ON TURBULENCE IN NONLINEAR SCHRÖDINGER EQUATION

Sergei B. Kuksin

To professor M.I. Vishik on his 75^{th} *birthday*

Abstract. We consider the small-dispersion and small-diffusion nonlinear Schrödinger equation

$$-i\,\dot u = -\delta_1\Delta u - i\delta_2\Delta u + |u|^2u + \zeta_\omega(t,x), \quad 1\ge\delta := \sqrt{\delta_1^2+\delta_2^2}>0,$$

where the space-variable x belongs to the unit n-cube ($n\le 3$) and u satisfies Dirichlet boundary conditions. Assuming that the force ζ is a zero-meanvalue random field, smooth in x and stationary in t with decaying correlations, we prove that the C^m-norms in x with $m\ge 3$ of solutions u, averaged in ensemble and locally averaged in time, are larger than $\delta^{-\kappa m}$, $\kappa\approx 1/5$. This means that the length-scale of a solution u decays with δ as its positive degree (at least, as δ^κ) and – in a sense – proves existence of turbulence for this equation.

0. Introduction

The turbulence phenomenon can be loosely described like this: Velocity field $u(t,x)$ of low-viscosity fluid which satisfies the forced Navier-Stokes equations

$$\dot u - \boldsymbol{\delta\Delta \mathrm{u}} + (u\cdot\nabla)u + \nabla p = \boldsymbol{\zeta(\mathrm{t},\mathrm{x})}, \qquad \operatorname{div} u = 0, \tag{NS}$$

develops well-observable short-scale wiggles ($\delta>0$ is small), cf. [F]. As a mathematical reformulation of the vague notion of "high-frequency wiggles" we suggest growth of derivatives of u as a negative degree of δ when δ decreases: $|\partial_x^\alpha u|\ge\delta^{-\kappa|\alpha|}$ if $\alpha\ge m_0$, or

$$|u(t,\cdot)|_m := |u(t,\cdot)|_{C^m} \geqslant \delta^{-\kappa m} \quad \text{for} \quad m\ge m_0. \tag{0.1}$$

The inequality (0.1) corresponds to wiggles with characteristic length-scale $\lesssim\delta^\kappa$.

If we cross off in (NS) the terms printed in bold, we get the Euler equations: $\dot u + (u\cdot\nabla)u + \nabla p = 0$, $\operatorname{div} u = 0$, well-known to be Hamiltonian. Accordingly, the (NS) equations can be viewed as

$$\langle\text{hamiltonian PDE}\rangle + \langle\text{small linear damping and/or small dispersion}\rangle + \langle\text{order-one forcing}\rangle. \tag{0.2}$$

In this paper we develop an approach which allows to prove (0.1) for some other nonlinear PDEs of the form (0.2) which are easier than (NS): We consider randomly forced damped

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nonlinear Schrödinger equation (as a popular model to study turbulence, cf. [LO]) and briefly discuss other equations which can be treated similar at the end of the paper.

So below we study the equation

$$-i\dot{u} = -\delta\nu\Delta u + |u|^2 u + \varepsilon\zeta_\omega(t,x), \quad u = u(t,x), \tag{0.3}$$

where u is a complex function of n-dimensional ($n = 1,2,3$) space-variable x, satisfying the odd periodic boundary conditions:

$$u(t,x) \equiv u(t,x_1,\dots,x_j+2,\dots,x_n) \equiv -u(t,x_1,\dots,-x_j,\dots,x_n),\ j = 1,\dots,n. \tag{0.4}$$

These conditions clearly imply Dirichlet boundary conditions on the cube K^n of half-periods:

$$u\mid_{\partial K^n} = 0, \quad K^n = \{0 \le x_j \le 1\}.$$

They are supposed everywhere below. In the equation (0.3) ν is a unit complex number, $\nu \in \mathbb{C}$, $|\nu| = 1$, δ is a real parameter, $0 < \delta \le 1$, and ε is a degree of δ,

$$\varepsilon = \delta^{c_\varepsilon}, \quad c_\varepsilon < \frac{3}{2}, \tag{0.5}$$

(e.g., $\varepsilon = 1$). We are mostly concerned with dissipative defocusing equations, corresponding to the case

$$\nu = \nu_R + i\nu_I, \quad \nu_R \ge 0,\ \nu_I > 0. \tag{0.6}$$

The force $\zeta_\omega(t,x)$ is a zero-meanvalue random field, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As a function of x it is smooth and satisfies (0.4) and as a function of t it is a stationary process with sufficiently fast decaying correlations. More specifically, the random field ζ has to have a version such that for a.a. ω the function $\zeta_\omega(\cdot,\cdot)$ is Lebesgue-measurable and bounded on compact subsets of $\mathbb{R}\times\mathbb{T}^n$. Besides, ζ has to satisfy the following hypothesis (H1), (H2):

(H1) (slow growth of ζ). There exist $c_1, c_2 \ge 0$ and $C_* > 0$ such that for each $0 < \gamma < 1$ and each t_0 we have:

$$\mathbf{P}\left\{|\zeta_\omega(t_0+t,\cdot)|_m \le C^*(\ln\gamma^{-1})^{c_1}\big(\frac{1}{2}\ln(1+|t|)\big)^{c_2} \quad \forall t\right\} \ge 1-\gamma.$$

We denote $c_1 + c_2 = c$.

(H2) (diffusion growth of $\int\nabla_x\zeta$). There exists $T_0 > 0, C > 0$ such that for each $1 \geqslant a > 0$, any t_0 and any $T \geqslant T_0$ we have

$$\mathbf{P}\left\{\left|\Big(\int_{t_0}^{t_0+T}\nabla_x\zeta_\omega(t,\cdot)\,dt\Big)\mid_{\partial K^n}\right|_{C^0(\partial K^n)} \le a\sqrt{T}\right\} \le Ca.$$

The assumption (H1) is obvious for bounded random fields and also holds for random fields with fast decaying tail probabilities (e.g., for the Gaussian ones, see Chapter 6). The assumption (H2) is close to validity of the central limit theorem for the stationary process $\nabla_x\zeta(t,\tilde{x})$ with some $\tilde{x}\in\partial K^n$; see examples in Chapter 6.

In this paper we study random fields $u_\omega(t,x)$ defined for $t>t'$ with some non-random $t'\geq-\infty$, which are classical solutions of (0.3), (0.4) for a.a. ω. We are not concerned with the existence problem for the solutions u_ω (under the assumption (0.6) there exists abundance of them since the initial-value problem $u\mid_{t=t'}=u_0(x)$ for the equation (0.3), (0.4) is well-posed, see e.g. [LO]).

The main result of this paper is Theorem 4 in Chapter 4 which states that with high probability the C^m-norm $|u(t,\cdot)|_m$ of a solution $u(t,x)$ of (0.3) is larger than a negative degree of δ, if $m\geqslant 3$ (even if $m>2$, see Chapter 9.2); thus, Theorem 4 proves property (0.1) for equation (0.3). To state the result we suppose for simplicity that $\varepsilon=1$ (for the general case see Chapter 4 – roughly, in the theorem below δ has to be replaced by $\tilde{\delta}=\delta\varepsilon^{-\frac{2}{3}}$):

Theorem A. *For any $m\geqslant 3$, for sufficiently small $\delta>0$ and any t_0 there exists a subset $\Omega_{t_0}\in\mathcal{F}$ such that $\mathbf{P}(\Omega\setminus\Omega_{t_0})<\delta^{1/35}$ and for $\omega\in\Omega_{t_0}$ the solution u_ω of (0.3) with $\varepsilon=1$ satisfies the estimate $|u_\omega(t_*,\cdot)|_m>\delta^{-\frac{3}{17}m}$ for some random $t_*\in[t_0,t_0+\delta^{-11/17}]$.*

(The exponent 3/17 can be slightly improved, see the main text.)

In Chapter 5 we derive some corollaries from Theorem A. In particular, we consider the random function Ψ_m, $\Psi_m(t)=|u(t,\cdot)|_m+\delta^{-d}|\dot{u}(t,\cdot)|_m$, $d=11/17$, which measures space-time oscillations of u. We prove that Ψ_m becomes large as δ to a negative degree after averaging in ensemble and local averaging in time:

Corollary 1. *For $m\geqslant 3$ and any t_0 we have $L^{-1}\int_{t_0}^{t_0+L}\mathbf{E}\Psi_m(t)\,dt\geq\frac{1}{4}\delta^{-\frac{3}{17}m}$, if $L\geq 3\delta^{-d}$.*

Estimating $\dot{u}(t,x)$ via $u(t,x)$ from equation (0.3), we can bound from below averaged Sobolev norms $\|u\|_m$ of solutions: Let us denote $U_m(t)=\mathbf{E}\|u(t,\cdot)\|_m^3$. Then

Corollary 2. *If $m\geq 6$, then $U_m(t)$ becomes bigger than $C\delta^{1-\frac{3m}{17}}$ after averaging along any time-interval longer than $3\delta^{-d}$. If $n=1$, then the average of $U_4(t)$ is bigger than $C\delta^{-3/55}$.*

If the assumption (0.6) holds, if the noise ζ is bounded and the solution u_ω is bounded for $t=0$,

$$\|u_\omega(t,\cdot)\|_0\leq C\delta^{-1}\quad\text{for a.a. }\omega$$

(e.g., the solution u_ω satisfies the zero initial condition), then the result of Corollary 2 can be improved for large m: In Proposition 2 (Chapter 5) we show that the averaging of $\mathbf{E}\|u(t)\|_m^2$ with any $m\geq 12$ along any time-interval $[t_0,t_0+L]$ with $t_0\geq 0$, $L\geq 3\delta^{-d}$ is bigger than $\delta^{2-3m/17}$.

The last statement and the assertion of Corollary 1, in general, both fail without time-averaging: if the solution of (0.3) satisfies the zero initial condition $u(0,x)=0$, then clearly $\Psi_m(0)=0$. But we can omit time-averaging if the random field ζ_ω is stationary in t and u_ω is a stationary solution of (0.3), so $u_\omega(0,\cdot)$ defines an invariant measure μ_δ in a space of complex functions $u(x)$ (we do not discuss the existence problem for the invariant measure and its smoothness; this problem is nontrivial in dimensions $n\geq 2$, see [VF, JM]). Now $\mathbf{E}\|u(t,\cdot)\|_m^3=\int\|u(\cdot)\|_m^3\mu_\delta(du)$ is a time-independent constant and we get

Corollary 3. *If μ_δ is an invariant measure for the equation (0.3) with $\varepsilon=1$, then for $m\geq 6$ we have $\mathcal{E}_m^{(3)}:=\int\|u\|_m^3\mu_\delta(du)\geq C^{-1}\delta^{1-\frac{3m}{17}}$, and $\mathcal{E}_4^{(3)}\geq C^{-1}\delta^{-3/55}$ for $n=1$. If the field ζ_ω is bounded, $\|\zeta_\omega(t)\|_m\leq C_m$ for a.a. ω, then for $m\geq 12$ we also have $\mathcal{E}_m^{(2)}\geq C\delta^{2-3m/17}$.*

Theorem A and Corollaries 1-3 show that solutions of the equation (0.3) develop length-scales $\lesssim \delta^{3/17}$ (since their m^{th} derivatives grow like $\delta^{-\frac{3}{17}m}$). Certainly the exponent 3/17 which initially appeared in Theorem A is not optimal, but qualitatively the assertion that "solutions of (0.3) develop length-scales equal to δ in a positive degree" is right: In Chapter 7 we study the equation with $n = 1$ [1] and apply elementary techniques of a priori estimates to prove that solutions of equation (0.3) with $\varepsilon = 1$, with bounded forcing ζ and with ν as in (0.6) has their m^{th} Sobolev norms bounded by $C\delta^{-3m-1}$ and the moments $\int \|u\|_m^2 \mu_\delta(du)$ of the corresponding invariant measure μ_δ grow with m no faster than $C\delta^{-6m-8}$. Thus – at least for $n = 1$ and bounded ζ – length-scales of solutions of (0.3) decay with δ as δ^κ, $\frac{3}{17} \le \kappa \le 3$, and Fourier transforms $\hat{u}(t,\xi)$ of the solutions u are essentially supported in the domain $\delta^{-3/17} \lesssim |\xi| \lesssim \delta^{-3}$ (see (7.14)). For $n = 2, 3$ we get similar one-side estimates and prove that solutions are mostly supported by frequencies $|\xi| \gtrsim \delta^{-\gamma}$ with some positive γ. These estimates justify the energy-transition to high frequencies phenomenon for solutions of the equation (0.3). This phenomenon is obviously nonlinear: if in (0.3) we replace $|u|^2u$ by u, then the length-scale of the solution will be of order one and no energy-transition to high frequencies will happen.

It was conjectured by A.N.Kolmogorov (see e.g. [Kol]) that for an invariant measure P_δ for equation (NS) one has $\mathcal{E}_1^{(2)} := \int \|u\|_1^2 P_\delta(du) \sim C_1\delta^{-1}$ with a nonzero C_1. See [VK1] where the *equality* $\mathcal{E}_1^{(2)} = C\delta^{-1}$ is proven for *some* invariant measures of the two-dimensional (NS) equations with a Gaussian random field ζ which is a white noise in time. Formally this result follows from Ito's formula applied to the functional $\|u(t)\|_0^2$ and averaged over ensemble; for $n = 2$ these calculations can be justified. For $n = 3$ only inequalities are known (see [VK2] for the estimate $\mathcal{E}_1^{(2)} \le C\delta^{-2}$ in case when the force ζ is similar to the fields we consider in our work), see [F, VF] for discussions.

Kolmogorov's insight to turbulence in case of equation (0.3) with stationary in time Gaussian noise ζ_ω predicts that some characteristics of the invariant measure μ_δ and of the corresponding stationary solution u_ω asymptotically (as δ goes to zero) behave as δ in a *fixed* degree, see [Z]. This assertion is much stronger than the lower and upper bounds constructed in our paper.

The "turbulence-limit" $\delta \to 0$ has no analogy for finite-dimensional stochastic equations. Classical for finite-dimensional systems is the large-deviation limit: $\varepsilon \to 0$ with δ fixed. This limit also can be studied for equation (0.3) with $n = 1, 2$, see [JM].

In Chapters 9.1-9.4 we briefly discuss generalizations of Theorem A to other classes of differential equations and to function spaces of lower smoothness. In Chapter 9.5 we discuss some relations of the theorem with Gromov's non-squeezing property and in Chapter 9.6 show how it can be used to explain difficulties with numerical investigations of the equation (0.3) with $\delta \ll 1$ and $n \ge 2$: due to the theorem, to calculate corresponding solutions with high accuracy the numerical scheme has to have huge dimension.

In Chapter 8 we consider deterministic version of equation (0.3) with non-random and time-independent forcing:

$$-i\dot{u} = -\delta\nu\Delta u + |u|^2u + w(x).$$

Techniques developed in Chapters 1-7 to study equation (0.3) are applicable to this equation

[1] to get a priori estimate for solutions of equation (0.3) with $n = 2, 3$, $\varepsilon = 1$ and with a smooth bounded force ζ one should exploit J. Bourgain's techniques [B]. We conjecture that in this way one can prove the estimate $\|u(t)\|_k \le C_k\delta^{-c^1k-c^2}$, similar to the one obtained in Chapter 7 for $n = 1$.

to get some results similar to Theorem A.

The techniques we apply to equation (0.3) to prove Theorem A develop the techniques from [K1, K2] where we studied the deterministic nonlinear Schrödinger equation

$$-i\dot{u} = -\delta\Delta u + |u|^{2p}u, \ \ p \in \mathbb{N}, \tag{0.7}$$

and showed that a solution of (0.7) will fast increase its C^m-norm ($m \geq 3$) till the level $\delta^{-m/4}$, unless the initial condition $u(0,x)$ is a small-amplitude function.[2] In this work we use the diffusive growth property (H2) to show that the noise ζ will increase the supremum norm of the solution u till some prescribed level, after this we use the slow-growth property (H1) to prove that the extra term ζ in the r.h.s. of the equation can not prevent the deterministic mechanism from [K1, K2] to increase the C^m-norm till the level $\delta^{-\frac{3}{17}m}$ (smaller than the "deterministic" level $\delta^{-m/4}$). The proof is elementary but rather technical. For a reader who prefer to get ideas without going through technicalities we presented in Appendix 2 its scheme, neglecting all details.

Notations. By C, C_1 etc. we denote different positive constants, independent of δ and ε. By $|\cdot|_m$ we denote the homogeneous norm in the function space $\{u \in C^m(K^n, \mathbb{C}) \mid u\mid_{\partial K^n} = 0\}$, $|u|_m = \max_{|\alpha|=m} \max_{x\in K^n} |\partial_x^\alpha u|$; by $|\cdot|_{L_p}$ – the norm in $L_p(K^n;\mathbb{C})$; by $\|\cdot\|_m$ – the homogeneous norm in $\overset{\circ}{H}{}^m(K^n,\mathbb{C})$. Given a complex odd periodic function $u(x)$ (i.e., a complex function which satisfies (0.4)), we denote by $|u|_m$ the C^m-norm of its restriction to K^n, etc. For any odd periodic function $u(x)$ we obviously have

$$|u|_l \leq |u|_m \quad \text{if} \quad 0 \leq l \leq m. \tag{0.8}$$

We systematically use the Landau - Hadamard - Kolmogorov interpolation inequality (see [Ad]) which states that for integer a, b, c such that $0 \leq a \leq b \leq c$, one has

$$|u|_b \leq C_c |u|_a^{\frac{c-b}{c-a}} |u|_c^{\frac{b-a}{c-a}}. \tag{0.9}$$

We write a function $u(t,x)$ as $u(t)$ if we treat it as a curve in one of the function spaces as above; so $|u(t)|_m = |u(t,\cdot)|_m$, etc.

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[2] the main theorems of [K1, K2] are close to Theorem 1 below where one sets $\mathcal{D} = 0$. Also see [Ily] (especially its Chapter 2.2), where similar ideas were applied to study the Kuramoto - Sivashinsky equation.

1. Equations with forcing

Let us denote by ν any complex number such that $|\nu| = 1$. In this part we study the equation

$$-i\dot{v}(t,x) = -\tilde{\delta}\nu\Delta v + |v|^2 v + \xi(t,x), \tag{1.1}$$

where $\tilde{\delta} \in]0,1]$. We fix an integer $m \geq 3$ and suppose that ξ is a C^m-smooth in x, measurable in (t,x) function such that

$$|\xi(t,\cdot)|_m \leq \mathcal{D} \quad \forall t \in [0,T') \tag{1.2}$$

with some $T' \leq \infty$, where $\mathcal{D} \geq 1$ is a parameter of the problem. We study the equation for $0 \leq t < T'$ under the odd periodic boundary conditions

$$v(t,x) = v(t,x_1,\dots,x_j+2,\dots) = -v(t,x_1,\dots,-x_j,\dots), \quad j = 1,\dots,n,$$

and suppose that the function ξ also satisfies them. These boundary conditions are assumed everywhere below, even if we do not mention them explicitly. We supplement the equation with an initial condition

$$v(0,x) = v_0(x),$$

such that

$$|v_0|_m = r \geq 1, \quad |v_0|_0 = \rho. \tag{1.3}$$

If we rescale v and time t:

$$v = r\tilde{v}(\tilde{t},x), \quad t = r^{-2}\tilde{t}, \tag{1.4}$$

then for $\tilde{v}(t,x)$ we get the equation: $-ir^3\tilde{v}'_{\tilde{t}} = -\tilde{\delta}r\nu\Delta\tilde{v} + r^3|\tilde{v}|^2\tilde{v} + \xi(r^{-2}\tilde{t},x)$. We denote

$$\lambda = \tilde{\delta}r^{-2}, \; \eta = \rho r^{-1}, \; \kappa = r^{-3};$$

omit the tilde (so below t stands for $\tilde{t}$ and v – for $\tilde{v}$) and rewrite the initial-value problem for $v = \tilde{v}$ as

$$-i\dot{v} = -\lambda\nu\Delta v + |v|^2 v + \kappa\xi(r^{-2}t,x), \quad v(0,x) = v_0(x), \tag{1.5}$$

$$|v_0|_m = 1, \quad |v_0|_0 = \eta. \tag{1.6}$$

Lemma 1. *Let $v(t,x)$ be any solution of (1.5) with $|v_0|_m \leq 2$, $|v_0|_0 = \eta$, where*

$$\eta \geq K_1\lambda^\mu, \quad \eta \geq K_2(\mathcal{D}\kappa)^M, \tag{1.7}$$

with some $\mathcal{D}, \lambda, \kappa$-independent constants K_1, K_2 and with

$$\mu = \mu(m) = \frac{m(m-1)}{2m^2+2m-3}, \quad M = M(m) = \frac{m(m-1)}{3m^2-m-1}.$$

Suppose that $T \in (0, r^2T')$ is such that for all $t \in [0,T]$ we have

$$|v(t)|_m \leq 2. \tag{1.8}$$

Then either a) there exists $t_1 \leq T/2$ such that $|v(t_1)|_0 = 2\eta$, or b) $T \leq \tilde{\Theta}(\eta) = K\eta^{-2-\frac{1}{m}}$.

This statement is the main lemma of our work. After it is proven we easily show that under its assumptions always holds the assertion b):

Lemma 2. *Under the assumptions (1.7), (1.8) of Lemma 1 we have $T \le \tilde\Theta(\eta)$.*

Proof of Lemma 2 given Lemma 1. Let l be the minimal integer such that $2^{-l} \le \eta$. We shall prove the assertion by induction in l. Since $\eta \le |v_0|_m$, then $\eta \le 2$ and $l \ge -1$ – so for $l < -1$ the assertion is empty.

Assuming that the assertion is proven for $l \le k$ we shall prove it for $l = k+1$. We apply Lemma 1. If happens b), then the estimate $T \le \tilde\Theta$ is proven. If a) – then we found $t_1 \le T/2$ such that $|v(t_1)|_0 = 2\eta$. Let us take for the new initial time $t = t_1$. Since $|v(t_1)|_0 = 2\eta$, then (1.7) holds and Lemma 1 is applicable. Now $l_1 = l - 1 = k$ and by induction for the new doubling-time T_1 we have $T_1 \le \tilde\Theta(2\eta) \le \frac{1}{2}\tilde\Theta(\eta)$.

Since $T/2 \le T - t_1$, then $T/2 \le T_1 \le \tilde\Theta(2\eta) \le \tilde\Theta(\eta)/2$. So $T \le \tilde\Theta(\eta)$, the induction-step is proven and the lemma follows. $\square$

Proof of Lemma 1. We can suppose that

$$|v(t)|_0 \le 2\eta \quad \text{for} \quad 0 \le t \le T/2 \tag{1.9}$$

otherwise the assertion a) holds trivially. We should derive from (1.8), (1.9) that $T \le \tilde\Theta(\eta)$. Interpolating (1.8), (1.9) (see (0.9)) we get that

$$|v(t)|_k \le C_*\eta^{1-k/m} \quad \text{for} \quad 0 \le t \le T/2, \;\; 0 \le k \le m. \tag{1.10}$$

Below we shall derive a contradiction from the assumption that the estimates (1.10) hold with some $T \ge \tilde\Theta(\eta)$. We shall do it in four steps.

Step 1 (*an integral equation for v*). Since $\big|\,|v(t)|^2\big|_m \le C_1 \sup_{|\alpha+\beta|=m} |\partial_x^\alpha v \partial_x^\beta \overline{v}|_0$, then (1.10) implies the estimate

$$\big|\,|v(t)|^2\big|_m \le C_2\eta \quad \text{for} \quad 0 \le t \le T/2. \tag{1.11}$$

Let us denote by w the increment of $|v|^2$:

$$w(t,x) = |v(t,x)|^2 - |v_0(x)|^2.$$

Since

$$\frac{1}{2}\frac{d}{dt} w(t,x) = \operatorname{Re} \dot v \overline{v} \le \lambda |v|\,|\Delta v| + \kappa |v|\,|\xi| \le C(\lambda \eta^{2-2/m} + \kappa\eta\mathcal{D})$$

(we use (1.2), (1.10)), then

$$|w(t)|_0 \le Ct(\lambda\eta^{2-2/m} + \kappa\eta\mathcal{D}).$$

Interpolating this estimate with (1.11) we find that

$$|w(t)|_1 \le Ct^{\frac{m-1}{m}} \big(\lambda^{\frac{m-1}{m}} \eta^{2(\frac{m-1}{m})^2 + \frac{1}{m}} + (\kappa\mathcal{D})^{\frac{m-1}{m}}\eta\big) \tag{1.12}$$

(we used the elementary inequality $(a+b)^\theta \le a^\theta + b^\theta$, valid for $a, b \ge 0$ and $0 < \theta \le 1$).

Now we recast (1.5) as a linear ODE for the complex function $t \mapsto v(t,x)$, depending on the parameter $x \in K^n$:

$$\dot v - ia(t)v = b(t), \quad a = |v|^2, \quad b = -i\lambda\nu\Delta v + i\kappa\xi.$$

Accordingly we write v as a solution of this equation:

$$v(t)=\int_0^t \big(\exp i(c(t)-c(\tau))b(\tau)\big)\,d\tau+v(0)\exp ic(t),\quad c(t)=\int_0^t a.$$

Or as a solution of the nonlinear integral equation:

$$v(t,x)=i\int_0^t e^{i\int_\tau^t |v(s,x)|^2\,ds}(-\lambda\nu\Delta v+\kappa\xi)\,d\tau+e^{it|v_0|^2}e^{i\int_0^t w(\tau,x)\,d\tau}v_0(x)=:v_1(t,x)+v_2(t,x).$$

Next we shall estimate the norm $|v_1|_1$ from above and the norm $|v_2|_1$ from below.

Step 2 (*an estimate for v_1 from above*). Let us rewrite v_1 as

$$v_1=i\int_0^t A(\tau,x)B(\tau,x)\,d\tau,\ A=\exp\left(i\int_\tau^t |v|^2\,ds\right),\ B=-\lambda\nu\Delta v+\kappa\xi.$$

By (1.2), (1.10) for $0\le k\le m$ we have:

$$|B(\tau)|_k\le C_k\left(\lambda\eta^{\frac{m-2-k}{m}}+\kappa\mathcal{D}\right).$$

Clearly $|A(\tau)|_0\equiv 1$. Besides, since by (1.11), and (1.9) we have $\big|\,|v(t)|^2\,\big|_1\le C\eta^{2\frac{m-1}{m}+\frac{1}{m}}=C\eta^{2-1/m}$, then

$$|\nabla A(t)|_0\le\int_\tau^t |\,|v(s)|^2|_1\,ds\le C_1(t-\tau)\eta^{2-\frac{1}{m}}.$$

Since

$$|v_1(t)|_1\le\int_0^t \big(|\nabla A(s)|_0|B(s)|_0+|\nabla B(s)|_0\big)\,ds,$$

then we get:

$$|v_1(t)|_1\le Ct^2\eta^{2-\frac{1}{m}}\left(\lambda\eta^{\frac{m-2}{m}}+\kappa\mathcal{D}\right)+Ct\left(\lambda\eta^{\frac{m-3}{m}}+\kappa\mathcal{D}\right).\tag{1.13}$$

Step 3 (*an estimate for v_2 from below*). Let us observe that

$$|v_2(t)|_1\ge t|(\nabla|v_0|^2)v_0|_0-\left|\left(\int_0^t\nabla w(s)\,ds\right)v_0\right|_0-|v_0|_1=:V^1-V^2-V^3.$$

We shall estimate the three terms in the r.h.s.:

i) the first term V^1 equals to $t\frac{2}{3}\left|\nabla|v_0|^3\right|_0$. Since v_0 vanishes at ∂K^n, then

$$V^1\ge C_1t|v_0|_0^3=C_1t\eta^3.$$

ii) By (1.12),

$$V^2\le Ct\eta\sup|w(s)|_1\le Ct^{2-\frac{1}{m}}\eta\left(\lambda^{\frac{m-1}{m}}\eta^{2(\frac{m-1}{m})^2+\frac{1}{m}}+\eta(\kappa\mathcal{D})^{\frac{m-1}{m}}\right).$$

iii) By (1.10), $V^3\le C\eta^{\frac{m-1}{m}}$.

Step 4 (*deriving a contradiction*). Since $|v|_1 \geq |v_2|_1 - |v_1|_1 \geq V^1 - V^2 - V^3 - |v_1|_1$, then

$$|v(T_*)|_1 \geq 2C_*\eta^{\frac{m-1}{m}}, \tag{1.14}$$

if $\frac{1}{4}V^1 \geq |v_1(T_*)|_1$, $\frac{1}{4}V^1 \geq V^2$, $\frac{1}{4}V^1 \geq V^3$ and $\frac{1}{4}V^1 \geq 2C_*\eta^{\frac{m-1}{m}}$. That is, if the following four inequalities are satisfied:

$$\begin{aligned}
&\frac{1}{4}C_1T_*\eta^3 \geq CT_*^2\eta^{2-\frac{1}{m}}\left(\lambda\eta^{\frac{m-2}{m}} + \kappa\mathcal{D}\right) + CT_*\left(\lambda\eta^{\frac{m-3}{m}} + \kappa\mathcal{D}\right),\\
&\frac{1}{4}C_1T_*\eta^3 \geq CT_*^{2-\frac{1}{m}}\eta\left(\lambda^{\frac{m-1}{m}}\eta^{2(\frac{m-1}{m})^2+\frac{1}{m}} + \eta(\kappa\mathcal{D})^{\frac{m-1}{m}}\right),\\
&\frac{1}{4}C_1T_*\eta^3 \geq C\eta^{\frac{m-1}{m}},\\
&\frac{1}{4}C_1T_*\eta^3 \geq 2C_*\eta^{\frac{m-1}{m}}.
\end{aligned}$$

The first and the second estimates hold if $C_1T_*\eta^3/16$ is larger than all the terms in the corresponding r.h.s. So (1.14) holds if

$$\left.\begin{aligned}
&T_* \leq C_1'\lambda^{-1}\eta^{3/m},\\
&T_* \leq C_2'\kappa^{-1}\mathcal{D}^{-1}\eta^{1+1/m},\\
&\eta^{2+3/m} \geq C_3'\lambda,\\
&\eta^3 \geq C_4'\kappa\mathcal{D},\\
&T_* \leq C_5'\lambda^{-1}\eta^{\frac{3m-2}{m(m-1)}},\\
&T_* \leq C_6'\eta^{\frac{m}{m-1}}\kappa^{-1}\mathcal{D}^{-1},\\
&T_* \geq C_7'\eta^{-2-1/m}.
\end{aligned}\right\}$$

Let us observe that since $\eta \leq 1$ by (1.6), (0.8) and $m/(m-1) > (m+1)/m$, $3/m < (3m-2)/(m(m-1))$, then the sixth inequality implies the second and the fifth implies the first. Therefore this system is equivalent to the following five inequalities:

$$\left.\begin{aligned}
\eta^{2+3/m} &\geq C_3'\lambda,\\
\eta^3 &\geq C_4'\kappa\mathcal{D},
\end{aligned}\right\} \tag{1.15}$$

$$\left.\begin{aligned}
&T_* \leq C_5\lambda^{-1}\eta^{\frac{3m-2}{m(m-1)}},\\
&T_* \leq C_6\eta^{\frac{m}{m-1}}\kappa^{-1}\mathcal{D}^{-1},\\
&T_* \geq C_7\eta^{-2-1/m}.
\end{aligned}\right\} \tag{1.16}$$

The first and the last inequalities in (1.16) are consistent if $C_7\eta^{-2-1/m} \leq C_5\lambda^{-1}\eta^{\frac{3m-2}{m(m-1)}}$, or if

$$\eta^{\frac{2m^2+2m-3}{m(m-1)}} \geq C\lambda. \tag{1.17}$$

The second and the last are consistent if $C_7\eta^{-2-1/m} \leq C_6\eta^{\frac{m}{m-1}}\kappa^{-1}\mathcal{D}^{-1}$, or if

$$\eta^{\frac{3m^2-m-1}{m(m-1)}} \geq C\kappa\mathcal{D}. \tag{1.18}$$

Since $\eta \leq 2$ and $\frac{3m^2-m-1}{m(m-1)} \geq 3$, $\frac{2m^2+2m-3}{m(m-1)} \geq \frac{2m+3}{m}$, then (1.17) and (1.8) imply (1.15).

We have seen that the inequality (1.14) holds with

$$T_* := C_7 \eta^{-2-1/m}$$

if (1.17), (1.18) and (1.8) are fulfilled. But (1.14) contradicts (1.10) with $k = 1$ if $T_* \leq T/2$. Since (1.17) holds if $\eta \geq K_1 \lambda^\mu$ with $K_1 = C^\mu$ and (1.18) holds if $\eta \geq K_2(\mathcal{D}\kappa)^M$ with $K_2 = C^M$, then under the lemma's assumption we must have $T \leq 2T_*$. The lemma is proven with $K = 2C_7$. $\square$

Reformulating the assertion of Lemma 2 in terms of the initial equation (1.1) we get

Theorem 1. *Let $m \geq 3$ and the function $\xi(t,x)$ satisfies (1.2). Let $r \geq 1$ and $\rho > 0$ be such that a) $\rho \geq K_1 \tilde{\delta}^\mu r^{1-2\mu}$, b) $\rho \geq K_2 \mathcal{D}^M r^{1-3M}$.*
Then for any solution $v(t,x)$ of (1.1), (1.3) there exists $t_1 \leq \Theta := K r^{1/m} \rho^{-2-1/m}$ such that $|v(t_1)|_m = 2r$, provided that $\Theta < T'$.

Let us note that

$$3M - 2\mu = \frac{m(m-1)(8m-7)}{(2m^2+2m-3)(3m^2-m-1)}$$

is positive and that the r.h.s. of b) is larger than the r.h.s. of a) if

$$r^{3M-2\mu} \leq C\mathcal{D}^M \tilde{\delta}^{-\mu}, \tag{1.19}$$

So a), b) hold if hold (1.19) and b).

Suppose that $r \leq R$ where

$$R = C_R \mathcal{D}^{\frac{m}{4}} \tilde{\delta}^{-\gamma m}, \tag{1.20}$$

with some $C_R > 0$ and some $\gamma = \gamma(m)$ such that

$$0 < \gamma \leq \frac{3}{8} \tag{1.21}$$

and sufficiently small C_R. Then r meets (1.19) (one should note that $\frac{M}{3M-2\mu} = \frac{2m^2+2m-3}{8m-7} > \frac{m}{4}$ and $\frac{\mu}{3M-2\mu} = \frac{3m^2-m-1}{8m-7} > \frac{3}{8}m$).

We have got a corollary from Theorem 1 which will be used later:

Proposition 1. *Let in (1.3) ρ and r are such that $1 \leq r \leq R$ and*

$$\rho \geq \rho_0 := K_2 R^{1-3M} \mathcal{D}^{\frac{1}{3}}. \tag{1.22}$$

Then for any solution $v(t,x)$ of (1.1), (1.3) there exists $t_1 \leq \Theta(r,\rho) = K r^{1/m} \rho^{-2-1/m}$ such that $|v(t_1)|_m = 2r$.

2. Cascade of doublings.

In this part we study solutions $v(t,x)$ of equation (1.1)-(1.3) beyond the doubling-time t_1 under the a priori assumption that C^0-norm ρ of the initial condition is not too small.

We define R as in (1.20) and suppose that in (1.3)

$$\rho \geq \rho_1 = 2C_R^{1-3M} K_2 \tilde{\delta}^{-\gamma \frac{2m^2-m}{3m^2-m-1}} \mathcal{D}^{\frac{1}{2}}. \tag{2.1}$$

Since for the number ρ_0 defined in (1.22) we have

$$\rho_0 = K_2 C_R^{1-3M} \tilde{\delta}^{-\gamma \frac{2m^2-m}{3m^2-m-1}} \mathcal{D}^{(1-3M)\frac{m}{4}+\frac{1}{3}} \leq K_2 C_R^{1-3M} \tilde{\delta}^{-\gamma \frac{2m^2-m}{3m^2-m-1}} \mathcal{D}^{\frac{1}{6}+\frac{1}{3}},$$

then $\frac{1}{2}\rho_1 \geq \rho_0$.

Theorem 2. *Let $v(t,x)$ solves (1.1)-(1.3) with $1 \leq r \leq R$, ρ as in (2.1) and with $1 \leq \mathcal{D} \leq (\ln \delta^{-1})^c$. Then there exists t_*,*

$$t_* \leq T_1 := C_1 \tilde{\delta}^{-\frac{2}{3}\gamma}, \tag{2.2}$$

such that $|u(t_*)|_m = R$, provided that $\tilde{\delta}$ is sufficiently small and $T_1 < T'$.

Proof. Let us suppose that – on the contrary –

$$|v(t)|_m \leq R \quad \text{for} \quad 0 \leq t \leq T, \tag{2.3}$$

where $T \leq T_1$ is some specific number defined below. We shall derive from (2.3) a contradiction. To do it we make the a priori assumption

$$\frac{1}{2}\rho \leq |v(t)|_0 \leq \frac{3}{2}\rho \quad \text{for} \quad 0 \leq t \leq T, \tag{2.4}$$

which will be finally justified.

Since $\frac{1}{2}\rho_1 \geq \rho_0$, then Proposition 1 is applicable to prove existence of the doubling-time t_1,

$$t_1 \leq K|v_0|_0^{-2-1/m} |v_0|_m^{1/m} \leq K(\rho/2)^{-2-1/m} R^{1/m} =: \theta.$$

If still $|v(t_1)|_m = 2|v_0|_m = 2r$ is smaller than R, we can iterate these arguments to find a second doubling-time t_2, $t_2 \leq K|v(t_1)|_0^{-2-1/m} |v(t_1)|_m^{1/m} \leq 2^{1/m}\theta$, etc. Finally we find $N \geq 1$ such that

$$2^{N-1} r < R \leq 2^N r. \tag{2.5}$$

For $t' = t_1 + \cdots + t_N$ we have $|v(t')|_m \geq R$ (so $|v(t_*)|_m = R$ for some $t_* \leq t'$) and

$$t' \leq \theta(1 + 2^{\frac{1}{m}} + \cdots + 2^{\frac{N}{m}}) \leq C\theta m 2^{\frac{N+1}{m}}.$$

By (2.5), $2^{(N+1)/m} \leq (4R)^{1/m}$. So

$$t' \leq C_1 \rho^{-2-\frac{1}{m}} R^{\frac{2}{m}} =: T < T_1$$

and $t_* \leq t' \leq T$ which proves the theorem if we can check the a priori assumption (2.4).

To check (2.4) we note that since at any point $x \in K^n$ we have

$$\left| \frac{1}{2} \frac{d}{dt} |v(t,x)|^2 \right| \leq \tilde{\delta} |\Delta v||v| + |\xi||v|,$$

then $\left|(d/dt)|v(t,x)|\right| \leq \tilde{\delta}|\Delta v| + |\xi|$. Since $|\Delta v| \leq C|v|_2 \leq C'|v|_0^{(m-2)/m}|v|_m^{2/m}$, then for $0 \leq t \leq T$ we have $|\Delta v| \leq C_1 R^{2/m} \rho^{(m-2)/m}$. The a priori assumption (2.4) holds if $T\tilde{\delta}|\Delta v| + T|\xi| \leq \rho/2$ or if

$$\left. \begin{aligned} & C_1 \rho^{-2-1/m} R^{2/m} \tilde{\delta} R^{2/m} \rho^{1-2/m} \leq \rho/4, \\ & C_2 \rho^{-2-1/m} R^{2/m} \mathcal{D} \leq \rho/4. \end{aligned} \right\} \tag{2.6}$$

By (2.1) these inequalities hold if

$$\left. \begin{aligned} & C_1 \tilde{\delta}^{-4\gamma+\gamma(2+\frac{3}{m})\frac{2m^2-m}{3m^2-m-1}+1} \mathcal{D}^{-\frac{3}{2m}} \leq 1, \\ & C_2 \tilde{\delta}^{\gamma\frac{m+1}{3m^2-m-1}} \mathcal{D}^{-\frac{1}{2m}} \leq 1. \end{aligned} \right\}$$

Since $0 < \gamma \leq \frac{3}{8}$ (see (1.21)), then the exponents for $\tilde{\delta}$ in both inequalities are positive. So the relations (2.6) are satisfied if $\tilde{\delta}$ is sufficiently small, and the theorem is proven. $\square$

3. Small solutions

In this part we take the initial equation (0.3):

$$-i\dot{u}(t,x) = -\delta\nu\Delta u + |u|^2 u + \varepsilon\zeta_\omega(t,x), \tag{3.1}$$

where $\varepsilon = \delta^{c_\varepsilon}$ with $c_\varepsilon < \frac{3}{2}$ (e.g., $\varepsilon = 1$) and the force ζ possesses the properties (H1), (H2). Let $u_\omega(t,x)$ be a measurable random field, satisfying (3.1) for a.a. ω. Our final goal is to estimate functional norms of the solution u from below. We suppose that u is "small" on some time-segment $[t_0, t_0 + T'(\omega)]$ and construct an upper estimate for T'. Since all the information on the random field ζ_ω we possess is translationary invariant in time (see (H1), (H2)), we can take for this segment $[0, T']$.

So let us suppose that T' is a non-negative random variable such that

$$|u(t)|_m \leq R\varepsilon^{1/3} \quad \text{for} \quad 0 \leq t \leq T'(\omega) \tag{3.2}$$

and

$$|u(t)|_0 \leq \rho_1 \varepsilon^{1/3} \quad \text{for} \quad 0 \leq t \leq T'(\omega), \tag{3.3}$$

where R and ρ_1 are defined in (1.20), (2.1) (with $\tilde{\delta} = \delta\varepsilon^{-2/3}$). We shall show that the estimates (3.2), (3.3) can not last "for too long". To state the result we take any a such that $\delta < a < 1$. We suppose that

$$C\tilde{\delta}^{\frac{1}{2}-2\gamma-\gamma\frac{2m^2-5m+2}{3m^2-m-1}} \mathcal{D} \leq \varepsilon^{\frac{1}{3}} a \tag{3.4}$$

and that $1 \leq \mathcal{D} \leq (\ln \delta^{-1})^c$.

Theorem 3. *If the assumption (3.4) holds as well as the estimates (3.2), (3.3), then there exists a set $\Omega_a \in \mathcal{F}$ such that $\mathbf{P}(\Omega \setminus \Omega_a) \le a$ and for $\omega \in \Omega_a$ we have*

$$T'(\omega) \le C_1 a^{-2} \varepsilon^{-\frac{4}{3}} \tilde{\delta}^{-\frac{10}{3}\gamma} \mathcal{D}^{\frac{3}{2}} =: T_2.$$

Proof. For $\tilde{x} \in \partial K^n$ let us denote $U(t, \tilde{x}) = \nabla_x u(t, \tilde{x})$. Since $\nabla |u|^2 u = 0$ on ∂K^n, then we get from (3.1) that

$$iU(0, \tilde{x}) - iU(t, \tilde{x}) = -\delta\nu \int_0^t \nabla \Delta u(s, \tilde{x})\, ds + \varepsilon \int_0^t \nabla \zeta_\omega(s, \tilde{x})\, ds.$$

By (3.2), (3.3) for $0 \le t \le T'$ we have

$$|\nabla \Delta u(t)|_0 \le C|u(t)|_3 \le C_1 R^{3/m} \rho_1^{1-3/m} \varepsilon^{1/3},$$
$$|U(t)|_0 \le C|u(t)|_1 \le C_2 R^{1/m} \rho_1^{1-1/m} \varepsilon^{1/3}.$$

Therefore for any $T \le T'$ we have

$$\left| \int_0^T \nabla \zeta_\omega(t, \cdot)\, dt \right|_{C^0(\partial K^n)} \le 2C_2 \varepsilon^{-2/3} R^{1/m} \rho_1^{1-1/m} + \tilde{\delta} T C_1 R^{3/m} \rho_1^{1-3/m}. \tag{3.5}$$

We note that (3.5) clearly contradicts to the system of two inequalities (3.6), (3.7):

$$\left| \int_0^T \nabla \zeta_\omega(t, \cdot)\, dt \right|_{C^0(\partial K^n)} \ge 4C_2 \varepsilon^{-2/3} R^{1/m} \rho_1^{1-1/m}, \tag{3.6}$$

$$\left| \int_0^T \nabla \zeta_\omega(t, \cdot)\, dt \right|_{C^0(\partial K^n)} \ge 2\tilde{\delta} T C_1 R^{3/m} \rho_1^{1-3/m}. \tag{3.7}$$

To prove that $T' \le T_2$ we shall show that under the lemma's assumptions both (3.6) and (3.7) hold with some $T = T_0 \le T_2$.

By the assumption (H2), for any $T \ge T_0$ we can find a set $\Omega_a^T \in \mathcal{F}$, $\mathbf{P}(\Omega \setminus \Omega_a^T) < a$, such that

$$\left| \int_0^T \nabla \zeta_\omega(t, \cdot)\, dt \right|_{C^0(\partial K^n)} \ge aC\sqrt{T} \quad \text{for} \quad \omega \in \Omega_a^T.$$

Therefore for $\omega \in \Omega_a^T$ both relations (3.6), (3.7) are satisfied if

$$a\sqrt{T} \ge C_1' \varepsilon^{-\frac{2}{3}} R^{\frac{1}{m}} \rho_1^{1-\frac{1}{m}} \;\text{ and }\; a\sqrt{T} \ge C_2' \tilde{\delta} T R^{\frac{3}{m}} \rho_1^{1-\frac{3}{m}},$$

or

$$C_1' \varepsilon^{-\frac{2}{3}} a^{-1} R^{\frac{1}{m}} \rho_1^{1-\frac{1}{m}} \le \sqrt{T} \le (C_2')^{-1} a \tilde{\delta}^{-1} R^{-\frac{3}{m}} \rho_1^{\frac{3}{m}-1}. \tag{3.8}$$

This system is consistent if

$$\varepsilon^{\frac{2}{3}} a^2 \ge C\tilde{\delta} R^{\frac{4}{m}} \rho_1^{2-\frac{4}{m}} = C_1 \tilde{\delta}^{1-4\gamma-(2-\frac{4}{m})\gamma \frac{2m^2-m}{3m^2-m-1}} \mathcal{D}^{2-\frac{2}{m}};$$

that is, if (3.4) is satisfied.

Finally, if (3.4) holds then $T = T_0$ satisfies the inequalities (3.8), where

$$T_0 = \Big(C_1'\varepsilon^{-\frac{2}{3}}a^{-1}R^{\frac{1}{m}}\rho_1^{1-\frac{1}{m}}\Big)^2 \le T_2. \tag{3.9}$$

For $\omega \in \Omega_a := \Omega_a^{T_0}$ we must have $T'(\omega) \le T_0 < T_2$, which proves the theorem. $\square$

Remark. If $m = 3$ and $\varepsilon = 1$, then in (3.9) T_2 can be replaced by the better bound $T_2' = C\varepsilon^{-2}\mathcal{D}^{\frac{3}{2}}\delta^{-\gamma\frac{66}{23}}$.

4. The main theorem

We continue to study the initial equation (0.3) (=(3.1)), where the random field ζ_ω possesses the properties (H1), (H2). Let us take any (non-random) t_0 and for a solution $u = u_\omega(t,x)$ of (3.1) define the random times T_R and T_ρ as follows:

$$T_R(\omega) = \sup\{\tau \ge 0 \mid |u_\omega(t_0+\tau)|_m \le R\varepsilon^{1/3}\},$$
$$T_\rho(\omega) = \sup\{\tau \ge 0 \mid |u_\omega(t_0+\tau)|_0 \le \rho_1\varepsilon^{1/3}\},$$

where $T_R = 0$ ($T_\rho = 0$) if the corresponding set is empty. In this part we shall derive an upper bound for T_R on a subset $\Omega_0 \subset \Omega$ of large measure.

Let us fix the a priori time-segment $[t_0, t_0+T^*]$ where we shall examine the solution u,

$$T^* = \tilde{C}\delta^{-\tilde{c}} \tag{4.1}$$

with $\tilde{C}$, $\tilde{c} > 0$ to be specified later.

By (H1) we can find $\Omega_1 \in \mathcal{F}$ such that $\mathbf{P}(\Omega \setminus \Omega_1) \le \delta$ and for $t_0 \le t \le t_0 + T^*$ we have:

$$|\zeta_\omega(t)|_m \le C^*(\ln\delta^{-1})^{c_1}\Big(\frac{1}{2}\ln(1+T^*)\Big)^{c_2} \le CC^*(\ln\delta^{-1})^{c} \quad \text{if} \quad \omega \in \Omega_1. \tag{4.2}$$

Let δ satisfies (3.4) and Ω_2 be equal to the set Ω_a from Theorem 3. Then $\mathbf{P}(\Omega\setminus\Omega_2) \le a$ and by Theorem 3,

$$\min(T_R, T_\rho) \le T_2 \quad \text{for} \quad \omega \in \Omega_2.$$

Since $\gamma \le 3/8$ and $a \ge \delta$, then

$$T_2 \le C\mathcal{D}^{\frac{3}{2}}\tilde{\delta}^{-3-\frac{1}{4}-\frac{4}{3}c_\varepsilon} \le \frac{1}{2}T^*$$

if $\tilde{C} \ge C$, $\tilde{c} \ge (1-\frac{2}{3}c_\varepsilon)(4+\frac{4}{3}c_\varepsilon)$ and δ is sufficiently small.

Let us define the set Ω_{t_0} as

$$\Omega_{t_0} = \Omega_1 \cap \Omega_2.$$

Then $\mathbf{P}(\Omega\setminus\Omega_{t_0}) \le a+\delta$. If $\omega \in \Omega_{t_0}$ is such that $\min(T_R,T_\rho) = T_R$, then $T_R \le T_2$ and our goal to estimate T_R on Ω_{t_0} is achieved.

If for some $\omega \in \Omega_{t_0}$ the minimum $\min(T_R,T_\rho)$ equals T_ρ, then for $t_1 = t_1(\omega) = t_0 + T_\rho$ we have

$$|u(t_1)|_0 = \varepsilon^{1/3}\rho_1, \quad |u(t_1)|_m \le \varepsilon^{1/3}R.$$

Besides

$$|\zeta_\omega(t)|_m \le CC^*(\ln\delta^{-1})^c \quad \text{for} \quad t_1 \le t \le t_1 + \frac{1}{2}T^*$$

because (4.2) since $T_\rho < \frac{1}{2}T^*$.

Now we rescale equation (3.1):

$$-i\frac{\partial v}{\partial \tilde{t}} = -\tilde{\delta}\nu\Delta v + |v|^2 v + \zeta_\omega^\varepsilon(\tilde{t}, x),$$

where

$$\tilde{\delta} = \delta\varepsilon^{-2/3},\ v = u\varepsilon^{-1/3},\ \tilde{t} = t\varepsilon^{2/3} - t_1\varepsilon^{2/3},\ \zeta^\varepsilon(\tilde{t}, x) = \zeta(\varepsilon^{-2/3}\tilde{t} + t_1, x),$$

and

$$|v(0)|_0 = \rho_1, \quad |v(0)|_m \le R.$$

For $0 \le \tilde{t} \le \frac{1}{2}\varepsilon^{2/3}T^*$ we have

$$|\zeta^\varepsilon(\tilde{t})|_m \le C'(\ln\delta^{-1})^c.$$

Since $\gamma \le 3/8$, then the time T_1 as in (2.2) is no larger than $C\delta^{-\frac{1}{4}(1-\frac{2}{3}c_\varepsilon)}$ and is smaller than $\frac{1}{2}\varepsilon^{2/3}T^*$, if $\tilde{C}$ in (4.1) is chosen accordingly. Now Theorem 2 can be applied to prove that the relation $|u(\tilde{t})|_m \le R$ can not last longer than the time $\tilde{t} = T_1$. It proves that

$$T_R - T_\rho \le \varepsilon^{-\frac{2}{3}}T_1 = C_1\varepsilon^{-\frac{2}{3}}\tilde{\delta}^{-\frac{2}{3}\gamma},$$

and that by the time

$$C\left(\varepsilon^{-\frac{2}{3}}\tilde{\delta}^{-\frac{2}{3}\gamma} + a^{-2}\varepsilon^{-\frac{4}{3}}\tilde{\delta}^{-\frac{10}{3}\gamma}\mathcal{D}^{\frac{3}{2}}\right) \tag{4.3}$$

the C^m-norm of the solution u_ω, $\omega \in \Omega_{t_0}$, will become larger than $R\varepsilon^{1/3}$.

To prove the result we have imposed only one additional restriction – we supposed that δ satisfies the assumption (3.4). Clearly it holds with some $a = \frac{1}{2}\delta^{c_a}$, $c_a > 0$, if

$$2\gamma\left(2 + \frac{2m^2 - 5m + 2}{3m^2 - m - 1}\right) < \frac{3 - 4c_\varepsilon}{3 - 2c_\varepsilon} \tag{4.4}$$

and δ is sufficiently small.

We also note that (for δ small) the number in (4.3) can be estimated by δ^{-c_t} with any c_t such that

$$c_t > \frac{2}{3}\max\left(c_\varepsilon + \gamma(1 - \frac{2}{3}c_\varepsilon),\ 3c_a + 2c_\varepsilon + 5\gamma(1 - \frac{2}{3}c_\varepsilon)\right). \tag{4.5}$$

We have got the main result of this work:

Theorem 4. *For an integer $m \ge 3$ let γ be a positive number satisfying (4.4). Then there exists $c_a > 0$ such that for any t_0 and any c_t as in (4.5) one can find a set $\Omega_{t_0} \in \mathcal{F}$ such that $\mathbf{P}(\Omega \setminus \Omega_{t_0}) < \delta^{c_a}$ and, provided that δ is sufficiently small, for each $\omega \in \Omega_{t_0}$ there exists a $t_*(\omega)$, $t_0 \le t_*(\omega) \le t_0 + \delta^{-c_t}$, such that the solution $u_\omega(t, x)$ of (0.3) satisfies the estimate $|u_\omega(t_*)|_m \ge C_R\varepsilon^{\frac{1}{3}}\tilde{\delta}^{-\gamma m}$.*

The theorem's assertion becomes less awkward if we consider equation (0.3) with $\varepsilon = 1$:

$$-i\dot{u}(t, x) = -\delta\nu\Delta u + |u|^2 u + \zeta_\omega(t, x). \tag{4.6}$$

If $m = 3$, we can take in (4.4) $\gamma = \frac{1}{5}$ and choose $a = \frac{1}{20}$. Now clearly $T_2 > T_1$ and we use the Remark from Chapter 3 to see that any $c_t > 66/115$ is admissible. Choosing $c_t = 3/5$ we get

Corollary 1. *If δ is sufficiently small, then for any t_0 there exists a set $\Omega_{t_0} \in \mathcal{F}$ such that $\mathbf{P}(\Omega \setminus \Omega_{t_0}) < \delta^{1/20}$ and for $\omega \in \Omega_{t_0}$ the solution u_ω of (4.6) satisfies the estimate $|u_\omega(t_0 + t_*)|_3 \geq \delta^{-3/5}$ with some random $t_* \in \left[t_0, t_0 + \delta^{-3/5}\right]$.*

Now we take any $m \geq 3$. Since $(2m^2 - 5m + 2)/(3m^2 - m - 1) < 2/3$, then (4.4) (with $\varepsilon = 1$) holds if $\gamma < 3/17$. We can take $\gamma = 3/17$, $c_a = 1/35$ and $c_t = 11/17$:

Corollary 2. *For any $m \geq 3$, any $\delta \leq \delta_m$ with some $\delta_m > 0$ and for any t_0 there exists a set $\Omega_{t_0,m} \in \mathcal{F}$ such that $\mathbf{P}(\Omega \setminus \Omega_{t_0,m}) < \delta^{1/35}$ and for $\omega \in \Omega_{t_0,m}$ the solution u_ω of (4.6) satisfies the estimate $|u_\omega(t_*)|_m > \delta^{-\frac{3}{17}m}$ with some random $t_* \in [t_0, t_0 + \delta^{-11/17}]$.*

Another admissible choice is $\gamma = 3/16$ with some m-dependent c_a, c_t.

5. Low estimates for averaged norms

To simplify the presentation, in this part we discuss only the equation (4.6).

Let us take the segment $\Sigma = [t_0, t_1 = t_0 + \delta^{-d}]$, $d = 11/17$, as in Corollary 2, a point $\omega \in \Omega_{t_0,m}$ and the corresponding time $t_* = t_*(\omega) \in \Sigma$, $|u(t_*)|_m \geq \delta^{-3m/17}$. Since

$$\frac{1}{t_1 - t_*} \int_{t_*}^{t_1} |u(t)|_m \, dt \geq |u(t_*)|_m - \int_{t_*}^{t_1} \left| \frac{d}{dt} |u(t)|_m \right| dt,$$

then

$$\frac{1}{t_1 - t_*} \int_{t_*}^{t_1} \left(|u(t)|_m + \delta^{-d} \Big| \frac{d}{dt} |u(t)|_m \Big| \right) dt \geq \delta^{-\frac{3}{17}m}.$$

Similar estimate holds for the integral from t_0 to t_* and we get that

$$\delta^d \int_{t_0}^{t_0 + \delta^{-d}} \left(|u(t)|_m + \delta^{-d} \Big| \frac{d}{dt} |u(t)|_m \Big| \right) dt \geq \frac{1}{2} \delta^{-\frac{3}{17}m} \tag{5.1}$$

for each $\omega \in \Omega_{t_0,m}$. We can suppose that $\delta^{1/35} \leq \frac{1}{2}$. Then $\mathbf{P}\Omega_{t_0,m} \geq \frac{1}{2}$ and we get from (5.1) that

$$\mathbf{E} \left(\delta^d \int_{t_0}^{t_0 + \delta^{-d}} \left(|u(t)|_m + \delta^{-d} \Big| \frac{d}{dt} |u(t)|_m \Big| \right) dt \right) \geq \frac{1}{4} \delta^{-\frac{3}{17}m}.$$

Now let us take any $L > \delta^{-d}$. Then $L = m\delta^{-d} + l$ with some $l < \delta^{-1}$ and we can apply the last estimate to the segments $[t_0, t_0 + \delta^{-d}], \dots, [t_0 + (m-1)\delta^{-d}, t_0 + m\delta^{-d}]$ to get that for each t_0 and each $L > \delta^{-d}$ we have:

$$\mathbf{E} \left(\frac{1}{L} \int_{t_0}^{t_0 + L} \left(|u(t)|_m + \delta^{-d} \Big| \frac{d}{dt} |u(t)|_m \Big| \right) dt \right) \geq \frac{1}{8} \delta^{-\frac{3}{17}m}. \tag{5.2}$$

Since $\left| \frac{d}{dt} |u(t)|_m \right| \leq |\dot{u}(t)|_m$,[3] then the integrand in (5.2) is bounded by $\Psi_m(t)$ where the function

$$\Psi_m(t) = |u(t)|_m + \delta^{-d} |\dot{u}(t)|_m, \quad d = \frac{11}{17},$$

measures space-time oscillations of the solution. Applying to (5.2) the Fubini theorem we get:

[3]Proof: For $\varepsilon > 0$, $\varepsilon^{-1} \big| |u(t+\varepsilon)|_m - |u(t)|_m \big| \leq |\varepsilon^{-1}(u(t+\varepsilon) - u(t))|_m = |\dot{u}(t)|_m + O(\varepsilon^2)$. So the function $|u(t)|_m$ is absolutely continuous and its derivative satisfies the estimate.

Theorem 5. *For $m \geq 3$, $\delta \leq \delta_m$, any $L \geq \delta^{-d}$ and any t_0 we have:*

$$L^{-1} \int_{t_0}^{t_0+L} \mathbf{E}\Psi_m(t)\, dt \geq \frac{1}{8}\delta^{-\frac{3}{17}m}.$$

For $m = 3$ one can take $d = \frac{6}{11}$ and the average of $\mathbf{E}\left(|u(t)|_3 + \delta^{-6/11}|\dot{u}(t)|_3\right)$ along any time-segment longer than $\delta^{-6/11}$ is bigger than $\frac{1}{8}\delta^{-3/5}$.

The second assertion of the theorem follows from Corollary 1.

In particular, time-average of $\mathbf{E}\big(|u(t)|_3 + |\dot{u}(t)|_3\big)$ is bigger than $\frac{1}{8}\delta^{-3/55}$. Let $\|u\|_l$ denotes the l^{th} Sobolev norm of a function $u(x)$. Since $|u|_{m-2} \leq C\|u\|_m$, then repeating the arguments which we used to prove (5.1), we get for $m \geq 5$ the estimate:

$$\delta^d \int_{t_0}^{t_1} \|u\|_m\, dt + \int_{t_0}^{t_1} \left|\frac{d}{dt}\|u\|_m\right| dt \geq A := C\delta^{-\frac{3}{17}(m-2)}.$$

For a function $f(t)$ we denote

$$f(t)_+ = f(t)\chi(f(t)), \;\; f(t)_- = -f(t)\chi(-f(t)),$$

where χ is the Heaviside function, i.e., $\chi(f) = 1$ for $f \geq 0$ and $\chi(f) = 0$ for $f < 0$. Now we rewrite the last estimate as

$$\delta^d \int_{t_0}^{t_1} \|u\|_m\, dt + \int_{t_0}^{t_1} \left(\left(\frac{d}{dt}\|u\|_m\right)_+ + \left(\frac{d}{dt}\|u\|_m\right)_-\right) dt \geq A. \tag{5.3}$$

We need a consequence of this inequality:

Lemma 3. *For any t_0 and for each $\omega \in \Omega_{t_0,m}$ we have*

$$\frac{1}{3\delta^{-d}} \int_{t_-}^{t_+} \|u\|_m\, dt + \int_{t_-}^{t_+} \left(\frac{d}{dt}\|u\|_m\right)_+ dt \geq \frac{A}{9}, \tag{5.4}$$

where $t_- = t_0 - \delta^{-d}$, $t_+ = t_0 + 2\delta^d$.

Proof. First we note that if $\|u(t)\|_m \geq \frac{A}{3}$ for all $t \in [t_-, t_0]$ or all $t \in [t_1, t_+]$ then the estimate is obvious. Therefore we can suppose that $\|u(t_2)\|_m \leq \frac{A}{3}$ and $\|u(t_3)\|_m \leq \frac{A}{3}$ for some $t_2 \in [t_-, t_0]$ and $t_3 \in [t_1, t_+]$. Then $\left|\int_{t_2}^{t_3} \frac{d}{dt}\|u\|_m\, dt\right| \leq \frac{2}{3}A$, or

$$\left|\int_{t_2}^{t_3} \left(\frac{d}{dt}\|u\|_m\right)_+ dt - \int_{t_2}^{t_3} \left(\frac{d}{dt}\|u\|_m\right)_- dt\right| \leq \frac{2}{3}A.$$

Now because of (5.3) we have:

$$\delta^d \int_{t_2}^{t_3} \|u\|_m\, dt + 2\int_{t_2}^{t_3} \left(\frac{d}{dt}\|u\|_m\right)_+ dt \geq A - \frac{2}{3}A = \frac{1}{3}A.$$

The l.h.s. of (5.4) is bigger than one-third of the l.h.s. of the last inequality. So it is $\geq A/9$ and (5.4) follows. $\square$

Suppose now that the complex coefficient ν in equation (4.6) is as in (0.6). Then

$$i\delta\nu\left((-\Delta)u,(-\Delta)^m\bar{u}\right)_{L_2}=\delta(i\nu_R-\nu_I)\|u\|_{m+1}^2. \tag{5.5}$$

Since $\frac{1}{2}(d/dt)\|u\|_m^2=\operatorname{Re}\left(\dot{u},(-\Delta)^m\bar{u}\right)_{L_2}$, then replacing $\dot{u}$ by i times the r.h.s. of (4.6) and using (5.5) we get:

$$\frac{1}{2}\frac{d}{dt}\|u\|_m^2=-\delta\nu_I\|u\|_{m+1}^2+\operatorname{Re}\left(i|u|^2u+i\zeta_\omega,(-\Delta)^m\bar{u}\right)_{L_2}\leq\Big(\big\||u|^2u\big\|_m+\|\zeta_\omega(t)\|_m\Big)\|u\|_m.$$

Therefore, $\frac{d}{dt}\|u\|_m\leq\||u|^2u\|_m+\|\zeta_\omega(t)\|_m$ and we get from (5.4) that

$$\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\left(\|u\|_m+\delta^{-d}\||u|^2u\|_m+\delta^{-d}\|\zeta_\omega\|_m\right)dt\geq\frac{A}{27} \tag{5.6}$$

for $\omega\in\Omega_{t_0,m}$. The set $\Omega_{t_0,m}$ is contained in a set Ω_1 as in (4.2). Therefore $\|\zeta_\omega(t,\cdot)\|_m\leq C(\ln\delta^{-1})^c$ for $|t-t_0|\leq 3\delta^{-1}$ and $\omega\in\Omega_{t_0,m}$. Because of this estimate and (5.6) we have

$$\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\left(\|u\|_m+\delta^{-d}\||u|^2u\|_m\right)dt\geq C^{-1}A, \tag{5.7}$$

if δ is sufficiently small. Since $\||u|^2u\|_m\leq C\|u\|_m^3$ (see [Ad]), then

$$\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\left(\|u\|_m+\delta^{-d}\|u\|_m^3\right)dt\geq C_1^{-1}A. \tag{5.8}$$

Using the elementary estimate $a\leq\frac{1}{3}a^3\delta^{-d}+\frac{2}{3}\delta^{d/2}$ with $a=\|u\|_m$ we get that

$$\frac{4}{3}\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\delta^{-d}\|u\|_m^3\,dt\geq C_1^{-1}A-\frac{2}{3}\delta^{\frac{1}{2}}\geq C_2^{-1}A.$$

Now arguing as in the beginning of this part we find that for $L\geq 3\delta^{-d}$,

$$\mathbf{E}\left(\frac{1}{L}\int_{t_0}^{t_0+L}\|u(t)\|_m^3\,dt\right)\geq C^{-1}\delta^{\frac{6-3m}{17}+d}=C^{-1}\delta^{1-\frac{3m}{17}}.$$

We have proved:

Theorem 6. *Suppose that (0.6) holds. Then for $m\geq 6$ and $\delta\leq\delta_m$ the quantity $U_m(t)=\mathbf{E}\|u(t)\|_m^3$ becomes bigger than $C^{-1}\delta^{1-3m/17}$ after averaging along any time-interval longer than $3\delta^{-11/17}$. For $m=5$, $U_5(t)$ becomes bigger than $C^{-1}\delta^{-3/55}$ after averaging along time-intervals longer than $3\delta^{-6/11}$. If $n=1$, then $U_4(t)$ satisfies the same estimate.*

The second assertion of the theorem follows from Corollary 1; the third holds since $|u|_3\leq C\|u\|_4$ for $n=1$.

Results of Theorem 6 can be improved if the random field ζ_ω is L_2-bounded in x:

$$\|\zeta_\omega(t,\cdot)\|_0 \le C_0 \quad \forall\, t,\omega \tag{5.9}$$

(cf. Examples 3,4 in Chapter 6) and the solution u_ω of (4.6) is L_2-bounded at $t=0$:

$$\|u_\omega(0)\|_0 \le C\delta^{-1} \quad \text{for a.a.} \quad \omega \tag{5.10}$$

(e.g., $u_\omega(0)=0$). Suppose that (0.6) holds. Then

$$\frac{1}{2}\frac{d}{dt}\|u_\omega\|_0^2 \le -\delta\nu_I\|u_\omega\|_1^2 + \|\zeta_\omega(t)\|_0\|u_\omega(t)\|_0$$

and $(d/dt)\|u_\omega\|_0 \le -\delta\nu_I\|u_\omega\|_0 + \|\zeta_\omega(t)\|_0$. Using (5.10) we find that

$$\|u_\omega(t)\|_0 \le \delta^{-1}\big(C + C_0/\nu_I\big) \tag{5.11}$$

for $t\ge 0$ and almost all ω.

Now we come back to the estimates (5.7), (5.8). The Banach algebra inequality $\|u^2\bar u\|_m \le C\|u\|_m^3$ which was used to derive (5.8), can be specified as follows:

$$\|u^2\bar u\|_m \le C_\gamma\|u\|_m\|u\|^2_{n/2+\gamma/2} \le C_\gamma\|u\|_m^{1+\frac{n}{m}+\frac{\gamma}{m}}\|u\|_0^{2-\frac{n}{m}-\frac{\gamma}{m}}$$

with any $\gamma>0$ (see e.g., Appendix 1 in [K2]). Combining this estimate with (5.11) we find that $\|u^2\bar u\|_m \le C_\gamma\delta^{-2+\frac{n}{m}+\frac{\gamma}{m}}\|u\|_m^{1+n/m+\gamma/m}$ and – after substitution to (5.7) – that

$$\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\|u\|_m + \delta^{-d-2+\frac{n}{m}+\frac{\gamma}{m}}\|u\|_m^{1+\frac{n}{m}+\frac{\gamma}{m}}\,dt \ge C_1^{-1}A.$$

Choosing $\gamma = m-n$ we get:

$$\frac{1}{3\delta^{-d}}\int_{t_-}^{t_+}\|u\|_m + \delta^{-d-1}\|u\|_m^2\,dt \ge C_1^{-1}A.$$

Using these estimates instead of (5.8), we get the results which improve Theorem 6 for large m:

Proposition 2. *If the assumptions (0.4), (5.9), (5.10) hold, then for $t_0\ge 0$ and $L\ge 3\delta^{-\frac{11}{17}}$ we have: $L^{-1}\int_{t_0}^{t_0+L}\mathbf{E}\|u\|_m^2\,dt \ge C\delta^{2-\frac{3m}{17}}$ if $m\ge 12$, and $L^{-1}\int_{t_0}^{t_0+L}\mathbf{E}\|u\|_m^{1+\frac{n}{m}+\gamma_0}\,dt \ge C_\gamma^{-1}\delta^{3-\frac{3m}{17}}$ if $m\ge 18$ with any positive γ_0.*

6. On properties (H1), (H2)

Below we present examples of random fields $\zeta_\omega(t,x)$ which satisfy (H1), (H2).

Example 1. Let $\zeta(t)$ be a stationary complex Gaussian process with zero meanvalue and the correlation function $B(t)$. So

$$B(t) = \mathbf{E}\big(\zeta(s)\bar{\zeta}(s+t)\big), \quad 0 = \mathbf{E}\big(\zeta(s)\zeta(s+t)\big)$$

for all s, t, see [CL, IL, R] (cf. the Remark at the end of this chapter). We assume that

$$\int_{-\infty}^{\infty} |B(t)|\, dt < \infty$$

and that the spectral density of the process exists and is positive at zero.

Let $w(x)$ be any smooth odd periodic complex function such that

$$\frac{\partial w}{\partial x_1}(0, \tilde{x}_2, \dots, \tilde{x}_n) = 1 \quad \text{for some} \quad \hat{x} = (\tilde{x}_2, \dots, \tilde{x}_n) \in K^{n-1}. \tag{6.1}$$

We define the random field $\zeta_\omega(t,x)$ as $\zeta_\omega(t,x) = \zeta_\omega(t)w(x)$. This process satisfies (H1), (H2):

1) (H1) with $c_1 = c_2 = \frac{1}{2}$ follows from the estimate

$$\mathbf{P}\left\{|\zeta(t_0+t)| \le C\sqrt{\ln\gamma^{-1}}\,\sqrt{\ln(1+|t|)} \quad \text{for all } t\right\} \ge 1-\gamma \tag{6.2}$$

which holds with an appropriate $C > 0$. See Theorem A2 in the Appendix 1 (cf. Chapter 12.3 of [CL]).

2) (H2) with $\tilde{x} = (0, \hat{x})$ is a corollary of (6.1) and the estimate

$$\mathbf{P}\left\{\left|\frac{1}{\sqrt{T}}\int_{t_0}^{t_0+T} \zeta(t)\, dt\right| \le a\right\} \le C_1 a$$

which loosely follows from the central limit theorem. See Lemma A3 in the Appendix 1 for an elementary direct proof.

Example 2. Let $\zeta^s(t)$, $s \in \mathbb{N}^n$, be independent complex processes, distributed as the process ζ from the previous example and let $\{\varphi_s(x) = \sqrt{2}\sin\pi x_1 \dots \sqrt{2}\sin\pi x_n \mid s \in \mathbb{N}^n\}$ be the L_2-normalized *sin*-basis of the space of odd periodic functions. We define ζ as

$$\zeta_\omega(t,x) = \sum_s \zeta^s_\omega(t)\varphi_s(x) z_s, \quad z_s \in \mathbb{R},\ 0 < z_s \le |s|^{-M},$$

where $M \ge m + 2 + (n+1)/2$. Now

1) (H1) follows from the estimate (6.2), applied to ζ^s, with γ replaced by γ_s, $\gamma_s = C_*^{-1}\gamma|s|^{-n-1}$. Indeed, denoting by Ω_0 the set

$$\Omega_0 = \left\{\omega \;\middle|\; |\zeta^s(t_0+t)| \le C\sqrt{\ln\gamma_s^{-1}}\,\sqrt{\ln(1+|t|)} \quad \forall\, t, s\right\},$$

we have from (6.2) that $\mathbf{P}(\Omega_0) \ge 1 - \sum\gamma_s > 1-\gamma$, if C_* is sufficiently large. Since

$$|\zeta_\omega(t_0+t,\cdot))|_m^2 \le C\|\zeta_\omega(t_0+t,\cdot)\|_{m+2}^2 \le \sum |\zeta^s_\omega(t_0+t)|^2|s|^{-2M}|s|^{2m+4}$$

and $\ln\gamma_s^{-1} \le C\ln\gamma^{-1}\ln(|s|+1)^{n+1}$, then for $\omega\in\Omega_0$ we have

$$|\zeta_\omega(t_0+t,\cdot)|_m^2 \le C_1 \ln\gamma^{-1}\ln(1+|t|)\sum_s |s|^{-2M+2m+4+\frac{1}{2}} \le C_2 \ln\gamma^{-1}\ln(1+|t|).$$

So (H1) is proven.

2) To check (H2) we note that for $\hat{x}\in K^{n-1}$ as in (6.1) the point $(0,\hat{x})$ belongs to ∂K^n. We have:

$$\frac{\partial}{\partial x_1}\zeta(t,0,\hat{x}) = \sum_{s\in\mathbb{T}^n}\zeta_\omega^s(t) z_s \frac{\partial}{\partial x_1}\varphi_s(0,\hat{x}).$$

Supremum-norm of any function on K^{n-1} is no smaller than its norm in $L_2(K^{n-1})$. Since the coefficient in front of $\sqrt{2}\sin\pi x_2\ldots\sqrt{2}\sin\pi x_n$ in the L_2-decomposition of the function $(\partial/\partial x_1)\zeta(t,0,\cdot)$ is $\xi(t)$,

$$\xi(t) = \pi\sqrt{2}\big(z^{1,\mathbf{1}}\zeta_{1,\mathbf{1}} + 2z^{2,\mathbf{1}}\zeta_{2,\mathbf{1}} + 3z^{3,\mathbf{1}}\zeta_{3,\mathbf{1}} + \dots\big), \quad \mathbf{1} = (1,\dots,1),$$

then we have:

$$\mathbf{P}\Big\{\big|\frac{1}{\sqrt{T}}\int_{t_0}^{t_0+T}\nabla_x\zeta(t,0,\cdot)\,dt\big|_{C(K^{n-1})} \le a\Big\} \le \mathbf{P}\Big\{\big|\frac{1}{\sqrt{T}}\int_{t_0}^{t_0+T}\xi(t)\,dt\big| \le a\Big\} \le Ca.$$

– We applied Lemma A3 to the stationary process $\xi(t)$.

Example 3. Let ζ^j, $j\in\mathbb{Z}$, be a sequence of i.i.d. complex random variables, valued in a bounded complex disc, such that their common distribution function in $\mathbb{C}\simeq\mathbb{R}^2$ has bounded density. Let $w(x)$ be as in the first example and $\zeta(t,x) = \zeta^{[t]}w(x)$, where $[t]$ stands for the integer part of t.

Now (H1) is obvious. To prove the property (H2) it is sufficient to check it for integer Ts. To do it we have to estimate the probabilities $p_N = \mathbf{P}\left\{N^{-1/2}|\zeta^1+\dots+\zeta^N| \le a\right\}$ uniformly in $N\ge T_0$. But under our restrictions the densities of distributions of the random variables $(\zeta^1+\dots+\zeta^N)/\sqrt{N}$ converge in the supremum-norm to a Gaussian density (this is the central limit theorem for densities, see [IL, P]). So $p_N\le Ca$ and (H2) follows.

The last example can be generalized:

Example 4. Let $\zeta_\omega(t)$ be a real zero-meanvalue stationary process such that $|\zeta_\omega(t)|\le 1$ for a.a. ω and the distribution of $\zeta_\omega(t)$ has a bounded density (with respect to the Lebesgue measure). Suppose that the correlation function $B(t)$, $B(t) = \mathbf{E}\zeta(t+s)\zeta(s)$, is integrable with a non-zero meanvalue. Moreover, suppose that the random variables $\zeta(t+s)$ and $\zeta(s)$ with large t are weakly dependent in the sense that the mixing coefficient $\alpha(t)$ (see [IL, R]) is fast decaying:

$$\alpha(t) = O(|t|^{-d}), \quad d>1. \tag{6.3}$$

Let us take any complex function $w(x)$ as in the first example and define $\zeta_\omega(t,x)$ as $\zeta_\omega(t,x) = \zeta_\omega(t)w(x)$.

The property (H1) for ζ is obvious. To check (H2) we examine the random process $\nabla_x\zeta_\omega(t,\tilde{x})$, $\tilde{x} = (0,\tilde{x}_2,\dots,\tilde{x}_n)$. Because the assumption (6.1), to check we (H2) we shall study the event

$$\Big\{\Big|\frac{1}{\sqrt{T}}\int_{t_0}^{t_0+T}\zeta_\omega(\tau)\,dt\Big| \le C'a\Big\}. \tag{6.4}$$

Due to the qualified strong mixing condition (6.3), the process $\zeta(t)$ satisfies the central limit theorem (see [R], cf. [IL]). Thus the random variable $T^{-1/2}\int_{t_0}^{t_0+T}\zeta\,dt$ asymptotically has Gaussian distribution and the set defined by (6.4) has measure $\le C_1 a$ for all $T \ge T(a)$. Since the distribution of each $\zeta(s)$ has bounded density, then $T(a)$ can be chosen uniformly in $0 < a \le 1$, and (H2) follows.

Remark. The examples given above have many obvious generalizations. In particular, the process $\zeta(t)$ in Examples 3,4 could be complex-valued and processes $\zeta(t)$, $\zeta^s(t)$ in Examples 1,2 could be stationary processes valued in $\mathbb{R}^2$ (the correlation function $B(t)$ has to be replaced by a correlation 2×2-matrix).

7. Upper estimates for one-dimensional equations. Energy cascade

In this part we derive some elementary upper estimates for solutions of the one-dimensional equation (0.3) with $\varepsilon = 1$ and ν as in (0.6):

$$\dot u = \delta\nu_I u_{xx} - i\delta\nu_R u_{xx} + i|u|^2u + i\zeta_\omega(t,x), \quad \nu_R \ge 0,\ \nu_I > 0. \tag{7.1}$$

We suppose that the process $\zeta_\omega(t,\cdot)$ is bounded in H^m for all m:

$$\|\zeta_\omega(t,\cdot)\|_m \le C_m \quad \text{for all } t, \text{ a.a. } \omega, \tag{7.2}$$

and we also assume that u is bounded for $t=0$:

$$\|u_\omega(0,\cdot)\|_k \le C'_k\delta^{-3k} \quad \text{for a.a. } \omega \text{ and } k \le m \tag{7.3}$$

(e.g., $u(0)=0$). Multiplying (7.1) by $u(t)$ in $L_2(K^n,\mathbb{C})$ we get that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_0^2 \le -\delta\nu_I\|u\|_1^2 + \|\zeta(t)\|_0\,\|u(t)\|_0 \tag{7.4}$$

and

$$\frac{d}{dt}\|u\|_0 \le -\delta\nu_1\|u\|_0 + \|\zeta\|_0.$$

Using (7.2), (7.3) we find the uniform estimate for $\|u\|_0$:

$$\|u(t)\|_0 \le C_1\delta^{-1} \quad \text{for } t \ge 0 \text{ and a.a. } \omega. \tag{7.5}$$

Now let us multiply (7.1) by $(-\Delta)^k u$. We get that

$$\frac{1}{2}\frac{d}{dt}\|u\|_k^2 \le -\delta\nu_I\|u\|_{k+1}^2 + \sum_{\substack{|\alpha_1|+\dots+|\alpha_4|=2k\\ 0\le|\alpha_j|\le k}} \int \partial_x^{\alpha_1}u\,\partial_x^{\alpha_2}u\,\partial_x^{\alpha_3}\bar u\,\partial_x^{\alpha_4}\bar u\,dx + \|u\|_k\,\|\zeta_\omega\|_k. \tag{7.6}$$

Because the Hölder inequality, any integral in the sum in r.h.s. of (7.6) can be estimated by

$$|\partial_x^{\alpha_1}u|_{L_4}\dots|\partial_x^{\alpha_4}u|_{L_4},\ \ l_1+\dots+l_4 = 2k,\ l_j := |\alpha_j|. \tag{7.7}$$

Since now $n=1$, then $|v|_{L_4} \le C\|v\|_{1/4}$ and (7.7) is no bigger than

$$C\|u\|_{l_1+1/4}\dots\|u\|_{l_4+1/4} \le C_1\|u\|_0^{1-\frac{l_1+1/4}{k+1}}\|u\|_{k+1}^{\frac{l_1+1/4}{k+1}}\dots\|u\|_0^{1-\frac{l_4+1/4}{k+1}}\|u\|_{k+1}^{\frac{l_4+1/4}{k+1}} =$$
$$= C_1\|u\|_0^{4-\frac{l_1+\dots+l_4+1}{k+1}}\|u\|_{k+1}^{\frac{l_1+\dots+l_4+1}{k+1}} = C_1\|u\|_0^{2+\frac{1}{k+1}}\|u\|_{k+1}^{2-\frac{1}{k+1}}.$$

Now we use the Young inequality $ab \le a^p/p + b^q/q$ $(1/p+1/q=1)$ with $p=2k+2$, $q=(2k+2)/(2k+1)$ to estimate the r.h.s. by

$$\frac{C_1}{2k+2}\left(C_*\delta^{-\frac{2k+1}{2k+2}}\|u\|_0^{\frac{2k+3}{k+1}}\right)^{2k+2} + C_1\frac{2k+1}{2k+2}\left(C_*^{-1}\delta^{\frac{2k+1}{2k+2}}\|u\|_{k+1}^{\frac{2k+1}{k+1}}\right)^{\frac{2k+2}{2k+1}} =$$
$$= \frac{C_1C_*^{2k+2}}{2k+2}\delta^{-2k-1}\|u\|_0^{4k+6} + C_1C_*^{-\frac{2k+2}{2k+1}}\frac{2k+1}{2k+2}\delta\|u\|_{k+1}^2.$$

Choosing C_* sufficiently large (but δ-independent) and using (7.5) we find that the sum in the r.h.s. of (7.6) is no bigger than $C\delta^{-6k-7} - \frac{1}{3}\delta\nu_I\|u\|_{k+1}^2$. Therefore

$$\frac{1}{2}\frac{d}{dt}\|u\|_k^2 \le -\frac{\delta\nu_I}{3}\|u\|_{k+1}^2 + C\delta^{-6k-7} + C_1\delta^{-1}\|\zeta_\omega\|_k^2. \tag{7.8}$$

Using (7.2), (7.3) we get that

$$\|u(t)\|_k \le C_k\delta^{-3k-4} \quad \text{for} \quad t\ge 0 \quad \text{and a.a. } \omega. \tag{7.9}$$

We have proved

Proposition 3. *If $n=1$, if the random fields ζ_ω and $u(0,\cdot)$ are bounded as in (7.2), (7.3), then for any k and any $t\ge 0$ the norm $\|u(t)\|_k$ of a solution $u(t,x)$ of (7.1) is bounded by $C_k\delta^{-3k-4}$.*

Remark. Interpolating (7.5) and (7.9) with $k:=m$ we get that $\|u(t)\|_k \le C_k\delta^{-1-3k-3k/m}$. So $\|u(t)\|_k \lesssim C_k'\delta^{-3k-1}$ for $k \ll m$ (and $t\ge 0$).

Now we suppose that ζ_ω is stationary in t such that

$$|\zeta_\omega(t,x)| \le C \quad \text{for all } t,x \quad \text{and a.a. } \omega \tag{7.10}$$

and

$$E\|\zeta(t)\|_m^2 \le C \quad \text{for all} \quad t. \tag{7.11}$$

Repeating the arguments from the beginning of this part (and using (7.10)) we find that

$$\|u(t)\|_0 \le C\delta^{-1} + e^{-\delta t}\|u(0)\|_0 \quad \text{for a.a. } \omega.$$

Therefore any invariant measure μ_δ of (7.1) has to be supported in the $C\delta^{-1}$-ball with respect to the L_2-norm:

$$\mu_\delta(\|u\|_0 \le C\delta^{-1}) = 1. \tag{7.12}$$

Now we integrate over μ_δ both parts of (7.4). The average of the l.h.s. vanishes and we have:

$$0 \le -\delta\nu_I\int\|u\|_1^2\mu_\delta(du) + \int\|\zeta\|_0\|u\|_0\mu_\delta(du).$$

Using (7.11), (7.12) we get:

$$\int\|u\|_1^2\mu_\delta\,du \le C_1\delta^{-2}. \tag{7.13}$$

Arguing as before we prove:

Proposition 4. *If $n=1$, if ζ is a stationary process as in (7.10), (7.11) and μ_δ is an invariant measure for (7.1), then $\int \|u\|_k^2 \mu_\delta(du) \le C_k \delta^{-6k-8}$ for all $k \le m$. For $k=0,1$ the estimate can be improved as in (7.12), (7.13).*

Propositions 3, 4 and Theorem 4 jointly prove that length-scales of solutions of the equation (7.1) decay with δ as its positive degree since their m^{th} norms grow as $\delta^{-m\kappa}, \kappa > 0$. This assertion can be reformulated in spectral terms. For example, in the following way: Let $n=1$. By Theorem 4, for any solution $u = \sum \hat{u}(t,\xi)\exp(i\xi x)$ and for typical (in probability) ω we have:

$$\|u(t_*,\cdot)\|_m^2 = \sum_\xi |\hat{u}(t_*,\xi)|^2 |\xi|^{2m} \ge C\delta^{-2Am} =: K_m$$

(for $m \ge 4$, $A \approx \frac{1}{5}$ due to the theorem since $\|u\|_m \ge C^{-1}|u|_{m-1}$; this estimate remains valid for $m > 2.5$ with some $A = A_m > 0$, see Chapter 9.2). Take any $\epsilon > 0$. By (7.5), $\sum_{|\xi|\le\eta_1} |\hat{u}(t_*,\xi)|^2|\xi|^{2m} \le \eta_1^{2m} C_1 \delta^{-1} \le \epsilon K_m$ if $\eta_1 = C_{1,\epsilon}\delta^{-A+1/m}$. By Proposition 3 with $l = m+\mu$,

$$\sum_{|\xi|\ge\eta_2} |\hat{u}(t_*,\xi)|^2|\xi|^{2m} \le C_l \delta^{-2B(m+\mu)} \eta_2^{-2\mu}, \quad B = 3 + \frac{4}{m+\mu}.$$

Therefore $\sum_{|\xi|\ge\eta_2} |\hat{u}(t_*,\xi)|^2|\xi|^{2m} \le \epsilon K_m$ if $\eta_2 = C_{\mu,\epsilon}\delta^{-\frac{m}{\mu}(B-A)-B}$. Taking $\mu \gg m$ we find that

$$\begin{gathered}\text{for typical } \omega,\ H^m\text{-norm of the function } u(t_*,\cdot) \text{ with } m \ge 4 \text{ is mostly concen-}\\ \text{trated in frequencies from the segment } [C_1\delta^{-A+1/m}, C_\gamma\delta^{-3-\gamma}] \text{ for any } \gamma > 0\end{gathered} \tag{7.14}$$

("mostly" means "up to its ϵ-part").

The a priori estimate (7.5) remains true for a solution u of the n-dimensional equation (4.6) if $\|\xi_\omega(t)\|_0 \le C$ for all t and $\|u(0)\|_0 \le C\delta^{-1}$ for a.a. ω (provided (0.6)). Arguing as above we find that for $n=2,3$ the norms $\|u(t_*)\|_m$ are mostly supported by the high frequencies ξ, $|\xi| \gtrsim \delta^{-A+1/m}$.

We have shown that "from time to time" the H^m-norms of solutions are concentrated in high frequencies. Proposition 2 can be used to get obvious versions of these results for the time- and ensemble-averaged squared norms $L^{-1}\int_{t_0}^{t_0+L} \mathbf{E}\|u\|_m^2\,dt$: For $n=2,3$ they are mostly supported by the frequencies $|\xi| \gtrsim \delta^{-3/34+2/m}$, etc.

If u is a stationary in time solution, then the time-averaging can be omitted.

The estimates we have just obtained justify the "energy cascade to high frequencies" for solutions of the equation (4.6). Cf. [Ily] where for a damped *non-forced* Kuramoto - Sivashinsky equation the energy cascade was studied as an intermediate phenomenon which occur to solutions on their way to an attractor.

8. Equations with time-independent forcing

In this part we consider the deterministic equation (0.7):

$$-i\dot{u} = -\delta\nu\Delta u + |u|^2 u + w(x), \tag{8.1}$$

where $w(x)$ is a smooth odd periodic complex function, nondegenerate in the following sense:

$$|\nabla w(\tilde{x})| = C_0 > 0 \quad \text{for some} \quad \tilde{x} \in \partial K^n. \tag{8.2}$$

This is a special case of equation (1.1) with $\mathcal{D}=1$, $\tilde{\tilde{\delta}}=\delta$ and time-independent forcing ξ. Theorem 2 is applicable to prove that if $\|u(t)\|_m \le R = C\delta^{-\gamma m}$ for $t_0 \le t \le t_0+T$ and $\|u(t_1)\|_0 \ge \rho = C\delta^{-\gamma\frac{2m^2-m}{3m^2-m-1}} < C\delta^{-\frac{2}{3}\gamma}$
for some $t_0 \le t_1 < T$, then

$$T - t_1 \le C_1\delta^{-\frac{2}{3}\gamma}. \tag{8.3}$$

So to estimate the length T of an interval $[t_0, t_0+T]$ where the solution $u(t,x)$ keeps its C^m-norm smaller than R, it remains to estimate the length $\tau = t_1 - t_0$ of an interval $[t_0, t_1]$ such that

$$|u(t)|_m \le R, \quad |u(t)|_0 \le \rho \quad \text{for all } t \in [t_0, t_1]. \tag{8.4}$$

To estimate τ we repeat the arguments of Chapter 3 with $\varepsilon = 1$, $\tilde{\tilde{\delta}} = \delta$. Denoting $U(t) = \nabla u(t,\tilde{x})$ (see (8.2)) we obtain a contradiction with (8.4) if

$$\tau C_0 \ge 4C_2 R^{\frac{1}{m}}\rho^{1-\frac{1}{m}} \quad \text{and} \quad \tau C_0 \ge 2\delta\tau C_1 R^{\frac{3}{m}}\rho^{1-\frac{3}{m}}.$$

These inequalities are satisfied if

$$\tau \ge C_1 R^{\frac{1}{m}}\rho \quad \text{and} \quad \delta^{-1} \ge C_2 R^{\frac{3}{m}}\rho. \tag{8.5}$$

Since $(2m^2-m)/(3m^2-m-1) < 2/3$, then (8.5) holds if

$$\gamma \le \frac{3}{11}, \quad \tau \ge \delta^{-\frac{5}{3}\gamma} \tag{8.6}$$

and δ is small. Now (8.6) together with (8.3) imply the following result:

Theorem 7. *Let $u(t,x)$ be any solution of (8.1), where smooth $w(x)$ satisfies (8.2). Then in any time-segment $[t_0, t_0+\delta^{-1/2}]$ one can find a point t_* such that $|u(t_*,\cdot)|_m \ge \delta^{-\frac{3}{11}m}$, provided that $\delta > 0$ is sufficiently small.*

The results of Chapter 5 admit immediate straightforward reformulation for the equation (8.1) (the symbol $\mathbf{E}$ for expectation should be dropped and the exponents for δ have to be replaced by $-\frac{3}{11}m$).

9. Generalizations and discussions

9.1. Other equations. The approach to study small-dissipative and small-dispersion equations (0.3) we proposed in this paper is applicable to other equations (0.2). In particular, to the nonlinear wave equation

$$u_{tt} - \delta_1\Delta u + \delta_2 u_t + u^3 = \varepsilon\zeta_\omega(t,x), \quad \delta_1,\ \delta_2 > 0, \tag{9.1}$$

and the Burgers equation

$$u_t - \delta u_{xx} + \frac{\partial}{\partial x}u^{2p} = \varepsilon\zeta_\omega(t,x), \quad p = 1, 2, \ldots,\ \delta > 0. \tag{9.2}$$

The equation (9.1) is similar to (0.3) – see [K3] for its non-forced and non-dissipative version with $\delta_2 = \varepsilon = 0$.

The Burgers equation suits well our techniques since the equation (9.2) with $\delta = 0$, $\varepsilon = 0$ is the first-order equation $u_t + (\partial/\partial x)u^{2p}$, integrable by the method of characteristics. During a finite time this equation develops a singularity for each non-zero odd-periodic initial data. Accordingly, representing the solution of (9.2) in the form $u = u_1 + u_2$ as in the Step 1 of the proof of Lemma 1, we shall get for $|u_2(t)|_1$ a better estimate from below than in Chapter 1 since $|u_2|_1$ goes to infinity when t tends to the blowing-up time. Therefore, C^m-norms of solutions of (9.2) with $\delta < 1$ and $m > 2$ are larger than $\delta^{-\mu m}$ ($\mu > 0$) after averaging in time and ensemble. Multidimensional in x version of the equation (9.2) can be treated as well.

The equation (9.2) with $p = 1$ is integrable and the limit $\delta \to 0$ gives rise to an "integrable turbulence": the limiting (as $\delta \to 0$) characteristics of the invariant measure μ_δ for the equation (9.2) can be calculated explicitly, see [KhS]. In the same time our results show that some other characteristics of this measure diverge as a negative degree of δ (see Corollary 3 in the Introduction).

To apply our techniques to the Navier-Stokes equations (NS) we need some information on the Euler equations (which are the non-damped, nor forced (NS)-equations). – We have to know that for nonzero (or "typical") initial data some functional norm of the solution of the Euler equations grows with time linearly or faster

It is believed that in the three-dimensional case (typical?) solutions of the Euler equation develop singularities during a finite time. If it is so then the corresponding norm of the solution has a finite-time blowing up (and the Burgers equation indeed is a good model for the Navier-Stokes).

9.2. Function spaces of lower smoothness. It is important to understand if the phenomenon of growth of the m^{th}-norm of a solution of (0.3) as δ in a negative degree remains true for smaller m (we recall that in our theorems $m \geq 3$). The parts of our proof which are critical to the smoothness of solutions are Step 2 from the proof of Lemma 1 in Chapter 1 and Chapter 2. – In Step 2 to estimate the C^1-norm of v_1 we used an estimate for the C^1-norm of Δv and in Chapter 3 we considered the equation for $\nabla_x u$ restricted to ∂K^n and use an estimate for $|\nabla_x \Delta u|_0$. So both parts of the proof demand $m \geq 3$. In fact, we can easily adapt our arguments to work in a Hölder space $C^{2+\alpha}$ with any $\alpha > 0$: If we use this space, then in Steps 2-4 we should estimate the C^α-norms of v, v_2, v_2 (rather than their C^1-norms). In Chapter 3 we should examine not the function $\nabla_x u\mid_{\partial K^n}$, but $u(t, \tilde{x})$ where $\tilde{x} \in K^n$ is a point at a distance $y \ll 1$ from ∂K^n. Now $|u| \leq y|u|_1$, $\left||u|^2 u\right| \leq y^3 |u|_1^3$, $|\Delta u| \leq y^\alpha |u|_{2+\alpha}$ and $\left|\int_0^T \zeta(t, \tilde{x})\, dt\right|$ grows with T as $C\sqrt{T}y$ (if the hypothesis (H2) was modified accordingly). Choosing y sufficiently small, we can argue as before to prove that a solution u can not meet for "too long" the estimates $|u|_{2+\alpha} \leq R\varepsilon^{1/3}$, $|u|_0 \leq \rho\varepsilon^{1/3}$.

Modifying slightly our arguments we also can estimate from below by a negative degree of δ Sobolev norms $\|u\|_m$ of solutions with $m > 2.5$ (see [K2] for a non-random version of this result). The restriction $m > 2.5$ is needed to be able to restrict Δu to a hypersurface, close to ∂K^n, in order to modify the arguments from Chapter 3 as it was explained above.

Our techniques fail to estimate the norms $\|u\|_m$ with $0 \leq m \leq 2.5$ including – unfortunately – the norm $\|u\|_1$, distinguished by the Kolmogorov theory of turbulence.

Remark. It is quite probable that expectation of the L_2-norm of the solution $\mathbf{E}\|u(t)\|_0$ *can be bounded* when $\delta \to 0$, since we can easily construct L_2-bounded uniformly in $\delta > 0$ solutions of the equation (0.3) with $\varepsilon = 1$ and with some stationary in t forcing $\zeta = \zeta^\delta$ such that

$C^{-1} \le |\zeta^\delta(t)|_{L_2} \le C$ for all $\delta > 0$. Indeed, let us consider the equation

$$\dot{u} = \delta\nu_I \Delta u - i\delta\nu_R \Delta u - u + i|u|^2 u + i\xi_\omega(t,x),$$

where ξ is a real valued random field, stationary bounded in t and smooth in x. Let u be any stationary solution. Then $|u(t)|_{L_2} \le C_1$ for a.a. ω (cf. (7.12)) and u solves (0.3) with $\zeta = \zeta^\delta = \xi + iu$. The L_2-norm of ζ^δ remains bounded when $\delta \to 0$. So the nonlinear operator which corresponds to a forcing ζ a stationary solution of (0.3) *does not* increase L_2-norms by the factor δ in a negative degree as it does increase the higher Sobolev norms due to Theorem 4.

9.3. Other nonlinearities. The proofs of Chapters 1–5 run without changes for the equation (0.3) with the nonlinearity $|u|^2 u$ replaced by $|u|^{2p}u$, $p = 1, 2, \dots$; and with small changes – for equations with a nonlinearity $f(|u|^2)u$, where $f(t)$ is a smooth function equal zero for $t = 0$ and monotonically increasing for $t > 0$.

9.4. Higher dimensions. The results of Theorems 1-5 remain true for n-dimensional equations ($x \in K^n$) with any $n \ge 1$. Proofs are the same.

9.5. Turbulence as "random squeezing". In this part we describe the intuition we have behind the main results of this work. Let us consider the non-perturbed and non-damped nonlinear Schrödinger equation:

$$-i\dot{u} = -\delta\Delta u + |u|^2 u, \quad 0 < \delta < 1. \tag{9.3}$$

In [K1, K2] we studied the dynamical system defined by this equation in the spaces $C^m = C^m(K^n, \mathbb{C}), m \ge 3$ (and in $H^m, m > 2$). We proved that each space C^m contains a relatively small recursion subset $\mathfrak{A} = \mathfrak{A}^m(\delta)$ with good control for the recursion time $\tilde{t}$. More specifically, we proved that each solution of (9.3) with an initial condition $u(0) = u_0 \in C^m$ will visit $\mathfrak{A}$ at some time $\tilde{t} \ge 0$ such that $\tilde{t} \le \Theta$, where Θ depends only on m, δ and $\|u_0\|_m$. We checked that $\Theta \le C\delta^{-1}$ for all u_0, so by the time $C\delta^{-1}$ the flow of the equation will pull the whole space C^m through $\mathfrak{A}$. The set $\mathfrak{A}$ looks like a narrow tube which at a distance r from the origin has the diameter d_r ,

$$d_r \sim \delta^\mu r^\nu. \tag{9.4}$$

Here $\mu(m) \sim \frac{1}{2}$ and $\nu(m) \sim \frac{2}{m}$ for large m.

Now let us consider the equation (0.3) with a locally constant in time forcing ζ (cf. Example 3 in Chapter 6):

$$-i\dot{u} = -\delta\Delta u + |u|^2 u + \zeta_\omega^{[t]}(x) \tag{9.5}$$

(not much will change if we replace here δ by $\delta\nu$). For any integer j and for $t \in [j, j+1)$ the forcing is time independent and – similar to the zero-forcing case – the flow of the equation will start to pull the phase-space through a tube-like set $\mathfrak{A}_j$. For $t \in [j+1, j+2)$ the set $\mathfrak{A}_j$ will be replaced by $\mathfrak{A}_{j+1}$, etc. Each set $\mathfrak{A}_j$ equals to a deformed set $\mathfrak{A}$, shifted by a vector of order one. By (9.4) points in the intersection of any two different sets $\mathfrak{A}_j, \mathfrak{A}_k$ must have C^m-norms r such that $\delta^\mu r^\nu \gtrsim 1$, or $r \gtrsim \delta^{-\mu/\nu}$ and $r \gtrsim \delta^{-m/4}$ for large m. Therefore either the norm $|u(t)|_m$ is as big as $\delta^{-m/4}$, or $u(t)$ moves fast from $\mathfrak{A}_j$ to $\mathfrak{A}_{j+1}$, etc.

So the flow of the randomly forced equation (9.5) try to squeeze the whole C^m to the randomly moving slit $\mathfrak{A}_{[t]}$, thus producing high values of the C^m-norm as is described in our theorems.

To complete, we note that this intuition well agree with the fact that L_2-norms of solutions can be of order one as $\delta \to 0$ (see the Remark in Chapter 9.2): in striking difference with the case of smooth spaces C^m with $m \geq 3$, due to deep symplectic reasons the flow of (9.3) can not squeeze L_2-balls with radii of order one to narrow cylinders (this is Gromov's non-squeezing property, see [K3, K4]).

9.6. On numerical investigations of the equation. Theorem 4 explains difficulties in numerical evaluation solutions of the equation (0.3) with $\delta \ll 1$. Indeed, let us suppose that $\varepsilon = 1$ (so the equation takes the form (4.6)) and decompose a solution $u(t, x)$ to Fourier series: $u = \sum u_s(t) \exp(\pi i s \cdot x)$. If to solve (4.6) we use Fourier method, then we replace u by a truncated Fourier sum $S^N u$,

$$S^N u(t,x) = \sum_{\|s\| \leq N} u_s(t) e^{\pi i s \cdot x}, \quad \|s\| := \max_{1 \leq j \leq n} |s_j|,$$

neglecting the modes with $\|s\| > N$. The cut-off level N has to be sufficiently large to guarantee that

$$|R^N u(t,x)| \leq \epsilon, \quad R^N u := u - S^N u, \tag{9.6}$$

with some sufficiently small ϵ.[4] Without extra information on the solution u, the best we can do is to estimate $R^N u$ via the C^m-norm of u. Say, via its C^3-norm:

$$|R^N u(x)| \leq C_N |u|_3. \tag{9.7}$$

Optimal constant C_N obviously is $\geq CN^{-3}$ as shows an example of the function $u = e^{\pi i s \cdot x}$ with $\|s\| = N + 1$. In fact, it is $\sim CN^{-2}$. Indeed, $S^N u$ equals to the convolution of $u(x)$ with the Dirichlet kernel $D_N(x) = \prod_{j=1}^n \frac{\sin \pi (N + \frac12) x_j}{2 \sin \frac{\pi}{2} x_j}$. Since D_N has unit integral over the n-torus, then

$$R^N u(x) = \int_{\mathbb{T}^n} \big(u(x-y) - u(x)\big) D_N(y)\, dy = 2^{-n} \int_{\mathbb{T}^n} \frac{u(x-y) - u(x)}{\prod \sin \frac{\pi}{2} y_j} \prod \sin \pi \big(N + \frac12\big) y_j \, dy.$$

The function $y \to (u(x-y) - u(x)) / \prod \sin \frac{\pi}{2} y_j$ is not C^3- but only C^2-smooth and its C^2-norm is bounded by $C|u|_3$. Since $R^N u(x)$ decays with N as N-th Fourier coefficient of this function, then $C_N \sim CN^{-2}$.

Hence, to guarantee (9.6) we should have $CN^{-2}|u|_3 \leq \epsilon$, or $N \geq \epsilon^{-1/2} |u|_3^{1/2}$. Now we use Corollary 1 from Theorem 3 to find that $N \geq C \epsilon^{-\frac12} \delta^{-\frac32 \gamma}$ with $\gamma = \frac15$. Thus, the total number D of Fourier modes we have to keep (=dimension of the problem) is

$$D = (2N)^n \geq C \epsilon^{-\frac{n}{2}} \delta^{-\frac32 n \gamma}, \quad \gamma = \frac15. \tag{9.8}$$

(It is very likely that the optimal estimate for γ is better than $1/5$ and is at least $1/2$, see the discussions in the Introductuon and Chapter 7.3 of [K2].)

[4] we have tacitly suppose that the solution u is of order one (cf. the Remark in Chapter 9.2).

The estimate (9.8) with $\gamma = \frac{1}{5}$ (or $\gamma = \frac{1}{2}$) shows that for $\delta \sim 10^{-3}$, $\epsilon = 10^{-2}$ and $n = 3$ we have $D \gtrsim 10^{5.7}$ – honest numerical evaluation of solutions of three-dimensional equation (0.3) is beyond reach of modern computers.

Our pessimistic conclusion was based on the assumption that estimate (9.7) is optimal. This assumption is rather realistic but it can be repealed by additional a priori information on the solutions. For example, if we take the non-perturbed non-damped equation (9.3) (or more general equation (0.9)) and consider its solution corresponding to an analytic initial data $u_0(x)$ then – due to rather reliable numerics – the solution remains analytic in x for long time with some positive radius of analyticity r. Now

$$|u_s| \lesssim e^{-r|s|} \tag{9.9}$$

and $|u|_m \sim r^{-m}$, so by Theorem 4 $r < \delta^\gamma$. The estimate (9.9) implies better than (9.8) estimate for the dimension D: if $\epsilon \geq \delta$, then $D \gtrsim \delta^{-\gamma n}$. For $n = 2$ and $\delta = 10^{-3}$ problems of this dimension are within powers of modern computers.[5]

Appendix 1. On stationary Gaussian processes

Let $\zeta_\omega(t)$ be a stationary complex Gaussian process with zero meanvalue and with the correlation function $B(t)$, i.e.,

$$B(t) = \mathbf{E}(\zeta(s)\bar{\zeta}(s+t)), \quad 0 = \mathbf{E}(\zeta(s)\zeta(s+t)).$$

So

$$B(t) = \int e^{i\lambda t} F(d\lambda), \tag{A1}$$

where F is the spectral measure of the process. We assume that F has a spectral density, $F(d\lambda) = f(\lambda)\, d\lambda$. The density f is a real function since $B(-t) = \overline{B}(t)$. We suppose that

$$\text{the function } \; B(t) \; \text{ is Lipschitz and } \quad \int_{-\infty}^{\infty} |B(t)|\, dt < \infty. \tag{A2}$$

(Note that (A2) holds if $f(\lambda)\left(1+|\lambda|^r\right) \in L_2(\mathbb{R})$ for some $r > \frac{3}{2}$.) Because (A2) an appropriate realization of ζ_ω has continuous sample trajectories – this is an immediate consequence of Kolmogorov's theorem (see e.g., [CL, Chapter 9]).

Lemma A1. *For each $R > 1$ there exists a set $\Omega_R \in \mathcal{F}$ such that $\mathbf{P}(\Omega_R) \leq C_1 R^{-1} e^{-R^2/C^2}$ and $|\zeta_\omega(x) - \zeta_\omega(0)| \leq R$ for each $0 \leq x \leq 1$, if $\omega \notin \Omega_R$.*

Proof. By (A2), $\zeta^\tau = \zeta(t+\tau) - \zeta(t)$ is a complex Gaussian function with a dispersion $\leq C\tau$. Therefore

$$\mathbf{P}\left\{|\zeta^\tau| \geq \delta\right\} \leq \mathbf{P}\left\{|\mathrm{Re}\,\zeta^\tau| \geq \frac{\delta}{2}\right\} + \mathbf{P}\left\{|\mathrm{Im}\,\zeta^\tau| \geq \frac{\delta}{2}\right\} \leq$$
$$\leq C'\left(\frac{\delta}{C_1\sqrt{\tau}}\right)^{-1} \exp\left(-\frac{1}{2}\left(\delta/(C_1\sqrt{\tau})\right)^2\right).$$

[5] I wish to thank Robert Almgren for acquainting me with results of his very convincing numerical investigations of the two-dimensional equation (9.3) with analytic initial data.

If we denote
$$\mathcal{O}_{l,r} = \left\{\omega \mid \left|\zeta(l2^{-r}) - \zeta((l-1)2^{-r})\right| \geq Rr^{-2}\right\},$$
then due to the last estimate with $\tau = 2^{-r}$ and $\delta = Rr^{-2}$ we find that
$$P(\mathcal{O}_{l,r}) \leq C\frac{C_1 r^2}{R2^{r/2}} \exp\left(-\frac{1}{2}\left(\frac{R2^{r/2}}{C_1 r^2}\right)^2\right).$$
Therefore,
$$\mathbf{P}\left(\bigcup_{l=1}^{2^r} \mathcal{O}_{l,r}\right) \leq CC_1 r^2 R^{-1} 2^{r/2} \exp\left(-\frac{1}{2}\left(\frac{R2^{r/2}}{C_1 r^2}\right)^2\right).$$
Since $R > 1$, then for the set $\mathcal{O}_R = \bigcup_{r=1}^{\infty} \bigcup_{l=1}^{2^r} \mathcal{O}_{l,r}$ we have
$$\mathbf{P}(\mathcal{O}_R) \leq C_2 R^{-1} \exp\left(-R^2/C_1^2\right).$$

Now let us take any $x \in [0,1]$ and write it as a binary fraction: $x = \alpha_1 2^{-1} + \alpha_2 2^{-2} + \dots$, where $\alpha_j = 0$ or 1. Let x^N be the sum of the first N terms of this series. Since for $\omega \notin \mathcal{O}_R$ we have $|\zeta(l \cdot 2^{-r}) - \zeta((l-1)2^{-r})| \leq Rr^{-2}$, then $|\zeta(x_N) - \zeta(x_{N-1})| \leq RN^{-2}$ and $|\zeta(x_N) - \zeta(0)| \leq R\sum N^{-2} \leq CR$. Since sample trajectories of ζ are continuous, then also $|\zeta(x) - \zeta(0)| \leq CR$ and the lemma follows if we redenote $CR = R$ (and change the constants C_1, C). $\square$

By the lemma, for $j = 0, \pm 1, \pm 2 \dots$ we can find a set Ω_j such that
$$|\zeta_\omega(x) - \zeta_\omega(j)| \leq C_* \sqrt{\mu}\sqrt{\ln(|j|+3)} \quad \text{for } x \in [j, j+1],\ \omega \notin \Omega_j, \tag{A3}$$
and $\mathbf{P}(\Omega_j) \leq \frac{C_1}{C_*\sqrt{\mu}} (|j|+3)^{-C_*^2 \mu C^{-2}}$. Suppose that $\mu \geq 1$. Then
$$\mathbf{P}(\cup \Omega_j) \leq \frac{C_2}{C_*\sqrt{\mu}} 3^{-C_*^2 \mu C^{-2}} \leq \frac{1}{2} e^{-\mu},$$
if $C_* \geq C$, $C_* \geq 2C_2$.

For $j \in \mathbb{Z}$, let us also consider the sets $Q_j = \left\{\omega \mid |\zeta_j| \geq \tilde{C}\sqrt{\mu}\sqrt{\ln(|j|+3)}\right\}$. We have
$$\mathbf{P}(Q_j) \leq (C'/\tilde{C}\sqrt{\mu})(|j|+3)^{-C''\tilde{C}\mu}$$
and $\mathbf{P}(\cup Q_j) \leq \frac{1}{2}e^{-\mu}$, if $\tilde{C}$ is sufficiently large.

Now we denote $\Omega_\mu = (\cup \Omega_j) \cup (\cup Q_j)$. Then $\mathbf{P}(\Omega_\mu) \leq e^{-\mu}$ and – by (A3) and the definition of the sets Q_j – for $\omega \notin \Omega_\mu$ and any real t we have:
$$|\zeta_\omega(t)| \leq (C_* + \tilde{C})\sqrt{\mu}\sqrt{\ln([t]+3)} \leq C\sqrt{\mu}\sqrt{\ln(|t|+1)}.$$
Denoting $e^{-\mu} = p$ we get

Theorem A2. *For each $p > 0$ there exists a measurable subset Ω, $\mathbf{P}(\Omega) \le p$, such that $|\zeta_\omega(t)| \le C\sqrt{\ln p^{-1}}\sqrt{\ln(|t|+1)}$ for all t if $\omega \notin \Omega$.*

Now let us consider the integral $\xi(t)$ of $\zeta(t)$, $\xi_\omega(t) = \int_0^t \zeta_\omega(\tau)\, d\tau$. Clearly ξ_ω is a zero-meanvalue Gaussian process and

$$\mathbf{E}|\xi(t)|^2 = \mathbf{E}\Big(\int_0^t \int_0^t \zeta_\omega(u)\bar{\zeta}_\omega(v)\, dudv\Big) = \int_0^t \int_0^t B(u-v)\, dudv = \int_0^t dw \int_{-w}^{w} B(a)\, da.$$

The spectral density $f(\lambda)$ is real nonnegative. Let us assume that it is positive at zero: $f(0) = f_0 > 0$. Since the correlation $B(t)$ is integrable, then by (A1) $\int_{-\infty}^{\infty} B(a)\, da = 2\pi f_0$ and $\int_{-w}^{w} B(a)\, da \ge \frac{3}{2}\pi f_0$ for $w \ge w_*$. Therefore there exists t_* such that $\mathbf{E}|\xi(t)|^2 \ge t\pi f_0$ for $t \ge t_*$. Since $\xi(t)$ is a Gaussian random variable, we get:

Lemma A3. *If the spectral density $f(\lambda)$ is positive at zero and the correlation function $B(t)$ is integrable, then for sufficiently large T and all a we have:*

$$\mathbf{P}\Big\{\Big|\frac{1}{\sqrt{T}}\int_0^T \zeta_\omega\, dt\Big| \le a\Big\} \le Ca$$

with some T-independent $C > 0$.

Appendix 2. Scheme of the Main Theorem's proof

Our arguments in this appendix are imprecise. To convey them we use loose language. In particular, we say that a probability of an event is *high* if it is bigger than $1 - \delta^a$ with some $a > 0$, asymptotically as $\delta \to 0$, and a probability of an event is *positive* if asymptotically it is bigger than δ^b for any $b > 0$.

Below we discuss Theorem 4 in its simplified version, presented in the Introduction as Theorem A. That is, we study a solution of equation (0.3) with $\varepsilon = 1$ on any time-segment of length $\ge T_1 = \delta^{-11/17}$; say on the segment $[0, T_1]$, and show that with high probability somewhere on this segment C^m-norm (in x) of the solution becomes bigger than $R = R_m = \delta^{-\frac{3}{17}m}$. To prove this result we assume that the opposite holds:

$$\text{with a positive probability, } \ |u(t,\cdot)|_m \le R \ \text{ for all } \ 0 \le t \le T_1. \tag{A4}$$

We shall deduce from (A4) a contradiction, using two following assertions:

Statement 1 ("deterministic"). *Under the a priory assumption (A4), if $|u(t_1,\cdot)|_0 = \rho_0 \ge \rho = \delta^{-3/17}$ for some $t_1 \ge 0$, then with high probability there exists $t_2 \le t_1 + T_1/2$ such that $|u(t_2,\cdot)|_m > R$. Here t_1, t_2 and ρ_0 are are random variables.*

Statement 2 ("random"). *Under the a priori assumption (A4) there exists a random time $t_1 \ge 0$ such that with high probability $t_1 \le T_1/2$ and $|u(t_1,\cdot)|_0 \ge \rho$.*

These two statements jointly contradict (A4) since with high probability the random mechanism of Statement 2 makes C^0-norm of a solution $u(t,\cdot)$ bigger than ρ and next by the time $t_1 + t_2 \le T_1$ the deterministic mechanism of Statement 1 increases the C^m-norm above the level R. This contradicts (A4) and proves Theorem A.

The two statement are loose versions of theorems proven in Chapters 1-3. Here we briefly explain how the proofs go, abbreviating a function $u(t,x)$ to $u(t)$ or u.

To prove Statement 1 we rewrite the equation (0.3) with $\varepsilon = 1$ as

$$-i\dot{u} - |u|^2 u = \eta(t,x) := -\delta\nu\Delta u + \zeta_\omega(t,x). \tag{A5}$$

Let $\tilde{u}(t,x)$ solves this equation with $\eta = 0$ subject to the initial condition $\tilde{u}(t_1,x) = u(t_1,x)$:

$$-i\dot{\tilde{u}} - |\tilde{u}|^2\tilde{u} = 0, \quad \tilde{u}(t_1,x) = u(t_1,x) =: u_1(x).$$

Then $|\tilde{u}(t,x)|^2$ is a t-independent constant and $\tilde{u}(t,x) = e^{i(t-t_1)|u_1(x)|^2}u_1(x)$. In partucular,

$$\nabla_x\tilde{u}(t,x) = i(t-t_1)(\nabla|u_1|^2)u_1 e^{i(t-t_1)|u_1(x)|^2} + O(1) \ \text{ as } \ (t-t_1)\to\infty,$$

and C^1-norm in x of $\tilde{u}(t,x)$ grows with t at least linearly:

$$|\tilde{u}(t)|_1 \geq C_1(t-t_1)|u_1|_0^3 - C_2 \geq C_1(t-t_1)\rho_0^3 - C_2. \tag{A6}$$

If $|u(t)|_m \leq R$ for $0 \leq t \leq T_1$, then $|\tilde{u}(t)|_1 \leq C|\tilde{u}(t)|_0^{\frac{m-1}{m}}|\tilde{u}(t)|_m^{\frac{1}{m}} \leq C\rho_0^{\frac{m-1}{m}}R^{\frac{1}{m}}$ by the interpolation inequality (0.9). Since $t-t_1$ can be made as big as $T_1/2$, this estimate would contradict (A6) if T_1 is sufficiently big compare to $R^{\frac{1}{m}}$ and if ρ_0 is not too small; the choice of ρ_0, R and T_1 made in assumptions of Statement 1 is quite sufficient to guarantee this contradiction.

The above arguments would prove Statement 1 if η in (A5) was zero. Of course, this is not the case. Still, it can be checked that under the assumption (A4) one has $|u(t)|_0 \leq C\rho_0$ for all $0 \leq t \leq T_1$ (see in Chapter 2). Therefore, first,

$$|\delta\nu\Delta u(t)|_1 \leq \delta|u(t)|_3 \leq C\delta|u(t)|_0^{\frac{m-3}{m}}|u(t)|_m^{\frac{3}{m}} \leq C_1\delta\rho^{\frac{m-3}{m}}R^{\frac{3}{m}},$$

and, second, by assumption (H2) with high probability everywhere on the segment $[0,T_1]$ we have $|\zeta_\omega(t,\cdot)|_1 \leq C(\ln\delta^{-1})^c$ with some finite c. Thus,

$$|\eta(t)|_1 \leq C(\delta\rho^{\frac{m-3}{m}}R^{\frac{3}{m}} + (\ln\delta^{-1})^c).$$

This estimate allows us to treat $\eta(t,x)$ as a perturbation and to obtain for $u(t,x)$ an estimate from below, similar to estimate (A6) for the unperturbed solution $\tilde{u}$. Due to the above arguments, this estimate contradicts (A4) (again, under the specific choice of R, T and ρ we made). This proves Statement 1.

To prove Statement 2 we assume that

$$|u(t)|_0 \leq \rho \quad \forall\, 0 \leq t \leq T_1/2$$

on a set of positive measure and prove that under this additional (exept (A4)) assumption the random force ζ_ω will swing the solution u. Here the main idea is to examine the gradient $\nabla_x u(t,y)$ of the solution at some point $y \in \partial K^n$. For $v(t) = \nabla u(t,y)$ we have the equation:

$$-i\dot{v} = -\delta\nu\nabla\Delta u(t,y) + \nabla\zeta_\omega(t,y), \tag{A7}$$

where

$$|\delta\nu\nabla\Delta u(t,y)| \le \delta|u(t)|_3 \le C\delta\rho^{\frac{m-3}{m}}R^{\frac{3}{m}}$$

by our assumptions. Then the integral in dt over the segment $[0,t]$ of the l.h.s. of (A7) is bounded by $2|v| \le 2|u|_1 \le C\rho^{\frac{m-1}{m}}R^{\frac{1}{m}}$ and the integral of the first term in the r.h.s. is bounded by $Ct\delta\rho^{\frac{m-3}{m}}R^{\frac{3}{m}}$. By the assumption (H2), with high probability the integral of the second term in the r.h.s. is bigger than $C\sqrt{t}$. Thus, (A7) implies that

$$\sqrt{t} \le C_1(\rho^{\frac{m-1}{m}}R^{\frac{1}{m}} + t\delta\rho^{\frac{m-3}{m}}R^{\frac{3}{m}})$$

on a set of positive measure. It turns out that under our choice of ρ, R and T_1 the last estimate can not hold for all $0 \le t \le T_1/2$. This contradiction proves Statement 2.

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Steklov Mathematical Institute, Section for Geometry & Topology, Vavilova St. 42, 117966 Moscow

E-mail address: `kuksin@sci.lpi.ac.ru`