Analyticity of solutions for quasilinear wave equations and other systems

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Abstract

We prove the persistence of analyticity for classical solution of the Cauchy problem for quasilinear wave equations with analytic data. Our results show that the analyticity of solutions, stated by the Cauchy-Kowalewski and Ovsiannikov-Nirenberg theorems, lasts till a classical solution exists. The approach applies to other quasilinear equations and implies the persistence of space-analyticity of their classical solutions.

0 Introduction

Consider a quasilinear wave equation:

$$\Box u + f(t, x, u, \nabla u, \dot{u}) = 0, \quad \dim x = d, \;\; t \in \mathbb{R}, \tag{0.1}$$

$$u_{t=0} = u_0, \qquad \dot{u}_{t=0} = u_1, \tag{0.2}$$

where f is a real-analytic function of all its arguments and the Cauchy data u_0, u_1 are real-analytic functions of x. To begin with we assume the periodic boundary conditions:

$$x \in \mathbb{T}^d = \mathbb{R}^d/\Gamma \qquad (\Gamma \;\text{ is a lattice}).$$

Regarding the solvability of the Cauchy problem (0.1), (0.2) two facts are well known: the life-span of its classical solution is non zero, i.e., there is $T > 0$ such that on $[0, T)$ the problem (0.1), (0.2) has a C^2-solution u, e.g. see [Hör97] and Proposition 1.1 below. On the other hand by the Cauchy-Kowalewski theorem [Kow75] there is a positive ε_1 such that for $t \in [0, \varepsilon_1)$ the solution u is real-analytic. Ovsiannikov and Nirenberg gave a beautiful generalization of the latter theorem to equations (0.1), where the nonlinearity f is continuous in t (and still is analytic in other variables), see in [Nis77]. By their result, for any real-analytic u_0 and u_1 there is a positive ε_2 such that for $t \in [0, \varepsilon_2)$ the solution u is real-analytic in x.

From the proofs of the Cauchy-Kowalewski and Ovsiannikov-Nireberg theorems all what one can say about the life-spans of analyticity ε_1 and ε_2 is just their positivity. However, for

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the different classes of quasilinear wave equations, supplemented with sufficiently smooth initial data, the life-span T of classical (smooth) solutions is often fairly large, sometimes $T = \infty$. The natural question is if the range of analyticity is extendable up to T.

By a general result of Alinhac and Metivier [AM84] the life-span of analyticity ε_1 in the Cauchy-Kowalewski theorem is equal to T. The proof of their theorem is very technical and involves complicated recombination of the Taylor's coefficients. (In [BB77] a similar result was obtained earlier for solutions of the 2d Euler equation, using hyperbolic features of that equation.) The aim of this paper is to give a short and transparent proof of this (and actually more general) properties of solutions of quasilinear wave equations. We also show that the life-span of analyticity ε_2 in the Ovsiannikov-Nirenberg theorem equals the life-span time T.

Theorem 0.1. *Let $u(t, x)$, where $0 \leq t \leq T, x \in \mathbb{T}^d$, be a solution of the Cauchy problem* (0.1), (0.2)*, H^{m+1}-smooth in x and C^1-smooth in t. Then*

i) if f and u_0, u_1 are real-analytic in $(x_1, \ldots, x_k)$, $1 \leq k \leq d$, then u also is real-analytic in these variables,

ii) if f and u_0, u_1 are real-analytic in all their arguments, then u also is.

Note that the first assertion of the theorem and the local in time existence of a classical solution imply that if u_0, u_1 and f are sufficiently smooth in x, continuous in t and analytic in $x_1, \ldots, x_k, u, \nabla u$ and $\dot{u}$, then the problem (0.1), (0.2) has a unique local in time solution, analytic in $x_1, \ldots, x_k$ (see below Corollary 1.9). This generalises the Ovsiannikov-Nirenberg theorem for equations of such class.

Theorem 0.1 is proved in Section 1; its proof is based on properties of the nonlinear semigroup, generated by the problem (0.1), (0.2). In Theorem 2.2, Section 2, we show that the assertion holds for solutions of (0.1), (0.2) defined locally, in a characteristic cone in $\mathbb{R} \times \mathbb{R}^d$. The local result on the analyticity implies the analyticity of global solutions defined on the whole torus. It straightforwardly generalizes to equations on homogeneous spaces and implies the corresponding global results. For example, Theorem 0.1.ii) remain true for equations in the standard sphere S^d, see Section 3.1. It also generalises to more general hyperbolic systems in real-analytic manifolds; we will consider this problem in a separate paper.

We preface Theorem 0.1 to local Theorem 2.2 since the assertion i) of the former and its proof remain true for other classes of equations for which the latter is no more valid. E.g. see Section 3 for quasilinear parabolic equations, the 3d Navier-Stokes system and NLS equations. In the same time, the proof of assertion ii) does not generalise to non-hyperbolic equations (and for quasilinear parabolic equations its claim is wrong). So while the Cauchy-Kowalewski theorem is an assertion about hyperbolic equations, the Ovsiannikov-Nirenberg theorem describes a general property of a large class of quasilinear systems.

We note that similar C^∞-smooth properties of solutions for (0.1), (0.2) are known, see [Koc93, Sog08].

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1 Global results: quasilinear wave equation on $\mathbb{T}^d$

1.1 Single equation.

Here we study the Cauchy problem for a quasilinear wave equation (0.1), (0.2) on $\mathbb{T}^d = \mathbb{R}^d/\Gamma$, where the function f is continuous in all variables, is H^m smooth in x, where

$$m > d/2 \qquad (\text{or } m > d/2 - 1 \text{ if } f \text{ does not depend on } \nabla u \text{ and } \dot{u}),$$

and is (real-)analytic in the arguments $u, \nabla u, \dot{u}$. [1] We denote by H^m the Sobolev spaces $H^m(\mathbb{T}^d)$ with the norm $\|u\|_m = \left(|\nabla^m u|_{L_2}^2 + |u|_{L_2}^2\right)^{1/2}$, and abbreviate $H^{m+1} \times H^m = \mathcal{H}^m$. Consider the Cauchy operator for the linear wave equation:

$$\widetilde{\Box} : u \mapsto (u_{t=0}, \dot{u}_{t=0}, \Box u). \tag{1.1}$$

It is well known that for any reasonable domain of definition this map is an embedding. For any $T > 0$ consider the spaces

$$X_m^T = C(0, T; H^{m+1}) \cap C^1(0, T; H^m), \qquad Y_m^T = \mathcal{H}^m \times C(0, T; H^m).$$

It is also well known (e.g., see [Tem97]) that the inverse operator defines a continuous mapping

$$\widetilde{\Box}^{-1} : Y_m^T \to X_m^T. \tag{1.2}$$

(but certainly $\widetilde{\Box}$ does not map X_m^T to Y_m^T).

The spaces Y_m^T and X_m^T suit well to study solvability of the problem (0.1), (0.2). Indeed, denote $\varkappa = |(u_0, u_1)|_{\mathcal{H}^m}$, assume that u_0, u_1 are smooth and that $u(t, x)$ is a smooth solution of the problem such that

$$|U(t)|_{\mathcal{H}^m} \le 3\varkappa \qquad \forall\, 0 \le t \le T'$$

with some $T' > 0$, where $U(t) = (u(t), \dot{u}(t))$. Taking the H^m scalar-product of (0.1) with $\dot{u}(t)$, we get that

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}\|_m^2 + C\frac{d}{dt}\|u\|_{m+1}^2 \le C_1\|u\|_m\|\dot{u}\|_m + C_2\|f(t, x, u, \nabla u, \dot{u})\|_m\|\dot{u}\|_m.$$

By the apriori assumption and since the space H^m is an algebra, for $0 \le t \le T'$ the r.h.s. is bounded by $C(\varkappa)$. Therefore $|U(t)|_{\mathcal{H}^m}^2 \le \varkappa^2 + tC_1(\varkappa)$. So there exists $T(\varkappa) > 0$ such that $|U(t)|_{\mathcal{H}^m} \le 2\varkappa$ for $0 \le t \le T(\varkappa)$. In the usual way the obtained apriori estimate implies

Proposition 1.1. *There exists $T' > 0$, depending only on f and $|(u_0, u_1)|_{\mathcal{H}^m}$, such that the problem* (0.1), (0.2) *has a unique solution $u(t, x)$, $0 \le t \le T'$, belonging to the space $Y_m^{T'}$.*

It is well known that in general the local solution $u(t)$ cannot be extended to all $t \ge 0$. The construction below gives a convenient implicit description of the set of initial data for which a solution exists for $0 \le t \le T$. This construction is a part of the PhD thesis of the first author [Kuk81].

[1] More precisely, the function $f(t, x, u, \xi, \eta)$ defines a real-analytic mapping $\mathbb{R}^{d+2} \to C(0, T; H^m)$, $(u, \xi, \eta) \mapsto f(\cdot, \cdot, u, \xi, \eta)$.

Denote $\widetilde{\Box}^{-1}Y_m^T = Z_m^T$ and provide Z_m^T with a norm, induces from Y_m^T by $\widetilde{\Box}^{-1}$. This is a Banach space,

$$\widetilde{\Box} : Z_m^T \to Y_m^T \qquad \text{is an isomorphism,} \tag{1.3}$$

$$Z_m^T \subset X_m^T \qquad \text{continuously} \tag{1.4}$$

by (1.2), and $X_{m+1}^T \subset Z_m^T$. Denote by F the nonlinear differential operator $F(u) = f(t, x, u, \nabla u, \dot{u})$ and by Φ – the operator of the Cauchy problem (0.1), (0.2). That is

$$\Phi(u) = \widetilde{\Box}(u) + (0, 0, F(u)). \tag{1.5}$$

Since $m > d/2$, then the space $C(0, T; H^m)$ is a Banach algebra. Using (1.3) and (1.4) we see that the mapping

$$\Phi : Z_m^T \to Y_m^T \qquad \text{is analytic.} \tag{1.6}$$

It is well known that the Cauchy problem (0.1), (0.2) with zero in the r.h.s. replaced by any function from $C(0, T; H^m)$ has at most one solution in X_m^T. So Φ is an embedding. Consider its differential in any point $u \in Z_m^T$:

$$d\Phi(u)(v) = \big(v_{t=0}, \dot{v}_{t=0}, \Box v + d_3 f[u]v + d_4 f[u]\nabla v + d_5 f[u]\dot{v}\big). \tag{1.7}$$

Here $f[u] = f(x, u, \nabla u, \dot{u})$ and d_j denotes the differential with respect to the j-th variable.

Lemma 1.2. *For any $u \in Z_m^T$ the map $d\Phi(u) : Z_m^T \to Y_m^T$ is an isomorphism.*

Proof. For any $(v_0, v_1, g) \in Y_m^T$ consider the corresponding Cauchy problem which we write as

$$\Box v + v + \big(d_3 f[u]v + d_4 f[u]\nabla v + d_5 f[u]\dot{v} - v\big) = g, \quad v(0) = v_0,\ \dot{v}(0) = v_1. \tag{1.8}$$

Let v_0, v_1 and g be smooth and v be a smooth solution of the problem. Multiplying the equation by $\dot{v}$ in H^m and using that the space $H^m, m > d/2$, is an algebra, we get:

$$\frac{1}{2}\frac{d}{dt}\|\dot{v}\|_m^2 + \frac{1}{2}\frac{d}{dt}\|v\|_{m+1}^2 \le C_2\|v\|_m^2 + C_3\|v\|_{m+1}\|v\|_m + C_4\|\dot{v}\|_m\|v\|_m + \|g\|_m\|v\|_m,$$

where the constants C_j are continuous functions of $\|u\|_{X_m^T}$. We immediately get from this relation that

$$\|v\|_{X_m^T} \le C(\|u\|_{X_m^T})\|(v_0, v_1, g)\|_{Y_m^T}.$$

In particular, $d\Phi(u)$ is an embedding. In the usual way this apriori estimate implies that (1.8) has a unique solution $v \in X_m^T$. Then we see from (1.8) that $\widetilde{\Box}v \in Y_m^T$. So $v \in Z_m^T$ and $\|v\|_{Z_m^T} \le C'(\|u\|_{Z_m^T})\|(v_0, v_1, g)\|_{Y_m^T}$. $\Box$

Since Φ is an embedding, then Lemma 1.2 jointly with the inverse function theorem (see [PT87], Appendix B) imply

Lemma 1.3. *The mapping Φ is an analytic diffeomorphism of the space Z_m^T and a domain $\mathcal{O} \in Y_m^T$.*

Therefore if for some $(u_1, u_2, g) \in Y_m^T$ the problem (0.1), (0.2) with zero in the r.h.s. of (0.1) replaced by g has a solution $u \in X_m^T$, then $(u_1, u_2, g) \in \mathcal{O}$, u belongs to Z_m^T and analytically depends on (u_1, u_2, g). Denote

$$\mathcal{O}^0 = \{(u_0, u_1) \in \mathcal{H}^m : (u_0, u_1, 0) \in \mathcal{O}\} \tag{1.9}$$

Then for $0 \le t \le T$ the flow-maps

$$S_0^t : \mathcal{O}^0 \to \mathcal{H}^m, \qquad (u_0, u_1) \to (u(t), \dot{u}(t)),$$

are well defined and analytic.

We recall that T is any positive number and that the domain $\mathcal{O}^0$ depends on the time-interval $[0,T]$, $\mathcal{O}^0 = \mathcal{O}^0([0,T])$. Similar we may study solutions of (0.1), (0.2) on negative time-intervals $[-T, 0]$. The assertions above remain true for operators S_0^t with $t \in [-T, 0]$ and with the domain $\mathcal{O}^0 = \mathcal{O}^0([-T, 0])$. Finally, we may consider eq. (0.1) with the Cauchy data given not at $t = 0$, but at $t = t_1$, for arbitrary $t_1 \in [0,T]$. In this way we find that the flow-maps $S_{t_1}^{t_2}$, where $t_1, t_2 \in [0,T]$, are analytic operators with domains $\mathcal{O}^{t_1}([0,T])$.

It is clear from our construction that the operators $S_{t_1}^{t_2}$ and $S_{t_2}^{t_1}$ with domains $\mathcal{O}^{t_1}([0,T])$ and $\mathcal{O}^{t_2}([0,T])$ are inverse to each other, and that an operator $S_{t_1}^{t_2}$ analytically extends to the bigger domain $\mathcal{O}^{t_1}([t_1, t_2])$. This is its maximal domain of definition.

1.2 Families of equations.

We fix any $k \in \{1, \dots, d\}$ and assume that

$$\mathbb{T}^d = \mathbb{T}^k \times \mathbb{T}^{d-k}$$

(that is, $\Gamma = \Gamma_k \oplus \Gamma_{d-k}$ and $\mathbb{T}^k = \mathbb{R}^k/\Gamma_k$, $\mathbb{T}^{d-k} = \mathbb{R}^{d-k}/\Gamma_{d-k}$). We make the torus $\mathbb{T}^k = \{\theta = (\theta_1, \dots, \theta_k)\}$ to act on $\mathbb{T}^d$ by the shifts ${}_\theta R$,

$${}_\theta R(x) = (x^I + \theta, x^{II}),$$

where $x^I = (x_1, \dots, x_k)$ and $x^{II} = (x_{k+1}, \dots, x_d)$. Then the torus acts on the operators F by shifting their coefficients: $({}_\theta RF)(u) = f(t, {}_\theta Rx, u, \nabla u, \dot{u})$. Clearly we have

$$(\Box + {}_\theta RF)({}_\theta Ru) = {}_\theta R\big((\Box + F)(u)\big). \tag{1.10}$$

The operator of the shifted Cauchy problem ${}_\theta\Phi(u) = \widetilde{\Box} u + (0, 0, ({}_\theta RF)u)$ defines a mapping

$$\bar{\Phi}^1 : \mathbb{T}^k \times Z_m^T \to Y_m^T, \qquad (\theta, u) \to {}_\theta\Phi(u). \tag{1.11}$$

Lemma 1.4. *Assume that the function $f(t, x, u, \nabla u, \dot{u})$ is analytic in x^I. Then the mapping $\bar{\Phi}^1$ is analytic.*

Proof. By (1.6) we only have to check that the mapping is analytic in θ. Since f is analytic in x^I, then ${}_\theta\Phi(u)$ complex-analytically depends on θ from the complex vicinity of $\mathbb{T}^k$. This implies the assertion. $\square$

By the results of Section 1.2, for any θ the operator $\bar\Phi^1(\theta,\cdot)$ defines an analytic diffeomorphism of Z_m^T and a domain ${}_\theta\mathcal{O}\subset Y_m^T$. By the implicit function theorem

$$\text{the mapping}\quad {}_\theta\mathcal{O}\ni\xi\to\big(\bar\Phi^1(\theta,\cdot)\big)^{-1}\in Z_m^T\quad\text{is analytic in } \xi \text{ and } \theta. \tag{1.12}$$

Denoting ${}_\theta\mathcal{O}^0=\{(u_0,u_1):(u_0,u_1,0)\in{}_\theta\mathcal{O}\}$ we see that for $\theta\in\mathbb{T}^k$ and $0\le t\le T$ the time-t flow-mapping, corresponding to the nonlinearity ${}_\theta RF$, is an analytic transformation

$$_\theta S_0^t:{}_\theta\mathcal{O}^0\to\mathcal{H}^m\qquad\text{which analytically depends on}\quad\theta. \tag{1.13}$$

Relation (1.10), where $(\Box+F)u=0$, implies that

$$_\theta S_0^t\circ{}_\theta R={}_\theta R\circ S_0^t. \tag{1.14}$$

In particular, ${}_\theta R\mathcal{O}^0={}_\theta\mathcal{O}^0$.

Similar for any $\delta\in\mathbb{R}$ we define

$$({}^\delta RF)(u)=f(t+\delta,x,u,\nabla u,\dot u),\qquad {}^\delta\Phi=\widetilde{\Box}+(0,0,{}^\delta RF).$$

Assume that there exists $\rho>0$ such that for each value of $(x,u,\nabla u,\dot u)$ the function $t\mapsto f(t,x,u,\nabla u,\dot u)$ analytically extends to the segment $[-\rho,T+\rho]$. Then the mapping

$$\bar\Phi^2:(-\rho,\rho)\times Z_m^T\to Y_m^T,\qquad(\delta,u)\mapsto{}^\delta\Phi(u),$$

is analytic, and for any $|\delta|<\rho$ it defines an analytic isomorphism

$$^\delta\Phi:Z_m^T\to{}^\delta\mathcal{O}\subset Y_m^T\,,$$

which analytically depends on δ. We set ${}^\delta\mathcal{O}^0=\{(u_0,u_1)\in\mathcal{H}^m:(u_0,u_1,0)\in{}^\delta\mathcal{O}\}$ and denote by ${}^\delta S_0^t$ the mapping S_0^t, corresponding to the operator ${}^\delta RF$. Then

$$\text{the operator}\quad{}^\delta S_0^t:{}^\delta\mathcal{O}^0\to\mathcal{H}^m\quad\text{is analytic and analytically depends on }\delta\in(-\rho,\rho). \tag{1.15}$$

1.3 Analyticity of solutions

There is a delicate difference between the smoothness (or analyticity) of solutions for a nonlinear wave equation in time and in space. For instance, there is a number of results which imply for a solution a high smoothness in x and only a limited smoothness in t, see [Hör97]. Another example is given by the Ovsiannikov-Nirenberg theorem. Accordingly below we consider smoothness of solutions for (0.1), (0.2) in x and in t separately.

Space-analyticity. Assume that the function f as above is analytic in x^I, as well as the initial data u_0 and u_1. Assume also that the problem (0.1), (0.2) has a solution $u\in Y_m^T$. Then $(u_0,u_1)\in{}_\theta\mathcal{O}^0$ and by (1.13) ${}_\theta S_0^tu_0$ with $0\le t\le T$ is well defined for θ from a small ball $B_\varepsilon=\{|\theta|<\varepsilon\}$ and is analytic in θ. Using (1.14) we have

$$u(t,x^I+\theta,x^{II})=({}_\theta R\circ S_0^t)(u_0,u_1)(x)=({}_\theta S_0^t\circ{}_\theta R)(u_0,u_1)(x).$$

Since u_0 is analytic in x^I, then the mapping $\mathbb{T}^k\to\mathcal{H}^m,\quad B_\varepsilon\ni\theta\mapsto{}_\theta R(u_0,u_1)$, is analytic. Using (1.13) we get

Theorem 1.5. *Assume that the nonlinearity f and the initial data u_0, u_1 are analytic in $x^I \in \mathbb{T}^k$ and the problem* (0.1), (0.2) *has a solution $u(t,x) \in Y_m^T$, $0 \le t \le T, x \in \mathbb{T}^d$. Then u is analytic in x^I.*

Time-analyticity. Now assume that the function $f(t,x,u,\nabla u,\dot u)$, where $0 \le t \le T$, is analytic in all its arguments, that the Cauchy data $u_0(x)$ and $u_1(x)$ are analytic and that the problem (0.1), (0.2) has a solution $u \in Y_m^T$. Denote $U(t) = (u(t), \dot u(t))$. By the Cauchy-Kowalewski theorem, the function $u(t,x)$ is analytic for $|t| < \varepsilon$ and $x \in \mathbb{T}^d$ with a suitable $\varepsilon > 0$. Therefore the curve $[0,T] \to \mathcal{H}^m$, $t \mapsto U(t)$, also is analytic for $|t| < \varepsilon$. For any $t_* \in [0,T]$ we write the solution $U(t)$ for t close to t_* as $U(t_* + \tau) = {}^\tau S_0^{t_*} \circ U(\tau)$. Using (1.15) we get

Lemma 1.6. *Under the above assumptions the curve $[0,T] \to \mathcal{H}^m$, $t \to (u,\dot u)(t)$, is analytic. In particular, $u(t,x)$ is analytic in $t \in [0,T]$ for each x.*

This result and Theorem 1.5 imply

Theorem 1.7. *Assume that the nonlinearity $f(t,x,u,\nabla u,\dot u)$, where $0 \le t \le T$, and the initial data u_0, u_1 are analytic in all variables and the problem* (0.1), (0.2) *has a solution $u(t,x) \in Y_m^T$. Then $u(t,x)$ is an analytic function in all its arguments.*

We recall that for any $t_1, t_2 \in [0,T]$ the flow-map $S_{t_1}^{t_2}$ defines an analytical isomorphism $\mathcal{O}^{t_1}([0,T]) \rightleftarrows \mathcal{O}^{t_2}([0,T])$. Denote by $\mathcal{A}(\mathbb{T}^d)$ the space of analytic functions on T^d.

Corollary 1.8. *If the nonlinearity f is analytic in $x, u, \nabla u, \dot u$, then the mapping $S_{t_1}^{t_2}$ defines a bijection $\mathcal{O}^{t_1}([0,T]) \cap \mathcal{A}(\mathbb{T}^d) \rightleftarrows \mathcal{O}^{t_2}([0,T]) \cap \mathcal{A}(\mathbb{T}^d)$. If f is also analytic in t, then for each $u \in \mathcal{O}^{t_1}([0,T]) \cap \mathcal{A}(\mathbb{T}^d)$ the curve $(t_1,t_2) \mapsto S_{t_1}^{t_2}(u)$ is analytic in t_1 and t_2.*

Combining Theorem 1.5 with Proposition 1.1 we get

Corollary 1.9. *Let the nonlinearity f and the Cauchy data u_0, u_1 be as in Theorem 1.5. Then there exists $T' > 0$ such that for $0 \le t \le T'$ the Cauchy problem has a unique solution $u \in Y_m^{T'}$, which is analytic in x^I.*

The global results above generilise to other classes of quasilinear PDE. E.g., to quasilinear parabolic and Schrödinger equations, see Section 3. Moreover, they remain true for strongly nonlinear hyperbolic equations. This will be shown in a separate publication

2 Local results

2.1 Equations in characteristic cones.

In this section we consider the problem (0.1), (0.2) defined in a characteristic cone in $\mathbb{R}^{d+1}$.

Let $T > 0$ and $0 < a < T$. Denote by K a truncated characteristic cone:

$$K = K(T,a) = \{(t,x) \in [0, T-a] \times \mathbb{R}^d : |x| \le T - t\}$$

(below for short we call it cone). Denote by $B_r \subset \mathbb{R}^d$ the closed ball of radius r centered in the origin and denote $b^t = B_{T-t}$. In this section we study the problem (0.1), (0.2) in the cone K, where the Cauchy data are given on the ball $K \cap \{t = 0\}$, identified with b^0. The nonlinearity

f is assumed to be analytic in all its variables and analytically extendable to $U_\varepsilon(K)\times\mathbb{R}^{d+2}$, where $U_\varepsilon(K)$ is the ε-vicinity of K in $\mathbb{R}^{d+1}$, $\varepsilon>0$. Then for a given Cauchy data the problem (0.1), (0.2) has at most one classical solution, [Hör97]. Our goal is to prove for this solution Theorem 0.1, assuming that the Cauchy data also are analytic.

Denote by $Tr_\rho(g)$ the restriction of a function $g(x)$ to the ball B_ρ, $\rho>0$. It is well known that there exists an integral operator L_1 which for any $s\ge 0$ defines a bounded linear map

$$L_1: H^s(B_1)\to H_0^s(B_2)$$

such that $Tr_1\circ L_1=\mathrm{id}$ and $L_1u(x)=0$ for $|x|\ge r$ and each u, where $r<2$ depends only on L_1. For $\rho>0$ we denote by L_ρ the linear operator $L_\rho: H^s(B_\rho)\to H_0^s(B_{2\rho})$, obtained from L by the dilation (so $Tr_\rho\circ L_\rho=\mathrm{id}$). For integer $k\ge 0$ denote by $C([0,T-a];H^k(b^t))$ the space of functions $u(t,x)$ on the cone K such that[2]

$$\mathcal{L}u\in C([0,T-a];H^k(B_{2T}))\quad\text{where}\quad \big(\mathcal{L}u\big)(t)=L_t(u(t)),$$

and provide this space with the norm, induced from $C([0,T-a];H^k(B_{2T}))$. Next, for $m>d/2$ denote $\mathcal{H}^m=H^{m+1}(b^0)\times H^m(b^0)$ and set

$$\begin{aligned}X'_m&=\{u\in C([0,T-a];H^{m+1}(b^t)):\dot u\in C([0,T-a];H^m(b^t))\},\\ Y'_m&=\mathcal{H}^m\times C([0,T-a];H^m(b^t)).\end{aligned}$$

We denote by $\widetilde{\Box}'$ the operator of the Cauchy problem for $\Box$ which sends any function $u(t,x)\in X'_m$ to $\widetilde{\Box}'u=(u(0),\dot u(0),\Box u)\in Y'_{m-1}$.

Consider the torus $T^d=\mathbb{R}^d/(4T)\mathbb{Z}^d$, the corresponding spaces X_m^T,Y_m^T,Z_m^T and the operator $\widetilde{\Box}$. We will identify functions on T^d with $4T$-periodic functions on $\mathbb{R}^d$. Denoting by $\mathbf{Tr}$ the operator of restricting a function on $[0,T-a]\times\mathbb{R}^d$ to K, we get the mappings

$$\mathbf{Tr}: X_m^T\to X'_m,\quad \mathbf{Tr}: Y_m^T\to Y'_m.$$

For $\rho\le T$ and any $u(x)\in H_0^s(B_{2\rho})$ denote by ι the operator which first extends $u(x)$ to the cube $[-2T,2T]^d$ by zero outside the ball $B_{2\rho}$ and next extends it to a $4T$-periodic function. This is a bounded linear operator from $H_0^s(B_{2\rho})$ to $H^s(T^d)$, for any $s\ge0$ and any $0<\rho\le T$. For a function $u(t,x)$ we set $(\iota u)(t,x)=\iota(u(t,\cdot)(x)$, if the r.h.s. is defined. Clearly $\mathbf{Tr}\circ\iota=\mathrm{id}$ on the space $C([0,T-a];H^m(b^t))$.

Lemma 2.1. *The inverse operator $(\widetilde{\Box}')^{-1}$ equals*

$$(\widetilde{\Box}')^{-1}=\mathbf{Tr}\circ\widetilde{\Box}^{-1}\circ\iota. \tag{2.1}$$

It defines a continuous mapping $(\widetilde{\Box}')^{-1}: Y'_m\to X'_m$.

Proof. For $(u_0,u_1,g)\in Y'_m$ consider functions $\hat u_j(x)=\iota\big(L_T(u_j(x))\big), j=0,1$ and $\hat g(t,x)=\iota\big(\mathcal{L}(g)\big)$. Then $|(\hat u_0,\hat u_1,\hat g)|_{Y_m^T}\le C|(u_0,u_1,g)|_{Y'_m}$. The solution of the Cauchy problem for $\Box$ in T^d, $U(t,x)=\widetilde{\Box}^{-1}(\hat u_0,\hat u_1,\hat g)$, satisfies $|U|_{X_m^T}\le C|(\hat u_0,\hat u_1,\hat g)|_{Y_m^T}$. Since solutions of the wave

[2]This space is formed by restrictions to K of functions from $C([0,T-a];H^k(B_{2T}))$.

equation in K depend only on the data in the characteristic cone, then $(\widetilde{\Box}')^{-1}(u_0,u_1,g)$ equals to the restriction of U to K. This implies the assertions. $\Box$

As above, we define a Banach space Z'_m as $(\widetilde{\Box}')^{-1}Y'_m$, $Z'_m \subset X'_m$. Due to (2.1) $Z'_m = \mathbf{Tr}\,(Z^T_m)$. Denote by Φ' the operator of the Cauchy problem (0.1), (0.2) on K (cf. (1.5)). Then the following diagram is commutative

$$\begin{array}{ccc} Z_m & \xrightarrow{\Phi} & Y_m \\ {\scriptstyle \mathbf{Tr}}\downarrow & & \downarrow \\ Z'_m & \xrightarrow{\Phi'} & Y'_m \end{array}$$

where the second vertical line stands for the mapping $Tr_T \times Tr_T \times \mathbf{Tr}$. So we derive from Lemma 1.2 that for each $u \in Z_m$ the mapping $d\Phi'(u)$ is an isomorphism. Hence, Φ' is an analytic isomorphism

$$\Phi' : Z_m \rightleftarrows O,$$

where $O = \Phi'(Z_m)$ is a domain in Y'_m. We define $\mathcal{O}^0$ by the relation (1.9). This is a domain in $\mathcal{H}^m$ such that the problem (0.1), (0.2) has a solution $u \in X'_m$ if and only if $(u_0,u_1) \in \mathcal{O}^0$. The mapping

$$S^\tau_0(u_0,u_1) \mapsto (u(\tau),\dot u(\tau)), \qquad \mathcal{O}^0 \to \mathcal{H}^m,$$

is analytic since Φ'^{-1} is an analytic mapping on $\mathcal{O}$.

Similar we may consider eq. (0.1) on the smaller cone

$$K^\tau = \{(t,x) \in [\tau, T-a] \times \mathbb{R}^d : |x| \le T-t\} = K \cap ([\tau,T-a]\times\mathbb{R}^d), \quad 0 \le \tau < T-a.$$

In this way we get a domain $\mathcal{O}^\tau \subset \mathcal{H}^m(K^\tau)$ such that eq. (0.1) has a solution $u(t,x)$, $(t,x) \in K^\tau$, which is a trace on K^τ of a function from X'_m, if and only if $(u(\tau),\dot u(\tau)) \in \mathcal{O}^\tau$. Clearly the flow-map S^τ_0 is an analytic operator $S^\tau_0 : \mathcal{O}^0 \to \mathcal{O}^\tau$. In difference with Section 1 this is not an embedding.

Now we define families of the Cauchy problems and of the corresponding operators Φ'. Since the function f analytically in (t,x) extends to $U_\varepsilon(K)$, then eq. (0.1) analytically extends to the bigger cone

$$K^+ = \{(t,x) \in [-\varepsilon, T+\varepsilon-a] \times \mathbb{R}^d : |x| \le T+\varepsilon-t\}.$$

For any $\theta \in B_\varepsilon = \{|\theta| \le \varepsilon\}$, as before, denote by ${}_\theta R$ the shift ${}_\theta R(t,x) = (t, x+\theta)$, and set

$$K^\theta = {}_\theta R(K) \subset K^+, \qquad ({}_\theta Rf)(t,x,u\nabla u,\dot u) = f(t,{}_\theta Rx,u\nabla u,\dot u).$$

As before, ${}_\theta\Phi'$ is the operator of the Cauchy problem with the nonlinearity ${}_\theta Rf$. The mapping

$$\bar\Phi^1 : B_\varepsilon \times Z'_m \to Y'_m, \qquad (\theta,u) \to {}_\theta\Phi'(u)$$

is analytic. For each $\theta \in B_\varepsilon$ it defines an analytic diffeomorphism

$$\bar\Phi^1(\theta,\cdot) : Z'_m \to {}_\theta\mathcal{O} \subset Y'_m,$$

which analytically depends on θ, as well as its inverse.

Considering eq. (0.1) with the shifted nonlinearity ${}_\theta Rf$ in a smaller cone K^τ we define the corresponding domain ${}_\theta\mathcal{O}^\tau \subset \mathcal{H}^m(b_\tau)$, formed by the initial data $(u(\tau), \dot u(\tau))$ for which the shifted equation has a solution in K^τ, extendable to a function from X'_m. Then the flow-map of the shifted equation ${}_\theta S_0^\tau$ is an analytical mapping

$$ {}_\theta S_0^\tau : {}_\theta\mathcal{O}^0 \to {}_\theta\mathcal{O}^\tau $$

which analytically depends on θ, and ${}_\theta S_0^\tau \circ {}_\theta R = {}_\theta R \circ S_0^\tau$. That is, denoting by $u(t,x)$ a solution of (0.1), (0.2) we have

$$ u(t, x+\theta) = ({}_\theta S_0^t)(u_0(x+\theta, u_1(x+\theta)). $$

In particular, if the functions u_0 and u_1 are analytic in the closed ball b_0 (i.e. analytically extend to its vicinity in $\mathbb{R}^d$), then $u(t,x)$ is analytic in x.

By the Cauchy-Kowalewski theorem the solution $u(t,x)$ is analytic in the vicinity of the disc $b_0 \times \{0\}$ in $\mathbb{R}^{d+1}$. Considering shifts of the nonlinearity f by the time-translations and arguing as above (cf. Section 1.3) we find that u is analytic in t. We have proven

Theorem 2.2. *Assume that f is an analytic function on $U_\varepsilon(K) \times \mathbb{R}^{d+2}$ and that the Cauchy data u_0, u_1 are analytic in b^0. Let the Cauchy problem* (0.1), (0.2) *has a solution $u(t,x) \in X_m$. Then u is analytic.*

An obvious local version of Theorem 1.5 also is true.

Since the open non-truncated characteristic cone $K^o = \{(t,x) \in [0,T) \times \mathbb{R}^d : |x| < T - t\}$ is the union of the closed truncated cones $K(T', a)$ with $T' < T$ and $a > 0$, then a natural version of Theorem 2.2 holds for the cone K^o.

2.2 Global problems

In the assumptions of Theorems 1.7 let R be such that any two points of the ball B_R are not equivalent modulo the lattice Γ (which defines the torus T^d). Define $T' = \min\{2R, 2T\}$ and cover the layer $[0, T'] \times \mathbb{T}^d$ by finitely many truncated cones $K(2T', T')$, shifted by vectors $(0, \xi)$, $\xi \in \mathbb{R}^d$. Then by Theorem 2.2 the solution u is analytic in the layer. Iterating this construction (if $T' < T$) we see that u is analytic in $[0,T] \times \mathbb{T}^d$. So the local results of this section provide another proof of Theorems 1.5 and 1.7.

They also straightforwardly generalise to quasilinear wave equations in a connected open domain in an analytic Riemann homogeneous space. In this case $\Box = \partial^2/\partial t^2 - \Delta$, where Δ is the corresponding Laplace-Beltrami operator. Now the straight cone K should be replace by the characteristic cone, constructed in terms of the geodesics of the non-flat metric, and the translations ${}_\theta R$ – by the local isometies. This generalisation implies that Theorem 1.5 with $k = d$ and Theorem 1.7 remains true for quasilinear wave equations on a compact homogeneous analytic Riemann manifold M. For example, on the standard sphere S^d.

3 Related results

3.1 Quasilinear parabolic equations

The approach to study analyticity and partial analyticity of solutions in the space-variables, explained above, applies to other equations (to which the Cauchy-Kowalewski and Ovsiannikov-

Nirenberg theorems do not apply). For example, to quasilinear parabolic equations

$$\dot{u} - \Delta u + f(t, x, u, \nabla u) = 0, \quad x \in \mathbb{T}^d, \;\; t \geq 0, \qquad u_{t=0} = u_0, \tag{3.1}$$

where f is sufficiently smooth in t, x and is analytic in u and ∇u. As in Section 1.1, one can find suitable space Z_m^T and Y_m^T such that the operator Φ of the Cauchy problem (3.1) defines an analytic diffeomorphism between Z_m^T and a subdomain of the space Y_m^T, see [Kuk81, Kuk82]. In the same way as in Section 2 we prove that if f is analytical in its space-variables, then classical solutions of (3.1) with analytical initial data are space-analytic. This is a well known result, which holds true for $t > 0$ without assuming analyticity of $u_0(x)$. But we also can prove that if f is analytic in $u, \nabla u$ and in a part of the space-variables, as well as the function u_0, then the solution $u(t, x)$ is analytic in these space variables. This result seems new. Note that the assertion of Theorem 1.7 does not hold for the problem (3.1), even when $f = 0$, since a solution of the Cauchy problem $(3.1)|_{f=0}$ with analytic $u_0(x)$ may be non-analytic in t when $t = 0$.

The approach applies to the Navier-Stokes system on the d-torus with $d = 2$ or $d = 3$, perturbed by a sufficiently smooth force $h(t, x)$, see [Kuk82]. It implies that if the initial data and the force h are analytical in space-variables $x_1, \dots, x_k$, where $1 \leq k \leq d$, then a corresponding strong solution $u(t, x)$ remains analytical in this space-variables till it exists. See [DG95] for a proof that a strong solution, corresponding to an analytical force h, is analytic. Similar consider the 3d NSE in the thin layer $M \times (0, \varepsilon) = \{(\varphi, r\})$, where $M = S^2$ or $M = \mathbb{T}^2$. At the boundary $M \times \{0\} \cup M \times \{\varepsilon\}$ impose the Dirichlet our Navier boundary conditions. Let the force and initial data are
i) analytic in φ,
ii) bounded uniformly in $t \geq 0$, uniformly in $\varepsilon \in (0, 1)$.
Due to Raugel-Sell (see [TZ97] and references therein), if $\varepsilon > 0$ is sufficiently small, then there exists a unique strong solution $u(t, \varphi, r), \;\; t \geq 0$. Our result implies that this solution is analytic in φ.

3.2 NLS equations

The result of Theorem 1.5 remains true for the nonlinear Schrödinger equation

$$\dot{u} - i\Delta u + f(t, x, \operatorname{Re} u, \operatorname{Im} u) = 0, \qquad u_{t=0} = u_0, \;\; x \in \mathbb{T}^d,$$

where the complex function f is continuous in t, H^m-smooth in x ($m > d/2$) and real analytic in $\operatorname{Re} u, \operatorname{Im} u$. The proof of the theorem remains literally the same if we choose $X_m^T = C(0, T; H^m)$, $Y_m^T = H^m \times C(0, T; H^m)$ and proceed as in Section 1 (cf. [Kuk81]). As before, we can replace $\mathbb{T}^d$ by any homogeneous Riemann space, analytic and compact.

3.3 Smooth and partially smooth solutions

Results of Sections 1 and 3.1 concerning spatial analyticity and partial spacial analyticity of solutions remain true, with the same proof, for their spacial smoothness. For example if the nonlinearity f and the initial data u_0, u_1 of the problem (0.1), (0.2) are smooth in the variables $x_1, \dots, x_k$, $1 \leq k \leq d$, and the problem has a solution $u(t, x) \in Y_m^T$, then u also is smooth in $x_1, \dots, x_k$. Similar, if for the Navier-Stokes system on $\mathbb{T}^3$ the initial data and the force are smooth in some variable x_l, then a corresponding strong solution is smooth in x_l till it exists.

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