This is the first version of the paper. It contains some flaws, corrected in the final version "KAM for the Nonlinear Schroedinger Equation". I keep this text on the web since its presentation differs a bit from that in the final version and gives some additional details.

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KAM for NLS

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Abstract

We consider the d-dimensional ($d \geq 1$) nonlinear Schrödinger equation (NLS) under the periodic boundary conditions:

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon_0 |u|^2 u; \quad u = u(t,x),\ x \in \mathbb{T}^d, \qquad (0.1)$$

where $V(x) = \sum \hat{V}(a) e^{ia\cdot x}$ is an analytic function with $\hat{V}$ real. [1] For $\varepsilon = 0$ the equation is linear and has the time–quasiperiodic solutions u,

$$u(t,x) = \sum_{a\in\mathcal{A}} \hat{u}_0(a) e^{i(|a|^2+\hat{V}(a))t} e^{ia\cdot x}, \quad 0 < |\hat{u}_0(a)| \leq 1,$$

where $\mathcal{A}$ is any finite subset of $\mathbb{Z}^d$, $n := |\mathcal{A}| \geq 1$. We shall treat $\omega_a = |a|^2 + \hat{V}(a),\ a \in \mathcal{A}$ as free parameters in some domain $\Omega \subset \mathbb{R}^n$ and we shall prove the following KAM–result:

If $|\varepsilon|$ is sufficiently small, then there is a large subset Ω' in Ω such that for all $\omega \in \Omega'$ the solution u persists as a time–quasiperiodic solution of (0.1) *which has all Lyapounov exponents equal to zero and whose linearized equation is reducible to constant coefficients.*

0 Introduction.

If we write

$$\begin{cases} u(x) = \sum_{a\in\mathbb{Z}^d} \sqrt{2} u_a e^{i<a,x>} \\ \overline{u(x)} = \sum_{a\in\mathbb{Z}^d} \sqrt{2} v_a e^{i<a,x>}, \end{cases}$$

then, in the symplectic space

$$\begin{cases} \{(u_a, v_a) : a \in \mathbb{Z}^d\} = \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d} \\ i \sum_{a\in\mathbb{Z}^d} du_a \wedge dv_a, \end{cases}$$

the equation (0.1) becomes a Hamiltonian system with Hamiltonian

$$H_0 = \frac{1}{2} \sum_{a\in\mathbb{Z}^d} (|a|^2 + \hat{V}(a)) u_a v_a \quad + \frac{1}{4}\varepsilon_0 \sum_{a_1+a_2-b_1-b_2=0} u_{a_1} u_{a_2} v_{b_1} v_{b_2}.$$

[1] This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu. Considering this model we remove some technical difficulties, which are not related to the main ones.

For $a \in \mathcal{A}$ we introduce the action angle variables (q_a, p_a), defined through the relations

$$u_a = \sqrt{2(p_a - |\hat{u}_0(a)|^2)}e^{iq_a},\ v_a = \sqrt{2(p_a - |\hat{u}_0(a)|^2)}e^{-iq_a}.$$

In order to write it in real form we introduce $\zeta = (\xi, \eta)$ through

$$u_a = \frac{1}{\sqrt{2}}\left(\xi_a + i\eta_a\right),\ v_a = \frac{1}{\sqrt{2}}\left(\xi_a - i\eta_a\right).$$

The integrable part of the Hamiltonian now becomes

$$\mathcal{H}(p, \zeta) = \sum_{a\in\mathcal{A}}(|a|^2 + \hat{V}(a))p_a + \frac{1}{2}\sum_{a\in\mathcal{L}=\mathbb{Z}^d\setminus\mathcal{A}}(|a|^2 + \hat{V}(a))(\xi_a + \eta_a),$$

while the perturbation $\varepsilon_0 h$ will be a function of

$$\{(q_a, p_a) : a \in \mathcal{A}\} \text{ and } \{\zeta_\mathrm{a} : \mathrm{a} \in \mathcal{L}\}.$$

$\omega_a = |a|^2 + \hat{V}(a),\ a \in \mathcal{A}$ are the basic frequencies and $\lambda_a = |a|^2 + \hat{V}(a),\ a \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$ are the normal frequencies. The ω's will be our free parameters belonging to a set $\Omega \in \mathbb{R}^n$.

We shall assume that $\hat{V}$ is real and

$$|\hat{V}(a)| \le C_1 e^{-C_2|a|} \quad \forall\, a \in \mathcal{L}\,,$$

and that

$$\Omega \subset \{|\omega| \le C_3\}$$

We also assume

$$\begin{aligned} &|\lambda_a + \lambda_b| \ge C_4 \quad \forall\, a, b \in \mathcal{L},\ \forall\,\omega\,,\\ &|\lambda_a - \lambda_b| \ge C_4 \quad \forall\, a, b \in \mathcal{L}, |a| \ne |b|. \end{aligned}$$

We define the complex domain

$$O^\gamma(r, \mu) = \begin{cases} |\Im q| < r \\ |p| < \mu^2 \\ \|\zeta\|_\gamma = \sqrt{\sum_{a\in\mathcal{L}} |\zeta_a|^2|a|^{2m^*}e^{2\gamma|a|}} < \mu. \end{cases}$$

Theorem 0.1. *Under the above assumptions, for ε_0 sufficiently small there exist a Borel subset $\Omega' \subset \Omega$,*

$$Leb\,(\Omega \setminus \Omega') \le \mathrm{const}\,.\varepsilon_0^{exp_1}\,,$$

and for each $\omega \in \Omega'$, a real analytic symplectomorphisms

$$\Sigma_\omega : O^{\gamma/2}(r/2, \mu/2) \to O^{\gamma/2}(r, \mu)\,,$$

such that $|\Sigma_\omega - id| \leq \text{const}\,.\varepsilon_0^{exp_2}$ *and*

$$(\mathcal{H} + \varepsilon_0 h) \circ \Sigma_\omega = \omega' \cdot p + \frac{1}{2} \langle \zeta, A'\zeta \rangle + h' \,,$$

where the quadratic form $\frac{1}{2} \langle \zeta, A'\zeta \rangle$ *has the form*

$$\langle u, Qv \rangle,$$

with Q *Hermitian and block-diagonal with finite-dimensional blocks, and where*

$$h' \in \mathcal{O}(p^2, p\zeta, \zeta^3).$$

The constant const *only depends on the dimensions* d *and* n *and on* $C_1, \ldots, C_4$. *The exponents* exp_1 *and* exp_2 *only depends on the dimensions* d *and* n.

Every torus $\Sigma_\omega(\mathbb{T}^n \times \{0\} \times \{0\})$ is invariant for the Hamiltonian equations and is filled in with the time-quasiperiodic solutions $t \to \Sigma_\omega(q + \omega_\infty t)\,,\ q \in \mathbb{T}^n$. The linearised map $D\Sigma_\omega(q + \omega_\infty t)$ reduces the linearized equation on the torus in the ζ-direction to the constant coefficient system

$$\frac{d}{dt}\,\zeta = JA'\zeta\,.$$

Due to the form of A' all Lyapunov exponents of the solutions vanish.

Theorem 0.1 follows from a bit more general result, proved in Section 2.

Some references. For finite dimensional Hamiltonian systems the first proof of stable (i.e. vanishing of all Lyapunov exponents) was obtained by Eliasson [Eli85, Eli88]. This has been improved in many works and the situation in finite dimension is pretty well understood. Not so, however, in infinite dimension.

If $d = 1$, the space-variable x belongs to a finite segment and the equation is supplemented by the Dirichlet or Neumann boundary conditions, this result was obtained by Kuksin in [Kuk88] (also see [Kuk93, Pös96]). The case of 1d periodic boundary conditions was treated later by Bourgain in [Bou96], using another multi–scale scheme, suggested by Fröhlich–Spencer in their work on the Anderson localisation [FS83], and later exploited by Craig–Wayne in [CW93] to construct time–periodic solutions of nonlinear PDEs. Due to these and other publications, the perturbation theory for quasiperiodic solutions of 1d Hamiltonian PDE is now sufficiently well developed, e.g. see books [Kuk93, Cra00, Kuk00]. Study of the corresponding problems for space–multidimensional equations is now at its early stage. Developing further the scheme, suggested by Fröhlich–Spencer, Bourgain managed to prove Theorem 0.1 for the 2d case [Bou98]. Finally, he has recently announced (e.g. in [Bou04]) that the new techniques, invented by him with collaborators in their works on spectral theory of (linear) Schrödinger operators with quasiperiodic coefficients, allow to establish existence of quasi-periodic solutions for any d. (A detailed proof has not been given yet.) It should be mentionned the multi–scale-scheme developped by these authors does not (at least not immediately) give neither vanishing of the Lyapunov exponents nor reducibility of the linearized equation.

Main ideas. Very briefly, our main idea is to put under strict control the linear parts of the transformations, forming the KAM–procedure, defined by the homological equation.The solution, with estimates, of this equation requires control of the "small divisors" which imposes conditions on $\omega \in \Omega$. These conditions are relatively easy to fulfill when $\mathcal{L}$ is a finite set in $\mathbb{Z}^d$ or when $\mathcal{L} \subset \mathbb{Z}^1$ because then the equation imposes on finitely many conditions on ω on every scale. In the case when $\mathcal{L}$ is an infinite subset of $\mathbb{Z}^d$, $d \geq 2$, the equation imposes infinitely many conditions on ω on every scale.

To verify that these conditions can be fulfilled in the n-parameter family $\omega \in \Omega$, we make use a special property of infinite–dimensional matrices — the Töplitz-Lipschitz property. This property has two nice features. These matrices is an algebra: one can multiply them and solve linear differential equations [EK05b]. They permit a "compactification of the dimensions: if the Hessian (with respect to ζ) of the Hamiltonian is Töplitz-Lipschitz then the infinitely many small divisor conditions needed to solve the homological equation reduce to finitely many conditions [EK05a].

In this paper we prove that, if the Hessian (with respect to ζ) of the Hamiltonian is Töplitz-Lipschitz, then this is also true of the linear part of our KAM–transformations and of the Hessian of the transformed Hamiltonian. This will permit us to formulate an inductive statement which, as usual in KAM, gives Theorem 0.1.

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1 Domains, functions and Hamiltonian equations.

1.1 Constants

Let us take a real number $m_* > d/2$ and integers n, $d \geq 1$. They are fixed in our work, and the dependence on then of the objects which we consider will not be indicated. The domains and functions we will construct also depend on the following real parameters:

$$\Lambda \geq 6, \quad \gamma \in (0,1], \quad \mu \in (0,1), \quad \varepsilon \in (0,1).$$

These parameters will change from one KAM-step to another, and we shall control how our objects depend on them. By C, C_1 etc and c, c_1 etc we denote different positive constants, independent of Λ, γ, μ and ε (but they may depend on m_*, n and d).

1.2 Linear spaces.

Let

$$\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A} \quad \mathcal{A} = \text{a finite set.}$$

We fix any constant $m_* > \max\{2, n/2\}$ and denote by Y_γ, $\gamma \in [-1, 1]$, the following weighted l_2-spaces:

$$Y_\gamma = \{\zeta = (\zeta_s \in \mathbb{R}^2, s \in \mathcal{L}) \mid \|\zeta\|_\gamma < \infty\}.$$

Here

$$\|\zeta\|_\gamma^2 = \sum_{a \in \mathcal{L}} |\zeta_a|^2 e^{2\gamma|a|} \langle a \rangle^{2m_*}, \quad \langle a \rangle = |a| \vee 1.$$

In the spaces Y_γ acts the linear operator J,

$$J : \{\zeta_s\} \mapsto \{\sigma_2 \zeta_s\}, \tag{1.1}$$

where $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It provides the spaces $Y_\gamma, \gamma \geq 0$, with the symplectic structure $J\, d\zeta \wedge d\zeta$. [2] To any C^1-smooth function, defined on a domain $O \subset Y_\gamma$, this structure corresponds the Hamiltonian equation

$$\dot{\zeta} = J \nabla f(\zeta),$$

where $\nabla f \in Y_{-\gamma}$ is the gradient with respect to the scalar product in Y (i.e., $\langle \nabla f(\zeta), \eta \rangle = df(\zeta)\eta$) for all $\eta \in Y_\gamma$).

1.3 Infinite matrices – quadratic forms

Details of the definition and results below see in [EK05b].

Consider a matrix $A : \mathcal{L} \times \mathcal{L} \to M(2 \times 2)$ with values in the space of real 2×2-matrices. We assume it is

symmetric, i.e.

$$As, s' = A_{s',s} \quad \forall s, s'.$$

To such an A we associate in a unique way a real quadratic form

$$q(\zeta, \zeta) = \sum_{a,b \in \mathcal{L}} \langle \zeta_a, A_{a,b} \zeta_b \rangle.$$

Let us abbreviate $M(2 \times 2) = X$, and consider the following four real matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

[2] $J\, d\zeta \wedge d\zeta(\xi, \eta) = \langle \xi, \eta \rangle$ for any $\xi, \eta \in Y_\gamma$.

We denote by X_p the linear subspace of X, generated by σ_0, σ_2, denote by X_q the subspace, generated by σ_1, σ_2, and denote by p (by q) the projection of X to X_p along X_q (respectively, the projection to X_q along X_p). Finally, we define

$$M_\gamma^+ = \{A \in M_\gamma \mid |||A|||_\gamma^+ < \infty\},$$

where

$$|||A|||_\gamma^+ = \sup_{s,\, s'} \{|pA_{s\,s'}|e^{\gamma|s-s'|} \vee |qA_{s\,s'}|e^{\gamma|s+s'|}\}.$$

Remark 1.1. This supremum norm is denoted $|\ |^\gamma$ in [EK05a].

We have for $\gamma' \geq \gamma$

$$\|Ay\|_\gamma \leq C(\gamma'-\gamma)^{-d_1} |||A|||_{\gamma'}^+ \|y\|_\gamma, \tag{1.3}$$

where we denoted

$$d_1 = d + m_*.$$

The spaces M_γ^+ contain the identity matrix. The union $\cup_{0<\gamma\leq 1} M_\gamma^+$ is a stratified algebra, and

$$|||AB|||_\gamma^+ + |||BA|||_\gamma^+ \leq C(\gamma'-\gamma)^{-d} |||A|||_{\gamma'}^+ |||B|||_\gamma^+ \tag{1.4}$$

if $\gamma < \gamma' \leq 1$.

We also note that since the multiplication by σ_2 preserves the spaces X_p and X_q, then $JM_\gamma^+ = M_\gamma^+$ and the mapping

$$M_\gamma^+ \to M_\gamma^+, \quad A \mapsto JA \tag{1.5}$$

is an isometry for each γ.

A matrix A is called *Töplitz at* ∞ if for all $a \neq 0, b_1, b_2 \in \mathbb{Z}^d$ the two limits

$$A_{a,b_1,b_2}^{\infty\pm} = \lim_{t\to\infty} A_{(ta+b_1)\pm(ta+b_2)}$$

exist (here and in similar situations below, t goes to ∞ along the set $\{t \geq 0 \mid ta+b_1 \in \mathcal{L},\, ta+b_2 \in \mathcal{L}\}$).

Note that for $A \in M_\gamma^+$ we have

$$(pA)_{a,b_1,b_2}^{\infty-} \equiv 0\,, \quad (qA)_{a,b_1,b_2}^{\infty+} \equiv 0\,.$$

For a Töplitz at ∞ matrix A and for $\Lambda \geq 0$ we define

$$A^\pm(a, b_1, b_2;\ \Lambda) = \sup\{t|A_{(ta+b_1)\pm(ta+b_2)} - A_{a,b_1,b_2}^{\infty\pm}|\} \leq \infty\,,$$

where the supremum is taken over all $t > 0$ such that

$$|ta + b_j| \geq \Lambda(1 + |a| + |b_j|)|a|, \quad j = 1, 2.$$

Next for $A \in M_\gamma^+$ we set

$$\langle A\rangle^+_{\gamma,\,\Lambda} = \max\big(\sup_{a\neq 0, b_1, b_2\in\mathbb{Z}^d} e^{\gamma|b_1-b_2|}(pA)^+(a,\, b_1,\, b_2;\ \Lambda)\,,$$
$$\sup_{a\neq 0, b_1, b_2\in\mathbb{Z}^d} e^{\gamma|b_1-b_2|}(qA)^-(a,\, b_1,\, b_2;\ \Lambda)\big)\,,$$

$$|||A|||_{\gamma,\,\Lambda} = |||A|||^+_\gamma + \langle A\rangle^+_{\gamma,\,\Lambda}.$$

A matrix is called *Töplitz–Lipschitz* if this norm is
finite for some γ, Λ.

Remark 1.2. This Lipschitz norm is denoted $\langle\cdot\rangle^{\gamma,\Lambda}$ in [EK05a], and is denoted as $|||\cdot|||^+_{\gamma,\,\Lambda}$ in [EK05b].

Example 1.3. If $\zeta^1, \zeta^2 \in Y_\gamma$, then $|||\zeta^1\otimes\zeta^2|||_{\gamma,\Lambda} \le C\|\zeta^1\|_\gamma\|\zeta^2\|_\gamma$ for any $0\le\gamma\le 1$ and any $\Lambda\ge 6$. See [EK05b].

The space of all Töplitz–Lipschitz matrices is an algebra, and the following inequality holds:

$$|||AB|||_{\gamma,\,\Lambda} + |||BA|||_{\gamma,\,\Lambda} \le C(\gamma'-\gamma)^{-d-1}\Lambda^2|||A|||_{\gamma',\,\Lambda'}|||B|||_{\gamma,\,\Lambda'} \tag{1.6}$$

if $\gamma'\ge\gamma$ and $\Lambda\ge\Lambda'+6$. See [EK05b], Theorem 2.7′.

Denote

$$M^c_{\gamma,\,\Lambda} = \{A\in M_\gamma^{+c} \mid |||A|||_{\gamma,\,\Lambda} < \infty\}.$$

Since the map $A\mapsto JA$ obviously preserves the semi-norms $\langle A\rangle_{\gamma,\,\Lambda}$, then by (1.5)

$$\text{the map } M^c_{\gamma,\Lambda}\to M^c_{\gamma,\Lambda},\ \ A\mapsto JA\,,\ \text{is an isometry.} \tag{1.7}$$

1.4 Domains and functions on them

For $r>0$ and a Banach space B (real or complex) we denote

$$O_r(B) = \{x\in B \mid \|x\|_B < r\}$$

and

$$\mathbb{T}^n_r = \{q\in\mathbb{C}^n/2\pi\mathbb{Z}^n \mid |\mathrm{Im}\, q| < r\}.$$

Now, for $r\in(0,\,1]$, $\gamma\in(0,\,1]$, $\mu\in(0,\,1)$ and $\Lambda\ge 6$ we set

$$O^\gamma(r,\,\mu) = \mathbb{T}^n_r\times O_{\mu^2}(\mathbb{C}^n)\times O_\mu(Y^c_\gamma)\,,$$
$$O^{\gamma\mathbb{R}}(r,\,\mu) = O^\gamma(r,\,\mu)\cap\mathbb{T}^n\times\mathbb{R}^n\times Y_\gamma\,.$$

We denote points in $O^\gamma(r,\mu)$ as $\mathfrak{h} = (q,\,p,\,\zeta)$ and abbreviate $O^0(r,\mu) = O(r,\mu)$. A function, defined on a domain $O^\gamma(r,\mu)$, is called *real* if it takes real values on $O^{\gamma\mathbb{R}}(r,\mu)$.

Töplitz–Lipschitz functions. Let $\Omega \subset \mathbb{R}^n$ be an open domain and $h : O(r,\mu) \times \Omega \to \mathbb{C}$ be a C^1–function, analytic in the first variable. We define

$$\begin{aligned} |h(\mathfrak{h},\cdot)|_\Omega &= \sup_{\omega\in\Omega, j=0,1} |\partial_\omega^j h(\mathfrak{h},\omega)|\,, \\ \Big|\frac{\partial h}{\partial \zeta}(\mathfrak{h},\cdot)\Big|_\Omega^\gamma &= \sup_{\omega\in\Omega, j=0,1} \|\partial_\omega^j \nabla_\zeta h(\mathfrak{h},\omega)\|_\gamma\,, \\ \Big|\frac{\partial^2 h}{\partial \zeta^2}(\mathfrak{h},\cdot)\Big|_\Omega^{\gamma,\Lambda} &= \sup_{\omega\in\Omega, j=0,1} |||\partial_\omega^j \nabla_\zeta^2 h(\mathfrak{h},\omega)|||_{\gamma,\Lambda}\,. \end{aligned}$$

Here $\nabla_\zeta h = (\frac{\partial h}{\partial \zeta_a} \in \mathbb{C}^2\ a \in \mathcal{L})$ and $\nabla_\zeta^2 h$ is the matrix, formed by the 2×2-blocks $\frac{\partial^2 h}{\partial \zeta_a \partial \zeta_b}$, $a,b \in \mathcal{L}$. Now, for any $0 \le \gamma \le 1$ we denote

$$\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu} = C\,,$$

where $C \le \infty$ is the infimum of all $C' \ge 0$ such that for all $\gamma' \le \gamma$ and all $\mathfrak{h} \in O^{\gamma'}(r,\mu)$ we have

$$\begin{aligned} &|h(\mathfrak{h},\cdot)|_\Omega \le C' \\ &\Big|\frac{\partial h}{\partial \zeta}(\mathfrak{h},\cdot)\Big|_\Omega^{\gamma'} \le \mu^{-1} C' \\ &\Big|\frac{\partial^2 h}{\partial \zeta^2}(\mathfrak{h},\cdot)\Big|_\Omega^{\gamma',\Lambda} \le \mu^{-2} C' \end{aligned}$$

(as usual, $\inf \emptyset = \infty$). We denote by

$$\mathcal{T}^{\gamma,\Lambda}(\Omega,r,\mu)$$

the space, formed by functions h as above such that $\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu} < \infty$. Elements of spaces $\mathcal{T}^\gamma(\Omega,r,\mu)$ are called *Töplitz–Lipschitz functions.*

Note that the sets $\mathcal{T}^{\gamma,\Lambda}(U,r,\mu)$ grows with Λ and decays with γ.

Jets of Töplitz–Lipschitz functions. For any function $h \in \mathcal{T}^{\gamma,\Lambda}(\Omega,r,\mu)$ we define its *jet* $h^T = h^T(\mathfrak{h};\omega)$ as the Taylor polynomial of h at $p=0$, $\zeta = 0$:

$$\begin{aligned} h^T =& h_q + h_p\cdot p + \langle h_\zeta,\zeta\rangle + \frac{1}{2}\langle h_{\zeta\zeta}\zeta,\zeta\rangle \\ &:= h(q,0;\omega) + \nabla_p h(q,0;\omega)\cdot p + \langle \nabla_\zeta h(q,0;\omega),\zeta\rangle + \frac{1}{2}\langle \nabla^2_{\zeta\zeta} h(q,0;\omega)\zeta,\zeta\rangle. \end{aligned} \tag{1.8}$$

Choosing $\mathfrak{h} = (q,0,0)$ in the definition of the norm $\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}$ we immediately get that

$$\begin{aligned} &|h_q(q;\cdot)|_\Omega \le \big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\,, \quad |h_p(q;\cdot)|_\Omega \le \mu^{-2}\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\,, \\ &|h_\zeta(q;\cdot)|^\gamma_\Omega \le \mu^{-1}\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\,, \quad |h_{\zeta\zeta}(q;\cdot)|^{\gamma,\Lambda}_\Omega \le \mu^{-2}\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\,, \end{aligned} \tag{1.9}$$

for any $q \in T^n_r$.

Lemma 1.4. *For $h \in \mathcal{T}^{\gamma\Lambda}(U, r, \mu)$ and any $0 < \gamma' < \gamma$, $0 < \mu' \le \frac{1}{2}\mu$ we have*

$$\big[h^T\big]^{\gamma',\Lambda}_{\Omega,r,\mu} \le C(\gamma-\gamma')^{-d_1}\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}$$

and

$$\big[h - h^T\big]^{\gamma,\Lambda}_{\Omega,r,\mu'} \le 2\Big(\frac{\mu'}{\mu}\Big)^3\big[h\big]^{\gamma,\Lambda}_{\Omega,r,\mu}.$$

Proof. The first assertion follows from (1.9) due to (1.3). To prove the second, we have to estimate $|h - h^T|_\Omega, |\nabla_\zeta(h - h^T)|^{\gamma'}_\Omega$ and $|\nabla^2_\zeta(h - h^T)|^{\gamma',\Lambda}_\Omega$ for $\mathfrak{h} = (q,\, p,\, \zeta) \in O^{\gamma'}(r,\, \mu')$, $\gamma' \le \gamma$. Let us denote $m = \mu'/\mu$. Then for $|z| \le 1$ we have $(q,\, (z/m)^2 p,\, (z/m)\zeta) \in O^{\gamma'}(r,\, \mu)$. Therefore the function

$$\{|z| < 1\} \ni z \mapsto \nabla^2_\zeta h\big(q,\, (\frac{z}{m})^2 p,\, \frac{z}{m}\zeta\big) = h_0 + h_1 z + \dots \in M^c_{\gamma,\,\Lambda}$$

is holomorphic and is bounded in norm by $\varepsilon\mu^{-2}$. So, by the Cauchy estimate, $|||h_j|||_{\gamma',\,\Lambda} \le \varepsilon\mu^{-2}$. Since $\nabla^2_\zeta h_3 = h_1 m + h_2 m^2 + \dots$, then

$$|||\nabla^2_\zeta h_3|||_{\gamma',\,\Lambda} \le \varepsilon\mu^{-2}(m + m^2 + \dots) \le 2\,\varepsilon\mu^{-2}\frac{\mu'}{\mu}$$

(since $\mu' < \frac{1}{2}\mu$). Same arguments apply to estimate the norm of $\partial_\omega\nabla^2_\zeta(h - h^T)$, as well as $|h - h^T|_\Omega$ and $|\nabla_\zeta(h - h^T)|^{\gamma'}_\Omega$. □

A Töplitz–Lipschitz function h is called a *jet-function* if $h^T = h$.

Poisson brackets of jet-functions. For given jet-functions f and g let us consider

$$h(\mathfrak{h}) := \{f(\mathfrak{h}),\, g(\mathfrak{h})\} = \nabla_p f \cdot \nabla_q g - \nabla_q f \cdot \nabla_p g + \langle J\nabla_\zeta f,\, \nabla_\zeta g\rangle. \tag{1.10}$$

Lemma 1.5. *If $f, g \in \mathcal{T}^{\gamma,\Lambda}(U, r, \mu)$, then for any $0 < \gamma' < \gamma$, $0 < r' < r$ and $\Lambda' \ge \Lambda + 6$ we have*

$$\big[h\big]^{\gamma',\Lambda'}_{\Omega,r',\mu} \le C(\gamma-\gamma')^{-d-1}\big((r-r')^{-1} + \Lambda^2\mu^{-2}\big)\big[f\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\big[g\big]^{\gamma,\Lambda}_{\Omega,r,\mu}. \tag{1.11}$$

Proof. Let us denote the three terms in the r.h.s. of (1.10) by h_1, h_2 and h_3. It is a straightforward consequence of the Cauchy inequality and (1.3) that

$$\big[h_1 + h_2\big]^{\gamma',\Lambda'}_{\Omega,r',\mu} \le C(\gamma-\gamma')^{-d-1}(r-r')^{-1}\big[f\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\big[g\big]^{\gamma,\Lambda}_{\Omega,r,\mu}.$$

Now consider the term h_3. Since $\nabla_\zeta f = f_\zeta + f_{\zeta\zeta}\zeta$ and similar with $\nabla_\zeta g$, then

$$h_3 = \langle Jf_\zeta, g_\zeta\rangle - \langle \zeta, f_{\zeta\zeta}Jg_\zeta\rangle + \langle g_{\zeta\zeta}Jf_\zeta, \zeta\rangle + \langle g_{\zeta\zeta}Jf_{\zeta\zeta}\zeta, \zeta\rangle.$$

It is clear that $|h_3(\mathfrak{h}, \cdot)|_\Omega$ is bounded by

$$C_* = C(\gamma-\gamma')^{-d-1}\Lambda^2\mu^{-2}\big[f\big]^{\gamma,\Lambda}_{\Omega,r,\mu}\big[g\big]^{\gamma,\Lambda}_{\Omega,r,\mu},$$

for any $\mathfrak{h}\in O(r,\mu)$. Since

$$\nabla_\zeta h_3=-f_{\zeta\zeta}Jg_\zeta+g_{\zeta\zeta}Jf_\zeta+g_{\zeta\zeta}Jf_{\zeta\zeta}\zeta-f_{\zeta\zeta}Jg_{\zeta\zeta}\zeta\,,$$

then for any $\mathfrak{h}\in O^{\hat\gamma}(r,\mu)$, $\hat\gamma\le\gamma'$, the norm $\mu|\nabla_\zeta h_3|^{\hat\gamma}_\Omega$ is bounded by C_*. Finally, since $\nabla^2 h_3=g_{\zeta\zeta}Jf_{\zeta\zeta}-f_{\zeta\zeta}Jg_{\zeta\zeta}$, then $\mu^2|\nabla^2_\zeta h_3(\mathfrak{h},\cdot)|^{\hat\gamma,\Lambda'}_\Omega\le C_*$, due to (1.7) and (1.6). This implies the lemma's assertion. □

1.5 Hamiltonian equations in domains $O^\gamma(r,\mu)$.

Any C^1–smooth function f on a domain $O^\gamma(r,\mu)$ defines there the Hamiltonian equations, corresponding to the symplectic form $dp\wedge dq+J\,d\zeta\wedge d\zeta$ [3] :

$$\dot{\mathfrak{h}}^t=\mathcal{J}\nabla f(\mathfrak{h})^t=:V_f(\mathfrak{h}),\quad \mathcal{J}=\begin{pmatrix}0&E&0\\-E&0&0\\0&0&J\end{pmatrix},\tag{1.12}$$

where $\nabla f=(\nabla_q f,\nabla_p f,\nabla_\zeta f)$. We denote by S^t, $t\in\mathbb{R}$, the corresponding flow-maps. These maps C^1-smoothly depend on the parameter ω.

Now let us assume that $f=f^T$ is a jet-function

$$f=f_q(q;\,\omega)+f_p(q;\,\omega)\cdot p+\langle f_\zeta(q;\,\omega),\,\zeta\rangle+\frac12\langle f_{\zeta\zeta}(q;\,\omega)\zeta,\,\zeta\rangle\,,$$

such that

$$|f_q(\mathfrak{h};\cdot)|_\Omega\le\varepsilon'\,,\ |f_p(\mathfrak{h};\cdot)|_\Omega\le\mu^{-2}\varepsilon'\,,|f_\zeta(\mathfrak{h};\cdot)|^\gamma_\Omega\le\mu^{-1}\varepsilon'\,,|f_{\zeta\zeta}(\mathfrak{h};\cdot)|^{\gamma,\Lambda}_\Omega\le\mu^{-2}\varepsilon'\,,\tag{1.13}$$

for all $\mathfrak{h}\in O^\gamma(r,\mu)$, with some $\Lambda\ge 6$. Then the Hamiltonian equations take the form

$$\dot q=f_p(q),\tag{1.14}$$

$$\dot p=-\nabla_q f(q,\,p,\,\zeta),\tag{1.15}$$

$$\dot\zeta=J\big(f_\zeta(q)+f_{\zeta\zeta}(q)\zeta\big)\tag{1.16}$$

(here and below we often suppress the argument ω). Let us fix any

$$\gamma'\in(0,\gamma),\ \ r'\in(0,r)\ \ \mu'\in(0,\frac12\mu]\,,\tag{1.17}$$

denote $r_\Delta=\frac13(r-r')$, $\mu_\Delta=\frac13(\mu-\mu')$, $\gamma_\Delta=\frac13(\gamma-\gamma')$, and for $j=0,1,2,3$ set

$$O^{\hat\gamma}_j=O^{\hat\gamma}(r'+jr_\Delta,\mu'+j\mu_\Delta),\quad 0\le\hat\gamma\le\gamma'\,.\tag{1.18}$$

We supplement the equations with initial conditions

$$\mathfrak{h}(0)=\mathfrak{h}_0=(q_0,\,p_0,\,\zeta_0)\in O^{\hat\gamma}_1\,.$$

[3] $d\zeta\wedge d\zeta$ is the form which sends any pair of vectors (ξ,η) to the number $\langle J\xi,\eta\rangle$.

Assume that the solution $\mathfrak{h}(t)$ exists for $0 \le t \le 1$ and satisfies

$$\mathfrak{h}(t) \in O_2^{\hat{\gamma}} \quad \text{for } 0 \le t \le 1. \tag{1.19}$$

Then

$$|q(t) - q_0| \le \varepsilon' \mu^{-2}, \quad 0 \le t \le 1\,, \tag{1.20}$$

due to (1.13). The estimates (1.13) imply that $|f(\mathfrak{h};\cdot)|_\Omega \le C\gamma_\Delta^{-d}\varepsilon'$ for $\mathfrak{h} \in O^{\hat{\gamma}}(r,\mu)$. Therefore $|\nabla_q f(\mathfrak{h};\cdot)|_\Omega \le C\gamma_\Delta^{-d} r_\Delta^{-1}\varepsilon'$ for $\mathfrak{h} \in O_2^{\hat{\gamma}}$ by the Cauchy inequality. So

$$|p(t) - p_0| \le C\varepsilon' r_\Delta^{-1}\gamma_\Delta^{-d}, \quad 0 \le t \le 1. \tag{1.21}$$

Now, let $\{\Phi_{t_1}^{t_2}\}$ be the flow of the linear equation $\dot{\zeta} = Jf_{\zeta\zeta}(q(t))\zeta$. Then

$$\zeta(t) = \Phi_0^t \zeta_0 + \int_0^t \Phi_\tau^t J f_\zeta(q(\tau))\, d\tau\,. \tag{1.22}$$

Assume that ε' satisfies

$$\varepsilon' \le C^{-1}\mu^2 r_\Delta \gamma_\Delta^{d_1+d}. \tag{1.23}$$

Then

$$|||\Phi_{t_1}^{t_2} - \mathrm{id}\,|||_{\gamma'+2\gamma_\Delta} \le C\varepsilon'\mu^{-2} \quad \text{and} \quad |||\Phi_{t_1}^{t_2} - \mathrm{id}\,|||_{\gamma'+\gamma_\Delta,\Lambda} \le C\varepsilon'\mu^{-2}\gamma_\Delta^{-d-1}\Lambda^2 \tag{1.24}$$

for $0 \le t_1, t_2 \le 1$ due to (1.13) and Theorems 3.3, 3.4 in [EK05b]. So that

$$\|\zeta(t) - \zeta_0\|_{\hat{\gamma}} \le C\varepsilon'\mu^{-1}\gamma_\Delta^{-d_1}\,, \quad 0 \le t \le 1\,, \tag{1.25}$$

by (1.3).

Consider the $n \times n$–matrix $\nabla_q f_p(q(t))$. Due to (1.13), its norm is bounded by $Cr_\Delta^{-1}\mu^{-2}\varepsilon'$. Therefore if (1.23) holds, then the flow-maps $\{\Sigma_{t_1}^{t_2}\}$ of equation $\dot{p} = -\nabla_q f_p(q(t))p$ satisfy

$$\|\Sigma_{t_1}^{t_2} - \mathrm{id}\,\| \le C_1 r_\Delta^{-1}\mu^{-2}\varepsilon'\,, \quad 0 \le t_1, t_2 \le 1.$$

By (1.15), the vector $p(t)$ can be written as

$$p(t) = \Sigma_0^t p_0 - \int_0^t \Sigma_s^t \pi(s)\, ds, \tag{1.26}$$

where

$$\pi(s) = \nabla_q f_q(q(s)) + \langle \nabla_q f_\zeta(q(s)), \zeta(s)\rangle + \frac{1}{2}\langle \nabla_q f_{\zeta\zeta}(q(s))\zeta(s), \zeta(s)\rangle\,.$$

Lemma 1.6. *Let the numbers* γ, r, μ *and* γ', r', μ' *satisfy* (1.17), (1.23), *and let* $\hat{\gamma} \in [0, \gamma']$. *Define the domains* $O_j^{\hat{\gamma}}$ *as above. Then for any* $0 \le t \le 1$

i) the flow-map S^t *defines an analytic diffeomorphism* $S^t : O_1^{\hat{\gamma}} \to O_2^{\hat{\gamma}}$ *and defines a symplectomorphism* $S^t : O_1^{\hat{\gamma}\mathbb{R}} \to O_2^{\hat{\gamma}\mathbb{R}}$.

ii) The map $\Pi_\zeta S^t$ is affine in ζ and $\Pi_\zeta S^t \mathfrak{h}_0 = \zeta(t)$ can be written in the form (1.22)*. The map $\Pi_p S^t$ is affine in p and $\Pi_p S^t \mathfrak{h}_0 = p(t)$ can be written in the form* (1.26)*.*

iii) The map S^t analytically extends to a map

$$\mathbb{T}^n_{r'} \times \mathbb{C}^n \times Y^c_{\hat\gamma} \to \mathbb{T}^n_r \times \mathbb{C}^n \times Y^c_{\hat\gamma}$$

such that for any $\mathfrak{h} = (q, p, \zeta) \in \mathbb{T}^n_{r'} \times \mathbb{C}^n_p \times Y^c_{\hat\gamma}$ we have

$$\begin{aligned}
&|\Pi_q S^t(\mathfrak{h}) - q)| \le \varepsilon' \mu^{-2} ,\\
&|\Pi_p S^t(\mathfrak{h}) - p)| \le C(r - r')^{-1}\varepsilon' \Big(1 + \mu^{-2}|p_0| + \gamma'^{-d_1}\mu^{-2}\big(\mu^{-1}\varepsilon' + 2\|\zeta_0\|_{\hat\gamma}\big)^2\Big) ,\\
&\|\Pi_\zeta S^t(\mathfrak{h}) - \zeta\|_{\hat\gamma} \le C\varepsilon'(\gamma - \gamma')^{-d_1}\mu^{-2}\|\zeta_0\|_{\hat\gamma} + \mu^{-1}\varepsilon' .
\end{aligned} \tag{1.27}$$

Moreover, ω–derivatives of these maps satisfy same estimates: $|\partial_\omega \Pi_q S^t(\mathfrak{h}; \omega)| \le \varepsilon'\mu^{-2}$, etc.

Proof. The maps S^t send $O^{\hat\gamma}(r', \mu')$ to $O^{\hat\gamma}(r, \mu)$ since the estimates (1.20), (1.21), (1.25) and (1.23) imply (1.19).The fact that these maps are analytical symplectomorphisms is classical. The assertion ii) follows from (1.22) and (1.26).

The first assertion in iii) is a consequence of ii) since the map $T^n_{\hat\gamma} \to \mathbb{T}^n_\gamma$, $q_0 \mapsto q(t)$ is analytic and independent of p_0 and ζ_0. The first estimate in (1.27) follows from (1.20), and second one – from (1.22) and (1.24). Due to the estimates for $\Pi_\zeta S^t$ and (1.23),

$$\|\zeta(t)\|_{\hat\gamma} \le C\varepsilon'\mu^{-2}\gamma_\Delta^{-d_1}\|\zeta_0\|_{\hat\gamma} + \mu^{-1}\varepsilon' + \|\zeta_0\|_{\hat\gamma} \le 2\|\zeta_0\|_{\hat\gamma} + \mu^{-1}\varepsilon' =: B .$$

Therefore $|\pi(s)| \le Cr_\Delta^{-1}\varepsilon'\big(1+\gamma'^{-d_1}B^2\mu^{-2}\big)$. Now the estimate for $\Pi_p S^t$ follows from (1.26) and (1.23).

The estimates for the ω–derivatives follow from similar arguments. □

Next we study how the flow–maps S^t as in Lemma 1.6 transform Töplitz–Lipschitz functions. Let us take any function g such that $[g]^{\gamma,\Lambda}_{\Omega,r,\mu} = 1$, and for $0 \le t \le 1$ denote $g_t(\mathfrak{h}; \omega) = g(S^t(\mathfrak{h}; \omega); \omega)$.

Lemma 1.7. *Under the assumptions of Lemma 1.6 we have*

$$[g_t]^{\gamma',\Lambda+12}_{\Omega,r',\mu'} \le C\Lambda^8 .$$

Proof. i) By Lemma 1.6, any function g_t is analytic in $\mathfrak{h} \in O(r', \mu')$ and is real for real $\mathfrak{h}$. Clearly, it is ≤ 1. It is easy to estimate $\partial_\omega g_t$ and see that $|g_t(\mathfrak{h}; \cdot)|_\Omega \le 2$ for $\mathfrak{h} \in O(r', \mu')$.

ii) To estimate $\nabla_\zeta g_t$ we note that

$$\frac{\partial g_t}{\partial \zeta_a} = \sum_{k=1}^n \frac{\partial g(\mathfrak{h}(t)}{\partial p_k}\,\frac{\partial p_k(t)}{\partial \zeta_a} + \sum_b \frac{\partial g(\mathfrak{h}(t))}{\zeta_b}\,\frac{\partial \zeta_b(t)}{\partial \zeta_a} =: \Xi^1_a + \Xi^2_a ,$$

where $S^t(\mathfrak{h}) = \mathfrak{h}(t) = (p(t), q(t), \zeta(t))$. By Lemma 1.6, $\mathfrak{h}(t) \in O_2^{\hat{\gamma}}$. Therefore

$$\frac{\partial g}{\partial p_k}(\mathfrak{h}(t)) \leq C\mu^{-2} \tag{1.28}$$

(recall that $\mu' \leq \frac{1}{2}\mu$). By (1.22), the matrix $\partial\zeta(t)/\partial\zeta$ is

$$\Big(\frac{\partial \zeta_b(t)}{\partial \zeta_a}\Big) = \Phi_0^t\,. \tag{1.29}$$

Let us denote $p'_a(t) = \partial p(t)/\partial\zeta_a$, etc. Then, due to (1.26),

$$p'_a(t) = -\int_0^t \Sigma_s^t\big(\langle \nabla_q f_\zeta, \zeta'_a(s)\rangle + \langle \nabla_q f_{\zeta\zeta}\zeta(s), \zeta'_a(s)\rangle\big)\, ds\,.$$

Due to (1.13), (1.24), (1.3) and (1.23),

$$\|\langle \nabla_q f_\zeta, \zeta'_\cdot(s)\rangle\|_{\hat{\gamma}} \leq C\mu^{-1}\varepsilon' r_\Delta^{-1}(1 + C\varepsilon'\mu^{-2}\gamma_\Delta^{-d_1}) \leq C\mu^{-1}\varepsilon' r_\Delta^{-1}\,.$$

Similar, using (1.3) and (1.6) we get that

$$\begin{aligned}\|\langle \nabla_q f_{\zeta\zeta}\zeta(s), \zeta'_\cdot(s)\|_{\hat{\gamma}} =& \|(\Phi_0^s)^t\big(\nabla_q f_{\zeta\zeta}\big)\zeta(s)\|\|_{\hat{\gamma}} \\ &\leq C\mu^{-2}\varepsilon' r_\Delta^{-1}(1 + C\varepsilon'\mu^{-2}\gamma_\Delta^{-d_1}\mu \leq C\varepsilon'\mu^{-1}r_\Delta^{-1}\,.\end{aligned}$$

Therefore

$$\|\nabla_\zeta p(t)\|_{\hat{\gamma}} \leq C\varepsilon'\mu^{-1}r_\Delta^{-1}\,, \tag{1.30}$$

and we see that

$$\|\Xi^1\|_{\hat{\gamma}} \leq C\varepsilon'\mu^{-3}r_\Delta^{-1} \leq C\mu^{-1}\,.$$

Using (1.29), (1.24), (1.3) and (1.23) we get

$$\|\Xi^2\|_{\hat{\gamma}} \leq \mu^{-1} + C\varepsilon'\mu^{-2}\mu^{-1}\gamma_\Delta^{-d_1} \leq C\mu^{-1}\,.$$

Estimating similar $\frac{\partial}{\partial\omega}\frac{\partial g_t}{\partial\zeta_a}$ we see that

$$|\nabla g_t(\mathfrak{h};\cdot)|_\Omega^{\hat{\gamma}} \leq C_1\mu^{-1}\,.$$

iii) To estimate $\nabla_\zeta^2 g_t$, we write

$$\frac{\partial^2 g_t}{\partial\zeta_a\partial\zeta_b} = d^2 g\big(S^t(\mathfrak{h}(t))\big)\Big(\frac{\partial S^t}{\partial\zeta_a}, \frac{\partial S^t}{\partial\zeta_b}\Big) + dg\big(S^t(\mathfrak{h}(t))\big)\frac{\partial^2 S^t}{\partial\zeta_a\partial\zeta_b} =: D_{ab}^1 + D_{ab}^2\,. \tag{1.31}$$

As before, we denote $\partial S^t/\partial\zeta_a = \big(p'_a(t), q'_a(t), \zeta'_a(t)\big)$. Clearly, $q_a(t) \equiv 0$. To estimate in D^1 the term $(d^2_{\zeta\zeta}g)(\zeta'_a, \zeta'_a)$ we apply (1.29), (1.24) and (1.6) to get

$$\|\|(d^2_{\zeta\zeta}g)(\zeta'_a, \zeta'_b)\|\|_{\hat{\gamma},\Lambda+12} \leq C\mu^{-2}(1 + \Lambda^4\gamma_\Delta^{-2d-2}(\varepsilon'\mu^{-2}\gamma_\delta^{-d-1}\Lambda^2)^2) \leq C\Lambda^8\mu^{-2}.$$

To estimate $(d^2_{pp}g)(p'_a, p'_b)$, we use (1.30), Example 1.3 and (1.23) to find that

$$\|\|(d^2_{pp}g)(p'_a, p'_b)\|\|_{\hat{\gamma},\Lambda} \leq C\mu^{-2}(\varepsilon'\mu^{-2}r_\Delta^{-1}\gamma_\Delta^{-d-d_*})^2 \leq C\mu^{-2}.$$

We got that

$$|||D^1|||_{\hat\gamma,\Lambda+12} \le C\Lambda^8\mu^{-2}.$$

To estimate D^2 we note that due to Lemma 1.6, $\partial^2 q(t)/\partial\zeta^2 = 0$ and $\partial^2\zeta(t)/\partial\zeta^2 = 0$. Denoting $p''_{ab} = \partial^2 p(t)/\partial\zeta_a\partial\zeta_b$, we see from (1.26) that

$$p''_{ab}(t) = -\int_0^t \Sigma_s^t \langle \nabla_q f_{\zeta\zeta}\zeta'_a(s), \zeta'_b(s)\rangle\, ds\,.$$

Since the numbers $\langle \nabla_q f_{\zeta\zeta}\zeta'_a(s), \zeta'_b(s)\rangle$, where the indexes $a, b \in \mathcal{L}$, form the matrix $\left(\Phi_0^s\right)^t (\nabla_q f_{\zeta\zeta})\,\Phi_0^s$, then (1.13), (1.24) and iterative application of (1.6) result in the estimate

$$|||p''(t)|||_{\hat\gamma,\Lambda+12} \le C\mu^{-2}r_\Delta^{-1}\varepsilon'\big(1 + (C\varepsilon'\mu^{-2}\gamma_\Delta^{-d-1}\Lambda^2\Lambda^2\gamma_\Delta^{-d-1})^2\big) \le C\varepsilon'\mu^{-2}r_\Delta^{-1}\Lambda^8$$

(we use (1.23), where $m_* > 2$, see in section 1.3). This estimate and (1.28)give us that

$$|||D^2|||_{\hat\gamma,\Lambda+12} \le C\Lambda^8\mu^{-2}\,.$$

We have estimated $\nabla_\zeta^2 g_t$. Estimating similar $(\partial/\partial\omega)\nabla_\zeta^2 g_t$, we have

$$|\nabla_\zeta^2 g_t(\mathfrak{h};\cdot)|_\Omega^{\hat\gamma,\Lambda+12} \le C_2\Lambda^8\mu^{-2}\,.$$

The lemma is proved. □

2 The main theorem.

Let $\Omega_0 \subset \mathbb{R}^n$ be an open domain such that

$$\Omega_0 \subset \{K_1^{-1} \le |\omega| \le K_1\}\,, \quad \operatorname{Leb}\Omega_0 = K_2$$

(here and below $K_1, K_2, \dots$ are fixed positive constants). Let $\lambda_a = |a|^2 + V_a(\omega)$, $a \in \mathcal{L}$, be real functions, where V_a satisfy

$$\sup_{\omega\in\Omega_0, j=0,1} |\partial_\omega^j V_a| \le K_3 e^{-K_4|a|} \quad \forall\, a\,,$$

and

$$|\lambda_a + \lambda_b| \ge K_5 \quad \forall\, a, b \in \mathcal{L},\ \forall\,\omega\,,$$
$$|\lambda_a - \lambda_b| \ge K_5 \quad \forall\, a, b \in \mathcal{L}, |a| \ne |b|,\ \forall\,\omega\,,$$

and

$$|\partial_\omega \lambda_a| \le \frac{1}{4}\min(1, K_5)\,.$$

Let $A = A(\omega)$ be the diagonal operator

$$A : (y_a, a \in \mathcal{L}) \mapsto (\lambda_a y_a, a \in \mathcal{L})$$

(we recall that each y_a is a a two–vector). We define the jet–function $\mathcal{H}$,

$$\mathcal{H}(p,\zeta;\omega) = \omega\cdot p + \frac{1}{2}\langle A\zeta,\zeta\rangle\,,$$

and the hamiltonian

$$H_0(\mathfrak{h};\omega) = \mathcal{H} + h_0\,,$$

where h_0 is a real Töplitz–Lipschitz function, such that

$$[h_0]^{\gamma_0,\Lambda_0}_{\Omega_0,r_0,\mu_0} \le \varepsilon_0 < 1$$

for some $\gamma_0 \in (0,1]$, $r_0,\mu_0 \in (0,1]$.

The hamiltonian H_0 defines the Hamiltonian equations

$$\dot{\mathfrak{h}}^t = \mathcal{J}\nabla H_0(\mathfrak{h})^t\,. \tag{2.1}$$

Theorem 2.1. *If the assumptions above hold, then for ε_0 sufficiently small there exist a Borel subset $\Omega_{\varepsilon_0}\subset\Omega_0$,*

$$Leb\,(\Omega_0\setminus\Omega_{\varepsilon_0}) \le C_1\varepsilon^c\,,\quad c>0\,,$$

and real analytic symplectomorphisms

$$\Sigma_\omega : O^{\gamma/2}(r_0/2,\mu_0/2)\to O^{\gamma/2}(r_0,\mu_0)\,,\quad \omega\in\Omega_{\varepsilon_0}\,,$$

such that $|\Sigma_\omega - id| \le C_2\varepsilon^c$. The map Σ_ω transforms the hamiltonian H_0 to

$$H_0\circ\Sigma_\omega = \omega_\infty\cdot p + \frac{1}{2}\,\langle A_\infty\zeta,\zeta\rangle + h_\infty\langle(\zeta;\omega)\,,$$

where $\omega_\infty = \omega_\infty(\omega)$, $A_\infty = A_\infty(\omega)$ and

$$|\omega_\infty-\omega|\le C\varepsilon_0^{1/3}\,,\quad |||A_\infty - A|||_{\gamma/2}\le C\varepsilon_0^{1/3}\,,\quad A_\infty\in\mathcal{NF}_{\mathcal{E}_0}(\Omega_{\varepsilon_0})\,,\quad h^T_\infty = 0\,. \tag{2.2}$$

Every torus $\Sigma_\omega(\mathbb{T}^n\times\{0\}\times\{0\})$ is invariant for the eq. (2.1) and is filled in with the time-quasiperiodic solutions $\mathfrak{z}(t;q,\omega) = \Sigma_\omega(q+\omega_\infty t)\,,\ q\in\mathbb{T}^n\,.$ The linearised map $\Sigma_\infty(q+\omega_\infty t)_*$ reduces eq. (2.1), linearised about a solution $\mathfrak{z}$, to the autonomous equation

$$\frac{d}{dt}\,\delta q = 0\,,\quad \frac{d}{dt}\,\delta p = 0\,,\quad \frac{d}{dt}\,\delta\zeta = JA_\infty\delta\zeta\,. \tag{2.3}$$

In particular, all Lyapunov exponents of the solutions $\mathfrak{z}$ vanish.

3 Homological equations

From now on we restrict our presentation to the $2d$ case. Our proof applies to the nd equations, $n\ge 3$, but in that case solvability of the homological equation (3.5) below is established in [EK05a] under the assumption that the involved linear matrices belong to a subclass of Töplitz–Lipschitz matrices, called in [EK05b] "double Töplitz–Lipschitz matrices $M(M^c_{\gamma,\Lambda})$". These matrices possess all the properties, needed for the arguments in Sections 2–4 (see [EK05b]), but the corresponding notations become more combersome. Now we are in a process of developing them. A proof of Theorem 0.1 for the nd case, $n \geq 3$, will be presented in the next version of this text.

Normal forms. A symmetric matrix $A = (A_{ab}) \in M_\gamma$, $A_{ab} \equiv A^t_{ba}$, $\gamma > 0$, defines on the l_2–space Y the continuous quadratic form $q(\zeta,\zeta) = \langle \zeta, A\zeta\rangle$ (see in [EK05b] Theorem 1.2 with $\gamma = m_* = 0$). In the complex variables $(w_a, a \in \mathcal{L})$, where

$$w_a = \begin{pmatrix} u_a \\ v_a \end{pmatrix}, \quad u_a = \frac{1}{\sqrt{2}}\left(\zeta^1_a + i\zeta^2_a\right),\ v_a = \frac{1}{\sqrt{2}}\left(\zeta^1_a - i\zeta^2_a\right),$$

we have the matrices $\nabla^2_u q$, $\nabla^2_v q$, $\nabla_u\nabla_v q$, and

$$q(\zeta,\zeta) = \langle u, \nabla^2_u q\, u\rangle + \langle v, \nabla^2_v q\, v\rangle + \langle u, \nabla_u\nabla_v q\, v\rangle .$$

The matrices $\nabla^2_u q$ and $\nabla^2_v q$ are symmetric and complex conjugate of each other, while $\nabla_u\nabla_v q$ is Hermitian. If $A \in M^+_\gamma$, then

$$\sup_{a,b} |(\nabla_u\nabla_v q)_{ab}| e^{\gamma|a-b|} + \sup_{a,b} \left|\left(\nabla^2_u q)_{ab}\right| + |(\nabla^2_v q)_{ab}|\right) e^{\gamma|a+b|} < \infty .$$

Moreover, the l.h.s. defines on the space of symmetric matrices in M_γ a norm, equivalent to $|||\cdot|||^+_\gamma$.

A quadratical form $f = \langle \zeta, f_{\zeta\zeta}(\omega)\zeta\rangle$ is called a *normal form* if it is block-diagonal over some decomposition (into finite subsets) $\mathcal{E}$ of $\mathcal{L}$, i.e.

$$f = \sum_{I\in\mathcal{E}} \langle \zeta_I, (\nabla^2 f)\zeta_I\rangle ,$$

and

$$\nabla^2_u f \equiv \nabla^2_v f \equiv 0$$

This property will be denoted

$$f \in \mathcal{NF}_{\mathcal{E}}(\Omega).$$

We say that a matrix $A(\omega) = \big(A_{ab}(\omega)\big)$ is a normal form,

$$A(\omega) \in \mathcal{NF}_{\mathcal{E}}(\Omega),$$

if it is symmetric, i.e. $A_{ab} = A^t_{ba}$ for all $a, b \in \mathcal{L}$, and the corresponding quadratic form belongs to $\mathcal{NF}_{\mathcal{E}}(\Omega)$.

Note that if a matrix A is a normal form, then the corresponding Hamiltonian operator JA has a discrete pure imaginary spectrum.

Block decomposition. For a non-negative integer Δ we define an equivalence relation on $\mathcal{L}$, generated by the pre-equivalence relation

$$a \sim b \iff \begin{cases} |a|^2 = |b|^2 \\ |a-b| \leq \Delta. \end{cases}$$

Let $[a] = [a]_\Delta$ denote the equivalence class (*block*) of a, and let $\mathcal{E}_\Delta$ denote the set of all blocks. The blocks are finite with a diameter

$$\leq \text{ const } \Delta^{\frac{(d+1)!}{2}}.$$

(See [EK05a].) If the decomposition above is $\mathcal{E}_\Delta$, then we say that

$$f \in \mathcal{NF}_\Delta(\Omega).$$

Each decomposition $\mathcal{E}_\Delta$ is a subdecomposition of the trivial decomposition $\mathcal{E}_\infty$, formed by the spheres $\{|a| = \text{const}\}$.

Homological equations. Let $A = A(\omega)$ be the diagonal operator

$$A : (y_a, a \in \mathcal{L}) \mapsto (\lambda_a y_a, a \in \mathcal{L}),$$

where the numbers $\lambda_a = |a|^2 + \hat{V}(a)$ are the same as in the Introduction (we recall that each y_a is a a two–vector). We define the jet–function $\mathcal{H}$,

$$\mathcal{H}(p, \zeta; \omega) = \omega \cdot p + \frac{1}{2}\langle A\zeta, \zeta\rangle,$$

and consider a hamiltonian

$$H = \mathcal{H} + g(p, \zeta; \omega) + h,$$

where $\omega \in \Omega \subset \Omega_0$ and $g = g_p(\omega) \cdot p + \frac{1}{2}\langle g_{\zeta\zeta}(\omega)\zeta, \zeta\rangle$ is a jet–function such that $\frac{1}{2}\langle g_{\zeta\zeta}(\omega)\zeta, \zeta\rangle \in \mathcal{NF}_\Delta(\Omega)$, and

$$\sup_{\omega \in \Omega, j=0,1} |\partial_\omega^j g_p| + |||\partial_\omega^j g_{\zeta\zeta}|||_{\gamma,\Lambda} \leq C_1 \varepsilon_0^{1/3}. \tag{3.1}$$

The function h is assumed to be Töplitz–Lipschitz and

$$[h]^{\gamma,\Lambda}_{\Omega,r,\mu} = \varepsilon \leq \varepsilon_0,$$

where $0 < \gamma \leq \gamma_0$, $0 < r \leq r_0$, $0 < \mu \leq \mu_0$ and $\Lambda \geq 6$.

Clearly, the hamiltonian H_0 in the Introduction has the form above with $\varepsilon = C\varepsilon_0$, $h = \varepsilon_0 h$ and $g = 0$.

We abbreviate

$$\omega' = \omega + g_p(\omega), \quad A' = A(\omega) + g_{\zeta\zeta}(\omega), \quad \Delta' = \max(\Lambda, \Delta^{(d+1)!}), \quad \bar{\Lambda} \geq C_2 \Delta^{\frac{(d+1)}{2}} \Delta'$$

(note that $\bar{\Lambda} > \Lambda$).

Let us calculate the jet h^T of h (see (1.8)) and denote

$$h_3 = h - h^T.$$

Consider the following four equations for four unknown maps $f_q(q;\ \omega)$, $f_p(q;\ \omega)$, $f_\zeta(q;\ \omega)$ and $f_{\zeta\zeta}(q;\ \omega)$, valued in the same spaces as h_q, h_p, h_y

and $h_{\zeta\zeta}$ respectively:

$$(\omega' \cdot \nabla_q) f_q = h_q - \langle h_q \rangle + \tilde{h}_q, \tag{3.2}$$

$$(\omega' \cdot \nabla_q) f_p = h_p - \langle h_p \rangle + \tilde{h}_p, \tag{3.3}$$

$$(\omega' \cdot \nabla_q) f_\zeta - JA' f_\zeta = h_\zeta + \tilde{h}_\zeta, \tag{3.4}$$

$$(\omega' \cdot \nabla_q) f_{\zeta\zeta} + (f_{\zeta\zeta} JA' - A' J f_{\zeta\zeta}) = h_{\zeta\zeta} - \tilde{h}_{\zeta\zeta} - \bar{g}_{\zeta\zeta}. \tag{3.5}$$

In these equations $\tilde{h}_q, \tilde{h}_p, \tilde{h}_\zeta, \tilde{h}_{\zeta,\zeta}$ and $\bar{g}_{\zeta\zeta}$ are admissible disparities, and for a vector–function k on the torus $\mathbb{T}^n$ we denote $\langle k \rangle = (2\pi)^{-n} \int k \, dq$. The set of four unknown vector–functions and five disparities is called a *solution* of eq. (3.2)–(3.5). Recall that the functions h_q, etc. are estimated in (1.9).

Lemma 3.1. *If the hamiltonian H satisfies the assumptions above, then for any $\varkappa \in (0, 1]$ there exists an open set $\Omega' \subset \Omega$,*

$$Leb\,(\Omega \setminus \Omega') \le \Delta'^{c_1} \varkappa\,,$$

such that for any $\bar{r} \in (0, r)$ and $\bar{\gamma} \in (0, \gamma)$ the equations have a real solution, satisfying for any $q \in \mathbb{T}^n_{\bar{r}}$ the estimates

i) $|f_q(q;\cdot)|_{\Omega'} + \mu^2 |f_p(q;\cdot)|_{\Omega'} + \mu |f_\zeta(q;\cdot)|^{\gamma}_{\Omega'} + \mu^2 |f_{\zeta\zeta}(q;\cdot)|^{\bar{\gamma},\bar{\Lambda}}_{\Omega'} \le N\varepsilon\,,$

ii) $|\tilde{h}_q(q;\cdot)|_{\Omega'} + \mu^2 |\tilde{h}_p(q;\cdot)|_{\Omega'} + \mu |\tilde{h}_\zeta(q;\cdot)|^{\gamma}_{\Omega'} + \mu^2 |\tilde{h}_{\zeta\zeta}(q;\cdot)|^{\bar{\gamma},\bar{\Lambda}}_{\Omega'} \le \bar{\varepsilon}\,,$

where $N = C_1 \varkappa^{-c_2} |r - r'|^{-c_3} \Delta'$ and $\bar{\varepsilon} = C_1 (e^{-\Delta' |\gamma - \gamma'|} + e^{-\Delta' |r - r'|}) |r - r'|^{c_3} \varepsilon$. The matrix $\bar{g}_{\zeta\zeta}$ is q–independent and satisfies $|\bar{g}_{\zeta\zeta}|^{\gamma,\bar{\Lambda}}_{\Omega'} \le \varepsilon C_1$. Moreover, $\bar{g}_{\zeta\zeta} \in \mathcal{NF}_{\Delta'}(\Omega')$.

The constants c_1, c_2, c_3 and C_1, C_2 depends on m_*, n, d and the constants $K_1 - K_5$.

Proof. Note that in view of (3.1) the map $\omega \mapsto \omega'$ is a C^1–smooth diffeomorphism that changes measures of sets no more than twice (if $\varepsilon_0 \ll 1$).

The fact that the equations (3.2) and (3.3) admit solutions f_q and f_p with disparities $\tilde{h}_q$ and $\tilde{h}_p$, satisfying the estimates in i) and ii), provided that the frequency vector ω' lies outside a set of small measure $\le \frac{1}{6} \Delta'^{c_1} \varkappa$, is classical for the KAM–theory. Similar, the solvability of eq. (3.4) for ω' outside a small set is well known in the 'KAM theory for PDE'. [4] The forth equation is far more difficult. Its solvability is established in [EK05a].

So the four homological equations can be solved for ω' outside a set of measure $\le \frac{1}{2} \Delta'^{c_1} \varkappa$. That is, for ω outside a set of measure $\le \Delta'^{c_1} \varkappa$. □

Given the solution of the homological equations, constructed in the lemma above, we denote by f and $\tilde{h}$ the following jet–functions:

$$\begin{aligned} f &= f_q + p \cdot f_p + \langle \zeta, f_\zeta \rangle + \frac{1}{2} \langle \zeta, f_{\zeta\zeta} \zeta \rangle\,, \\ \tilde{h} &= \tilde{h}_q + p \cdot \tilde{h}_p + \langle \zeta, \tilde{h}_\zeta \rangle + \frac{1}{2} \langle \zeta, \tilde{h}_{\zeta\zeta} \zeta \rangle\,, \end{aligned} \tag{3.6}$$

[4] See [Kuk93], p. 62. The assumption that there the eigenvalues λ_j satisfy $\lambda_j \sim C j^{d_1}$, $d_1 \ge 1$ (see (1.11)) is needed for the forth homological equation (2.33), while for solvability of the third equation (2.32) it is only needed that $d_1 > 0$. The eigenvalues λ_a, $a \in \mathcal{L}$, of the operator A, after we re-parameterise them as λ_j, $j \in \mathbb{N}$, satisfy $\lambda_j \sim C j^{d_1}$, $d_1 = 2/d$.

and denote

$$\bar{g}(p,\zeta;\omega) = \langle h_q \rangle + \langle h_p \rangle \cdot p + \frac{1}{2}\langle \zeta, \bar{g}_{\zeta\zeta}\zeta \rangle \,.$$

4 Proof of the theorem

In this section we study the Hamiltonian equation (2.1) and prove Theorem 2.1.

4.1 The KAM–step

Let us re–write the hamiltonian $H = \mathcal{H} + g + h$ as in Section 2 in the form

$$H = \mathcal{H}'(p,\zeta;\omega) + h^T(\mathfrak{h};\omega) + h_3(\mathfrak{h};\omega)\,, \quad \mathcal{H}' = \omega' \cdot p + \frac{1}{2}\langle A'(\omega)\zeta, \zeta \rangle \,.$$

Next, take any numbers

$$\gamma' \in (0,\gamma),\, r' \in (0,r),\, \mu' \in (0, \frac{1}{16}\mu)\,,$$

and for $\hat{\gamma} \le \gamma$ define the domanins $O_j^{\hat{\gamma}}$ as in (1.18). For the hamiltonian f, defined by (3.6), consider the corresponding Hamiltonian equation (1.12) and denote by S^t, $0 \le t \le 1$, its flow-maps. Let us assume that

$$\varepsilon N \le C^{-1}{\mu'}^2(r-r')(\gamma-\gamma')^{d_1} \tag{4.1}$$

(with a suitable $C \ge 1$). Then, by Lemma 1.6, $S^t : O_1^{\hat{\gamma}} \to O_2^{\hat{\gamma}}$.

Let us denote

$$Z_\gamma = \mathbb{R}^n_p \times \mathbb{R}^n_q \times Y_\gamma\,, \quad 0 \le \gamma \le 1\,.$$

Lemma 4.1. *Assume that the assumptions above hold. Then the map* $S = S^1 : O_1^{\hat{\gamma}} \times \Omega' \to O_2^{\hat{\gamma}}$, $0 \le \hat{\gamma} \le \gamma'$, *is real for real arguments, is a symplectomorphism as a function of the first variable, and is close to the identity, i.e. satisfies the estimates* (1.27) *with* $\varepsilon' = N\varepsilon$. *This map transforms the hamiltonian* H *to* $H'(\mathfrak{h};\omega) = H(S(\mathfrak{h};\omega);\omega)$, *which can be written as*

$$H' = \mathcal{H}(p,\zeta;\omega) + g'(p,\zeta;\omega) + h'(\mathfrak{h};\omega)\,. \tag{4.2}$$

Here $g' = g'_p(\omega)\cdot p + \frac{1}{2}\langle \zeta, g'_{\zeta\zeta}(\omega)\zeta\rangle$ *satisfies* (3.1) *with* C_1 *replaced by* $C_1 + C\varepsilon'\mu^{-2}\varepsilon_0^{1/3}$, *and* $g'_{\zeta\zeta} \in \mathcal{NF}_{\Delta'}(\Omega')$. *The function* h' *is Töplitz–Lipschitz and*

$$[h']^{\gamma',\Lambda'}_{\Omega',r',\mu'} \le C\,\bar{\varepsilon}\,\gamma_\Delta^{-d_1} + C\,\Lambda^8\,\varepsilon\left(\frac{\mu'}{\mu}\right)^3 + C\bar{\Lambda}^{10}\,\gamma_\Delta^{-d-2d_1-1}\,r_\Delta^{-1}N\,\varepsilon(\varepsilon+\bar{\varepsilon})\,,$$

where $\Lambda' = \bar{\Lambda} + 18$.

Proof. We have

$$\begin{aligned}\{\mathcal{H}',\, f\} = (\omega'\cdot\nabla_q)f_q + p\cdot(\omega'\cdot\nabla_q)f_q + \langle\zeta,\, (\omega'\cdot\nabla_q)f_\zeta\rangle \\ + \langle\zeta,\, (\omega'\cdot\nabla_q)f_{\zeta\zeta}\zeta\rangle + \langle JA'(\omega')\zeta,\, f_\zeta + 2f_{\zeta\zeta}\zeta\rangle.\end{aligned} \tag{4.3}$$

Using the relations (3.2)–(3.5) we get that

$$\{\mathcal{H}',\, f\} = h^T + \tilde h - \bar g$$

Therefore, $\frac{d}{dt}\mathcal{H}' \circ S^t = \{f,\, \mathcal{H}'\} = -(h^T + \tilde h - \bar g)\circ S^t$, and

$$\begin{aligned}\mathcal{H}' \circ S = \mathcal{H}' &+ \frac{d}{dt}\mathcal{H}' \circ S^t\,|_{t=0} + \int_0^1 (1-t)\frac{d^2}{dt^2}\mathcal{H}' \circ S^t\, dt \\ &= \mathcal{H}' - h^T - \tilde h + \bar g - \int_0^1 (1-t)\{f,\, h^T + \tilde h - \bar g\}\circ S^t\, dt\,.\end{aligned}$$

Similar,

$$h^T \circ S = h^T + \int_0^1 \{f,\, h^T\}\circ S^t\, dt.$$

So,

$$\begin{aligned}H \circ S = (\mathcal{H}' + \bar g) - \tilde h + h_3 \circ S + &\int_0^1 \{f,\, (t-1)(\tilde h - \bar g) + t h^T\}\circ S^t\, dt \\ &=: (\mathcal{H}' + \bar g) + {h'}_1 + {h'}_2 + {h'}_3\,.\end{aligned}$$

Removing from $\bar g$ the irrelevant constant $\langle h_q\rangle$, we see that

$$\mathcal{H}' + \bar g = \mathcal{H} + g' + \mathrm{Const}\,, \quad g' = g + \langle h_p\rangle \cdot p + \frac{1}{2}\langle \zeta, \bar g_{\zeta\zeta}\rangle\,.$$

The function g' satisfies the lemma's assertion since the required bound on $\langle h_p\rangle$ folows from (1.9), and the bounds on $\frac{1}{2}\langle\zeta, \bar g_{\zeta\zeta}\rangle$ follow from the estimates on $\bar g_{\zeta\zeta}$ in Lemma 3.1.

It remains to estimate the terms ${h'}_1$, ${h'}_2$ and ${h'}_3$. Let us denote

$$\gamma_j = \gamma' + j\,\frac{\gamma_\Delta}{4}\,, \quad r_j = r' + j\,\frac{r_\Delta}{4}\,, \quad \mu_j = 2^j\,\mu\,, \quad j = 1,2,3\,.$$

a) Choosing in Lemma 3.1 $\bar\gamma = \gamma_3$, we immediately get that

$$[h'_1]^{\gamma_2,\bar\Lambda}_{\Omega',r_2,\mu_2} = [\tilde h]^{\gamma_2,\bar\Lambda}_{\Omega',r_2,\mu_2} \le C\,\bar\varepsilon\,\gamma_\Delta^{-d_1}\,. \tag{4.4}$$

b) By Lemma 1.4, $[h^3]^{\gamma,\Lambda}_{\Omega,r,2\mu'} \le 16\bar\varepsilon(\mu'\mu)^3$. Due to (4.1), Lemma 4.1 applies to the map $S : O^{\gamma'}(r',\mu') \to O^{\gamma}(r,2\mu')$. Therefore

$$[h'_2]^{\gamma',\Lambda+12}_{\Omega,r',\mu'} \le C\,\Lambda^8\,\varepsilon\left(\frac{\mu'}{\mu}\right)^3.$$

c) The estimates in Lemma 3.1 imply that $[\bar g]^{\gamma,\bar\Lambda}_{\Omega',r,\mu} \le C\varepsilon$. This estimate jointly with (4.4) and Lemma 1.4 with $\gamma' := \gamma_2$ show that

$$[F_t]^{\gamma_2,\bar\Lambda}_{\Omega',r_2,\mu_2} \le C\,(\bar\varepsilon + \varepsilon)\,\gamma_\Delta^{-d_1}\,, \quad F_t = (t-1)(\tilde h - \bar g) + t h^T\,,$$

for $0 \le t \le 1$. Since $[f]^{\gamma_2,\bar\Lambda}_{\Omega',r_2\mu_2} \le CN\varepsilon\,\gamma_\Delta^{-d_1}$ by Lemma 3.1, then

$$[\{f,F_t\}]^{\gamma_1,\bar\Lambda+6}_{\Omega',r_1,\mu_1} \le C\bar\Lambda^2\,\gamma_\Delta^{-d-2d_1-1}\,r_\Delta^{-1}N\,\varepsilon(\varepsilon+\bar\varepsilon)$$

due to Lemma 1.5.

Now, applying Lemma 1.7, we get that

$$[h_3']^{\gamma',\bar\Lambda+18}_{\Omega',r',\mu'} \le C\bar\Lambda^{10}\,\gamma_\Delta^{-d-2d_1-1}\,r_\Delta^{-1}N\,\varepsilon(\varepsilon+\bar\varepsilon)\,.$$

Summing up the estimates a)-c) we get the lemma's assertion. □

4.2 Choice of parameters

To prove the main theorem we shall construct the transformation Σ_ω as the composition of infinitely-many transformations S as in Lemma 4.1. The first map S transforms the original system with hamiltonian H_0 and with $\varepsilon=\varepsilon_0$, $r=r_0$, etc. to the system with hamiltonian H_1 and with $\varepsilon_1=\varepsilon'$, $r_1=r'$, etc. The second transforms this one to the system, obtained by applying the lemma with $\varepsilon=\varepsilon_1$, $r=r_1,\dots$, and so on.

In this section we specify the parameters $(\varepsilon_j, r_j, \text{etc})$, $j=0,1,2,\dots$, study their asymptotic behaviour as $j\to\infty$ and check that this choice of parameters is consistent, i.e. that the assumption (4.1) holds at each step.

It is convenient to re-define μ_0 and replace it by $\varepsilon_0^{1/3}$. Let us denote $C_*=2(1^{-2}+2^{-2}+\dots)$, and for $j\ge1$ choose

$$\begin{aligned}
&r_{j-1}-r_j = C_*^{-1}j^{-2}r_0\,,\\
&\gamma_{j-1}-\gamma_j = C_*^{-1}j^{-2}\gamma_0\,,\\
&\mu_j=\varepsilon_j^{1/3}\,,\\
&\Delta_j=(\ln\varepsilon_j^{-1})\frac{1}{r_{j-1}-r_j}\,,\quad \Delta_0=1\,,\\
&\Lambda_j=C_2\Lambda_{j-1}\Delta_j^{(d+1)!/2}+18\,,\quad \Lambda_0=6\,,\\
&\varkappa=\varepsilon_j^{c_\varkappa}\,,\qquad \Delta_j'=\Delta_j^{(d+1)!}\,.
\end{aligned}$$

Here $C_2>0$ is the same as in the formular for $\bar\Lambda$ in Section 2, and $c_\varkappa>0$ is specified below.

The numbers above are defined in terms of ε_j's which are defined inductively (with given ε_0) through the relation

$$\begin{aligned}
\varepsilon_{j+1}\le\,& C\bar\varepsilon_j\big((j+1)^2\gamma_0^{-1}\big)^{d+1}+C\Lambda_j^8\varepsilon_j(\varepsilon_{j+1}/\varepsilon_j)^3\\
&+C\Lambda_{j+1}^{10}\big((j+1)^2\gamma_0^{-1}\big)^{d+2d_1+1}\big((j+1)^2r_\Delta\big)^{c_3+1}\varkappa^{-c_2}\Delta'_j\varepsilon_j(\bar\varepsilon_j+\varepsilon_j)\,,
\end{aligned}\tag{4.5}$$

$$\bar\varepsilon_j=C_1\Big(e^{-\Delta'C_*^{-1}(j+1)^2\gamma_0}+e^{-\Delta'C_*^{-1}(j+1)^2r_0}\Big)(j+1)^{2c_3}r_0^{-c_3}\varepsilon_j$$

(see Lemma 4.1).

The result below holds if $c_\varkappa=c_\varkappa(n,d)>0$ is sufficiently small.

Lemma 4.2. *For $j = 1, 2, \dots$ we have*

$$\varepsilon_j \le \varepsilon_0^{(5/4)^j}, \tag{4.6}$$

provided that $\varepsilon_0 > 0$ is sufficiently small (in terms of n, d, r_0, μ_0 and γ_0). Besides, the assumption (4.1) *holds for each j.*

Proof. It suffice to check that if

$$\varepsilon_k \le \varepsilon_0^{(5/4)^k} \quad \forall\, k \le j, \tag{4.7}$$

then all the three terms in the r.h.s. of (4.5) are $\le \frac{1}{3}\,\varepsilon_j^{5/4}$.

Let us abbreviate $\ln \varepsilon_0^{-1} = l_0$ and $c_d = (d+1)!$. It is easy to see that (4.7) implies that

$$\bar\varepsilon_j \le \varepsilon_j^2 \tag{4.8}$$

if $\varepsilon_0 \ll 1$. So the first term in the r.h.s. satisfies the desired estimate.

The formula for Δ_j and (4.7) imply that

$$\Delta_j \le l_0^j\, C_1^j\, e^{3j^2}.$$

So

$$\Lambda_j \le C_1\, e^{C_2 j^{C_3}}\, l_0^{c_d j^2}$$

(here and below in the lemma's proof C_1, C_2 etc are different constants, depending on n, d, r_0 and γ_0). Accordingly, the second term also satisfies the desired estimate, if ε_0 is small.

Due to (4.8) and the estimates for Δ_j and Λ_j above, the third term is bounded by

$$Ce^{C_2 j^{C_3}}\, l_0^{C_4 j^2}\, \varepsilon_j^{-c_2 c_\varkappa} \varepsilon_j^2.$$

We see that it is $\le \frac{1}{3}\varepsilon_j^{4/3}$, if ε_0 and $c_\varkappa$ are small.

The estimates (4.1) follows from (4.6) by strightforward arguments. ☐

4.3 Transition to the limit

For $m \ge 0$ let us denote

$$\mathcal{O}(m) = O^\gamma(r_m, \mu_m), \quad \mathcal{O}'(m) = O^\gamma(\tfrac{1}{3} r_{m+1} + \tfrac{2}{3} r_m, \tfrac{1}{2}\mu_m), \quad \gamma = \frac{1}{2}\gamma_0.$$

Applying Lemma 4.1 with the parameters $\varepsilon = \varepsilon_{m-1}$, $r = r_{m-1}$ etc we construct analytic symplectomorphisms

$$S_m(\cdot;\omega) : \mathcal{O}(m) \to \mathcal{O}'(m-1), \quad \omega \in \Omega_m, \quad m = 1, 2, \dots \tag{4.9}$$

(note that $\gamma_m > \gamma$ for all m). The domains $\dots \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 = \Omega$ satisfy $\operatorname{Leb}(\Omega_m \setminus \Omega_{m-1} \le {\Delta'_m}^{c-1} \varepsilon_{m-1}^c \le e_{m-1}^{c/2}$, if $\varepsilon_0 \ll 1$. Therefore $\Omega' = \cap \Omega_m$ is a Borel set such that

$$\operatorname{Leb}(\Omega \setminus \Omega') \le 2\varepsilon_0^{c/2}.$$

Next let us set

$$Q_l = O^l(r/l\,, \mu/l)\,, \quad \mathcal{Z}^c_\gamma = \mathbb{T}^n_{r_0} \times \mathbb{C}^n \times Y^c_\gamma\,, \quad \mathcal{Z}_\gamma = \mathbb{T}^n \times \mathbb{R}^n \times Y_\gamma\,,$$

where $l \geq 2$, and denote by $|\cdot|_\gamma$ the natural norm in $\mathbb{C}^n \times \mathbb{C}^n \times Y^c_\gamma$. It defines the distance in $\mathcal{Z}^c_\gamma$. By Lemma 1.6 for each $\omega \in \Omega'$ the map S_m extends to

$$S_m : Q_2 \to \mathcal{Z}^c_\gamma\,, \quad |S_m - \mathrm{id}|_\gamma \leq \varepsilon^c_m \tag{4.10}$$

(here and below $c > 0$ are different esponents, depending on n and d).

Now for $0 \leq r < N$ let us denote $\Sigma^r_N = S_{r+1} \circ \cdots \circ S_N$. Due to (4.9), it maps $\mathcal{O}(N)$ to $\mathcal{O}'(r)$. Due to (4.10), this map analytically extends to a map $\Sigma^r_N : Q_3 \to \mathcal{Z}^c_\gamma$, and when $N \to \infty$ the maps Σ^r_N converge to a limiting mapping

$$\Sigma^r_\infty : Q_3 \to \mathcal{Z}^c_\gamma\,, \quad |\Sigma^r_\infty - \mathrm{id}|_\gamma \leq 2\varepsilon^c_r \ \ \forall\, r \geq 1. \tag{4.11}$$

By the Cauchy estimate the linearised map satisfies

$$|\Sigma^r_\infty(\mathfrak{h})_* - \mathrm{id}|_{\gamma,\gamma} \leq C\varepsilon^c_r \quad \forall\, \mathfrak{h} \in Q_4,\ \forall\, r \geq 0\,. \tag{4.12}$$

By construction, the map Σ^0_N transforms the original hamiltonian H_0 to $H_N = H_0 \circ \Sigma^0_N$,

$$H_N = \omega_N \cdot p + \frac{1}{2}\langle A_N\zeta, \zeta\rangle + h_N(\mathfrak{h};\omega)\,.$$

Here $\omega_N = \omega + \langle h^0_p\rangle + \ldots \langle h^{N-1}_p\rangle$, $A_N = A + \bar g^1_{\zeta\zeta} + \ldots \bar g^N_{\zeta\zeta} \in \mathcal{NF}_{\Delta_N}(\Omega_N)\,,$ and

$$[h_N]^{\gamma,\Lambda_N}_{\Omega',r,\mu_N} \leq \varepsilon_N\,. \tag{4.13}$$

Clearly, $\omega_N \to \omega'$ and $A_N \to A'$, where the vector ω' and the operator A' satisfy the assertions of Theorem 0.1.

Let us denote $\Sigma_\omega = \Sigma^0_\infty$, consider the limiting hamiltonian $H' = H_0 \circ \Sigma_\omega$ and write it as

$$H'(\mathfrak{h}) = \omega' \cdot p + \frac{1}{2}\langle A'\zeta, \zeta\rangle + h'(\mathfrak{h})\,.$$

The function h' is analytic in the domain Q_3. Since $H' = H_l \circ \Sigma^l_\infty$, then for any $\mathfrak{h} = (q, 0, 0)$ we have

$$\nabla_{\mathfrak{h}} H'(\mathfrak{h}) = (\Sigma^l_\infty(\mathfrak{h}_q)_*)^t \nabla H_l(\mathfrak{h}_l)\,, \tag{4.14}$$

where $\mathfrak{h}_l = \Sigma^l_\infty(\mathfrak{h}) \in \mathcal{O}'(l)$. Due to (4.13), $\nabla H_l(\mathfrak{h}_l) = (0, \omega_l, 0)^t + O(\varepsilon^{1/4}_l)$. Since the map Σ^l_∞ satisfies (4.12), then $\nabla H'(\mathfrak{h}) = (0, \omega_l, 0)^t + o(\varepsilon^{c_1}_l)$, $c_1 > 0$, for each l. Hence, $\nabla H'(\mathfrak{h}) = (0, \omega_\infty, 0)^t$, and

$$\nabla h'(q, 0, 0) \equiv 0.$$

Now consider $\nabla_{\zeta_a}\nabla_{\zeta_b} H'(\mathfrak{h})$, $\mathfrak{h} = (q, 0, 0)$. To study this matrix let us write it in the form (1.31), where $g = H_l$ and S^t is replaced by the map Σ^l_∞. Repeating the arguments, used at the step iii) of the proof of Lemma 1.7 we get that

$$\nabla_{\zeta_a}\nabla_{\zeta_b} H'(\mathfrak{h}) = (A_l)_{ab} + o(\varepsilon^c_l)\,, \quad c > 0\,.$$

Sending l to ∞ we see that $\nabla_{\zeta_a}\nabla_{\zeta_b}H'(\mathfrak{h}) = (A')_{ab}$. That is, $\nabla^2_\zeta h'(q,0,0) \equiv 0$. Therefore

$$h' \in \mathcal{O}(p^2, p\zeta, \zeta^3)\,,$$

as states the theorem.

To complete the proof it remains to note that since $A_l \in \mathcal{NF}_{\Delta_l}(\Omega_l)$ for each l, then $A' \in \mathcal{NF}_{\mathcal{E}_\infty}(\Omega')$.

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