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Pseudoholomorphic 2-tori in $\mathbb{T}^4$

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Abstract: We study the Hilbert manifold formed by all pairs $\langle$*almost complex structure on the torus* $\mathbb{T}^4$, *pseudoholomorphic* 2*-torus in* $\mathbb{T}^4\rangle$ (the homotopy type of the two-tori is fixed). We prove that for a typical structure the number of the pseudoholomorphic two-tori is finite and even. The situations when these numbers are zero and non-zero both happen for open non-empty sets of almost complex structures.

Keywords: Pseudoholomorphic torus, Hilbert manifold, almost complex structure.

MS classification: 53C15.

Introduction

Starting with M. Gromov's work [3], compact pseudoholomorphic (PH) curves in almost complex symplectic manifolds became a powerful tool to study manifold's geometry. Correspondingly manifolds formed by the PH curves were an object of intensive investigation. However, the most attention was given to PH spheres (see e.g. [5]), since exactly the PH spheres and closely related PH discs are involved in the most important geometric applications (cf. [3, 5]).

Our paper is devoted to PH 2-tori. We restrict ourselves to the case when the symplectic manifold is a 4-torus with the standard symplectic structure, which is a good model comprising—we hope—the main difficulties of the problem. We study the Hilbert manifold $\mathfrak{A}$ formed by all pairs

$$\mathfrak{A} = \langle \text{tamed almost complex structure on } \mathbb{T}^4,\ \text{PH 2-torus corresponding to it} \rangle$$

and its projection π to the manifold of tamed almost complex structures (for a definition see below). We show that π is a Fredholm map of zero index such that the image and its complement both have nonempty interior. Therefore a tamed almost complex 4-torus typically has an even number of PH 2-tori. We also show that situations when this number is equal and is unequal to zero are both typical.

The approach to study PH curves for an individual almost complex structures via manifolds like $\mathfrak{A}$ they jointly form is not new—it was proposed in [3] (see [5] for details). Besides, in [9] a direct analogy to this manifold was used to study solutions of nonlinear elliptic equations with potentials of the equations playing the role of the almost complex structures.

Our interest to the subject was evoked by discussing with J. Moser his work [6], devoted to

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noncompact PH curves in $\mathbb{T}^{2n}$ which naturally led to some problems connected with compact PH curves in $\mathbb{T}^{2n}$. The results we present in this paper partially answer the questions which arose during the discussions.

1. Tamed almost complex structures

We supply the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ with the global coordinates $x = (x_1, \dots, x_4)$, with the usual Riemann metric dx^2 and with the symplectic structure $\omega_2 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. Recall that an almost complex structure $J(x)$ in $T\,\mathbb{T}^4$ is *tamed* by ω_2 if

$$\omega_2(\xi, J\xi) \geqslant \text{const}\,|\xi|^2 \quad \forall \xi \in T\mathbb{T}^4.$$

We denote by $\mathcal{C}$ the set of all tamed structures with the matrix coefficients of $J(x)$ in the Sobolev space $W^{M,2}(\mathbb{T}^4)$ with "sufficiently large M" (one can take, for example, $M \geqslant m + 3$, where $m \geqslant 3$ is a priori smoothness of PH curves—see below). The set $\mathcal{C}$ is contractible [3, p. 333] and carries a natural structure of a Hilbert manifold modeled by the space $H_{\mathcal{C}} = W^{M,2}(\mathbb{T}^4; \mathbb{R}^8)$.

Take any Riemann curve (W, j) (j is the complex structure). A map $f : (W, j) \to (\mathbb{T}^4, J)$ is called *pseudoholomorphic* (PH) if

$$f_* \circ j = J \circ f_*. \tag{1}$$

The main for our purposes property of PH curves is an a priori estimate for their areas in homological terms, converted by M. Gromov [3] to an effective tool to study the (nonlinear elliptic) equation (1), which defines the curves.

To derive the estimate we denote by $s + it$ the local conformal coordinate in W. Then by (1)

$$\frac{\partial f}{\partial t} = f_*(i) = Jf_*(1) = J\frac{\partial f}{\partial s}.$$

As J is tamed, then we have

$$\omega_2\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = \omega_2\left(\frac{\partial f}{\partial s}, J\frac{\partial f}{\partial s}\right) = \omega_2\left(-J\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) \geqslant \text{const}|\nabla f|^2.$$

So

$$\int_{f(W)} \omega_2 \geqslant \text{const area}\, f(W), \tag{2}$$

which is the a priory estimate we mentioned.

2. Complex 2-tori

Now take a 2-torus $\mathbb{T}^2$. To give it a complex structure, one can fix period vectors $(\nu, 1)$, where ν is taken from the upper half-plane U, and define $\mathbb{T}^2$ as

$$\mathbb{T}^2_\nu = \mathbb{C}/(\mathbb{Z}\nu + \mathbb{Z})$$

with the natural complex structure. Two ν's equivalent by the action of the unimodular group $SL(2; \mathbb{Z})$ define equivalent complex structures. So the equivalence classes of complex structures are parameterized by points of $M = U/SL(2;\mathbb{Z})$ (see more in [2]) and are represented by points of the fundamental domain $\mathcal{D}$,

$$\mathcal{D} = \{z \in \mathcal{C} \mid |z| \geqslant 1,\ \operatorname{Im} z > 0,\ -\tfrac{1}{2} < \operatorname{Re} z \leqslant \tfrac{1}{2}\}.$$

For our purposes it is convenient to renormalize periods $(\nu, 1)$ and replace them by

$$(\xi, \eta) \quad \text{where} \quad \xi = \nu/\sqrt{\operatorname{Im}\nu}, \qquad \eta = 1/\sqrt{\operatorname{Im}\nu}. \tag{3}$$

Now $\eta \in \mathbb{R}_+$ and the parallelogram of periods has unit area. So

$$\tfrac{2}{3}\pi > \operatorname{Arg}\xi > \tfrac{1}{3}\pi, \qquad |\xi| \geqslant |\eta| \quad \text{and} \quad 1 \leqslant |\xi|\cdot|\eta| \leqslant 2/\sqrt{3}. \tag{4}$$

Below we denote by τ a couple of periods (ξ, η) as above and denote by $\mathcal{T}$ the two-dimensional manifold they form (diffeomorphic to $\mathcal{D}$ via the mapping $(\xi, \eta) \mapsto \xi/\eta)$).

A PH 2-torus in $\mathbb{T}^4$

$$f : (\mathbb{T}^2, \tau) \to (\mathbb{T}^4, J), \qquad \tau = (\xi, \eta) \in \mathcal{T},$$

can be treated as a PH curve $f : \mathbb{C} \to (\mathbb{T}^4, J)$, $f_*(iv) = Jf_*(v)$, with periods ξ, η. The curve should satisfy the equation

$$\overline{D}_J f \equiv J(f)\frac{\partial f}{\partial s} - \frac{\partial f}{\partial t} = 0, \tag{5}$$

where $s + it$ is the complex coordinate in $\mathbb{C}$.

One can describe PH 2-tori (with fixed complex structure $\tau = (\xi, \eta)$) differently. Consider the linear (over reals) map $\Psi_\tau : \mathbb{C} \to \mathbb{C}$, $\xi \mapsto i$, $\eta \mapsto 1$, and the induced map

$$\mathbb{T}^2_\tau = \mathbb{C}/(\xi\mathbb{Z} + \eta\mathbb{Z}) \to \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}).$$

Define a complex structure j_τ in the standard torus $\mathbb{T}^2$ as an image of the standard complex structure in $\mathbb{T}^2_\tau$, $j_\tau = \Psi_{\tau *} i$. Under this representation PH tori can be treated as maps $f : \mathbb{C} \to (\mathbb{T}^4, J)$ with periods $(i, 1)$, but the operator $\overline{D}_J$ should be replaced by the according modification $\overline{D}^\tau_J$,

$$\overline{D}^\tau_J f = \frac{1}{\eta}(J(f) + \operatorname{Re}\xi/\operatorname{Im}\xi)\frac{\partial}{\partial s} - \frac{1}{\operatorname{Im}\xi}\frac{\partial}{\partial t}.$$

3. Manifolds of PH A-tori

Fix any nonzero non-multiple homology class $A \in H_2(\mathbb{T}^4;\mathbb{Z})$ and consider the set $\mathcal{F}_A$ of all maps $f \in W^{m+1,2}(\mathbb{T}^2, \mathbb{T}^4)$, representing the class A. This set is a Hilbert manifold modeled by the space $H_{\mathcal{F}} = W^{m+1,2}(\mathbb{T}^2;\mathbb{R}^4)$.

We call elements of $\mathcal{F}_A$ *A-tori*. An A-torus

$$f : \mathbb{T}^2 = \mathbb{C}/(\xi\mathbb{Z} + \eta\mathbb{Z}) \to \mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$$

admits a simple description in terms of its lift $\tilde{f} : \mathbb{C} \to \mathbb{C}^2$. A map $\tilde{f}$ covers some map f as above if

$$\tilde{f}(z + n\xi + m\eta) \equiv \tilde{f}(z) + n\Xi_1 + m\Xi_2, \qquad \Xi_j \in \mathbb{Z}^4.$$

Homological type of the map f is defined by the integer vectors Ξ_1, Ξ_2. (If two maps f_0, f_1 have lifts with the same vectors Ξ_j, then the map $\tilde{f}_t = t\tilde{f}_0 + (1-t)\tilde{f}_1$ is the lift of a homotopy $f_t : \mathbb{T}^2 \to \mathbb{T}^4$.)

The group $H_2(\mathbb{T}^4; \mathbb{Z})$ is a free abelian group generated by $c_k \otimes c_e$, $k < e$, where $c_1, \dots, c_4$ is the natural basis of the first homology group. The class $[f] \in H_2(\mathbb{T}^4, \mathbb{Z})$ of the map f equals

$$[f] = \sum_{k<e} \left(\Xi_{1k}\Xi_{2e} - \Xi_{1e}\Xi_{2k}\right) c_k \otimes c_e. \tag{6}$$

Indeed, for each $k < e$ we have $\langle [f],\ dx_k \wedge dx_e\rangle = \langle \mathbb{T}^2,\ f^*(dx_k \wedge dx_e)\rangle = \Xi_{1k}\Xi_{2e} - \Xi_{1e}\Xi_{2k}$, and (6) follows.

The group $G = \mathbb{T}^2$ acts on $\mathbb{T}^2$ by translations and acts on $\mathcal{F}_A$ by shifting the argument.

Lemma 1. *The group G acts on $\mathcal{F}_A$ freely, i.e., each orbit of G in $\mathcal{F}_A$ is in one-to-one correspondence with G.*

Proof. First we check that for each $f \in \mathcal{F}_A$ the map $g \mapsto f(\cdot + g) \in \mathcal{F}_A$ has rank two at each point $g \in G$. Suppose that it is not the case. Then there exists a nonzero vector $\xi \in T_g G \simeq \mathbb{R}^2$ such that $\partial/\partial t\ |_{t=0}\ f(z+t\xi) = f_*(z)\xi \equiv 0$ as a map from $\mathbb{T}^2$ to $\mathbb{R}^4$. So if the curve $t \mapsto z + t\xi \in \mathbb{T}^2$ is dense in $\mathbb{T}^2$, then $f(z) \equiv$ const. If this curve defines a loop in the torus, then the image of the map f is one-dimensional. In both cases the map f defines zero element of $H_2(\mathbb{T}^4; \mathbb{Z})$ in a contradiction with the choice of the class A.

Now we see that an isotropy group of f is a discrete lattice Γ_f in G. The cycle $f(\mathbb{T}^2)$ in $\mathbb{T}^4$ has multiplicity $\#\Gamma_f$. So $\#\Gamma_f = 1$ and the lemma is proven. $\square$

We consider the factor-space $\mathcal{F}_A^0 = \mathcal{F}_A/G$ and call its elements *nonparameterized A-tori.*

Denote by V the Sobolev space $V = W^{m,2}(\mathbb{T}^2; \mathbb{R}^4)$ and consider the map

$$\Phi : \mathcal{F}_A \times \mathcal{T} \times \mathcal{C} \to V, \quad (f, \tau, J) \mapsto \overline{D}_J^\tau f.$$

Its zero-set

$$\mathfrak{A} = \mathfrak{A}_A = \Phi^{-1}(0) \subset \mathcal{F}_A \times \mathcal{T} \times \mathcal{C}$$

is formed by parameterized PH A-tori jointly with the corresponding complex and almost complex structures. Observe that an application of a priori estimate (2) to $(f, \tau, J) \in \mathfrak{A}$ implies that

$$\langle A, \omega_2\rangle = \int_{f(\mathbb{T}^2)} \omega_2 \geqslant \text{const area } f(\mathbb{T}^2) \tag{7}$$

with a J-dependent const, which can be chosen J-independent for J from a compact part of $\mathcal{C}$.

The group G naturally acts on $\mathfrak{A}$. We denote by $\mathfrak{A}_0$ the factor-space $\mathfrak{A}_0 = \mathfrak{A}/G$ of nonparameterized PH A-tori and denote by π, π_0 the natural projections

$$\pi : \mathfrak{A} \to \mathcal{C}, \quad \pi_0 : \mathfrak{A}_0 \to \mathcal{C}.$$

Theorem 1. *The maps π and π_0 are proper.*

As $G = \mathbb{T}^2$ is compact, then the statement for π follows from the one for π_0. The latter is a particular case of Gromov's compactness theorem [3]. We present its proof (rather traditional by now, cf. [5, 8]) in Appendix.

Corollary 1. *The image $\mathcal{C}_0 = \pi(\mathfrak{A}) = \pi_0(\mathfrak{A}_0)$ is a closed subset of $\mathcal{C}$.*

The subset $\mathcal{C}_0$ turns out to be nontrivial:

Proposition 1. *The set $\mathcal{C}_0$ does not coincide with $\mathcal{C}$ (so $\mathcal{C} \setminus \mathcal{C}_0$ is an open nonempty set). It has a nontrivial interior.*

Proof. The proof is based on properties of two almost complex structures from $\mathcal{C}$, studied in [6]. Below we repeat corresponding arguments from this paper.

1) ($\mathcal{C}_0 \neq \mathcal{C}$). Take for $(\mathbb{T}^4, J)$ a 4-torus with the constant complex structure, $\mathbb{T}^4 = \mathbb{T}^4_\zeta = \mathbb{C}^2/\mathcal{L}$, $\mathcal{L} = \mathbb{Z}\zeta_1 + \cdots + \mathbb{Z}\zeta_4$, where $\zeta_1, \ldots, \zeta_4$ form a real basis of $\mathbb{C}^2$. Now take any PH 2-torus $f : (\mathbb{T}^2, \tau) \to \mathbb{T}^4_\zeta$, $\tau \in \mathcal{T}$. We renormalize periods $\tau = (\nu, \eta)$ to achieve $\eta = 1$ (see Part 1). The map f defines its suspension $\tilde{f} : \mathbb{C} \to \mathbb{C}^2$ which is a holomorphic map such that

$$\tilde{f}(z + m + n\nu) \equiv \tilde{f}(z) + m\Xi_1 + n\Xi_2, \qquad \Xi_j \in \mathcal{L}. \tag{8}$$

By (8) the map $\partial \tilde{f}(z)$ is bounded. So it is constant and $\tilde{f}(z) = a + \mu z$, $\mu \in \mathbb{C}^2$. Again due to (8) we have

$$\mu \in \mathcal{L}, \qquad \nu\mu \in \mathcal{L}. \tag{9}$$

Denote by W the manifold formed by all bases of $\mathbb{C}^2$ over reals and identify a lattice $\mathcal{L}$ with the corresponding point $\zeta \in W$. For different nonzero $s, l \in \mathbb{Z}^4$ the relation

$$(s_1\zeta_1 + \cdots + s_4\zeta_4) \text{ is parallel in } \mathbb{C}^2 \text{ to } (l_1\zeta_1 + \cdots + l_4\zeta_4) \tag{10}$$

defines a subset of W of codimension two. Thus for a typical (in the measure-sense) point of W the last relation is violated for all nonzero $s \neq l \in \mathbb{Z}^4$. For the corresponding lattice $\mathcal{L}$ the relation (9) is impossible. Therefore a torus $\mathbb{T}^4_\zeta$ with typical constant-coefficient complex structure has no PH 2-tori. This complex structure clearly belongs to $\mathcal{C}$—the form ω_2 is just its Kählerian form.

2) (interior of $\mathcal{C}_0 \neq \emptyset$). We can find an unimodular isomorphism of the torus $\mathbb{T}^4$ which sends the nonmultiple class A to the class given by the embedded 2-torus $\{(z_1, z_2) \in \mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2) \mid z_2 = 0\}$. With A normalized as above we shall produce $J_0 \in \mathcal{C}$ such that for all J close to J_0 the torus $(\mathbb{T}^4, J)$ has a PH A-torus. Moreover, we shall find this A-torus as a graph $\{(z, u(z))\} \subset \mathbb{T}^4 = \mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2)$, with some $u : \mathbb{T}^2 \to \mathbb{T}^2$. Such a torus is an image of the embedding f,

$$f : \mathbb{T}^2 \ni z \to (z, u(z)) \in \mathbb{T}^4. \tag{11}$$

This embedding is PH for *some* complex structure in $\mathbb{T}^2$ if and only if each subspace $f_*(T_z\mathbb{T}^2) \subset T_{f(z)}\mathbb{T}^4$ is J-invariant. One can check (or find the calculations in [6, Part 6b]) that the latter is equivalent to a first-order nonlinear differential equation for the map u:

$$F(u, Du; J) = 0. \tag{12}$$

The operator F defines a smooth map

$$\tilde{F} : W^{m+1,2}(\mathbb{T}^2; \mathbb{T}^2) \times \mathcal{C} \to W^{m,2}(\mathbb{T}^2; \mathbb{R}^2).$$

In a moment we shall produce a structure $J_0 \in \mathcal{C}$, calculate the corresponding equation $F(u, Du; J_0)$ and check that

i) $u_0 \equiv 0$ solves (12) with $J = J_0$;

ii) the map

$$\frac{\partial \tilde{F}}{\partial u}(0, J_0) : W^{m+1,2}(\mathbb{T}^2; \mathbb{R}^2) \to W^{m,2}(\mathbb{T}^2; \mathbb{R}^2) \tag{13}$$

is an isomorphism.

After this the statement we are proving will follow from the implicit function theorem.

Take for J_0 the complex structure given by the operator

$$J_0(z, u)(\delta z, \delta u) = (i\,\delta z, -2i\psi(u)\,\delta\bar{z} + i\,\delta u),$$

where ψ is a smooth complex function on $\mathbb{T}^2$ (clearly $J_0^2 = -1$). Now the map (11) is PH if and only if $\mathbb{T}^2$ is given the natural complex structure induced from $\mathbb{C}$ and

$$\frac{\partial u}{\partial \bar{z}} - \psi(u) = 0.$$

So $\tilde{F}(u, J_0) = \partial u/\partial\bar{z} - \psi(u)$. We specify $\psi(u)$ be equal to $\delta\bar{u}$, $\delta \neq 0$, for u close to $0 \in \mathbb{T}^2$. Then we can take for the solution of (12) $u_0 = 0$, and

$$\frac{\partial \tilde{F}}{\partial u}(u_0, J_0)v = \frac{\partial v}{\partial \bar{z}} - \delta\bar{v}.$$

Now the map (13) is an isomorphism. Indeed, it is Fredholm of zero index; if $v(z)$ belongs to its kernel, then $\partial v/\partial\bar{z} - \delta\bar{v} = 0$ and

$$\tfrac{1}{4}\Delta v = \frac{\partial^2}{\partial z \partial \bar{z}} v = \delta \frac{\partial}{\partial z}\bar{v} = \delta^2 v.$$

The only periodic solution of the equation $-\frac{1}{4}\Delta v + \delta^2 v = 0$ is $v \equiv 0$, so (13) has trivial kernel and cokernel.

If $\delta > 0$ is sufficiently small, then $J_0 \in \mathcal{C}$ and we got a structure with the properties we need. □

4. Fredholm property

Now we pass to smoothness of the manifold $\mathfrak{A}$ and the map π.

Lemma 2. *The map Φ is smooth and $0 \in V$ is its regular value.*

Proof. Smoothness of the map Φ results from smoothness of the composition map $(J, f) \mapsto J \circ f$ (the fact is valid because both J and f belong to Sobolev spaces embedded into the spaces of continuous functions). To prove the regularity we shall show that at each point $(\tilde{f}, \tilde{\tau}, \tilde{J}) \in \mathfrak{A}$ we have

$$\text{Image}\,\frac{\partial\Phi}{\partial f} + \text{Image}\,\frac{\partial\Phi}{\partial J} = V \tag{14}$$

with fixed $\tilde{\tau}$. It is convenient to pass to the first representation of a complex torus and treat it as $\mathbb{T}^2_v$ (see Part 2).

To prove (14) is equivalent to check that each vector $v \in V$ which is L_2-orthogonal both to the image of $\partial\Phi/\partial f$ and the image of $\partial\Phi/\partial J$, vanishes.

Observe that for $\Psi \in T_J\mathcal{C}$ one has

$$\frac{\partial\Phi}{\partial J}(\Psi) = \Psi(\hat{f})\frac{\partial\tilde{f}}{\partial s}. \tag{15}$$

Take any point $(t, s) \in \mathbb{T}^2$ such that $\partial\tilde{f}/\partial s \neq 0$ in its small neighbourhood Q. Then when Ψ varies in $T_J\mathcal{C}$, the r.h.s. of (15), restricted to Q, gives all sufficiently smooth maps $Q \to \mathbb{R}^4$. Thus, $v(t, s)$ as above must vanish in Q.

Observe next that

$$\frac{\partial\Phi}{\partial f}(v) = \tilde{J}\frac{\partial v}{\partial s} - \frac{\partial v}{\partial t} + \left(\frac{\partial J}{\partial f}v\right)\frac{\partial\tilde{f}}{\partial s} \equiv \overline{D}(v),$$

where $\tilde{J} = \tilde{J}(\tilde{f})$. As v is L_2-orthogonal to the image of $\overline{D}$, then v belongs to the kernel of the adjoint operator $\overline{D}^*$,

$$\overline{D}^*v = -\frac{\partial}{\partial s}\left(\tilde{J}v\right) + \frac{\partial v}{\partial t} + \left(\nabla J\frac{\partial f}{\partial s}, v\right) = \frac{\partial v}{\partial t} - \tilde{J}\frac{\partial v}{\partial s} + B(t, s)v = 0,$$

where the matrix B is sufficiently smooth. As v vanishes in the domain $Q \subset \mathbb{T}^2$, then it vanishes identically due to Aronshain's uniqueness theorem (see in [8]). $\square$

Remark. Aronshain's theorem we used immediately follows from its much more known counterpart for second-order elliptic equations. Indeed we can apply to the equality $\overline{D}^*v = 0$ the operator $(\partial/\partial t + \tilde{J}\partial/\partial s)$ to get

$$\Delta v + C_1(t, x)\frac{\partial v}{\partial t} + C_2(t, x)\frac{\partial v}{\partial s} + D(t, x)v = 0.$$

Now the usual "uniqueness theorem for the Cauchy problem" (see [1, 4]) implies that $v \equiv 0$ provided that $v|_Q \equiv 0$.

Lemma 3. *The set $\mathfrak{A}$ is a Hilbert manifold modeled by the space $\mathbb{R}^2 \times H_{\mathcal{C}}$. The map $\pi : \mathfrak{A} \to \mathcal{C}$ is Fredholm of index two.*

Proof. Take any $\tilde{\xi} = (\tilde{f}, \tilde{\tau}, \tilde{J}) \in \mathfrak{A}$ and denote by r the dimension of $\ker \partial\Phi/\partial f$ at this point. To study $\mathfrak{A}$ locally we can suppose that f (corr. J) varies in a ball $B_{\mathcal{F}} \subset H$ (corr. in $B_{\mathcal{C}} \subset H_{\mathcal{C}}$), centered at $\tilde{f}$ (at $\tilde{J}$). Observe that index of the operator $\overline{D} = \partial\Phi/\partial f$ vanishes because in the class of elliptic ifferential operators it can be deformed to $2i\,\bar{\partial} = i\,\partial u/\partial s - \partial u/\partial t$.

Denote $\mathfrak{H}_r = H_{\mathcal{F}} \ominus \ker \overline{D}$. The space $\overline{D}(\mathfrak{H}_r)$ has codimension r in V. By Lemma 2 there exists an r-dimensional subspace $\eta^r \subset H_{\mathcal{C}}$ such that

$$\overline{D}(\mathfrak{H}_r) + \frac{\partial \Phi}{\partial J}(\tilde{\xi})(\eta^r) = V,$$

so $\Phi_*(\tilde{\xi})$ defines an isomorphism of $\mathfrak{H}_r \oplus \{0\} \oplus \eta^r$ and V.

Locally (near $\tilde{\xi}$) the manifold $\mathcal{F}_A \times \mathcal{T} \times \mathcal{C}$ can be identified with a ball in the space

$$H_{\mathcal{F}} \times \mathbb{R}^2 \times H_{\mathcal{C}} = \left(\ker \overline{D} \times \mathbb{R}^2 \times (H_{\mathcal{C}} \ominus \eta^r)\right) \oplus (\mathfrak{H}_r \oplus \{0\} \oplus \eta^r).$$

By the implicit function theorem locally the equation $\Phi = 0$ defines the second summand in the r.h.s. as a smooth function of the first one. Thus near a point $\xi \in \mathfrak{A}$ we have constructed a chart which is diffeomorphic to a ball in the space

$$\ker \frac{\partial \Phi}{\partial u} \times \mathbb{R}^2 \times (H_{\mathcal{C}} \ominus \eta^r) \simeq \mathbb{R}^2 \times H_{\mathcal{C}}.$$

Clearly the transformations from one chart to another are smooth.

The last statement of the lemma readily follows from the structure of the coordinate charts (cf. [9]). □

The group G freely acts on $\mathfrak{A}$ (Lemma 1). So $\mathfrak{A}_0 = \mathfrak{A}/G$ is a Hilbert manifold modeled by the space $H_{\mathcal{C}}$ and $\pi_0 : \mathfrak{A}_0 \to \mathcal{C}$ is a Fredholm map of zero index.

Denote by $\mathcal{C}^* \subset \mathcal{C}$ the set of regular values of π_0. It is dense in $\mathcal{C}$ by the Sard–Smale theorem. As the map π_0 is proper, then $\#\pi_0^{-1}(J) < \infty$ for J in $\mathcal{C}^*$. Due to Proposition 1 the nontrivial part of $\mathcal{C}^*$,

$$\mathcal{C}_0^* = \mathcal{C}^* \cap \pi_0(\mathfrak{A}_0),$$

is nonempty and is not equal to $\mathcal{C}^*$.

For each $J \in \mathcal{C}^*$ a parity of the number $\#\pi_0^{-1}(J)$ equals degree modulo two of the map π_0 (the latter is well-defined since π_0 is Fredholm of zero index). So it is J-independent. As $\#\pi_0^{-1}(J) = 0$ for J in $\mathcal{C}_0^*$, then we get

Theorem 2. *The total number $\#\pi_0^{-1}(J)$ of nonparameterized PH A-tori is even for each regular structure $J \in \mathcal{C}^*$. This number is nonzero for J in $\mathcal{C}_0^*$ and is zero for J in $\mathcal{C}^* \setminus \mathcal{C}_0^*$. Both sets $\mathcal{C}_0^*$ and $\mathcal{C}^* \setminus \mathcal{C}_0^*$ have nonempty interiors with respect to the topology of $\mathcal{C}$ (where $\mathcal{C}^*$ is a dense subset).*

Unfortunately, the following problem due to M. Gromov is left without an answer:

Problem ("a closing lemma"). Is the union

$$\bigcup_{A \in H_2(\mathbb{T}^n, \mathbb{Z}) \setminus 0} \pi(\mathfrak{A}_A)$$

of tamed structures with at least one PH torus dense in the set $\mathcal{C}$ of all tamed structures? For a nonempty open $\tilde{\mathcal{C}} \subset \mathcal{C}$ is the union of PH A-tori with $J \in \tilde{\mathcal{C}}$ and $A \in H_2(\mathbb{T}^n, \mathbb{Z}) \setminus 0$ dense in $\mathbb{T}^4$?

Appendix

Proof of Theorem 1. We should check that if $J_n \in \mathcal{C}$, $(f_n, \tau_n, J_n) \in \mathfrak{A}_0$ and $J_n \to J \in \mathcal{C}$, then for a subsequence we have

$$f_n \to f \in \mathcal{F}_A^0, \quad \tau_n \to \tau \in \mathcal{T}, \tag{*}$$

where $(f, \tau, J) \in \mathfrak{A}_0$.

Denote by $\tilde{f}_n$ a lift of the map f_n,

$$\tilde{f}_n : \mathbb{C} \to \mathbb{C}^2, \quad \tilde{f}_n(0) = 0, \quad \tilde{f}_n(\xi_n), \ \tilde{f}_n(\eta_n) \in \mathbb{Z}^2 + i\mathbb{Z}^2.$$

As $[f] = A \neq 0$, then both the vectors $\tilde{f}_n(\xi_n)$, $\tilde{f}_n(\eta_n)$ are nonzero due to (6). Thus $|\tilde{f}_n(\xi_n)|$, $|\tilde{f}_n(\eta_n)| \geqslant 1$. As $|\xi_n| \geqslant |\eta_n|$, then

$$\sup |df_n| = \sup |d\tilde{f}_n| \geqslant |\eta_n|^{-1} \geqslant |\xi_n|^{-1}. \tag{**}$$

Step 1. The maps $f_n : \mathbb{T}^2 \to \mathbb{T}^4$ are uniformly bounded in C^1-norm.

To prove the statement we should check that $|df_n| \leqslant C$ for all n. Suppose it is not the case. Then for a subsequence we have $\sup |df_n| = |df_n(z_n)| = K_n \nearrow \infty$. As the PH tori f_n are nonparameterized, we can suppose that $z_n = 0$ for all n. Below we treat f_n as a periodic map $\mathbb{C} \to \mathbb{T}^2$.

Due to (**) $|\xi_n| \geqslant |\eta_n| \geqslant K_n^{-1}$. We shall distinguish two cases:

$$\overline{\lim}\, |\eta_n| K_n = \infty$$

and

$$K_n^{-1} \leqslant |\eta_n| \leqslant C\, K_n^{-1} \qquad \forall n.$$

In the first case for a subsequence we have $L_n \equiv |\eta_n| K_n \to \infty$. Denote $D_n = \{z \in \mathbb{C} \mid |z| \leqslant \frac{1}{4} L_n\}$ and

$$\psi_n : D_n \to \mathbb{C}, \quad z \mapsto K_n^{-1} z.$$

Consider the maps

$$g_n = f_n \circ \psi_n : D_n \to (\mathbb{T}^4, J_n).$$

They are PH and

i) $|dg_n| \leqslant 1$, $|dg_n(0)| = 1$.

Due to (4), $\psi_n(D_n)$ contains no different points equivalent modulo periods ξ_n, η_n. Therefore the maps $\psi_n : D_n \to \mathbb{C}/(\mathbb{Z}\xi_n + \mathbb{Z}\eta_n)$ are embeddings. So by (7)

$$\operatorname{area} g_n(D_n) \leqslant \operatorname{area} f_n(\mathbb{T}^2) \leqslant \text{const}.$$

Fix any m. For $n \geqslant m$ the maps g_n send D_m to $\mathbb{T}^4$ and due to i) their C^1-norms are uniformly bounded. As they are PH, they satisfy elliptic equations forming a compact family of equations. Therefore their $W^{2,m+1}$-norms are uniformly bounded (see Step 2 of the proof for more arguments). Thus after extracting a subsequence, g_n converge strongly in C^1-norm to a PH map $g_\infty : D_m \to (\mathbb{T}^4, J)$. We can apply these arguments with $m = 1, 2, \ldots$ to get a PH map

$$g : \mathbb{C} \to (\mathbb{T}^4, J)$$

such that

ii) $|dg(0)| = 1$, area $g(\mathbb{C}) < \infty$.

Due to the "length–area" arguments of Pansu [7] (see also [5, p.330], [8, Part 4.5]) the estimate ii) implies that g defines a continuous map $g : S^2 \to \mathbb{T}^4$, which is PH outside the north pole $N \in S^2$ (we identify $\mathbb{C}$ with $S^2 \setminus N$). This map defines a bubble in $\mathbb{T}^4$ which is nontrivial because $|dg(0)| = 1$. But su[illegible] bubble cannot exist since its area can be estimated from above (up to a constant facto[illegible] [illegible]plectic area $\int_{g(S^2)} \omega_2$, which vanishes because each 2-sphere in $\mathbb{T}^4$ defines zero element [illegible] $_2(\mathbb{T}^4; \mathbb{Z})$ (it bounds a 3-ball in $\mathbb{T}^4$ because $\pi_2(\mathbb{T}^4) = \pi_2(\mathbb{R}^4) = \{0\}$).

In the second case we [illegible]ine ψ_n, g_n as

$$\psi_n(z) = \eta_n z, \qquad g_n = f_n \circ \psi_n.$$

Then g_n has periods 1 and ξ_n/η_n. Besides,

i′) $|dg_n| \leqslant C,\ C^{-1} \leqslant |dg_n$ [illegible] C.

We treat g_n as a PH ma[illegible] the cylinder [illegible] to the torus $\mathbb{T}^4$:

$$g_n : \mathbf{C} \to (\mathbb{T}^4, J_n).$$

Due to (4), $|\xi_n| \geqslant C^{-1} K_n$. So the second period ξ_n/η_n of g_n is of order K_n^2. Denote $\mathbf{C}_m = \{z \in \mathbf{C} \mid |\operatorname{Im} z| \leqslant K_m\}$. The set $\psi_n(\mathbf{C}_m)$, $n \geqslant m$, contains no different points equivalent modulo periods of f_n (m is sufficiently large), so for the same arguments as above g_n's converge in the C^1-topology to a PH map $g : \mathbf{C} \to (\mathbb{T}^4, J)$, such that

ii′) $|dg(0)| \geqslant C^{-1}$, area $g(\mathbf{C}) < \infty$.

Consider the punctured unit disc $D \setminus \{0\}$. Logarithmic map defines its biholomorphic isomorphism with the upper semi-cylinder $\mathbf{C}_+$,

$$\mathrm{Ln} : D \setminus \{0\} \to \mathbf{C}_+.$$

So $g \circ \mathrm{Ln} : D \setminus \{0\} \to (\mathbb{T}^4, J)$ is a nontrivial PH map of finite area. Due to the same length-area arguments, $g \circ \mathrm{Ln}$ extends to a continuous map $D \to \mathbb{T}^4$, and g to a continuous map $\mathbf{C} \cup \{i\,\infty\} \to \mathbb{T}^4$. Similarly g is also continuous at the "minus infinity" and defines a continuous map

$$g : \mathbf{C} \cup \{i\,\infty\} \cup \{-i\,\infty\} \simeq S^2 \to \mathbb{T}^4,$$

which is PH outside the poles of the 2-sphere S^2. For the same arguments as in the first case such a PH bubble cannot exist.

Step 2. If $\{f_n\}$ are uniformly bounded in C^1-norm, then (*) holds.

The estimates (4) and (**) imply that for a subsequence the parallelograms of periods (ξ_n, η_n) converge to a nondegenerate parallelogram $\tau = (\xi, \eta)$ of unit area.

Now we treat $\mathbb{T}^2$ as $\mathbb{C}/(\xi\mathbb{Z} + \eta\mathbb{Z})$ and $\mathbb{T}^4$ as $\mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2)$. The lift of f_n is a map $\tilde{f}_n$,

$$\tilde{f}_n : \mathbb{C} \to \mathbb{C}^2, \quad \tilde{f}_n(z + m\xi + n\eta) \equiv \tilde{f}_n(z) + m\Xi_1 + n\Xi_2, \quad \Xi_j \in \mathbb{Z}^2 + i\mathbb{Z}^2.$$

We can apply the operator $D_J = \partial/\partial s + J_n(f_n)\,\partial/\partial t$ to the equality $\bar{D}_{J_n}\tilde{f}_n = 0$ to get

$$\Delta \tilde{f}_n + B_n(t, s)\frac{\partial \tilde{f}_n}{\partial t} = 0,$$

where

$$B_n = \frac{\partial J_n}{\partial \tilde{f}}\frac{\partial f_n}{\partial s} - J_n\frac{\partial J_n}{\partial f}\frac{\partial \tilde{f}_n}{\partial t}.$$

The matrices B_n are bounded uniformly in n. Thus the usual regularity theory for the Laplace operator implies that $\tilde{f}_n$ are bounded uniformly in $W^{2,p}_{\mathrm{loc}}(\mathbb{C})$ for all p.

We can iterate this estimate to prove that $\tilde{f}_n$ are bounded in $\mathcal{F}_A$. Now we can extract a subsequence converging to some $\tilde{f}$ strongly in the C^1-norm and weakly in the $H_{\mathcal{F}}$-norm. Clearly $f \in \mathcal{F}_A$ and $\Phi(f, \tau, J) = 0$. □

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