A family of balance relations for the two-dimensional Navier-Stokes equations with random forcing

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Abstract

For the 2D Navier-Stokes equation perturbed by a random force of a suitable kind we show that, if $g(\cdot)$ is an arbitrary real continuous function with (at most) polynomial growth, then the vorticity field $\omega(t,\mathbf{x})$ satisfies

$$\mathbb{E}\big(g(\omega(t,\mathbf{x}))|\nabla\omega(t,\mathbf{x})|^2\big)=\tfrac{1}{2}M_1\mathbb{E}(g(\omega(t,\mathbf{x}))),$$

where M_1 is a number, independent of g, which measures the strength of the random forcing.

Keywords: 2D Navier-Stokes equation, stationary measures, 2D turbulence.

1 Introduction

This paper concerns the Navier-Stokes equation for a fluid in the two-dimensional torus $\mathbb{T}^2=\mathbb{R}^2/(2\pi\mathbb{Z}^2)$ with a random applied force $\sqrt{\nu}\,\boldsymbol{\eta}$,

$$\partial_t\mathbf{u}(t,\mathbf{x})-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p(t,\mathbf{x})=\sqrt{\nu}\,\boldsymbol{\eta}(t,\mathbf{x}),\qquad \operatorname{div}\mathbf{u}=0, \tag{1.1}$$

$$(t,\mathbf{x})\in\mathbb{R}_0^+\times\mathbb{T}^2,\quad \mathbf{u}(t,\mathbf{x})\in\mathbb{R}^2,\quad p(t,\mathbf{x})\in\mathbb{R},\quad \boldsymbol{\eta}(t,\mathbf{x})\in\mathbb{R}, \tag{1.2}$$

where $\mathbf{u}$ is the velocity, p the pressure and ν the (non-zero) kinematic viscosity. The applied force is required to be divergence-free and to satisfy

$$\int_{\mathbb{T}^2}\boldsymbol{\eta}(t,\mathbf{x})d^2\mathbf{x}=0,$$

so that $\int\mathbf{u}(t,\mathbf{x})d^2\mathbf{x}$ is an invariant of the motion. We shall require the given initial velocity field in (1.1), $\mathbf{u}(0,\cdot)$, to be square-integrable over $\mathbb{T}^2$ and to satisfy $\int_{\mathbb{T}^2}\mathbf{u}(0,\mathbf{x})\,d^2\mathbf{x}=0$. The (weak) solution $\mathbf{u}(t,\mathbf{x})$ will therefore satisfy

$$\int_{\mathbb{T}^2}\mathbf{u}(t,\mathbf{x})\,d^2x=0\,,\quad \forall\, t\geq 0.$$

It is known that, subject to some further assumptions about the applied force which are detailed in the next section, the probability distribution of the solution

of the stochastic differential equation (1.1) converges as $t\to\infty$ to a unique stationary measure, [1] which we shall call the *equilibrium measure*; convergence results of this type were first obtained by Kuksin & Shirikyan [KS00, KS01] for a case where $\boldsymbol{\eta}$ consists of a random sequence of kicks. The extension to the 'white noise' form of $\boldsymbol{\eta}$ considered in the present paper (see eqn (2.1) below) was studied by Bricmont, Kupiainen & Lefevere [BKL02], by E, Mattingly & Sinai [EMS01], and others. For reviews, see [Bri02, Kuk02]. Below we use this result in the form, obtained in [KS02, KS03]. Moreover, it has been shown by Kuksin [Kuk04] that this equilibrium measure (which depends on ν) converges along subsequences $\nu_j\to 0$ to stationary measures of the Euler equation (i.e. of eqn (1.1) with $\nu=0$).

In the study of the $\nu\to 0$ limit an important part is played by a balance relation which equates the equilibrium rate of viscous dissipation of energy to the average rate at which the random force supplies energy. This relation, derived by applying the Ito formula to the total energy $\int_{\mathbb{T}^2}\mathbf{u}(t,\mathbf{x})^2d^2\mathbf{x}$ and taking the equilibrium expectation, can be written

$$\mathbb{E}\int_{\mathbb{T}^2}(\nabla\mathbf{u}(t,\mathbf{x}))^2d^2\mathbf{x}=\tfrac{1}{2}M_0, \tag{1.3}$$

where M_0 is related to the time autocorrelation function of $\boldsymbol{\eta}$. More precisely, if $\mathbb{E}[\boldsymbol{\eta}(t,\mathbf{x})\cdot\boldsymbol{\eta}(t',\mathbf{x})]=F_0(\mathbf{x})\delta(t-t')$, where $\delta(\cdot)$ is the Dirac distribution, then $M_0=\int_{\mathbb{T}^2}F_0(\mathbf{x})d^2\mathbf{x}$. For more detail about the derivation and meaning of (1.3) see section 3.

Another equally important balance relation is derived by applying the Ito formula to the enstrophy $\int_{\mathbb{T}^2}(\mathbf{curl}\,\mathbf{u}(t,\mathbf{x}))^2d^2\mathbf{x}$. This balance relation is

$$\mathbb{E}\int_{\mathbb{T}^2}(\Delta\mathbf{u}(t,\mathbf{x}))^2d^2\mathbf{x}=\tfrac{1}{2}M_1. \tag{1.4}$$

The left side is the equilibrium rate of viscous dissipation of enstrophy, and the right side is the average rate at which the random force creates enstrophy. That is to say, if $\mathbb{E}[\mathbf{curl}\,\boldsymbol{\eta}(t,\mathbf{x})\cdot\mathbf{curl}\,\boldsymbol{\eta}(t',\mathbf{x})]=F_1(\mathbf{x})\delta(t-t')$, then $M_1=\int_{\mathbb{T}^2}F_1(\mathbf{x})d^2\mathbf{x}$.

The purpose of the present note is to draw attention to an additional family of balance relations generalizing (1.4). They are related to the Helmholtz invariants for inviscid two-dimensional flow in the same way that the balance laws (1.3) and (1.4) are related to the energy and enstrophy invariants. These new balance relations are given in Theorem 3.1 below.

2 The invariant measure for eqn (1.1)

As in the papers cited earlier, we take $\boldsymbol{\eta}(t,\,\mathbf{x})$ to be the Gaussian random field

$$\boldsymbol{\eta}:=\frac{d}{dt}\boldsymbol{\zeta}\quad\text{with}\quad \boldsymbol{\zeta}:=\sum_{\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)}b_{\mathbf{s}}\beta_{\mathbf{s}}(t)e_{\mathbf{s}}(\mathbf{x}). \tag{2.1}$$

[1] Here and everywhere in this work a *measure* means a probability measure on the Borel σ-algebra of the relevant metric space.

where $\{b_{\mathbf{s}} : \mathbf{s} \in \mathbb{Z}^2 \setminus (0,0)\}$ is a set of real constants satisfying the conditions

$$b_{\mathbf{s}} = b_{-\mathbf{s}} \tag{2.2}$$

$$\sum |\mathbf{s}|^2 b_{\mathbf{s}} < \infty, \tag{2.3}$$

$\{\beta_{\mathbf{s}}(t) : \mathbf{s} \in \mathbb{Z}^2 \setminus (0,0)\}$ is a set of independent standard Wiener processes satisfying $\beta(0) = 0$, and $\{\mathbf{e}_{\mathbf{s}} : \mathbf{s} \in \mathbb{Z}^2 \setminus (0,0)\}$ denotes the orthonormal set of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\begin{aligned} \mathbf{e}_{\mathbf{s}}(\mathbf{x}) \quad &:= \quad \frac{\sin(\mathbf{s}\cdot\mathbf{x})}{\sqrt{2}\pi|\mathbf{s}|}\begin{bmatrix} -s_2 \\ s_1 \end{bmatrix} \quad \text{if} \quad s_1 + s_2\delta_{s_1,0} > 0 \\ \text{and} \quad &:= \quad \frac{\cos(\mathbf{s}\cdot\mathbf{x})}{\sqrt{2}\pi|\mathbf{s}|}\begin{bmatrix} -s_2 \\ s_1 \end{bmatrix} \quad \text{if} \quad s_1 + s_2\delta_{s_1,0} < 0 \end{aligned} \tag{2.4}$$

in which (s_1, s_2) are the elements of $\mathbf{s}$, $|\mathbf{s}| := \sqrt{s_1^2 + s_2^2}$, and $\delta_{s_1,0}$ is the Kronecker symbol. This definition makes the applied force white in time but (because of the convergence condition (2.3)) its spatial dependence at any given time has some smoothness.

We define the moments

$$M_n := \sum_{\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)} |\mathbf{s}|^{2n} b_s^2 \quad (n = 0, 1, 2, \dots). \tag{2.5}$$

This is consistent with the definitions of M_0, M_1 used in (1.3) and (1.4). It follows from (2.3) that M_0 and M_1 are finite, and from (2.1)–(2.4) that if M_n is finite then $\mathbb{E}\int_{\mathbb{T}^2} |(-\Delta)^{n/2}\boldsymbol{\zeta}(t,\mathbf{x})|^2\, d^2\mathbf{x} = M_n t$ for all t.

We can eliminate the pressure from eqn (1.1) by applying the Leray projector Π (see [CF88, VF88]) which removes the gradient part of any field it operates upon. Since the random force is divergence-free, eqn (1.1) simplifies to

$$\dot{\mathbf{u}}(t,\mathbf{x}) + \nu L(\mathbf{u})(t,\mathbf{x}) + B(\mathbf{u})(t,\mathbf{x}) = \sqrt{\nu}\,\boldsymbol{\eta}(t,\mathbf{x}), \tag{2.6}$$

where we have defined $L(\mathbf{u}) := -\Pi\Delta\mathbf{u}$ and $B(\mathbf{u}) := \Pi((\mathbf{u}\cdot\nabla)\mathbf{u})$. Let H denote the Hilbert space

$$H := \left\{ v \in L_2(\mathbb{T}^2) \times L_2(\mathbb{T}^2) \mid \int v\, d^2\mathbf{x} = 0,\ \operatorname{div} v = 0 \right\},$$

with the inner product

$$(v, w) := \int_{T^2} [v_1(\mathbf{x})w_1(\mathbf{x}) + v_2(\mathbf{x})w_2(\mathbf{x})] d^2\mathbf{x} \tag{2.7}$$

and the norm $\|v\| := (v,v)^{1/2}$. Let $u(t)$ denote the velocity field at time t, i.e. $u(t) := \mathbf{u}(t,\cdot)$. It is known [CF88] that if the initial velocity field $u(0) := \mathbf{u}(0,\cdot)$ lies in the space H then the stochastic Navier-Stokes equation (2.6) has a unique solution which almost surely remains in the space, i.e.

$$u(0) \in H \implies u(t) \in H \quad \text{for all} \quad t \geq 0, \quad \text{a.s.}$$

This solution is almost surely continuous in time (with respect to the Hilbert-space metric), and satisfies

$$\int_0^T \mathbb{E}\|\nabla u(t)\|^2\, dt + \sup_{0\leq t\leq T} \mathbb{E}\|u(t)\|^2 < \infty$$

for all positive T. Moreover, the above properties still hold if the initial condition $u(0)$ is selected at random from H according to a probability measure satisfying $\mathbb{E}\|u(0)\|^2 < \infty$.

One can think of H as the phase space for this model, and the time evolution of the velocity field as a continuous trajectory in H. The actual trajectory, for a given initial condition, depends on the realization of the stochastic processes β_s and is therefore itself a stochastic process in H. This stochastic process is Markovian; see [VF88, DZ92].

The result about convergence to an equilibrium measure mentioned in Section 1 is that if sufficiently many modes of the force $\boldsymbol{\eta}$ are excited i.e., if

$$b_{\mathbf{s}} \neq 0 \quad \forall \mathbf{s} : |\mathbf{s}| \leq N \tag{2.8}$$

for a suitable N (which depends on ν and goes to infinity when $\nu \to 0$), then as $t \to \infty$ the probability distribution of $u(t)$ converges to a uniquely defined measure μ over H:

$$\mathcal{D}\big(u(t)\big) \rightharpoonup \mu \quad \text{as } t \to \infty. \tag{2.9}$$

The arrow stands for the $*$-weak convergence of measures in H, and $\mathcal{D}$ signifies the distribution of a random variable. We note that the assumption (2.8) is fulfilled for all $\nu > 0$ if

$$b_{\mathbf{s}} \neq 0 \quad \forall \mathbf{s}. \tag{2.10}$$

The equilibrium measure μ is a stationary measure for the Markov process defined by eqn (2.6); that is to say if $\mathbf{u}(t,\mathbf{x})$ is a solution of (2.6) such that $\mathcal{D}u(0) = \mu$, then

$$\mathcal{D}u(t) = \mu \quad (t > 0).$$

In consequence of the symmetry condition (2.2), the Gaussian random field $\boldsymbol{\zeta}$ is *translationally invariant*, by which we mean that its measure is invariant under translations in $\mathbb{T}^2$. This measure also invariant under reflections in the origin. That is to say, if we define translation and reflection operators $T_{\mathbf{h}}, R$ by $T_{\mathbf{h}}\mathbf{x} := \mathbf{x} + \mathbf{h}, R\mathbf{x} := -\mathbf{x}, R\mathbf{u} := -\mathbf{u}$, with $\mathbf{h}$ any element of $\mathbb{T}^2$ (so that, for example, $T\boldsymbol{\zeta}(t,\mathbf{x}) = \boldsymbol{\zeta}(t,\mathbf{x}+\mathbf{h}), R\mathbf{u}(t,\mathbf{x}) = -\mathbf{u}(t,-\mathbf{x})$), then the Gaussian random fields $T_{\mathbf{h}}\boldsymbol{\zeta}$ and $R\boldsymbol{\zeta}$ have the same statistical properties as the original random field $\boldsymbol{\zeta}$. Because of these invariances of $\boldsymbol{\zeta}$ it follows, from the uniqueness of the equilibrium measure and the symmetry of the Navier-Stokes equation (1.1) with respect to translations and reflections, that this measure is also invariant under translations and reflections[2], i.e.

$$T_{\mathbf{h}}\mu = R\mu = \mu \quad (\mathbf{h} \in \mathbb{T}^2)$$

[2]For example, let $\mathbf{u}(t,\mathbf{x})$ be a solution of the equation (1.1); then because of the symmetry of this equation $T_{\mathbf{h}}\mathbf{u}(t,\mathbf{x}) := \mathbf{u}(t,\mathbf{x}+\mathbf{h})$ is also a solution; therefore, by the uniqueness of the stationary measure, $\mathcal{D}(T_{\mathbf{h}}\mathbf{u}) = \mu$.

These two invariances taken together imply that for each t and $\mathbf{x}$ the equilibrium distribution of the random variable $\mathbf{u}(t,\mathbf{x})$ is symmetrical under velocity reversal; the one-line proof of this is that if $\mathbf{u}(t,\mathbf{x})$ is distributed according to the equilibrium measure then

$$\mathcal{D}(-\mathbf{u}(t,\mathbf{x})) = \mathcal{D}(\mathbf{u}(t,-\mathbf{x}) = \mathcal{D}(\mathbf{u}(t,\mathbf{x}) \tag{2.11}$$

in which the first step follows from invariance under reflections and the second from invariance under translations (the shift being $2\mathbf{x}$). From this symmetry under velocity reversal it follows further that the mean of this random variable, $\mathbb{E}(\mathbf{u}(t,\mathbf{x}))$, is zero.

For further information about the above results see [Kuk02, Kuk04].

3 Balance equations

Let $u(t)$ denote a stationary solution of (2.6). The balance equation (1.3) is derived by applying the Ito formula to the process $\|u(t)\|^2 = \int_{\mathbb{T}^2} \mathbf{u}(t,\mathbf{x})\, d^2\mathbf{x}$, using (1.1) and taking the expectation. This gives

$$\frac{d}{dt}\,\mathbb{E}\|u(t)\|^2 = -\,2\nu\mathbb{E}(Lu,\ u) + \nu M_0, \tag{3.1}$$

see [VF88, EMS01], also [Kuk04] and references therein. Since the process is stationary and $(Lu,u) = (\nabla u, \nabla u)$, it follows that

$$\mathbb{E}\|\nabla u(t)\|^2 = \tfrac{1}{2}M_0 \quad (t>0), \tag{3.2}$$

with M_0 defined as in (2.5), and $\nabla u(t) := \nabla \mathbf{u}(t,\cdot)$. Eqn (3.2) is equivalent to (1.3). Similarly we can apply the Ito formula to $\|\nabla \mathbf{u}(t)\|^2$, and use the identity $(\nabla Bu,\ \nabla u) \equiv 0$, to get

$$\mathbb{E}\|\Delta u(t)\|^2 = \tfrac{1}{2}M_1 \quad (t>0), \tag{3.3}$$

which is equivalent to (1.4). The notation $\|\nabla u(t)\|$ used here is a generalization of that defined just after (2.7); it means the square root of the sum of the squares of the $L_2(\mathbb{T}^2)$ norms of the four components of the second-rank tensor $\nabla \mathbf{u}$.

The vorticity field ω is defined by

$$\omega(t,\ \mathbf{x}) := \mathbf{curl}\ \mathbf{u}(t,\ \mathbf{x}) := \partial u_2/\partial x_1 - \partial u_1/\partial x_2,$$

Taking the curl of (2.6) we find that the vorticity satisfies the equation

$$\dot\omega(t,\mathbf{x}) - \nu\Delta\omega(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\omega(t,\mathbf{x}) = \sqrt{\nu}\,\xi(t,\ \mathbf{x}), \tag{3.4}$$

where

$$\xi(t,\ \mathbf{x}) = \ \mathrm{curl}\ \boldsymbol{\eta}(t,\ \mathbf{x}) = \frac{d}{dt}\sum_{\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)} b_{\mathbf{s}}\beta_{\mathbf{s}}(t)\,\varphi_{\mathbf{s}}(\mathbf{x}), \tag{3.5}$$

and

$$\varphi_{\mathbf{s}} = \frac{|\mathbf{s}|}{\sqrt{2}\pi}\cos \mathbf{s}\cdot\mathbf{x}, \quad \varphi_{-\mathbf{s}} = -\frac{|\mathbf{s}|}{\sqrt{2}\pi}\sin \mathbf{s}\cdot\mathbf{x}, \tag{3.6}$$

for all $\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)$ such that $s_1 + s_2\delta_{s_1,0} > 0$. Because of (2.2), (2.8) the random field $\omega(t,\ \mathbf{x})$, like $\mathbf{u}(t,\ \mathbf{x})$, is stationary and translationary invariant.

The main result of this paper is the following theorem:

Theorem 3.1. *Suppose that* (2.2), (2.10) *hold, with* $M_3<\infty$, *and denote by* $\omega(t,\mathbf{x})$ *the vorticity of a stationary solution of* (2.6) *with positive* ν. *Let* $g(\varpi)$ *be a real-valued continuous function of the real variable* ϖ, *satisfying*

$$|g(\varpi)| \le A(1+|\varpi|^k) \quad \forall\,\varpi\in\mathbb{R}, \tag{3.7}$$

for some real A *and integer* k, *both positive. Then for all* $t\ge 0$, $\mathbf{x}\in\mathbb{T}^2$, *we have*

$$\mathbb{E}\big(g(\omega(t,\mathbf{x}))|\nabla\omega(t,\mathbf{x})|^2\big) = \tfrac{1}{2}(2\pi)^{-2}M_1\mathbb{E}(g(\omega(t,\mathbf{x}))). \tag{3.8}$$

Proof. Consider first the case where g has compact support. Define G to be its second integral, so that $G''(\varpi)=g(\varpi)$, $G(0)=G'(0)=0$. Since g has compact support, there exist constants A_1 and A_2 such that

$$|G(\varpi)| \le A_1|\varpi|, \quad |G'(\varpi)| \le A_2 \quad \forall\,\varpi\in\mathbb{R}. \tag{3.9}$$

Let us consider the stationary process

$$\int_{\mathbb{T}^2} G(\omega(t,\mathbf{x}))\, d^2\mathbf{x}.$$

The estimates (3.9) allow us to apply the Ito formula (see [DZ92]) to this process. After taking the expectation and using (3.4) together with the fact that the process is stationary, as in the derivation of (3.2), we find that for all $t\ge 0$ the random field $\omega(t,\ \mathbf{x})$ satisfies

$$\nu\mathbb{E}\int_{\mathbb{T}^2} G'(\omega(t,\mathbf{x}))(-\Delta)\omega(t,\mathbf{x})\, d^2\mathbf{x} = \tfrac{1}{2}\nu\mathbb{E}\sum_{\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)} b_{\mathbf{s}}^2\int_{\mathbb{T}^2} G''(\omega(t,\mathbf{x}))\varphi_{\mathbf{s}}^2(\mathbf{x})\, d^2\mathbf{x}. \tag{3.10}$$

Integrating by parts, we see that the left side of (3.10) equals

$$\nu\mathbb{E}\int_{\mathbb{T}^2} g(\omega(t,\mathbf{x}))|\nabla\omega(t,\mathbf{x})|^2\, d^2\mathbf{x}.$$

By (2.2) and the definition of G, the right side equals

$$\frac{1}{4}\nu\mathbb{E}\int_{\mathbb{T}^2}[g(\omega(t,\mathbf{x}))\sum_{\mathbf{s}\in\mathbb{Z}^2\setminus(0,0)} b_s^2(\varphi_{\mathbf{s}}^2+\varphi_{-\mathbf{s}}^2)]\, d^2\mathbf{x} = \tfrac{1}{2}(2\pi)^{-2}\nu M_1\mathbb{E}\int_{\mathbb{T}^2} g(\omega(t,\mathbf{x}))\, d^2\mathbf{x},$$

since $\varphi_{\mathbf{s}}^2+\varphi_{-\mathbf{s}}^2 = \frac{|\mathbf{s}|^2}{2\pi^2}$. Thus (3.10) implies that

$$\mathbb{E}\int g(\omega(t,\mathbf{x}))\,|\nabla\omega(t,\mathbf{x})|^2\, d^2\mathbf{x} = \tfrac{1}{2}(2\pi)^{-2}M_1\mathbb{E}\int g(\omega(t,\mathbf{x}))\, d^2\mathbf{x}, \tag{3.11}$$

when g has compact support. Eqn (3.11) can be rewritten

$$\int \mathbb{E}[g(\omega(t,\mathbf{x}))\,(|\nabla\omega(\mathbf{x})|^2 - \tfrac{1}{2}(2\pi)^{-2}M_1)]\,d^2\mathbf{x} = 0. \tag{3.12}$$

Since the random field $\omega(t,\mathbf{x})$ is translationally invariant, it follows that the integrand is zero for all $\mathbf{x}$, completing the proof of (3.8) for this case.

To deal with the case where g does not have compact support, we approximate g by a sequence of functions which do have compact support. Take any continuous function χ such that $0 \le \chi(t) \le 1$ for all t, $\chi = 1$ for $|t| \le 1$ and $\chi = 0$ for $|t| \ge 2$, and define the sequence of functions

$$g_n(\varpi) := \chi(\varpi n^{-1})\,g(\varpi) \quad (n = 1, 2, \ldots). \tag{3.13}$$

By (3.7) these functions have the properties

$$\left.\begin{array}{rcl} \lim_{n\to\infty} g_n(\varpi) &=& g(\varpi) \\ |g_n(\varpi)| &\le& A(1+|\varpi|^k) \end{array}\right\} \forall\, \varpi \in \mathbb{R} \tag{3.14}$$

Since g_n has compact support, the part of the theorem so far proved shows that

$$\mathbb{E}\int g_n(\omega(t,\mathbf{x}))\,|\nabla\omega(\mathbf{x})|^2\,d^2\mathbf{x} = \tfrac{1}{2}(2\pi)^{-2}M_1\mathbb{E}\int g_n(\omega(t,\mathbf{x}))\,d^2\mathbf{x}, \tag{3.15}$$

for each n. On each side of this equation the integrand converges pointwise, as $n \to \infty$, to the integrand in the corresponding term of (3.11).

To prove that the expectations of the integrals also converge, we consider the right side of (3.15) first. By analogy with the definition of the field $u(t) \in H$, we define the field $\omega(t) := \omega(t,\cdot)$. Since its space average is zero for all t, it satisfies the Sobolev-type inequality

$$|\omega(t,\mathbf{x})| < \text{const}\cdot\|\Delta\omega(t)\|_{L^2} \quad \forall\, \mathbf{x} \in \mathbb{T}^2 \tag{3.16}$$

It is shown in Proposition 2.4 of [KS03] that the condition $M_3 < \infty$, assumed in the statement of our theorem, implies the smoothness property

$$\mathbb{E}(\|\nabla^3 u(t)\|^m) < \infty, \tag{3.17}$$

for all positive integers m. The meaning of the notation $\|\nabla^3 u(t)\|$ (where $\nabla^3 u(t)$ is a fourth-rank tensor field) is analogous to that of $\|\nabla u(t)\|$, which is given just after eqn (3.3). Since $\omega(t)$ is the difference of two components of $\nabla u(t)$, eqns (3.16) and (3.17) imply

$$\mathbb{E}\int_{\mathbb{T}^2} |\omega(t,\mathbf{x})|^k\,d^2\mathbf{x} \le \text{const}\cdot\mathbb{E}\|\Delta\omega(t)\|_{L^2}^k \le \text{const}\cdot\mathbb{E}\|\Delta\nabla u(t)\|^k < \infty \tag{3.18}$$

It follows, by the second line of (3.14), that the integrands on the right side of (3.15) have a majorant whose integral has finite expectation, and consequently by Lebesgue's dominated convergence theorem that the right sides of (3.15) converge as $n \to \infty$ to the right side of (3.11).

We can apply a similar argument to the left side of (3.15). Eqn (3.17), with $m = 4$, implies

$$\mathbb{E}\|\nabla\nabla\omega\|_{L^2}^4 < \infty \tag{3.19}$$

Using the Sobolev-type inequality

$$\|\nabla\omega(t)\|_{L^4} < \text{const}\cdot\|\nabla\nabla\omega(t)\|_{L^2} \tag{3.20}$$

we obtain

$$\mathbb{E}\|\nabla\omega\|_{L^4}^4 < \infty. \tag{3.21}$$

The integrand is dominated by $A(1+|\omega(t,\mathbf{x})|^k)|\nabla\omega(t,\mathbf{x})|^2\,d^2\mathbf{x}$. By the Schwartz inequality, the expectation of the integral of this majorant is at most

$$A\sqrt{\mathbb{E}\int_{\mathbb{T}^2}(1+|\omega(t,\mathbf{x})|^k)^2\,d^2\mathbf{x}}\;\sqrt{\mathbb{E}\int_{\mathbb{T}^2}|\nabla\omega(t,\mathbf{x})|^4\,d^2\mathbf{x}} \tag{3.22}$$

An argument similar to that used for the right side of eqn (3.15) shows that the first factor is finite, and the second factor is finite because of (3.21). So we can again use Lebesgue's dominated convergence theorem, showing that the limit of the left side of (3.15) equals the left side of (3.11). The proof is then completed in the same way as for g with compact support. □

If g is an odd function, then both sides of (3.8) vanish because of (2.11). The special case of (3.8) when $g = 1$ is the same as (3.3).

Since $\|\Delta\mathbf{u}(t,\cdot)\|^2 = \|\nabla\omega(t,\cdot)\|_{L^2}^2$, eqn (3.3) and the translational invariance of the process $\omega(t)$ imply that $\mathbb{E}|\nabla\omega(t,\mathbf{x})|^2 = \frac{1}{2}(2\pi)^{-2}M_1$ for all t and x. Thus the right side of (3.8) equals $\mathbb{E}|\nabla\omega(t,\mathbf{x})|^2\,\mathbb{E}g(\omega(t,\mathbf{x}))$, and (3.8) means that the random variables $g(\omega(t,\,\mathbf{x}))$ and $|\nabla\omega(t,\,\mathbf{x})|^2$ are uncorrelated for every continuous g satisfying (3.7).

The theorem can be restated in terms of conditional expectations. For a fixed $(t,\,\mathbf{x})$ let $\mathbb{E}(\cdot|\mathcal{F}_{\omega(t,\mathbf{x})})$ denote conditional expectation with respect to the σ-algebra generated by the random variable $\omega(t,\,\mathbf{x})$. Then Theorem 3.1 implies:

Corollary 3.2. *Under the assumptions of Theorem 3.1 we have*

$$\mathbb{E}(|\nabla\omega(t,\,\mathbf{x})|^2 \mid \mathcal{F}_{\omega(t,\mathbf{x})}) = \tfrac{1}{2}(2\pi)^{-2}M_1 \quad \forall\, t, \mathbf{x} \tag{3.23}$$

To prove this, let us treat t and $\mathbf{x}$ as fixed and make the abbreviations $\omega(t,\,\mathbf{x}) = \xi$, $|\nabla\omega(t,\,\mathbf{x})|^2 = \eta$, and define $h(\xi) := \mathbb{E}(\eta \mid \mathcal{F}_\xi)$. Then

$$\mathbb{E}(g(\xi)\eta) = \mathbb{E}(g(\xi)h(\xi)).$$

Because of (3.8) this expression equals $\frac{1}{2}(2\pi)^{-2}M_1\mathbb{E}g(\xi)$ for any bounded continuous function g. Hence, $h(\xi) = \frac{1}{2}M_1$ a.e. with respect to the measure $\mathcal{D}(\xi)$. This proves (3.23).

4 The probability distribution of the vorticity field

We denote by ρ the probability measure of $\omega(t) = \mathbf{curl}\, u(t)$ when $u(t)$ is distributed in H according to the stationary measure μ. It is the measure induced in the space occupied by $\omega(t)$ by the mapping $u(t) \to \omega(t) = \mathbf{curl}\, u(t)$, and it is time-independent. We know already, because of (3.16), that $\omega(t)$ lies in $L^\infty(\mathbb{T}^2)$, but on the assumption that $M_4 < \infty$ we can show that it lies in the smaller space $C^1(\mathbb{T}^2)$. Indeed, it is shown in [KS03] that, if $M_4 < \infty$, the stationary solution $u(t)$ satisfies

$$\mathbb{E}\|\nabla^4 u(t)\|^r < \infty \quad \forall\, t > 0 \tag{4.1}$$

for all positive integers r. By the Sobolev-type inequality

$$\|\omega(t)\|_{C^1} \leq \text{const}\, \|\nabla^3 \omega(t)\|_{L^2}, \tag{4.2}$$

it follows that the vorticity satisfies

$$\mathbb{E}\big(\|\omega(t)\|_{C^1}^r\big) < \infty \quad \forall\, t > 0, \tag{4.3}$$

where $\|\omega(t)\|_{C^1}$ denotes the maximum of the $L^\infty(\mathbb{T}^2)$ norms of $\omega(t)$ and $\nabla\omega(t)$. Accordingly, the vorticity ω defines a (time-independent) probability measure $\rho := \mathcal{D}(\omega)$ on the space $C^1(\mathbb{T}^2)$. The relations (3.8) can be rewritten in terms of ρ:

$$\int_{C^1(\mathbb{T}^2)} g(\omega(t,\mathbf{x}))|\nabla\omega(t,\mathbf{x})|^2 \rho\,(d\omega) = \tfrac{1}{2}(2\pi)^{-2} M_1 \int_{C^1(\mathbb{T}^2)} g(\omega(\mathbf{x}))\rho\,(d\omega), \tag{4.4}$$

for all $\mathbf{x} \in \mathbb{T}^2$, all $t > 0$ and all g satisfying (3.7).

The information we have about the measure ρ can be codified in the following way. For every $M > 0$ define $\mathcal{B}_M$ to be the set of probability measures λ on $(C^1(\mathbb{T}^2), \Sigma)$ (Σ is the σ-algebra of Borel sets), which have the following properties:

(H1) λ is translationally invariant;

(H2) λ is reflection-invariant, i.e. $\lambda(A)=\lambda(RA)$ for any set $A \in \Sigma$, with R as defined at the end of section 2;

(H3) for every $r \geq 0$ we have

$$\int_{C^1(\mathbb{T}^2)} \|\omega\|_{C^1}^r \, \lambda(d\omega) < \infty;$$

(H4) $\lambda(d\omega)$ satisfies (4.4), with M_1 replaced by M for all g as in (3.7) and all $\mathbf{x} \in \mathbb{T}^2$.

Because of Corollary 3.2 and the arguments used to prove it, we can replace (H4) by

(H4′) $\mathbb{E}(|\nabla\omega(t,\mathbf{x})|^2 \mid \mathcal{F}_{\omega(t,\mathbf{x})})=\frac{1}{2}(2\pi)^{-2}M \quad \forall\, \mathbf{x} \in \mathbb{T}^2$, where $|\nabla\omega(t,\mathbf{x})|^2$ and $\omega(t,\mathbf{x})$ are viewed as random variables on $(C^1(\mathbb{T}^2),\ \Sigma,\ \lambda)$.

The properties of the stationary measure ρ set out in eqns (2.11), (4.3) and (4.4) can now be compressed into the formula

$$\rho \in \mathcal{B}_{M_1}. \tag{4.5}$$

Consider now the time-dependent distribution $\mathcal{D}(\omega(t))$ induced by an arbitrary probability distribution of the initial velocity field $u(0) \in H$, $\mathbb{E}\|u(0)\|^2 < \infty$. We would like to obtain a convergence result about the behaviour of this measure as $t \to \infty$, analogous the result (2.9) for $\mathcal{D}(u(t))$. Because of Theorem 3.1 of [KS03] our new assumption $M_4 < \infty$ implies that (2.9) holds not only in the sense of $*$-weak convergence of measures in H, but also in the sense of $*$-weak convergence of measures in the Sobolev space $H^3(\mathbb{T}^2;\mathbb{R}^2)$, so that the following result holds:

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold and also $M_4 < \infty$. Then for any solution $\mathbf{u}(t,\mathbf{x})$ of (2.6), with arbitrary initial condition $u(0) \in H$, we have*

$$\mathcal{D}\left(\mathbf{curl}\, u(t)\right) \rightharpoonup \rho \in \mathcal{B}_{M_1} \quad \textit{as} \quad t \to \infty. \tag{4.6}$$

where $\rightharpoonup$ stands for the $*$-weak convergence of measures on $H^1(\mathbb{T}^2)$, and ρ is interpreted as a measure on $H^1(\mathbb{T}^2)$.

This result shows that the sets $\mathcal{B}_M$ form a one-parameter family of attractors in the space of measures on $H^1(\mathbb{T}^2)$ for the dynamics, defined by the stochastic Navier–Stokes equation (2.6) with any viscosity $\nu > 0$ and any random force η as in (2.1). Because of the condition (H4), each of the attractors has infinite codimension.

A slightly awkward feature of this last result is that ρ has to be interpreted as a measure in $H^1(\mathbb{T}^2)$ rather than in the smaller space $C^1(\mathbb{T}^2)$ in which $\omega(t)$ almost surely lies. The awkwardness can be eliminated by requiring more smoothness in the applied force η: if we assume that $M_5 < \infty$, then Theorem 3.1 in [KS03] tells us that (2.9) holds in the sense of convergence of measures in $H^4(\mathbb{T}^2;\mathbb{R}^2)$. In that case (4.6) holds in the sense of measures in $H^3(\mathbb{T}^2)$, and the inequality (4.2) implies that $H^3(\mathbb{T}^2)$ is a subset of $C^1(\mathbb{T}^2)$, so that ρ in eqn (4.6) can now be interpreted in the more natural way, as a measure on the space $C^1(\mathbb{T}^2)$ in which $\omega(t)$ is known to lie.

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