac{\partial v}{\partial \bar z} + \tilde f(z, \tilde u)_* v = 0, \quad v : \mathbb{T}^2 \to T_{\tilde u}\mathfrak{M},$$

has only zero solution $v \equiv 0$.

Proof. If $\tilde u$ is nondegenerate, then the corresponding map $\Psi = \Psi_{(\tilde u, f)}$ is an isomorphism since this is a zero-index Fredholm map with a trivial kernel. Therefore by the implicit function theorem near $(\tilde u, f)$ a point (u, f) smoothly depends on f. So $(\tilde u, \tilde f)$ is regular for π. $\square$

Lemma 4.4. *For each Kählerian $\mathfrak{M}$ there exists*

1) a smooth vectorfield $f_1 = f_1(u)$ such that (2.2) has only one solution;

2) a smooth vectorfield f_2 such that (2.2) has exactly $m = |\chi(\mathfrak{M})|$ solutions $u_1, \dots, u_m$ and all of them are constant nondegenerate.

Proof. 1) Take any smooth vectorfield on $\mathfrak{M}$ with nondegenerate critical points and construct a diffeomorphism of $\mathfrak{M}$ which sends all the critical points to one chart (see [4], Part 5, Lemma 2.6). We identify this chart with a ball $B \subset \mathbb{C}^n$. After this we fix the vectorfield near ∂B and continue it in B in such a way that it has the only critical point m_* in the center of the ball (certainly, this critical point is degenerate). Denote this vectorfield g_1. By Lemma 4.2 the equation (2.2) with $f_1 = \varepsilon g_1$ has the only solution $u \equiv m_*$ if $\varepsilon \ll 1$.

2) Now we take for g_2 a smooth vectorfield which has exactly m nondegenerate critical points of index sgn $\chi(\mathfrak{M})$ (to construct g_2 one can argue as above). The equation with $f_2 = \varepsilon g_2$, $\varepsilon \ll 1$, has m solutions, $u = u_1, \dots, u = u_m$. The equation linearized about $u = u_j$ takes the form

$$D_\varepsilon v := \frac{\partial v}{\partial \bar z} + \varepsilon B v = 0,$$

where $B = (g_2)(u_j)_*$ is a nondegenerate linear (over reals) operator in $\mathbb{C}^n$.

We use an isomorphism $T_{u_j}\mathfrak{M} \simeq \mathbb{C}^n$ and treat D_ε as an operator $H^3 \to H^2$. Denote by H_0^2 the subspace of H^2 formed by functions with zero mean value. Then $H^2 \simeq H_0^2 \oplus E$, where $E \simeq \mathbb{C}^N$ is formed by constant vector-functions, and

$$D_0 : H_0^3 \underset{\sim}{\longrightarrow} H_0^2, \quad D_0 E = 0.$$

For $\varepsilon \ll 1$ the operator D_ε isomorphically sends H_0^3 to $D_\varepsilon(H_0^3)$ where the latter is a subspace, transversal to E, and $D_\varepsilon E \subset E$. As $D_0 \mid_E = 0$, then $D_\varepsilon \mid_E = B \mid_E$ is an isomorphism. So Ker $D_\varepsilon = \emptyset$ and the assertion follows from Lemma 4.3. $\square$

Let us take f_2 as in the second assertion of Lemma 4.4. Then

$$\pi^{-1}(f_2) = \{(u_1, f_2), \dots, (u_m, f_2)\} \subset \mathfrak{A}_V$$

and f_2 is a regular value of π since all points (u_j, f_2) are regular.

Each point $(u_j, f_2) \in \mathfrak{A}_V$ belongs to a chart on $\mathfrak{A}_V$ of the main type. To orient the chart we are free to choose the point $m_0 \in \mathfrak{M}$ (see Remark 3.3). Let us take $m_0 = u_j$. The orientation of the chart is given by $Sgn\, det\, \tilde{\Phi}'_u \cdot \mathcal{F}$. It turns out that $Sgn\, det\, \tilde{\Phi}'_u(u_j, f_2)$ agrees with $sgn\, det\, f_2(u_j)_* = sgn\, det\, (B := g_2(u_j)_*)$, if ε is small:

$$\text{sgn det } B = \text{Sgn det } \tilde{\Phi}'_u(u_j, f_2), \quad \text{if } \varepsilon \ll 1 \tag{4.1}$$

(here we treat B as an $2n \times 2n$-matrix with real entries).

To prove the formula we take $A_0 = (\partial/\partial\bar{z}) + \gamma$ with $\gamma \in (0, \frac{1}{2})$ (see Remark 3.1). Define the operator $D_\varepsilon = \tilde{\Phi}'_u$ and the spaces H^2_0, E as in the proof of Lemma 4.4. The operator $D_\varepsilon\mid_{H^3_0}: H^3_0 \underset{\sim}{\longrightarrow} D_\varepsilon(H^3_0)$ can be deformed to $A_0 \mid_{H^3_0}$ in the class of linear isomorphisms, if ε is small. Therefore D_ε is homotopic to

$$\Big(A_0\mid_{H^3_0} \oplus D_\varepsilon\mid_E\Big) : H^3_0 \oplus E \to H^2_0 \oplus E.$$

So $A_0^{-1} \circ D_\varepsilon = I \oplus \gamma^{-1}B$, and (4.1) follows since $det\; \gamma^{-1}B = \gamma^{-2n}\; det\, B$.

Finally, we see that

$$\text{sgn Jacobian}\, \pi(u_j, f_2) = \text{Sgn det}\, \tilde{\Phi}'_u(u_j, f_2) = \text{sgn det}\, (g_2)(u_j)_*$$

and

$$d(\pi) = \sum_j \text{sgn Jacobian}\, \pi(u_j, f_2) = \sum_j \text{sgn det}\, g_2(u_j)_* = \chi(M). \tag{4.2}$$

We have got

Theorem 4.1. *If $\mathfrak{M}$ is a compact Kählerian manifold, then regular values of π are open dense in $\mathcal{F}_V$ and deg $\pi = \chi(\mathfrak{M})$; besides there exist $f_1, f_2 \in \mathcal{F}_V$ such that equation (2.2) with $f = f_1$ has only one solution and with $f = f_2$ has exactly $|\chi(\mathfrak{M})|$ nondegenerate solutions. If $\chi(\mathfrak{M}) \neq 0$, then for each $f \in \mathcal{F}_V$ the equation has at least one solution.*

Indeed, the first group of the statements follows from Corollary 2.1, from (4.2) and Lemma 4.4. As regular values of π are dense in $\mathcal{F}_V$, then the last statement results from Lemma 2.1 since due to the first assertion of the theorem $\pi^{-1}(f) \neq \emptyset$ for each regular f if $\chi(\mathfrak{M}) \neq 0$.

5. Examples

5.1. $\mathfrak{M} = S^2$.

If $\mathfrak{M}$ is the normalized two-sphere, then $V = 1$, $\chi(\mathfrak{M}) = 2$ and by Theorem 4.1 the equation

$$u_{\bar{z}}(z) + f(z, u) = 0, \; u \in S^2, \; z \in \mathbb{T}^2, \tag{5.1}$$

has at least one contractible solution and at least two nondegenerate contractible solutions, if

$$\|f(z, u)\|_u < 1 \quad \forall z, u. \tag{5.2}$$

As in Example 2.3 we identify S^2 with the Riemann sphere $S^2 = \mathbb{C} \cup \infty$. Then a solution of (5.1) which does not equal ∞ identically is a solution with isolated poles of the equation written in the chart $\mathbb{C} \subset S^2$:

$$u_{\bar{z}} + f(z, u) = 0, \; z \in \mathbb{T}^2, \; u \in \mathbb{C}. \tag{5.3}$$

Indeed, due to the Carleman uniqueness theorem [15], for a solution u of (5.1) one has an alternative: either $u \equiv \infty$, or u has only a finite number of poles. We recall (Example 2.3) that in the chart $\mathbb{C}$ the condition (5.2) reeds as

$$\sup_{z,u} |f(z,u)|\,(1+|u|^2)^{-1} < \sqrt{\pi}. \tag{5.4}$$

Observe that in the chart $\mathbb{C}_* \cup \infty = \{w^{-1}\}$ the equation takes the form $w_{\bar z}(z) + F(z,w) = 0$, where $F(z,w) = -w^2 f(z,w^{-1})$. So if $F(z,0) \not\equiv 0$, then $u \equiv \infty$ (and $w \equiv 0$) is not a solution of (5.1). Therefore if (5.4) holds and

$$f, F \in H^{\tilde m}(\mathbb{T}^2 \times \mathbb{C}, \mathbb{C}), \quad F(z,0) \not\equiv 0, \tag{5.5}$$

then (5.4) has a contractible solution with poles and at least two solutions, if they are regular.

Example. If $f = \text{const}\ \neq 0$, then $F(z,0) \equiv 0$ and (5.5) is violated. Now the only solution of (5.1) is $u \equiv \infty$.

The equation (5.4) is closely related with the second-order quasilinear equation

$$\Delta\xi(x,y) = g(x,y;\xi_x,\xi_y), \quad \xi : \mathbb{T}^2 \to \mathbb{R}. \tag{5.6}$$

Indeed, if we denote $\xi_y = u_1$, $\xi_x = u_2$ and $x + iy = z$, then (5.6) can be written as $u_{\bar z} = f(z,u)$, where $u = u_1 + iu_2$ and $f = \frac{i}{2} g$. So (5.6) has a solution with poles, provided that (5.5) is fulfilled.

5.2. $\mathfrak{M} = \mathbb{T}^2$.

Now the manifold $\mathfrak{M}$ is aspherical. So $V = \infty$. We study contractible maps $u : \mathbb{T}^2 \to \mathbb{T}^2$ which satisfy (2.2) with given vectorfield f. Each contractible map $\mathbb{T}^2 \to \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ can be lifted to a map $U : \mathbb{T}^2 \to \mathbb{C}$. So we study solutions of the equation

$$\frac{\partial U}{\partial \bar z} + f(z,U) = 0, \quad U : \mathbb{T}^2 \to \mathbb{C}, \tag{5.7}$$

with a doubly-periodic in z and U complex function f. As $\chi(\mathbb{T}^2) = 0$, then Theorem 4.1 does not guarantee solvability of (5.7). To see that an equation without solutions does exist one can take $f \equiv -1$. Then (5.7) is a linear equation for the Cauchy-Riemann operator which can not be solved since $\int 1\, d^2z \neq 0$.

Still Theorem 4.1 can be used to get nontrivial results for some equations (5.7). Suppose that $f(z,0) \equiv 0$ and the solution $U \equiv 0$ is nondegenerate. Then (5.7) has a second solution. Indeed, if f is a regular value of π, then the statement is obvious since $\deg \pi = 0$. In general case we can approximate f by regular f_n's. By the implicit function theorem, (5.7) with $f = f_n$ has exactly one small solution. This equation must have another one which can not be small. When n grows, these second solutions converge to a nonzero solution of (5.7).

Example [11]. Zero-solution of (5.7) is nondegenerate if $f(z,u) = a\bar u + O(|u|^2)$, where a is a nonzero constant. Indeed, the linearized equation is $v_{\bar z} + a\bar v = 0$ and for its solution we have $(\partial/\partial z)v_{\bar z} = -a(\partial/\partial z)\bar v = -a(-\bar a)v = |a|^2 v$. So

$$-\int |v_{\bar z}|^2\, d^2z = \int (\partial/\partial z)v_{\bar z} \cdot \bar v\, d^2z = \int |a|^2|v|^2\, d^2z$$

and $v = 0$ as required.

5.3. $\mathfrak{M} = \mathbb{C}P^n$ (see Example 2.2).

Now $V = 1$ and $\chi(\mathfrak{M}) = n+1$. So if $f(z,u)$ is a (nonautonomous) $H^{\tilde m}$-smooth vectorfield in $\mathbb{C}P^n$ such that

$$\|f(z,u)\|_u < 1 \quad \forall\, z, u, \tag{5.8}$$

then the equation

$$u_{\bar z}(z) + f(z,u) = 0, \ \ u \in \mathbb{C}P^n, \ \ z \in \mathbb{T}^2, \tag{5.9}$$

has a contractible solution. Typically the number of solutions is at least $n+1$.

Equation (5.9) is related to a "deprojectivized" homogeneous equation in $\mathbb{C}^{n+1}$. To construct the equation we consider the Hopf fibering

$$p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n, \quad \xi \mapsto u.$$

The map p sends each one-homogeneous vectorfield $F(z,u)$ [7] on $\mathbb{C}^{n+1}$ to a well-defined vectorfield $f = p_* F$ on $\mathbb{C}P^n$. If $p_* F_1 = p_* F_2$, then $F_1(z,\xi) - F_2(z,\xi) \in \mathbb{C}\xi \quad \forall\, z, \xi$. Conversely, if we are given some vectorfield f on $\mathbb{C}P^n$, then there exists a one-homogeneous vectorfield F on $\mathbb{C}^{n+1}$ as smooth as f outside $\{\xi = 0\}$ and such that $p_* F = f$.

Due to the definition of the normalized Fubini–Studi metric [2], the assumption (5.8) on f is equivalent to

$$|F(z,\xi)|^2 |\xi|^2 - |\langle F(z,\xi), \xi\rangle|^2 < \pi |\xi|^4. \tag{5.10}$$

Lemma 5.1. *If u is a contractible solution of (5.9) and F is as above, then there exists a lift $U : \mathbb{T}^2 \to \mathbb{C}^{n+1} \setminus \{0\}$ of the map u and $\hat\lambda \in \mathbb{C}$ such that*

$$U_{\bar z} + F(z,U) = \hat\lambda U, \quad U \in \mathbb{C}^{n+1} \setminus \{0\}. \tag{5.11}$$

Proof. Since u is contractible it can be lifted to a map $w : \mathbb{T}^2 \to \mathbb{C}^{n+1} \setminus \{0\}$, $p \circ w = u$. Simple calculations using affine charts on $\mathbb{C}P^n$ show that $w(z)$ satisfies (5.11) with $\hat\lambda$ replaced by some (complex) function $\lambda(z)$.

Next we consider the map $U(z) = e^{-\rho(z)} w(z)$, where ρ is a function on the torus, and observe that

$$U_{\bar z} + F(z,U) = (\lambda(z) - (\partial/\partial \bar z)\rho(z))U.$$

We denote by $\hat\lambda$ the mean value of $\lambda(z)$ and find a function $\rho(z)$ such that $(\partial/\partial\bar z)\rho = \lambda - \hat\lambda$. Now $U(z)$ satisfies (5.11). □

Let $(U_1, \hat\lambda_1)$ be another solution of (5.11) such that $p \circ U_1 \equiv u$. Then $U_1 = e^\rho U$, where ρ is a map from $\mathbb{T}^2$ to $\mathbb{C}/2\pi i\mathbb{Z}$ and

$$(\partial/\partial\bar z)\rho = \hat\lambda_1 - \hat\lambda = \text{const}. \tag{5.12}$$

Let us consider the map $\hat\rho : \mathbb{C} \to \mathbb{C}$ which covers ρ. The function $\operatorname{Re}\hat\rho$ is harmonic and double-periodic. Therefore it is a constant and $\hat\rho(x+iy) = \text{const} + if(x,y)$, where $if'_x - f'_y \equiv 2(\hat\lambda_1 - \hat\lambda)$ by (5.12). Hence $f = \text{const} + i(ax+by)$ with some real a, b. Since ρ is the well-defined map from $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ to $\mathbb{C}/2\pi i\mathbb{Z}$, then we find that $a, b \in 2\pi\mathbb{N}$. So all solutions $(U_1, \hat\lambda_1)$ of (5.11) which cover the same u have the form

$$U_1 = U_{m,n} = Ce^{2\pi i(mx+ny)} U(x+iy), \quad \hat\lambda_1 = \hat\lambda + \pi(im - n), \tag{5.13}$$

where C is a nonzero constant.

Conversely, each solution U of (5.11) covers the solution $u = p \circ U$ of (5.9) and we get

[7] i.e., $F(z,\tau\xi) = \tau F(z,\xi)$ for each $\tau \in \mathbb{C}$ after we identify the tangent spaces with $\mathbb{C}^{n+1}$.

Proposition 5.1. *If the nonautonomous vectorfield $F(z,U)$, $U \in \mathbb{C}^{n+1}$, is one-homogeneous, belongs to $H^{\tilde m}_{\mathrm{loc}}(\mathbb{T}^2 \times \mathbb{C}^{n+1} \setminus 0, \mathbb{C}^{n+1})$ and meets (5.10), then the equation (5.11) has a family of solutions $(U_{m,n}(z),\ \hat\lambda - \pi(n-im))$ as in (5.13). For a typical F solutions of (5.11) form a finite system of families (5.13) and the number of the families is at least $n+1$.*

Example. Let us choose in (5.11) $F(z,U) = AU$, where A is a complex matrix. Now (5.11) is the spectral problem for the operator $(\partial/\partial\bar z)+A$ with periodic boundary conditions. The families of solutions (5.13) have the form $\big(Ce^{2\pi i(mx+ny)}\xi,\ \lambda+\pi(im-n)\big)$, where ξ is an eigenvector of A, $A\xi = \lambda\xi$. Corresponding solutions of (5.9) are constants, $u(z) \equiv p(\xi)$, in an agreement with Lemma 4.2 (and Lemma 4.1 if $A = 0$).

Finally we remark that if U solves (5.11), then $W = U/|U|$ solves the equation

$$W_{\bar z} + F(z,W) = \lambda(z)W, \quad |W| \equiv 1, \tag{5.14}$$

where $\lambda = \hat\lambda - |U|^{-1}(\partial/\partial\bar z)|U|$. If F_k is k-homogeneous, then $F := F_k|W|^{1-k}$ is one-homogeneous and each solution of (5.14) also solves the equation

$$W_{\bar z} + F_k(z,W) = \lambda(z)W, \quad |W| \equiv 1. \tag{5.15}$$

So our results imply solvability of (5.15), provided that the vectorfield F satisfies (5.10).

6. Other possibilities

1) Let $\mathfrak{M}$ be a compact manifold with a fixed tamed almost complex structure J. Now equation (1) takes the form

$$\frac{1}{2}(u_x + J(u)u_y) + f(z,u) = 0, \quad z \in \mathbb{T}^2.$$

An analogy of Theorem 1 holds for contractible solutions of this equation if the constant V is modified accordingly.

2) Let us replace the problem (1), (2) by the equation (1.3) for pseudoholomorphic tori in the almost complex manifold $(M = \mathbb{T}^2 \times \mathfrak{M}, J)$. Let J be tamed by a fixed symplectic form. Then the estimate for the number of pseudoholomorphic tori, homotopic to $\mathbb{T}^2 \times \{\mathrm{pt}\}$, is the same as in Theorem 1. Indeed, if $\tilde{\mathfrak{A}}$ is the manifold formed by all pairs (*pseudoholomorphic torus, tamed almost complex structure*) and $\tilde\pi$ is its projection to the almost complex structures, then $\tilde\pi^{-1}(J^f)$ is formed by the tori which are graphs of the maps from $\pi^{-1}(f)$ (see Proposition 1.1). The manifold $\tilde{\mathfrak{A}}$ can be oriented similar to $\mathfrak{A}$ and for f_2 as in Lemma 4.3 Jacobians of $\tilde\pi_*$ in points of $\tilde\pi_{-1}(J^{f_2})$ and of π_* in the corresponding points of $\pi^{-1}(f_2)$ are equal. See [6] for some details in the case $\mathfrak{M} = \mathbb{T}^2$.

3) Let us come back to periodic boundary value problem for an ordinary differential equation on a compact n-manifold $\mathcal{N}$ (equation (3) from the Introduction). Now we are interested in its contractible solutions. We can repeat constructions of Part 2 with $V := \infty$ to define the map Φ,

$$\Phi(v,R) = \dot v(\cdot) + R(\cdot, v(\cdot)),$$

the manifold $\mathfrak{A} = \Phi^{-1}(0)$ and the projection $\pi : \mathfrak{A} \ni (v, R) \mapsto R$. Next we choose any smooth Riemann metric on $\mathcal{N}$, construct the corresponding Levi-Civita connection and use it to define the maps $\tilde{\Phi}$ and $\Psi = \Psi_{(v,R)}$. An analogy of Theorem 2.1 holds with the same proof: the map π is proper and the manifold $\mathfrak{A}$ is quasi finite-dimensional. To orient $\mathfrak{A}$ we take for A_0 the operator $A_0 = \frac{\partial}{\partial t} + \gamma$, $\gamma > 0$, define $Sgn\, det\, \Phi'_v$ and $Sgn\, det_{\rm rel} \Phi'_v$ as in Definitions 3.1, 3.2 and orient the charts on $\mathfrak{A}$ as in (3.3), (3.3′). The orientations of different charts agree. If v is a nondegenerate solution of (3) (i.e., $\operatorname{Ker} \Phi'_v(v, R) = 0$), then $Sgn\, det\, \Phi'_v$ agrees with the *sgn det* of the Floquet map $\mathbf{F}$ (=the linearized period-shift map),

$$\mathbf{F} : T_{v(0)}\mathcal{N} \to T_{v(0)}\mathcal{N}, \;\; \xi(0) \mapsto \xi(1), \quad \text{where } \; \dot{\xi}(t) + R'_v(t, v(t))\xi(t) = 0 \; : \tag{6.1}$$

Lemma 6.1. *If v is a nondegenerate contractible solution, then $Sgn\, det\, \tilde{\Phi}'_v$ equals to $sgn\, det\, (\mathbf{F} - I)$.*

Sketch of the proof. The linear map $(\mathcal{U}(t))^{-1}$, where $\mathcal{U}(t) = \mathcal{U}(\hat{v}(t), v(t))$ as in Part 2, sends $\xi(t)$ as in (6.1) to a solution $\eta(t) = (\mathcal{U}(t))^{-1}\xi(t) \in \mathbb{R}^n$ of the equation

$$\dot{\eta}(t) + g(t)\eta(t) = 0, \tag{6.2}$$

where g is the same as in the formula for the map Ψ, $\Psi(w) = \dot{w} + gw$ (see calculations in the proof of Lemma 2.2). We denote by F_1 the Floquet map for (6.2) and observe that $1 \notin \sigma(F_1)$ since v is a nondegenerate solution and that $\det(\mathbf{F} - I) = \det(F_1 - I)$ since $\mathbf{F}$ and F_1 are conjugated.

Take any nondegenerate deformation $g_\lambda(t)$, $0 \le \lambda \le 1$, of $g_1(t) = g(t)$ to $g_0 \equiv \gamma > 0$ and denote by F_λ the Floquet map of equation (6.2) with g replaced by g_λ. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the set of all λ such that $1 \in \sigma(F_\lambda)$. Then $0 < \lambda_1$, $\lambda_N < 1$ and 1 is a single eigenvalue of each F_{λ_j} since the deformation is nondegenerate. For $\lambda = 0$ we have

$$\text{Sgn det}\left(\frac{\partial}{\partial t} + g_0\right) = \text{sgn det}\, F_0 = +1. \tag{6.3}$$

When λ crosses some λ_j, both *sgn det* and *Sgn det* change their signs – this is obvious for $sgn\, det\, F_\lambda$ and for $Sgn\, det\, \left(A_\lambda := \frac{\partial}{\partial t} + g_\lambda\right)$ the assertion follows from two observations:

a) $\operatorname{Ker} A_\lambda \neq 0$ if and only if λ equals to some λ_j, besides $\dim \operatorname{Ker} A_{\lambda_j} = 1$.

b) the L-S index of $A_0^{-1}A_\lambda$ equals $(-1)^\beta$, where β stands for the number of negative real eigenvalues of $A_0^{-1}A_\lambda$ (see [11]). When λ crosses some λ_j, β changes by one.

Now the lemma's assertion follows from (6.3). □

The degree $\deg \pi$ of the projection π is well-defined and the same arguments as in Part 4 show that it equals $\chi(\mathcal{N})$. By Lemma 6.1 $\deg \pi(R)$ equals to the algebraic number of contractible solutions of (3). So we get

Theorem 6.1. *For each nondegenerate vectorfield R the algebraic number of contractible solutions of (3) equals $\chi(\mathcal{N})$.*

In the theorem each solution is counted with the sign equal to the sign of determinant of the corresponding Floquet map minus identity and a vectorfield $R = R_0$ is nondegenerate if it is a regular value of π, i.e. if all contractible solutions of (3) smoothly depend on R close to R_0. The set of all regular R is open dense.

We recall that by the Lefschetz theorem the algebraic number of *all* solutions of (3) also equals $\chi(\mathcal{N})$.

Appendix 1. Some results on first-order elliptic equations and Sobolev spaces

1. Sobolev spaces (see [1]).

We systematically use Sobolev spaces $W^{m,p}(\mathfrak{N},\mathcal{N})$ $(m=0,1,\ldots,1<p<\infty)$ of maps defined in an N-dimensional manifold $\mathfrak{N}$ and valued in a linear space, $\mathcal{N}=\mathbb{R}^M$, or in a compact complex manifold, $\mathcal{N}=\mathfrak{M}$. The spaces $W^{m,p}(\mathfrak{N},\mathbb{R}^M)$ are classical. The easiest way to define $W^{m,p}(\mathfrak{N},\mathfrak{M})$ is to embed $\mathfrak{M}$ to some $\mathbb{R}^M$ and to set

$$W^{m,p}(\mathfrak{N},\mathfrak{M})=\{u\in W^{m,p}(\mathfrak{N},\mathbb{R}^M)\mid u(z)\in\mathfrak{M}\ \text{ for a.a. } z\in\mathfrak{N}\}. \tag{A1}$$

This definition provide $W^{m,p}(\mathfrak{N},\mathfrak{M})$ with a metric induced from $W^{m,p}(\mathfrak{N},\mathbb{R}^M)$.

We abbreviate the notation of a Sobolev space to $W^{m,p}$ (or $W^{m,p}(\mathfrak{N})$) if $\mathfrak{N}$ and $\mathcal{N}$ (or $\mathfrak{N}$) are fixed for a moment, and write H^m instead of $W^{m,2}$. For a bundle $E\to\mathfrak{N}$ by $W^{m,p}(E)$ we denote the subspace of $W^{m,p}(\mathfrak{N},E)$ formed by sections of the bundle (by $\Gamma(E)$ – the space of the sections and by $C^k(E)$ – the space of C^k-smooth sections).

By the *Sobolev embedding theorem*,

$$W^{m,p}(\mathfrak{N},\mathfrak{M})\subset C^k(\mathfrak{N},\mathfrak{M})\ \text{ if } m>k+\frac{n}{p}\,.$$

This result is classical in the linear case $\mathfrak{M}=\mathbb{R}^M$. The nonlinear case ($\mathfrak{M}$ is a manifold) follows from the linear one due to the definition (A1).

In particular, for the trivial bundle $E=\mathbb{T}^2\times\mathfrak{M}\to\mathbb{T}^2$ we have

$$H^{\tilde m}(E)\subset C^{m+2}(E)\ \text{ if }\ \tilde m-m-2>\frac{1}{2}(2n+2)$$

or, equivalently, if $\tilde m-m\geq n+4$ since both $\tilde m$, m are integer.

If $\mathfrak{N}$ is compact, then the embedding $W^{m+1,p}(\mathfrak{N},\mathfrak{M})\subset W^{m,p}(\mathfrak{N},\mathfrak{M})$ is compact by the *Rellich theorem* (again the nonlinear case follows from the linear one).

2. Linear equations.

Let $Q_1\Subset Q_2\Subset\mathbb{C}$ be open domains and

$$u_{\bar z}(z)=w(z) \tag{A2}$$

in Q_2, where u and w are valued in $\mathbb{C}$ or $\mathbb{C}^n$. The *Shauder estimate* for the Cauchy-Riemann operator states that

$$\|u\|^{Q_1}_{m+1,p}\leq C\left(\|w\|^{Q_2}_{m,p}+\|u\|^{Q_2}_{0,p}\right),\ 1<p<\infty, \tag{A3}$$

where $\|\cdot\|^{Q}_{m,p}$ is the norm in $W^{m,p}(Q)$ and $C=C(p,Q_1,Q_2)$. The easiest way to prove the estimate is to apply the operator $(\partial/\partial z)=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$ to (A2) to get

$\frac{1}{4}\Delta u = (\partial/\partial z)w(z)$. Now the estimate follows from its well-known analogy for the Laplace operator.

In particular, if $u(z)$ satisfies (A2) in $\mathbb{C}$ and is double-periodic in z, i.e., if $z \in \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, then

$$\|u\|_{m+1,p} \le C(\|w\|_{m,p} + \|u\|_{0,p}),\ 1 < p < \infty,$$

where now $\|\cdot\|_{m,p}$ stands for the norm in $W^{m,p}(\mathbb{T}^2)$. Direct decomposition in Fourier series shows that if $w \in L_2(\mathbb{T}^2)$ then (A2) is solvable if and only if $\int w\,d^2z = 0$. Besides, the solution $u(z)$ with zero mean-value is unique and $\|u\|_{1,2} \le C\|w\|_{0,2}$. The last two estimates jointly imply that if $w(z) \in W^{m,p}(\mathbb{T}^2)$ $(m \ge 0,\ p \ge 2)$ has zero mean-value, then (A2) has the only solution with zero mean-value and

$$\|u\|_{m+1,p} \le C\|w\|_{m,p}.$$

As $W^{1,p}(\mathbb{T}^2) \subset C(\mathbb{T}^2)$ for $p > 2$, then each function $u(z)$, $\int u\,d^2z = 0$, which satisfies (A2) meets the estimate

$$|u|_{L^\infty} \le C|w|_{L^\infty}.$$

3. Nonlinear equations.

Let $\mathfrak{M} = \mathfrak{M}^n$ be any complex manifold and $f : \mathbb{T}^2 \times \mathfrak{M} \ni (z,u) \mapsto f \in T_u\mathfrak{M}$ be a C^k-smooth nonautonomous vectorfield on $\mathfrak{M}$. Suppose that a C^1-smooth $u : \mathbb{T}^2 \to \mathfrak{M}$ solves the equation

$$u_{\bar z}(z) = f(z,u).$$

Then $u \in W^{k+1,p}(\mathbb{T}^2;\mathfrak{M})$ for each $p < \infty$.

Indeed, the assertion is a local statement. Locally near $z_0 \in \mathbb{T}^2$ and $u_0 = u(z_0) \in \mathfrak{M}$, we can treat u as a $\mathbb{C}^n$-valued map. As $u \in C^1$, then $g(z) := f(z,u(z)) \in W^{1,p}(Q_1)$ in a small neighborhood Q_1 of z_0. By (A2), $u \in W^{2,p}(Q_2), Q_2 \Subset Q_1$, and $f \in W^{2,p}(Q_2)$ (see the Lemma in Part 4 below). We can iterate these arguments to get that $u \in W^{k+1,p}(Q_{k+1})$ in a neighborhood Q_{k+1} of z_0, and the assertion follows.

If $\mathfrak{M}$ is compact then $\|u\|_{k+1,p}$ depends only on the C^k-norm of f. Therefore if

$$\frac{\partial u_n}{\partial \bar z} = f_n(z,u_n),\ \ |f_n|_{C^k} \le C, \tag{A4}$$

then the sequence $\{u_n\}$ is uniformly bounded in $W^{k+1,p} = W^{k+1,p}(\mathbb{T}^2,\mathfrak{M})$. So it is precompact in $W^{k,p}$ and $u_{n_j} \to u \in W^{k,p}$. We suppose $\mathfrak{M}$ be embedded to some $\mathbb{R}^M$ (see (A1)). Then also $u_{n_j} \rightharpoonup v$ weakly in $W^{k+1,p}(\mathbb{T}^2;\mathbb{R}^M)$. We must have $u = v$, so $u \in W^{k+1,p}$.

If $f_n \to f$ in the C^k-norm then one can go to a limit in (A4) to get $u_{\bar z} = f(z,u)$.

4. Nonlinear differential operators in Sobolev spaces.

Here we discuss smoothness of the map $\tilde\Phi$ we study in Parts 2-4,

$$\begin{aligned}&\tilde\Phi : W \times \mathcal{F} \to H^2(\mathbb{T}^2;\mathbb{C}^n),\\ &\qquad (u(z), f(z,u)) \mapsto \mathcal{U}(\hat u(z), u(z))u_{\bar z} + \mathcal{U}(\hat u(z), u(z))\, f(z,u)\end{aligned}$$

(the spaces $W, \mathcal{F}$ are defined in Part 2).

For an n-dimensional manifold $\mathfrak{N}$ we denote by $C^{k,l}(\mathfrak{N} \times \mathbb{R}^{N_1};\mathbb{R}^{N_2})$ $(0 \le k, l \le \infty)$ the space of maps $g(z,u) : \mathfrak{N} \times \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$ which are C^k-smooth in (z,u) with all derivatives in u up to the order l (so $C^{k,0} = C^k$). We endow this space with the natural Banach norm if both k, l are finite.

Lemma. *If $f(x,u)\in C^{2,k}(\mathfrak{N}\times\mathbb{R}^{N_1},\mathbb{R}^{N_2})$, then the map*

$$H^2(\mathfrak{N};\mathbb{R}^{N_1})\ni u(x)\mapsto f(x,u(x))\in H^2(\mathfrak{N};\mathbb{R}^{N_2})$$

is C^k-smooth. If $k<\infty$, then the map

$$C^{2,k}(\mathfrak{N}\times\mathbb{R}^{N_1};\mathbb{R}^{N_2})\times H^2(\mathfrak{N};\mathbb{R}^{N_1})\to H^2(\mathfrak{N};\mathbb{R}^{N_2}),\ (f,u)\mapsto f(\cdot,u(\cdot)),$$

also is C^k-smooth.

The first assertion is well-known and follows from the Sobolev embedding theorem. The second one results from the first since the map is linear continuous in f.

To prove that $\tilde{\Phi}$ is smooth we embed the compact complex manifold $\mathfrak{M}$ to some $\mathbb{R}^N$, denote by $\tilde{\mathfrak{M}}$ a tubular neighborhood of $\mathfrak{M}$ in $\mathbb{R}^N$ and by π the projection $\tilde{\mathfrak{M}}\to\mathfrak{M}$. We also denote by $\mathcal{Q}$ the neighborhood of $H^3(\mathbb{T}^2,\mathfrak{M})$ in $H^3(\mathbb{T}^2,\mathbb{R}^N)$ formed by maps which send $\mathbb{T}^2$ to $\tilde{\mathfrak{M}}$.

We identify $T\mathbb{R}^N$ with $\mathbb{R}^N\times\mathbb{R}^N$, extend the operators $\mathcal{U}(\hat u,u)$ to the linear maps $\tilde{\mathcal{U}}(\hat u,u):\mathbb{R}^N\to\mathbb{C}^n$ which vanish in the tangent spaces to fibers of π and treat $\tilde{\mathcal{U}}$ as a matrix-valued function, defined for $u\in\mathfrak{M}$ close to $\hat u$ and smooth in u.

Finally we smoothly extend $\tilde{\mathcal{U}}$ as a function of u from a neighborhood of $\tilde u(z)$ in $\mathfrak{M}$ to all $\mathbb{R}^N$ [8] to get

$$\tilde{\mathcal{U}}\in C^{2,\infty}(\mathbb{T}^2\times\mathbb{R}^N,\mathbb{R}^{2nN}),\quad \tilde{\mathcal{U}}:(z,u)\mapsto\tilde{\mathcal{U}}(\hat u(z),u).$$

We extend the complex-structure operator $\xi\mapsto i\xi$ defined in $T_m\mathfrak{M}\subset\mathbb{R}^N$ to the operator $J_{\tilde m}$ in $T_{\tilde m}\mathbb{R}^N\simeq\mathbb{R}^N$ which vanishes at the tangent space to the fiber of π and at the space orthogonal to this one[9] is conjugated with the complex-structure operator in $T_{\pi(\tilde m)}\mathfrak{M}$ by means of the map π_*. Clearly $J_{\tilde m}$ is smooth in $\tilde m\in\tilde{\mathfrak{M}}$. Now we can extend $(\partial/\partial\bar z)$ to the operator

$$\mathcal{Q}\ni u\mapsto\bar D_J u:=\frac{1}{2}(u_x+J(u)u_y)$$

and extend $\tilde{\Phi}$ to the map

$$\begin{aligned}&\Phi_1:\mathcal{B}\times\mathcal{F}\to H^2(\mathbb{T}^2;\mathbb{C}^n),\\ &\qquad(u,f)\mapsto\tilde{\mathcal{U}}((\hat u(z),u(z))\circ\bar D_J u+\tilde{\mathcal{U}}((\hat u(z),\pi u(z))\circ f(z,\pi(u(z))),\end{aligned}$$

where $\mathcal{B}$ is a neighborhood of $\hat u$ in $\mathcal{Q}$.

By the Sobolev theorem $\mathcal{F}\subset C^4(\pi^*T\mathfrak{M})$. Therefore the map

$$\mathcal{F}\to C^4(\mathbb{T}^2\times\tilde{\mathfrak{M}},\mathbb{R}^N),\quad f\mapsto g,\quad g(z,u)=f(z,\pi(u))$$

is a bounded embedding. Now we decompose Φ_1 as

$$\Phi_1:\mathcal{B}\times\mathcal{F}\subset\mathcal{B}\times C^4(\mathbb{T}^2\times\mathfrak{M},\mathbb{R}^N)\xrightarrow{\Phi_2}H^2(\mathbb{T}^2;\mathbb{C}^n),$$

[8] new $\tilde{\mathcal{U}}$ can be chosen equal to $\tilde{\mathcal{U}}(\hat u(z),\pi(u))$ times $\phi(u-\hat u(z))$, where $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ equals one near zero.

[9] with respect to the Euclidean scalar product in $\mathbb{R}^N$

where Φ_2 is given by the same formula as Φ_1 but with $f(z,\pi(u))$ replaced by $g(z,u)$.

By the lemma above the maps

$$(u,g) \mapsto \tilde{\mathcal{U}}(\hat{u}(z),u(z)),\quad \bar{D}_J u,\quad \tilde{\mathcal{U}}(\hat{u}(z),\pi u(z)),\quad g(z,u)$$

are C^2-smooth maps from $\mathcal{B}\times C^4$ to H^2. Since multiplication defines smooth maps in $H^2(\mathbb{T}^2)$ by the same lemma, then the map Φ_2 also is C^2-smooth.

Finally we get the result we use in Part 2:

$$\text{if } \tilde{m} \geq n+6, \text{ then the maps } \tilde{\Phi} \text{ and } \Phi_1 \text{ are } C^2\text{-smooth.}$$

Appendix 2. Proof of Lemma 2.3

Take any two charts $\mathcal{O}_1, \mathcal{O}_2$ centered at $\xi \in \mathfrak{A}_V$ and denote corresponding spaces in decomposition (3.1) as $H_1^j, H_2^j, Z_1^j, Z_2^j$ $j=1,2$, and corresponding spaces $\mathcal{F}_1, \mathcal{F}_2$ as $\mathcal{F}_1^j, \mathcal{F}_2^j$, $j=1,2$.

We define

$$\mathcal{F}_1^0 := \mathcal{F}_1^1 + \mathcal{F}_1^2,\quad Z_1^0 := Z_1^1 + Z_1^2 = \tilde{\Phi}'_f(\mathcal{F}_1^0),\quad H_1^0 := H_1^1 + H_1^2.$$

We set $\mathcal{F}_2^0 = \mathcal{F} \ominus \mathcal{F}_1^0$ and find spaces H_2^0 and Z_2^0 to meet (3.1) with $H_1 = H_1^0, Z_1 = Z_1^0$. Now we construct the corresponding chart $\mathcal{O}_0$, centered at ξ. Due to the transitivity it is sufficient to check that orientations of $\mathcal{O}_1$ and $\mathcal{O}_2$ agree with $\mathcal{O}_0$.

Denote

$$\mathcal{F}_2^{\triangle} = \mathcal{F}_2^1 \ominus \mathcal{F}_2^0,\quad H_1^{\triangle} = \left\{ v(x) \in H_1^0 \mid \operatorname{Re}\int v(z)\bar{w}(z)\,d^2z = 0 \quad \forall w(z) \in H_1^1 \right\}.$$

To construct coordinates in $\mathcal{O}^1$ (in $\mathcal{O}^0$) we express $(u,f) \in \mathfrak{A}_V$ in terms of ($u_1 \in H_1^1$, $f_2 \in \mathcal{F}_2^1 = \mathcal{F}_2^0 \oplus \mathcal{F}_2^{\triangle}$) (in terms of ($u_1 \in H_1^0 = H_1^1 \oplus H_1^{\triangle}$, $f_2 \in \mathcal{F}_2^0$)). Therefore to pass from $\mathcal{O}^1$ to $\mathcal{O}^0$ we should pass from $f^{\triangle} \in \mathcal{F}_2^{\triangle}$ to $u^{\triangle} \in H_1^{\triangle}$ with the rest of the variables fixed. To do so we shall use the equation $\Phi(u,f)=0$ to find $u^{\triangle}$ as a function of $f^{\triangle}$.

To make the calculations more transparent we make two simplifying assumptions:

1) $\xi = (0,f) \in \mathfrak{A}_V$,

2) $\mathcal{F}_1^1 = \{0\}$.

That is, both charts are centered at $(0,f)$ and we should pass from $f_2 \in \mathcal{F}_2^0 \oplus F_2^{\triangle}$ in the chart $\mathcal{O}_1$ of the main type to $(f_2 \in \mathcal{F}_2^0, u_1 \in H_1^{\triangle})$ in the chart $\mathcal{O}_0$.

Since $\mathcal{F}_1^1 = \{0\}$, then now the map

$$\tilde{\Phi}'_u(0,f) = \Psi : H^3(\mathbb{T}^2;\mathbb{C}^n) \underset{\sim}{\longrightarrow} H^2(\mathbb{T}^2;\mathbb{C}^n)$$

is an isomorphism.

We denote

$$H_1^{\triangle} = H^{\triangle},\quad \mathcal{F}_2^{\triangle} = \mathcal{F}^{\triangle}$$

and decompose the spaces $H^3 = H^3(\mathbb{T}^2;\mathbb{C}^n)$ and $H^2 = H^2(\mathbb{T}^2;\mathbb{C}^n)$ as follows:

$$H^3 = H^{\triangle} \oplus H^{\square},\quad H^2 = Z^{\triangle} \oplus Z^{\square},$$

where $Z^{\square} = \Psi H^{\square}$.

Lemma. *If in Definitions 3.1, 3.2 we replace the operator A_0 by any operator A_1 such that $(A_1 - (\partial/\partial\bar{z})) : H^3 \underset{\sim}{\longrightarrow} H^2$ is a compact map and A_1 defines an isomorphism of H^3 and H^2, then each $Sgn\, det_{\mathrm{rel}}$ will change by a factor $(-1)^\beta$ where β is the L-S index of $A_0^{-1}A_1$. Correspondingly orientations of all the charts will be changed by the same factor.*

Proof. The statement follows from the multiplicativity of the L-S index. Indeed,

$$\begin{aligned}\text{Sgn det } \tilde{\Phi}'_u = \text{L-S index } (A_0^{-1}\Psi) &= \text{L-S index } (A_0^{-1}A_1A_1^{-1}\Psi) = \\ &= \text{L-S index } A_0^{-1}A_1 \cdot \text{L-S index } A_1^{-1}\Psi.\end{aligned}$$

The index of $A_0^{-1}A_1$ is well-defined since this operator equals $I + A_0^{-1}(A_1 - A_0)$ and the second term is a compact operator. □

Due to the lemma to prove that the orientations of $\mathcal{O}^1$ and $\mathcal{O}^0$ agree we can replace in the definition of the orientation the operator A_0 by Ψ. Then the orientation of $\mathcal{O}_1$ is given by $Sgn\, det\, \Psi = L\text{-}S\, index\, I = 1$. The orientation of $\mathcal{O}_0$ is given by

$$\begin{aligned}&\text{L-S index}\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle} \times \frac{\partial\tilde{\Phi}^\square}{\partial u^\square}\right)^{-1} \cdot \left(L \times \frac{\partial\tilde{\Phi}^\square}{\partial u^\square}\right) = \\ &\text{L-S index}\left[\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle}\right)^{-1} \cdot L \times I\right] = \text{sgn}\left(\det\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle}^{-1} \cdot L\right)\right),\end{aligned}$$

where $\tilde{\Phi} = \tilde{\Phi}^\triangle \oplus \tilde{\Phi}^\square \in Z^\triangle \oplus Z^\square$, $L = -\frac{\partial\tilde{\Phi}^\triangle}{\partial f^\triangle} \cdot M$ and M is a linear isomorphism $M : H^\triangle \underset{\sim}{\longrightarrow} \mathcal{F}^\triangle$. Therefore the orientations differ by the factor

$$\det\left(-\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle}\right)^{-1}\frac{\partial\tilde{\Phi}^\triangle}{\partial f^\triangle}M\right). \tag{$*$}$$

To pass from coordinates in $\mathcal{O}_1$ to coordinates in $\mathcal{O}_0$ we should express $u^\triangle \in H^\triangle$ in terms of $f^\triangle \in \mathcal{F}^\triangle$. To do it we use the equation $\tilde{\Phi}^\triangle = 0$. Taking derivatives of $\tilde{\Phi}^\triangle \in Z^\triangle$ with respect to $f^\triangle$ and using that $\partial\tilde{\Phi}^\triangle/\partial u^\square(0, f) = 0$ since $\tilde{\Phi}'_u H^\square = Z^\square$, we find that

$$\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle} \cdot \frac{\partial u^\triangle}{\partial f^\triangle} + \frac{\partial\tilde{\Phi}^\triangle}{\partial f^\triangle} = 0$$

and

$$\frac{\partial u^\triangle}{\partial f^\triangle} = -\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle}\right)^{-1}\frac{\partial\tilde{\Phi}^\triangle}{\partial f^\triangle}.$$

The transition from the chart $\mathcal{O}_1$ to $\mathcal{O}_0$ is given by the map $f = (f^\triangle, f^\square) \mapsto (M \circ u^\triangle, f^\square)$. Its Jacobian equals

$$\det\left(M\frac{\partial u^\triangle}{\partial f^\triangle}\right) = \det\left(-M\left(\frac{\partial\tilde{\Phi}^\triangle}{\partial u^\triangle}\right)^{-1}\frac{\partial\tilde{\Phi}^\triangle}{\partial f^\triangle}\right).$$

This number is equal to $(*)$. So the transition map indeed adjusts the orientations. The lemma is proven. □

Appendix 3. Proof of Proposition 2.1 (break of compactness)

We represent the torus $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ as the square $K=\{z\in\mathbb{C}\mid |\mathrm{Re}z|\le 1/2,\ |\mathrm{Im}|\le 1/2\}$ with identified opposite sides. To prove that the map $\tilde\pi$ is not proper we shall construct a sequence $(u_n,f_n)\in\tilde{\mathfrak{A}}_V$, $n=3,4,\dots$ such that

1) $f_n:\mathbb{T}^2\times S^2\to TS^2$ are uniformly bounded in the $C^{\tilde m}$-norm,

2) $|du_n(0)|\to\infty$ as $n\to\infty$.

Let us denote $D_\varepsilon=\{z\mid |z|\le\varepsilon\}\subset K$, where $\varepsilon=n^{-1}$. We identify S^2 with $\mathbb{C}\cup\{\infty\}$ and define $u_n^0\mid_{D_\varepsilon}: D_\varepsilon\to S^2$ as

$$u_n^0(z)=\varepsilon^{-r-1}z\in\mathbb{C}\subset S^2,\quad z\in D_\varepsilon,$$

where $r=\tilde m+1$. Then for $z\in\gamma_\varepsilon=\partial D_\varepsilon$ we have $|u_n^0(z)|=\varepsilon^{-r}$. So to study $u_n(z)$ near ∂D_ε we should write u_n^0 in the chart $\mathbb{C}_*\cup\{\infty\}=\{\mathfrak{z}\}\subset S^2$. We have

$$\mathfrak{z}_n^0(z)=(u_n^0(z))^{-1}=\varepsilon^{r+1}z^{-1}.$$

Let $\tilde\Phi(r)$ be a smooth function, equal 0 for $r\ge 3$ and 1 for $r\le 2$. We extend $u_n^0: D_\varepsilon\to S^2$ to a map $u_n: K\to S^2$ in the following way:

$$\mathfrak{z}_n\equiv u_n^{-1}=\begin{cases}\tilde\Phi(\varepsilon^{-1}|z|)\,\mathfrak{z}_n^0, & |z|\le 4\varepsilon,\\ 0, \quad 4\varepsilon\le|z|\le\dfrac12, & \\ w(z)^{-1}, \quad \dfrac12\le|z|, & \end{cases}\tag{$*$}$$

where w is a smooth map $w: A=\{z\in K\mid |z|\ge\frac12\}\to S^2$, equal 0 near ∂K and equal ∞ near $|z|=\frac12$.

The map $u_n: K\to S^2$ is C^r-smooth in K, holomorphic in D_ε and

$$|du_n(0)|=\varepsilon^{-r-1},\quad \|u_n\mid_{K\setminus D_\varepsilon}\|_{C^r}\le C.$$

In particular, the maps $(\partial/\partial\bar z)u_n\in\Gamma(u_n^*TS^2)$ are uniformly bounded in C^{r-1}-norm.

Besides these maps are contractible to $0\in S^2$ by means of the homotopy $u_n^t(z)=u_n(tz)$, $t\in[0,1]$, where we extend $u_n(z)$ as zero to $\mathbb{C}\setminus K$.

Now we pass to construction of vectorfields $f_n(z,u)$ such that

$$\frac{\partial u_n}{\partial\bar z}=f_n(z,u_n).\tag{$**$}$$

Let us fix any smooth map $U: TS^2\to C^r(TS^2)$, linear in fibers of the tangent bundle TS^2 and such that

$$U(u,\xi)(u)=\xi\quad\forall\,(u,\xi)\in TS^2.$$ [10]

[10] one can use any smooth connection to spread the vector ξ to the spaces T_vS^2, v close to u, along the minimal geodesics, and next multiply $\xi_v\in T_vS^2$ by a smooth function $\varphi(v)$, supported near u and equal one at u.

Now we set $f_n(z,u) = U(u_n(z),\ (\partial/\partial\bar{z})u_n(z))u$. The relation $(**)$ clearly holds. As the maps $(\partial/\partial\bar{z})u_n$ are uniformly bounded in the $C^{r-1} = C^{\tilde{m}}$-norm, then the sequence $(u_n, f_n) \in \mathfrak{A}_V$ possesses the properties 1)-2) we need to prove Proposition 2.1.

Remark. In agreement with the general compactness theorem (see [4]), the graphs $\tilde{u}_n(\mathbb{T}^2)$, $\tilde{u}_n(z) = (z, u_n(z))$ converge to a cusp-curve in $\mathbb{T}^2 \times S^2$ (which is not a graph). Indeed, let us extend the map w as zero for $y \in K \setminus A$. Then – see $(*)$ – the tori $\tilde{u}_n(\mathbb{T}^2)$ converge to the cusp-curve $\tilde{w}(\mathbb{T}^2) \cup (\{0\} \times S^2)$, where $\tilde{w}$ is the graph of w.

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