

# Vey theorem in infinite dimensions and its application to KdV

Sergei Kuksin and Galina Perelman

## Abstract

We consider an integrable infinite-dimensional Hamiltonian system in a Hilbert space  $H = \{u = (u_1^+, u_1^-, u_2^+, u_2^-, \dots)\}$  with integrals  $I_1, I_2, \dots$  which can be written as  $I_j = \frac{1}{2}|F_j|^2$ , where  $F_j : H \rightarrow \mathbb{R}^2$ ,  $F_j(0) = 0$  for  $j = 1, 2, \dots$ . We assume that the maps  $F_j$  define a germ of an analytic diffeomorphism  $F = (F_1, F_2, \dots) : H \rightarrow H$ , such that  $dF(0) = id$ ,  $(F - id)$  is a  $\kappa$ -smoothing map ( $\kappa \geq 0$ ) and some other mild restrictions on  $F$  hold. Under these assumptions we show that the maps  $F_j$  may be modified to maps  $F'_j$  such that  $F_j - F'_j = O(|u|^2)$  and each  $\frac{1}{2}|F'_j|^2$  still is an integral of motion. Moreover, these maps jointly define a germ of an analytic symplectomorphism  $F' : H \rightarrow H$ , the germ  $(F' - id)$  is  $\kappa$ -smoothing, and each  $I_j$  is an analytic function of the vector  $(\frac{1}{2}|F'_j|^2, j \geq 1)$ . Next we show that the theorem with  $\kappa = 1$  applies to the KdV equation. It implies that in the vicinity of the origin in a functional space KdV admits the Birkhoff normal form and the integrating transformation has the form ‘identity plus a 1-smoothing analytic map’.

## 0 Introduction

In his celebrated paper [Vey78] J. Vey proved a local version of the Liouville-Arnold theorem which we now state for the case of an elliptic singular point.<sup>1</sup> Consider the standard symplectic linear space  $(\mathbb{R}_x^{2n}, \omega_0)$ ,  $\omega_0 = \sum_{j=1}^n dx_j \wedge dx_{n+j}$ . Let  $H(x) = O(|x|^2)$  be a germ of an analytic function<sup>2</sup> and  $V_H$  be the

---

<sup>1</sup>Vey’s result applies as well to hyperbolic singular points and to singular points of mixed type.

<sup>2</sup>Here and everywhere below ‘a germ’ means a germ at zero of a function or a map, defined in the vicinity of the origin.

corresponding Hamiltonian vector field. It has a singularity at zero and we assume that in a suitable neighbourhood  $\mathcal{O}$  of the origin,  $H$  has  $n$  commuting analytic integrals  $H_1 = H, H_2, \dots, H_n$  such that  $H_j(x) = O(|x|^2)$  for each  $j$ , the quadratic forms  $d^2H_j(0)$ ,  $1 \leq j \leq n$ , are linearly independent and for all sufficiently small numbers  $\delta_1, \dots, \delta_n$  we have  $\{x : H_j(x) = \delta_j \forall j\} \Subset \mathcal{O}$ . Then in the vicinity of the origin exist symplectic analytic coordinates  $\{y_1, \dots, y_{2n}\}$  (i.e.  $\sum_{j=1}^n dy_j \wedge dy_{n+j} = \omega_0$ ) such that each hamiltonian  $H_r(x)$  may be written as  $H_r(x) = \hat{H}_r(I_1, \dots, I_n)$ ,  $I_j = \frac{1}{2}(y_j^2 + y_{n+j}^2)$ , where  $\hat{H}_1, \dots, \hat{H}_n$  are germs of analytic functions on  $\mathbb{R}^n$ .

Vey's proof relies on the Artin theorem on a system of analytic equations, so it applies only to analytic finite-dimensional Hamiltonian systems. The theorem was developed and generalised in [Eli84, Ito89, Eli90, Zun05]. In [Eli84, Eli90] Eliasson suggested a constructive proof of the theorem, which applies both to smooth and analytic hamiltonians and may be generalised to infinite-dimensional systems. In this work we use Eliasson's arguments to get an infinite-dimensional version of Vey's theorem, applicable to integrable Hamiltonian PDE. Namely, we consider the  $l_2$ -space  $h^0$ , formed by sequences  $u = (u_1^+, u_1^-, u_2^+, u_2^-, \dots)$ , provide it with the symplectic form  $\omega_0 = \sum_{j=1}^{\infty} du_j^+ \wedge du_j^-$ , and include  $h^0$  in a scale  $\{h^j, j \in \mathbb{R}\}$  of weighted  $l_2$ -spaces. Let us take any space  $h^m$ ,  $m \geq 0$ , and in a neighbourhood  $\mathcal{O}$  of the origin in  $h^m$  consider commuting analytic hamiltonians  $I_1, I_2, \dots$ . We assume that  $I_j = O(\|u\|_m^2) \geq 0 \forall j$  and that this system of functions is *regular* in the following sense: There are analytic maps  $F_j : \mathcal{O} \rightarrow \mathbb{R}^2$ ,  $j \geq 1$ , such that  $I_j = \frac{1}{2}|F_j|^2$  and

i) the map  $F = (F_1, F_2, \dots) : \mathcal{O} \rightarrow h^m$  is an analytic diffeomorphism on its image,

ii)  $dF(0) = \text{id}$  and the mapping  $F - \text{id}$  analytically maps  $\mathcal{O} \rightarrow h^{m+\kappa}$  for some  $\kappa \geq 0$  (i.e.,  $F - \text{id}$  is  $\kappa$ -smoothing). Moreover, for any  $u \in \mathcal{O}$  the linear operator  $dF(u)^* - \text{id}$  continuously maps  $h^m$  to  $h^{m+\kappa}$ .

We also make some mild assumptions concerning Cauchy majorants for the maps  $F - \text{id}$  and  $dF(u)^* - \text{id}$ , see in Section 1. The main result of this work is the following theorem:

**Theorem 0.1.** *Let the system of commuting analytic functions  $I_1, I_2, \dots$  on  $\mathcal{O} \subset h^m$  is regular. Then there are analytic maps  $F'_j : \mathcal{O}' \rightarrow \mathbb{R}^2$ , defined on a suitable neighbourhood  $0 \in \mathcal{O}' \subset \mathcal{O}$ , such that the map  $F' = (F'_1, F'_2, \dots) : \mathcal{O}' \rightarrow h^m$  satisfies properties i), ii), it is a symplectomorphism, the functions  $I'_j = \frac{1}{2}|F'_j|^2$  commute and their joint level-sets define the same foliation of  $\mathcal{O}'$*

as level-sets of the original functions  $I_j$ . In particular, each  $I_j$  is an analytic function of the variables  $I'_1, I'_2, \dots$ .

See Section 1 for a more detailed statement of the result and see Section 4 for its proof. In Section 3 we develop some infinite-dimensional techniques, needed for our arguments.

Theorem 0.1 applies to study an integrable Hamiltonian PDE in the vicinity of an equilibrium. In Section 2 we apply it to the KdV equation under zero-meanvalue periodic boundary conditions

$$\dot{u}(t, x) = \frac{1}{4}u_{xxx} + 6uu_x, \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad \int_0^{2\pi} u dx = 0, \quad (0.1)$$

and to the whole KdV hierarchy. The equations are regarded as Hamiltonian systems in a Sobolev space  $H_0^m$ ,  $m \geq 0$ , of zero-meanvalue functions on  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . The space is given the norm  $\|u\|_m = |(-\Delta)^{m/2}u|_{L_2}$  and is equipped with the symplectic form  $\nu$ , where  $\nu(u(\cdot), v(\cdot)) = -\int_{S^1} (\partial/\partial x)^{-1}u(x) \cdot v(x) dx$ . If  $m \geq 1$ , then (0.1) is a Hamiltonian system in  $H_0^m$  with the analytic hamiltonian

$$h_{KdV}(u) = \int \left( -\frac{1}{8}u_x^2 + u^3 \right) dx.$$

To apply Theorem 0.1 we first normalise the symplectic form  $\nu$  to the canonical form  $\omega_0$ . To do this we write any  $u(x) \in H_0^m$  as Fourier series,  $u(x) = \pi^{-1/2} \sum_{s=1}^{\infty} (u_s^+ \cos sx - u_s^- \sin sx)$ , and consider the map

$$T : u(x) \mapsto v = (v_1^\pm, v_2^\pm, \dots), \quad v_j^\pm = u_j^\pm j^{-1/2} \quad \forall j.$$

Then  $T : H_0^m \rightarrow h^{m+1/2}$  is an isomorphism for any  $m$ , and  $T^*\omega_0 = \nu$ .

The Lax operator for the KdV hierarchy is the Sturm-Liouville operator  $L_u = -\partial^2/\partial x^2 - u(x)$ . Let  $\gamma_1, \gamma_2, \dots$  be the lengths of its spectral gaps. Then  $I_j(u) = \gamma_j^2(u)$ ,  $j \geq 1$ , are commuting analytic functionals which are integrals of motion for all equations from the hierarchy. In [Kap91] T. Kappeler suggested a way to use the spectral theory of the operator  $L_u$  to construct germs of analytic maps  $F_j : h^{m+1/2} \rightarrow \mathbb{R}^2$ ,  $j \geq 1$ , such that  $\frac{1}{2}|F_j(Tu)|^2 = \frac{\pi}{2j}I_j(u)$ . In Sections 5 we show that the maps  $F_j$  meet assumptions i), ii) with  $\kappa = 1$  (see Theorem 2.1). So the system of integrals  $I_j$  is regular. Accordingly, Theorem 0.1 implies the following result:

**Theorem 0.2.** *For any  $m \geq 0$  there exists a germ of an analytic symplectomorphism  $\bar{\Psi} : (H_0^m, \nu) \rightarrow (h^{m+1/2}, \omega_0)$ ,  $d\bar{\Psi}(0) = T$ , such that*

- a) the germ  $\bar{\Psi} - T$  defines a germ of an analytic mapping  $H_0^m \rightarrow h^{m+3/2}$ ;*
- b) each  $I_j = \gamma_j^2$  is an analytic function of the vector  $\bar{I} = (\frac{1}{2}|\bar{\Psi}^j(u)|^2, j \geq 1)$ . Similar, a hamiltonian of any equation from the KdV hierarchy is an analytic function of  $\bar{I}$  (provided that  $m$  is so big that this hamiltonian is analytic on the space  $H_0^m$ );*
- c) the maps  $\bar{\Psi}$ , corresponding to different  $m$ , agree. That is, if  $\bar{\Psi}_{m_j}$  corresponds to  $m = m_j$ ,  $j = 1, 2$ , then  $\bar{\Psi}_{m_1} = \bar{\Psi}_{m_2}$  on  $h^{\max(m_1, m_2)}$ .*

Assertion b) of the theorem means that the map  $\bar{\Psi}$  puts KdV (and other equations from the KdV hierarchy) to the Birkhoff normal form.

In a number of publications, starting with [Kap91], T. Kappeler with collaborators established existence of a global analytic symplectomorphism

$$\Psi : (H_0^m, \nu) \rightarrow (h^{m+1/2}, \omega_0), \quad d\Psi(0) = T,$$

which satisfies assertion b) of Theorem 0.2, see in [KP03]. Our work shows that a local version of Kappeler's result follows from Vey's theorem. What is more important, it specifies the result by stating that a local transformation which integrates the KdV hierarchy may be chosen '1-smoother than its linear part'. This specification is crucial to study qualitative properties of perturbed KdV equations, e.g. see [KP09].

A global symplectomorphism  $\Psi$  as above integrates the KdV equation, i.e. puts it to the Birkhoff normal form. Similar, the linearised KdV equation  $\dot{u} = u_{xxx}$  may be integrated by the (weighted) Fourier transformation  $T$ . An integrating transformation  $\Psi$  is not unique.<sup>3</sup> For the linearised KdV we do not see this ambiguity since  $T$  is the only linear integrating symplectomorphism. In the KdV case the best transformation  $\Psi$  is the one which is the most close to the linear map  $T = d\Psi(0)$  in the sense that the map  $\Psi - T$  is the most smoothing. Motivated by Theorem 0.2 and some other arguments (see in [KP09]), we are certain that there exists a (global) integrating symplectomorphism  $\Psi$  such that  $\Psi - T$  is 1-smoother than  $T$ .<sup>4</sup>

<sup>3</sup>in difference with the mapping to the action variables  $u \mapsto I \circ \Psi(u)$ , which is unique.

<sup>4</sup>We are cautious not to claim that the symplectomorphism  $\Psi$ , constructed in [KP03], possesses this extra smoothness since it is normalised by the condition

$$\text{if } u(x) \equiv u(-x), \text{ then } \Psi(u) = v = (v_j^\pm, j \geq 1), \text{ where } v_j^- = 0 \forall j.$$

In Proposition 2.2 we show that if a germ of an integrating analytic transformation  $\Psi$  is such that  $\Psi - T$  is  $\kappa$ -smoothing, then  $\kappa \leq 3/2$ . We conjecture that the 1-smoothing is optimal.

**Acknowledgment.** We thank H. Eliasson for discussion of the Vey theorem.

## 1 The main theorem.

Consider a scale of Hilbert spaces  $\{h^m, m \in \mathbb{R}\}$ . A space  $h^m$  is formed by complex sequences  $u = (u_j \in \mathbb{C}, j \geq 1)$  and is regarded as a real Hilbert space with the Hilbert norm

$$\|u\|_m^2 = \sum_{j \geq 1} j^{2m} |u_j|^2. \quad (1.1)$$

We will denote by  $\langle \cdot, \cdot \rangle$  the scalar product in  $h^0$ :  $\langle u, v \rangle = \sum u_j \cdot v_j = \operatorname{Re} \sum u_j \bar{v}_j$ . For any linear operator  $A : h^m \rightarrow h^n$  we will denote by  $A^* : h^{-n} \rightarrow h^{-m}$  the operator, conjugated to  $A$  with respect to this scalar product.

Below we study germs or real-analytic maps <sup>5</sup>

$$F : \mathcal{O}_\delta(h^m) \rightarrow h^n, \quad F(0) = 0,$$

where  $\mathcal{O}_\delta(h^m) = \{u \in h^m \mid \|u\|_m < \delta\}$  and  $\delta > 0$  depends on  $F$ . Abusing language we will say that  $F$  is an analytic germ  $F : h^m \rightarrow h^n$ . Any analytic germ  $F = (F^1, F^2, \dots)$  can be written as an absolutely and uniformly convergent series

$$F^j(u) = \sum_{N=1}^{\infty} F_N^j(u), \quad F_N^j(u) = \sum_{|\alpha|+|\beta|=N} A_{\alpha\beta}^j u^\alpha \bar{u}^\beta, \quad (1.2)$$

where  $\alpha, \beta \in \mathbb{Z}_+^\infty$ ,  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . We will write that  $F(u) = O(u^l)$  if in (1.2)  $F_N^j(u) = 0$  for  $N < l$  and all  $j$ .

---

It is not obvious that an optimal global symplectomorphism satisfies this condition, and we do not know if the local symplectomorphism  $\bar{\Psi}$  from Theorem 0.2 meets it.

<sup>5</sup>In Section 5 we mostly work with complex-analytic maps, so there *analytic* stands for *complex-analytic*.

Clearly,

$$|F(u)| \leq \underline{F}(|u|), \quad \underline{F}^j(|u|) = \sum_{N=1}^{\infty} \sum_{|\alpha|+|\beta|=N} |A_{\alpha\beta}^j| |u|^{\alpha+\beta} \leq \infty.$$

Here  $|F(u)| = (|F^1(u)|, |F^2(u)|, \dots)$ ,  $|u| = (|u_1|, |u_2|, \dots)$  and  $|u|^{\alpha+\beta} = \prod |u_j|^{\alpha_j+\beta_j}$ . The inequality is understood component-wise.

**Definition 1.** An analytic germ  $F$  as above is called normally analytic (n.a.) if  $\underline{F}$  defines a germ of a real analytic map  $h_R^m \rightarrow h_R^n$ , where the space  $h_R^m$  is formed by real sequences  $(u_j)$ , given the norm (1.1). That is, each  $N$ -homogeneous map  $\underline{F}_N^j(v) = \sum_{|\alpha|+|\beta|=N} |A_{\alpha\beta}^j| v^{\alpha+\beta}$ , where  $v \in h_R^m$ , satisfies  $\|\underline{F}_N(v)\|_n \leq CR^N \|v\|_m^N$  for suitable  $C, R > 0$ .

Take any  $m \geq 0$  and  $\kappa \geq 0$ .

**Definition 2.** A n.a. germ  $F : h^m \rightarrow h^{m+\kappa}$  belongs to  $\mathfrak{A}_{m,\kappa}$  if  $F = O(u^2)$  and the adjoint map  $dF(u)^*v$  is such that

$$\underline{dF}(|u|)^*|v| = \Phi(|u|)|v|. \quad (1.3)$$

Here the linear map  $\Phi(|u|) = \Phi_F(|u|) \in \mathcal{L}(h_R^m, h_R^{m+\kappa})$  has non-negative matrix elements and defines an analytic germ  $|u| \mapsto \Phi(|u|)$ ,  $h_R^m \rightarrow \mathcal{L}(h_R^m, h_R^{m+\kappa})$ .

The notion of a n.a. germ formalizes the method of Cauchy majorants in a way, convenient for our purposes. We study the class of n.a. germs and its subclass  $\mathfrak{A}_{m,\kappa}$  in Section 3.

We will write elements of the spaces  $h^m$  as  $u = (u_k \in \mathbb{C}, k \geq 1)$ ,  $u_k = u_k^+ + iu_k^-$ ,  $u_k^\pm \in \mathbb{R}$ , and provide  $h^m$ ,  $m \geq 0$ , with a symplectic structure by means of the two-form  $\omega_0 = \sum du_k^+ \wedge du_k^-$ . This form may be written as  $\omega_0 = idu \wedge du$ . Here and below for any antisymmetric (in  $h^0$ ) operator  $J$  we denote by  $Jdu \wedge du$  the 2-form

$$(Jdu \wedge du)(\xi, \eta) = \langle J\xi, \eta \rangle. \quad (1.4)$$

The form  $\omega_0$  is exact,  $\omega_0 = d\alpha_0$ , where

$$\alpha_0 = \frac{1}{2} \sum u_k^+ du_k^- - \frac{1}{2} \sum u_k^- du_k^+ = \frac{1}{2}(iu)du.$$

For a map  $f : h^m \rightarrow h^{-m}$ ,  $f(u)du$  stands for the one-form

$$(f(u)du)(\xi) = \sum_{j=1}^{\infty} f_j(u) \cdot \xi_j = \sum_{j=1}^{\infty} \operatorname{Re} f_j(u) \bar{\xi}_j.$$

By  $\{H_1, H_2\}$  we will denote the Poisson brackets of functionals  $H_1$  and  $H_2$ , corresponding to  $\omega_0$ :  $\{H_1, H_2\}(u) = \langle i\nabla H_1(u), \nabla H_2(u) \rangle$ . Functionals  $H_1$  and  $H_2$  commute if  $\{H_1, H_2\} = 0$ .

**Theorem 1.1.** *Assume that for some  $m \geq 0$  there exists a real analytic germ  $\Psi : h^m \rightarrow h^m$  such that*

i)  $d\Psi(0) = id$  and  $(\Psi - id) \in \mathfrak{A}_{m,\kappa}$ ,  $\kappa \geq 0$ ;

ii) the functionals  $I^j(\Psi(u)) = \frac{1}{2} |\Psi^j(u)|^2$ ,  $j \geq 1$ , commute with each other.

Then there exists a germ  $\Psi^+ : h^m \rightarrow h^m$  which satisfies i), ii) and such that

a) foliation of the vicinity of the origin in  $h^m$  by the sets

$$\{|\Psi^j|^2 = \text{const}_j, \forall j\}; \tag{1.5}$$

is the same as by the sets  $\{|\Psi^{+j}|^2 = \text{const}_j, \forall j\}$ .

b) the germ  $\Psi^+$  is symplectic:  $\Psi^{+*}\omega_0 = \omega_0$ .

The theorem is proved in Section 4.

*Remarks.* 1) The sets, forming the foliation (1.5), are tori of dimension  $\#\{\text{const}_j > 0\}$ , which is  $\leq \infty$ .

2) By the item a) of the theorem each  $I^j(\Psi(u))$  is a function of the vector  $I^+ = \{I^{+j} = \frac{1}{2} |\Psi^{+j}|^2, j \geq 1\}$ . In fact,  $I^j$  is an analytic function of  $I^+$  with respect to the norm  $\|I^+\| = \sum |I^{+j}| j^{2m}$ . E.g., see the proof of Lemma 3.1 in [Kuk00].

3) The map  $\Psi^+$  is obtained from  $\Psi$  in a constructive way, independent from  $m$ .

4) The theorem above is an infinite-dimensional version of Theorem C in [Eli90] which is the second step in Eliasson's proof of the Vey theorem. At the first step he *proves* that any  $n$  commuting integrals  $H_1, \dots, H_n$  as in Introduction can be written in the form ii). In difference with his work we have to *assume* that the integrals are of the form ii), where the maps  $\Psi_1, \Psi_2, \dots$  have additional properties, specified in i). Fortunately, we can check i) and ii) for some important infinite-dimensional systems.

## 2 Application to the KdV equation

To apply Theorem 1.1 we need a way to construct germs of analytic maps  $\Psi : h^m \rightarrow h^m$  which satisfy i) and ii). Examples of such maps may be obtained from Lax-integrable Hamiltonian PDEs

$$\dot{u}(t) = i\nabla H(u), \quad u(t) \in h^m. \quad (2.1)$$

(We normalized original Hamiltonian PDEs and wrote them as the Hamiltonian systems (2.1) in the symplectic space as in Section 1). The Lax operator  $L_u$ , corresponding to equation (2.1), is such that its spectrum  $\sigma(L_u)$  is an integral of motion for (2.1). Spectral characteristics of  $L_u$  may be used to construct (real)-analytic germs  $\Psi^j : h^m \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$  such that the functions  $\frac{1}{2}|\Psi^j|^2$ ,  $j \geq 1$ , are functionally independent integrals of motion. For some integrable equations these germs jointly define a germ of an analytic diffeomorphism  $u \mapsto \Psi = (\Psi^1, \Psi^2, \dots)$ , satisfying i) and ii). Below we show that this is the case for the KdV equation. Our construction is general and directly applies to some other integrable equations (e.g. to the defocusing Schrödinger equation).

Consider the KdV equation (0.1). This is a Hamiltonian equation in any Sobolev space  $H_0^m$ ,  $m \geq 1$ , given symplectic structure by the form  $\nu$ , see Introduction. It is Lax-integrable with the Lax operator  $L_u = -\partial^2/\partial x^2 - u(x)$ . Let  $\gamma_1(u), \gamma_2(u), \dots$  be the sizes of spectral gaps of  $L_u$  (e.g., see in [Kuk00, KP03]). It is well known that  $\gamma_1^2(u), \gamma_2^2(u), \dots$  are commuting analytic integrals of motion for (0.1), as well as for other equations from the KdV hierarchy, see in [KP03].

In Sections 5, 6 we show that the spectral theory of  $L_u$  may be used to construct an analytic germ  $\Psi : h^{1/2} \rightarrow h^{1/2}$ ,  $\Psi = (\Psi^1, \Psi^2, \dots)$ ,  $\Psi^j \in \mathbb{R}^2$ , with the following properties:

**Theorem 2.1.** *For any  $m' \geq 1/2$ ,  $\Psi$  defines a real-analytic germ  $\Psi : h^{m'} \rightarrow h^{m'}$  such that*

- i)  $d\Psi(0) = \text{id}$  and  $(\Psi - \text{id}) \in \mathfrak{A}_{m',1}$ ;*
- ii) for any  $j \geq 1$  and  $v \in h^{m'}$  we have  $\frac{1}{2}|\Psi^j(v)|^2 = \frac{\pi}{2j}\gamma_j(u)^2$ , where  $u(x) = \frac{1}{\sqrt{\pi}} \text{Re} \sum_{j=1}^{\infty} \sqrt{j} v_j e^{ijx}$ .*

Combining Theorems 2.1 and 1.1 we get Theorem 0.2, stated in the Introduction. Indeed, assertions a) and b) follow from the two theorems and



Remark 2 to Theorem 1.1 since the hamiltonian of any  $n$ -th KdV is a function of the lengths of spectral gaps. Assertion c) follows from Remark 3.

Towards the optimality of Theorems 0.2 and 2.1 we have the following partial results.

**Proposition 2.2.** *Assume that there exists a real-analytic germ  $\Psi : H_0^m \rightarrow h^{m+1/2} \forall m \geq 0$ ,  $d\Psi(0) = T$ , such that:*

*a) for each  $m \geq 0$ ,  $\Psi - T$  defines a germ of analytic mapping  $H_0^m \rightarrow h^{m+1/2+\kappa}$  with some  $\kappa \geq 0$ ;*

*b) the hamiltonian  $h_{KdV}$  of the KdV equation is a function of the variables  $\frac{1}{2}|\Psi^j(u)|^2$ ,  $j \geq 1$ , only.*

*Then  $\kappa \leq 3/2$ .*

*Proof.* We may assume that  $\kappa \geq 1$ . Denote by  $G$  the germ  $G = \Psi^{-1} \circ T : H_0^m \rightarrow H_0^m$ . We have  $dG(0) = \text{id}$  and  $G - \text{id} : H_0^m \rightarrow H_0^{m+\kappa}$ . So  $G(u) = u + \sum_{N=2}^{\infty} G_N(u)$ , where

$$\|G_N(u)\|_{H^{m+\kappa}} \leq C^N \|u\|_{H^m}^N \quad \forall N \geq 2, \quad (2.2)$$

for each  $m \geq 0$ , with some  $C = C(m)$ . Consider the functional  $K = h_{KdV} \circ G$ . It defines a germ of analytic mapping  $H_0^1 \rightarrow \mathbb{R}$  and can be written as an absolutely and uniformly convergent series  $K(u) = \sum_{n=2}^{\infty} K_n(u)$ , where  $K_n(\cdot)$  is an  $n$ -homogeneous functional on  $H_0^1$ . Then

$$K_2(u) = -\frac{1}{8} \int u_x^2 dx, \quad K_3(u) = \int \left( -\frac{1}{4} u_x \partial_x G_2(u) + u^3 \right) dx.$$

It follows from assumption b) that  $K_{2l+1}$ ,  $l = 1, 2, \dots$ , vanish identically. In particular,  $K_3(u) \equiv 0$ . Together with (2.2) this leads to the relations

$$\left| \int u^3 dx \right| = \frac{1}{4} \left| \int u_x \partial_x G_2(u) dx \right| \leq C \|u\|_{H_0^{2-\kappa}} \|G_2(u)\|_{H_0^\kappa} \leq C \|u\|_{H_0^{2-\kappa}} |u|_{L_2}^2,$$

valid for each  $u \in H_0^1$ . If  $\kappa \geq 2$  we have an obvious contradiction. It remains to consider the case when  $1 \leq \kappa < 2$ . Now  $\|u\|_{H_0^{2-\kappa}} \leq \|u\|_{H_0^1}^{2-\kappa} |u|_{L_2}^{\kappa-1}$  and the inequality above implies that

$$\left| \int u^3 dx \right| \leq C \|u\|_{H_0^1}^{2-\kappa} |u|_{L_2}^{1+\kappa} \quad \forall u \in H_0^1. \quad (2.3)$$

For  $0 < \varepsilon \leq 1$  we define  $v_\varepsilon(x)$  as the continuous piece-wise linear  $2\pi$ -periodic function, equal  $\varepsilon^{-2} \max(\varepsilon - |x|, 0)$  for  $|x| \leq 4$ . Then  $u_\varepsilon := v_\varepsilon - (2\pi)^{-1} \in H_0^1$  and

$$\int u_\varepsilon^3 dx \sim \varepsilon^{-2}, \quad \int u_\varepsilon^2 dx \sim \varepsilon^{-1}, \quad \int \left( \frac{\partial}{\partial x} u_\varepsilon \right)^2 dx \sim \varepsilon^{-3}.$$

Substituting  $u_\varepsilon$  in (2.3) we get that  $\varepsilon^{-2} \leq \text{const } \varepsilon^{-\frac{3}{2}(2-\kappa)} \varepsilon^{-\frac{1}{2}(1+\kappa)}$  for each  $\varepsilon$ . So  $\kappa \leq \frac{3}{2}$ , as stated.  $\square$

If a germ  $\Psi : H_0^m \rightarrow h^{m+1/2}$  is defined for a single value of  $m$  we have a weaker result:

**Proposition 2.3.** *Let for some  $m' \geq 1$  there exists a germ of real-analytic symplectomorphism  $\Psi : (H_0^{m'}, \nu) \rightarrow (h^{m'+1/2}, \omega_0)$ ,  $d\Psi(0) = T$ , satisfying a) and b) in Proposition 2.2 with  $m = m'$ . Then  $\kappa \leq 2$ .*

*Proof.* Assume that  $\kappa > 2$ . Keeping the notations above we still have  $K_3 = 0$ . So

$$0 = \nabla K_3(u) = \frac{1}{4} \Delta G_2(u) + \frac{1}{4} dG_2(u)^* \Delta u + 3u^2.$$

The first term in the r.h.s. clearly belongs to  $H^{m'+\kappa-2}$ . The germ  $G$  is a symplectomorphism of  $(H_0^{m'}, \nu)$ . Therefore  $dG(u)^* J dG(u) \equiv J$ ,  $J = -(\partial/\partial x)^{-1}$ , and  $dG(u)^*$  maps  $H_0^{m'+1}$  to itself. Since  $dG(u)^*$  also maps to itself  $H_0^{-m'}$ , then by interpolation  $dG(u)^* : H_0^s \rightarrow H_0^s$  for each  $s$  in  $[-m', m'+1]$ . So the second term in the r.h.s. also belong to  $H^{m'+\kappa-2}$ . Since  $\kappa - 2 > 0$ , then the sum of the first two terms cannot cancel identically the third, belonging to  $H_0^{m'}$ . Contradiction.  $\square$

### 3 Properties of normally analytic germs

**Lemma 3.1.** *If  $F : h^{n_1} \rightarrow h^{n_2}$  and  $G : h^{n_2} \rightarrow h^{n_3}$  are n.a. germs, then the composition  $G \circ F : h^{n_1} \rightarrow h^{n_3}$  also is n.a.*

*Proof.* Denote  $F(v) = w$  and  $G(w) = u$ . Then

$$w_j = G^j(v) = \sum A(G)_{\alpha\beta}^j v^\alpha \bar{v}^\beta, \quad v_l = \sum A(F)_{\alpha\beta}^l u^\alpha \bar{u}^\beta.$$

Substituting series in series, collecting similar terms and replacing  $u_j$  and  $\bar{u}_j$  by  $|u_j|$  we get  $\underline{G \circ F}(|u|)$ .

Next consider  $\underline{G} \circ \underline{F}(|u|)$ . This series is obtained by the same procedure as  $\underline{G} \circ \underline{F}(|u|)$ , but instead of calculating the modulus of an algebraical sum of similar terms we take the sum of their moduli. As  $|a+b| \leq |a|+|b|$ , we get  $\underline{G} \circ \underline{F} \leq \underline{G} \circ \underline{F}$ . Since both series have non-negative coefficients and  $\underline{G} \circ \underline{F}$  defines an analytic germ  $h_R^{n_1} \rightarrow h_R^{n_3}$ , the assertion follows.  $\square$

**Lemma 3.2.** *If  $F : h^m \rightarrow h^m$  is a n.a. germ such that  $F_1 = dF(0) = id$ , then the germ  $G = F^{-1}$  exists and is n.a.*

*Proof.* Write  $F(u) = u + F_2(u) + F_3(u) + \dots$ . We are looking for  $G(v)$  in the form  $G(v) = v - G_2(v) - G_3(v) - \dots$ . Then

$$\begin{aligned} F(G(v)) &= v - G_2(v) - G_3(v) - \dots \\ &\quad + F_2(v - G_2(v) - \dots, v - G_2(v) - \dots) \\ &\quad + F_3(v - G_2(v) - \dots, v - G_2(v) - \dots, v - G_2(v) - \dots) + \dots \end{aligned}$$

Here and below we freely identify  $n$ -homogeneous maps with the corresponding  $n$ -linear symmetric forms. Since  $F(G(v)) = v$ , we have the recursive relations

$$\begin{aligned} G_2(v) &= F_2(v, v), \\ G_3(v) &= F_3(v, v, v) - 2F_2(v, G_2(v)), \\ G_4(v) &= F_4(v, v, v, v) - 3F_3(v, v, G_2(v)) + F_2(G_2(v), G_2(v)) - 2F_2(v, G_3(v)), \\ &\dots \end{aligned}$$

For the same reasons as in the proof of Lemma 3.1 we have

$$\begin{aligned} \underline{G}_2(|v|) &\leq \underline{F}_2(|v|, |v|) =: \check{G}_2(|v|), \\ \underline{G}_3(|v|) &\leq \underline{F}_3(|v|, |v|, |v|) + 2\underline{F}_2(|v|, \check{G}_2(|v|)) =: \check{G}_3(|v|), \\ &\dots \end{aligned}$$

These recursive formulas define a germ of an analytic map  $h_R^m \rightarrow h_R^m$ ,  $|v| \mapsto \check{G}(|v|) = |v| + \check{G}_2(|v|) + \dots$ . Since  $\underline{G} \leq \check{G}$ , then the assertion follows.  $\square$

For a n.a. germ  $F : h^m \rightarrow h^n$  consider its differential, which we regard as a germ

$$dF(u)v : h^m \times h^m \rightarrow h^n. \quad (3.1)$$

**Lemma 3.3.** *Germ (3.1) is n.a. and  $\underline{dF}(|u|)|v| \leq d\underline{F}(|u|)|v|$ .*

*Proof.* Let us write  $F$  as series (1.2). For any  $u, v$  we have

$$\begin{aligned} dF^j(u)(v) &= \sum_{\alpha, \beta} \frac{\partial}{\partial t} \Big|_{t=0} A_{\alpha\beta}^j(u + tv)^\alpha (\bar{u} + t\bar{v})^\beta \\ &= \sum_{\alpha, \beta} \sum_r A_{\alpha\beta}^j (\alpha_r v_r u^{\alpha-1_r} \bar{u}^\beta + \beta_r \bar{v}_r u^\alpha \bar{u}^{\beta-1_r}), \end{aligned}$$

where  $1_r = (0, \dots, 0, 1, 0, \dots)$  (1 is on the  $r$ -th place). Therefore

$$\underline{dF^j(|u|)}(|v|) \leq \sum_{\alpha, \beta} \sum_r |A_{\alpha\beta}^j| |u|^{\alpha+\beta-1_r} |v_r| (\alpha_r + \beta_r) = d\underline{F^j}(|u|)|v|.$$

□

**Lemma 3.4.** *i) The class  $\mathfrak{A}_{m, \kappa}$  is closed with respect to composition of germs.*

*ii) If  $F \in \mathfrak{A}_{m, \kappa}$ , then  $(id + F)^{-1} = id + G$ , where  $G \in \mathfrak{A}_{m, \kappa}$ .*

*Proof.* i) If  $F, G \in \mathfrak{A}_{m, \kappa}$ , then  $F \circ G : h^m \rightarrow h^{m+\kappa}$  is n. a. by Lemma 3.1. It remains to verify that it satisfies (1.3). We have  $d(F \circ G(u))^* = dG(u)^* dF(G(u))^*$ . Arguing as when proving Lemma 3.1 we get

$$\underline{d(F \circ G(|u|))^*|v|} \leq \Phi_G(|G(u)|) \Phi_F(|u|)|v| \leq \Phi_G(\underline{G}(|u|)) \Phi_F(|u|)|v|.$$

So  $F \circ G$  meets (1.3).

ii) Relations, obtained in the proof of Lemma 3.2, imply that  $G : h^m \rightarrow h^{m+\kappa}$  is n. a. We have  $E + dG(u)^* = (E + dF(G(u))^*)^{-1}$ . Therefore  $dG(u)^* = \sum (-1)^k (dF(G(u))^*)^k$  and  $\underline{dG(|u|)^*|v|} \leq (\sum \Phi_G(\underline{G}(|u|)^k) |v|)$ . So  $dG(u)^*$  satisfies (1.3) and  $G \in \mathfrak{A}_{m, \kappa}$ . □

Let  $t \in [0, 1]$  and  $V^t(u) : [0, 1] \times \mathcal{O}_\delta(h^m) \rightarrow h^{m+\kappa}$  be a continuous map, analytic in  $u \in h^m$  and such that  $V^t \in \mathfrak{A}_{m, \kappa} \forall t$ , uniformly in  $t$ . Consider the equation

$$\dot{u}(t) = V^t(u(t)), \quad u(0) = v, \tag{3.2}$$

and denote by  $\varphi^t$ ,  $0 \leq t \leq 1$ , its flow maps. That is,  $\varphi^t(v) = u(t)$ .

**Lemma 3.5.** *For each  $0 \leq t \leq 1$  we have  $\varphi^t - id \in \mathfrak{A}_{m, \kappa}$ .*

*Proof.* Denote a solution for (3.2) as  $u = u(t; v)$ , and decompose  $u(t; v)$  in series in  $v$ :  $u(t; v) = u_1(t; v) + u_2(t; v) + \dots$ , where  $u_k(t; v)$  is  $k$ -homogeneous in  $v$ . Then  $u_1(t, v) \equiv v$ . Writing  $V^t(u) = V_2^t(u) + V_3^t(u) + \dots$ , we have

$$\dot{u}_2(t) = V_2^t(u_1, u_1) = V_2^t(v, v), \quad u_2(0) = 0.$$

Therefore  $u_2(t) = \int_0^t V_2^s(v, v) ds$ . Similar for  $k \geq 2$  we have

$$u_k(t) = \sum_{r=2}^{k-1} \sum_{k_1 + \dots + k_r = k} \int_0^t V_r^s(u_{k_1}(s), \dots, u_{k_r}(s)) ds.$$

Arguing by induction we see that the sum  $\sum_{k=1}^{\infty} u_k(t, v)$  defines a n.a. germ. This is the germ of the map  $\varphi^t(v)$ .

For any vector  $\xi$ ,  $d\varphi^t(v)\xi = w(t)$  is a solution of the linearized equation

$$\dot{w}(t) = dV^t(u(t))w(t), \quad w(0) = \xi.$$

So  $d\varphi^t(v)\xi = U(t)\xi$ , where the linear operator  $U(t)$  may be calculated as follows

$$U(t_0) = \text{id} + \sum_{n=1}^{\infty} \int_0^{t_0} \int_0^{t_1} \dots \int_0^{t_{n-1}} dV^{t_1}(u(t_1)) \dots dV^{t_n}(u(t_n)) dt_n \dots dt_1.$$

This series converges if  $\|u(0) = v\|_m \ll 1$ . Taking the adjoint to the integral above we see that  $d\varphi^t(u) - \text{id}$  satisfies (1.3) and the corresponding operator  $\Phi^t(|v|)$  meets the estimate

$$\Phi^{t_0}(|v|)|\xi| \leq \sum_{n=1}^{\infty} \int_0^{t_0} \dots \int_0^{t_{n-1}} \Phi_{V^{t_n}}(|u(t_n)|) \dots \Phi_{V^{t_1}}(|u(t_1)|)|\xi| dt_n \dots dt_1.$$

Replacing  $|u(t_n)|$  by  $\varphi^{t_n}(|v|)$  we see that the operator  $\Phi^t$  defines an analytic germ  $h_R^m \rightarrow \mathcal{L}(h_R^m, h_R^{m+\kappa})$ . So  $\varphi^t - \text{id} \in \mathfrak{A}_{m, \kappa}$ .  $\square$

Let  $G_0 \in \mathfrak{A}_{m, \kappa}$ . Denote  $G(u) = u + G_0(u)$ . The arguments in Section 4 use the map  $B(u) = dG_0(u)^*(iG(u))$ .

**Lemma 3.6.**  $B \in \mathfrak{A}_{m, \kappa}$ .

*Proof.* We have  $\underline{B}(|u|) \leq \Phi_{G_0}(|u|)|iG(u)| \leq \Phi_{G_0}(|u|)\underline{G}(|u|)$ . Since the map in r.h.s. defines an analytic germ  $h_R^m \rightarrow h_R^{m+\kappa}$ , then  $B$  is n.a.

It remains to check that  $B$  meets (1.3). We have  $dB(u)\xi = M_1\xi + M_2\xi$ , where  $M_1 = dG_0(u)^*idG(u)$  and

$$M_2 = dR(u), \quad R(u) = dG_0(u)^*U, \quad U = iG(u).$$

Since  $M_1^*v = -dG(u)^*idG_0(u)v$ , then by Lemma 3.3

$$\underline{M_1(|u|)^*|v|} \leq (\Phi_{G_0}(|u|) + E) d\underline{G_0}(|u|)|v|.$$

So  $M_1^*v$  has the required form. Now consider  $M_2$ . Let  $u(t)$  be a smooth curve in  $h^m$  such that  $u(0) = u$  and  $\dot{u}(0) = \xi$ . Then

$$\begin{aligned} \langle M_2\xi, v \rangle &= \left. \frac{\partial}{\partial t} \right|_{t=0} \langle dG_0(u(t))^*U, v \rangle = \\ &= \left. \frac{\partial}{\partial t} \right|_{t=0} \langle U, dG_0(u(t))v \rangle = \langle U, d^2G_0(u)(\xi, v) \rangle. \end{aligned}$$

Hence,  $M_2^*v = M_2v = dR(u)v$ . Due to (1.3) the map  $R$  is n.a. and  $\underline{R}(|u|) \leq \Phi_{G_0}(|u|)|U|$ . Now Lemma 3.3 implies that

$$\underline{dR(|u|)|v|} \leq (d_{|u|}\Phi_{G_0}(|u|)|v|)|U| \leq (d_{|u|}\Phi_{G_0}(|u|)|v|)(\underline{G_0}(|u|) + |u|).$$

This component of  $\underline{dB(|u|)^*|v|}$  also has the required form. So  $B$  satisfies (1.3).  $\square$

## 4 Proof of the main theorem

In this section we prove Theorem 1.1, following the scheme, suggested in Section VI of [Eli90]. To overcome corresponding infinite-dimensional difficulties we check recursively that all involved germs  $\Psi$  of transformations of the phase-space  $h^m$  are of the form  $id + \Psi_0$ , where  $\Psi_0 \in \mathfrak{A}_{m,\kappa}$ .

By Lemma 3.4 the germ  $G = \Psi^{-1}$  is n.a. and  $G = id + G_0$ , where  $G_0 \in \mathfrak{A}_{m,\kappa}$ . Denote

$$\omega_1 = G^*\omega_0, \quad \alpha_1 = G^*\alpha_0, \quad \omega_\Delta = \omega_1 - \omega_0, \quad \alpha_\Delta = \alpha_1 - \alpha_0.$$

Then  $d\alpha_1 = \omega_1$  and  $d\alpha_\Delta = \omega_\Delta$ . For any  $v$  and  $\xi$  we have

$$\alpha_1(v)(\xi) = \alpha_0(u)(G_*(v)\xi) = \alpha_0(v)(\xi) + \frac{1}{2} \langle iG_0(v), \xi \rangle + \frac{1}{2} \langle dG_0(v)^*iu, \xi \rangle,$$

where  $u = G(v)$ . Hence  $\alpha_\Delta = W(v)dv$ , where

$$W(v) = \frac{1}{2}iG_0(v) + \frac{1}{2}dG_0(v)^*iu.$$

By Lemma 3.6,  $W \in \mathfrak{A}_{m,\kappa}$ . Also we have  $\omega_1 = \bar{J}_1(v)dv \wedge dv$  (see (1.4)), where

$$\bar{J}_1(v) = i + dG_0(v)^*idG_0(v) + dG_0(v)^*i + idG_0(v) =: i + \bar{Y}_0(v).$$

Our goal is to find a transformation  $\Theta : h^m \rightarrow h^m$  which satisfies i), commutes with the rotations  $u_j \rightarrow e^{i\tau}u_j$  ( $j \geq 1$ ,  $\tau \in \mathbb{R}$ ), and which “kills” the form  $\alpha_\Delta$ , thus reducing  $\alpha_1$  to  $\alpha_0$  and  $\omega_1$  to  $\omega_0$ . Then  $\Psi^+ = \Theta \circ \Psi$  would satisfy the required properties. We will construct such  $\Theta$  in two steps.

**Step 1.** At this step we will achieve that the average in angles of the form  $\omega_1$  equal to  $\omega_0$ .

For  $j \geq 1$  and  $\tau \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$  we define  $\Phi_j^\tau : h^m \rightarrow h^m$  as the linear transformation of vectors  $(u_1, u_2, \dots)$  which does not change components  $u_l$ ,  $l \neq j$ , and multiplies  $u_j$  by  $e^{i\tau}$ . Clearly,  $(\Phi_j^\tau)^* = \Phi_j^{-\tau}$ . Therefore for a 1-form  $\alpha = F(u)du$  we have

$$(\Phi_j^\tau)^*\alpha(u) = (\Phi_j^{-\tau}F(\Phi_j^\tau(u))) du.$$

For any function  $f(u)$  we define its averaging with respect to  $j$ -th angle as

$$M_j f(u) = \frac{1}{2\pi} \int_0^{2\pi} f(\Phi_j^t u) dt,$$

and define its averaging in all angles as

$$Mf(u) = (M_1 M_2 \dots) f(u) = \int_{\mathbb{T}^\infty} f(\Phi^\theta u) d\theta,$$

where  $d\theta$  is the Haar measure on  $\mathbb{T}^\infty$  and  $\Phi^\theta u = (\Phi_1^{\theta_1} \circ \Phi_2^{\theta_2} \circ \dots)u$ . For a form  $\alpha$  we define  $M_j\alpha$  and  $M\alpha$  similarly. That is

$$M_j\alpha(u) = \frac{1}{2\pi} \int_0^{2\pi} ((\Phi_j^t)^*\alpha)(u) dt,$$

and  $M\alpha = (M_1M_2\dots)\alpha$ . In particular,

$$M_j(F(u)du) = \left( \frac{1}{2\pi} \int_0^{2\pi} \Phi_j^{-\tau} F(\Phi_j^\tau u) d\tau \right) du.$$

Since

$$\Phi_j^{\tau*} \omega_1 = (\Phi_j^{-\tau} \bar{J}_1(\Phi_j^\tau v) \Phi_j^\tau) dv \wedge dv,$$

then

$$(M\omega_1)(v) = (M\bar{J}_1)(v)dv \wedge dv, \quad (M\bar{J}_1)(v) = \int_{\mathbb{T}^\infty} \Phi^{-\theta} \bar{J}_1(\Phi^\theta v) \Phi^\theta d\theta.$$

Let us define

$$(M\bar{J})^\tau(v) = (1 - \tau)i + \tau(M\bar{J}_1)(v).$$

The operator  $\bar{J}_1(v)$  is  $i + O(v)$ . But the averaging in  $\theta$  cancels linear in  $v$  terms, so

$$(M\bar{J})^\tau(v) = i + \tau\tilde{\Upsilon}(v), \quad \tilde{\Upsilon}(v) = O(v^2).$$

The operator  $\tilde{\Upsilon}(v) \in \mathcal{L}(h^m, h^{m+\kappa})$  is analytic in  $v \in h^m$  and is antisymmetric,  $\tilde{\Upsilon}(v)^* = -\tilde{\Upsilon}(v)$ . So the germ  $v \rightarrow \tilde{\Upsilon}(v)\xi$  belongs to  $\mathfrak{A}_{m,\kappa}$  for any  $\xi \in h^m$ . Cf. the proof of Lemma 3.5.

Next we set

$$\hat{J}^\tau(v) = -((M\bar{J})^\tau(v))^{-1} = -(i + \tau\tilde{\Upsilon}(v))^{-1}.$$

Writing  $(i + \tau\tilde{\Upsilon}(v))^{-1}$  as a Neumann series we see that  $\hat{J}^\tau(v) = i + \hat{\Upsilon}^\tau(v)$ , where the operator-valued map  $v \mapsto \hat{\Upsilon}^\tau(v)$  enjoys the same smoothness properties as  $\tilde{\Upsilon}(v)$ .

Now consider the average of the 1-form  $\alpha_\Delta = W(v)dv$ . We have

$$M\alpha_\Delta = (MW)(v)dv, \quad (MW)(v) = \int_{\mathbb{T}^\infty} \Phi^{-\theta} W(\Phi^\theta v) d\theta.$$

Since  $W \in \mathfrak{A}_{m,\kappa}$ , then also  $(MW) \in \mathfrak{A}_{m,\kappa}$ . Let us define the mappings

$$V^\tau(v) = \hat{J}^\tau(v)(MW)(v), \quad 0 \leq \tau \leq 1.$$



Due to the properties of  $\hat{J}^\tau(v)$  and  $(MW)(v)$ ,  $V^\tau(v) \in \mathfrak{A}_{m,\kappa}$  for each  $\tau$ . Consider the equation

$$\dot{v}(\tau) = V^\tau(v(\tau))$$

and denote by  $\varphi^\tau$ ,  $0 \leq \tau \leq 1$ , its flow maps,  $\varphi^\tau(v(0)) = v(\tau)$ . By Lemma 3.5,  $\varphi^\tau - id \in \mathfrak{A}_{m,\kappa}$ . The operator  $\hat{J}^\tau(v)$  commutes with the rotations  $\Phi_j^\theta$ :

$$\hat{J}^\tau(\Phi^{\theta_0}v)\Phi^{\theta_0}\xi = \Phi^{\theta_0}\hat{J}^\tau(v)\xi.$$

The map  $(MW)(v)$  also commutes with them. Accordingly, the maps  $V^\tau(v)$  commute with  $\Phi_j^\theta$ , as well as the flow maps  $\varphi^\tau$ .

Let us denote  $\hat{\omega}^\tau = (M\bar{J})^\tau(v)dv \wedge dv$ . So  $\hat{\omega}^1 = M\omega_1$  and  $\hat{\omega}^0 = \omega_0$ . We claim that

$$(\varphi^\tau)^*\hat{\omega}^\tau = \text{const.}$$

To prove this we first note that

$$\frac{d}{d\tau}\hat{\omega}^\tau = M\omega_1 - \omega_0 = M(\omega_1 - \omega_0) = Md\alpha_\Delta = dM\alpha_\Delta.$$

Using the Cartan formula (e.g., see Lemma 1.2 in [Kuk00]) we have

$$\frac{d}{d\tau}((\varphi^\tau)^*\hat{\omega}^\tau) = (\varphi^\tau)^*\left(\frac{\partial\hat{\omega}^\tau}{\partial\tau} + d(V^\tau]\hat{\omega}^\tau)\right) = (\varphi^\tau)^*d(M\alpha_\Delta + V^\tau]\hat{\omega}^\tau).$$

The 1-form in the r.h.s. equals

$$(MW)dv + ((M\bar{J})^\tau V^\tau)dv = (MW)dv - (MW)dv = 0,$$

and the assertion follows.

Since the maps  $\varphi^\tau$  commute with the rotations  $\Phi_j^\theta$ , we have

$$\omega_0 = (\varphi^0)^*\hat{\omega}^0 = (\varphi^1)^*M\omega_1 = M(\varphi^1)^*\omega_1.$$

Denote  $\bar{\Psi} = (\varphi^1)^{-1} \circ \Psi$ . Then  $\bar{\Psi}$  satisfies assumptions i), ii) and in addition  $M((\bar{\Psi}^*)^{-1}\omega_0) = \omega_0$ . Since  $\varphi^1$  commutes with the rotations, then  $\bar{\Psi}$  satisfies assertion a) of Theorem 1.1.

We re-denote back  $\bar{\Psi} = \Psi$ . Then

iii)  $M\omega_1 = \omega_0$  for  $\omega_1 = (\bar{\Psi}^*)^{-1}\omega_0$ .

**Step 2.** Now we prove the theorem, assuming that  $\Psi$  meets i) – iii). Due to iii) we have  $dM\alpha_\Delta = d(M\alpha_1 - \alpha_0) = M\omega_1 - \omega_0 = 0$ . Therefore  $M\alpha_\Delta = dg$  for

a suitable function  $g$ . Since  $dg = Mdg = dMg$ , we may assume that  $g = Mg$ . Accordingly,  $\frac{\partial}{\partial \tau} g(\Phi_j^\tau(v)) = 0 \quad \forall j$ . Rotations  $\Phi_j^\tau$ ,  $\tau \in \mathbb{R}$ , correspond to the vector fields  $\chi_j(v) = (0, \dots, iv_j, 0, \dots)$ . So  $(dg, \chi_j) = 0$  and for each  $j$  we have

$$M(\alpha_\Delta, \chi_j) = (M\alpha_\Delta, \chi_j) = (dg, \chi_j) = 0. \quad (4.1)$$

Denote  $h_j(v) = (\alpha_\Delta, \chi_j)$  and consider the system of differential equations for a germ of a functional  $f : h^m \rightarrow \mathbb{R}$ :

$$(df, \chi_j) \equiv (iv_j \cdot \nabla_{v_j})f(v) = h_j(v), \quad j \geq 1. \quad (4.2)$$

First we will check that the vector of the r.h.s.'  $(h_1, h_2, \dots)$  satisfies certain compatibility conditions. Since  $d\alpha_0(\chi_i, \chi_j) = \omega_0(\chi_i, \chi_j) = 0$ , then

$$0 = d\alpha_0(\chi_i, \chi_j) = \chi_i(\alpha_0, \chi_j) - \chi_j(\alpha_0, \chi_i) - (\alpha_0, [\chi_i, \chi_j]),$$

where  $[\cdot, \cdot]$  is the commutator of vector-fields. Hence,

$$\chi_i(\alpha_0, \chi_j) = \chi_j(\alpha_0, \chi_i). \quad (4.3)$$

**Lemma 4.1.** *For any  $i$  and  $j$  we have  $\chi_i(\alpha_1, \chi_j) = \chi_j(\alpha_1, \chi_i)$ .*

*Proof.* Recall that  $\omega_1 = \bar{J}_1(v)dv \wedge dv$ , where  $\bar{J}_1(v) = i + \bar{\Upsilon}_0(v)$  and  $\bar{\Upsilon}_0(v) = O(v)$ . The operator  $J_1(v) = -\bar{J}_1(v)^{-1}$  exists for small  $v \in h^m$ , is antisymmetric and can be written as  $J_1(v) = i + \Upsilon_0(v)$ , where  $\Upsilon_0(v)$  belongs to  $\mathcal{L}(h^m, h^{m+\kappa})$ . By interpolation,  $\Upsilon_0(v) \in \mathcal{L}(h^0, h^\kappa)$ .

To prove the lemma it suffices to show that

$$\omega_1(\chi_i, \chi_j) = 0 \quad \forall i, j \quad (4.4)$$

since then the assertion would follow by the arguments, used to establish (4.3). Moreover, by continuity it suffices to verify the relation at a point  $v = (v_1, v_2, \dots)$  such that  $v_j \neq 0$  for all  $j$ .

Due to ii),  $\{I^j(v), I^k(v)\}_{\omega_1} = 0$  for any  $j$  and  $k$ . That is

$$0 = \langle J_1(v)\nabla I^j(v), \nabla I^k(v) \rangle = \langle J_1(v)v_j 1_j, v_k 1_k \rangle \quad \forall j, k. \quad (4.5)$$

Consider the space  $\Sigma_v = \text{span}\{v_j 1_j, j \geq 1\}$  (as before  $1_r = (0, \dots, 1, \dots)$ , where 1 is on the  $r$ -th place). Its orthogonal complement in  $h^0$  is  $i\Sigma_v = \text{span}\{iv_j 1_j, j \geq 1\}$ . Relations (4.5) imply that  $\langle J_1(v)\xi, \eta \rangle = 0$  for any  $\xi, \eta \in \Sigma_v$ . Hence,

$$J_1 : \Sigma_v \rightarrow i\Sigma_v.$$

Since  $J_1 - i = O(v)$ , then for small  $v$  this linear operator is an isomorphism. As  $\chi_i, \chi_j \in i\Sigma_v$ , then there exist  $\xi_i, \xi_j \in \Sigma_v$  such that  $J_1\xi_i = \chi_i, J_1\xi_j = \chi_j$ . So  $\omega_1(\chi_i, \chi_j) = \langle \bar{J}_1 J_1 \xi_i, J_1 \xi_j \rangle = -\langle \xi_i, J_1 \xi_j \rangle = 0$  and the lemma is proved.  $\square$

By (4.3) and the lemma above, relation (4.3) also holds with  $\alpha_0$  replaced by  $\alpha_\Delta$ . That is,

$$\chi_j(h_k) = \chi_k(h_j) \quad \forall j, k. \quad (4.6)$$

Also note that by (4.1)

$$Mh_j = 0 \quad \forall j. \quad (4.7)$$

For any function  $g(v)$  and for  $j = 1, 2, \dots$  denote

$$L_j g(v) = \frac{1}{2\pi} \int_0^{2\pi} t g(\Phi_j^t(v)) dt.$$

Due to (4.6), (4.7) the system of equations (4.2) is solvable and its solution is given by an explicit formula due to J. Moser (see in [Eli90]):

**Lemma 4.2.** *Consider the germ  $f$  of a function in  $h^m$ :*

$$f(v) = \sum_{l=1}^{\infty} f_l(v), \quad f_l = M_1 \dots M_{l-1} L_l h_l.$$

*If the series converges in  $C^0(h^m)$ , as well as the series for  $\chi_j(f)$ ,  $j \geq 1$ , then  $f$  is a solution of (4.2).*

*Proof.* For  $v = (v_1, v_2, \dots) \in h^m$  and  $j = 1, 2, \dots$  let us denote by  $\varphi_j$  the argument of  $v_j \in \mathbb{R}^2$ . Then  $\chi_j = \frac{\partial}{\partial \varphi_j}$ . Clearly,  $\frac{\partial}{\partial \varphi_j} M_j h_j = 0$ . By (4.6), for  $k \neq j$  we have

$$\frac{\partial}{\partial \varphi_k} M_j h_j = M_j \frac{\partial}{\partial \varphi_k} h_j = M_j \frac{\partial}{\partial \varphi_j} h_k = 0.$$

So  $M_j h_j$  is angle-independent and  $M_j h_j = Mh_j = 0$  by (4.7)

For any  $C^1$ -function  $g$  we have  $\frac{\partial}{\partial \varphi_j} L_j g = L_j \frac{\partial}{\partial \varphi_j} g = g - M_j g$ . Therefore

$$\frac{\partial}{\partial \varphi_j} f_k = \begin{cases} 0, & j < k, \\ M_1 \dots M_{k-1} h_k, & j = k, \\ M_1 \dots M_{k-1} h_j - M_1 \dots M_k h_j, & j > k, \end{cases}$$

(for  $k = 1$  we define  $M_1 \dots M_{k-1} h_j = h_j$ ). So  $\frac{\partial}{\partial \varphi_j} \sum f_k = h_j$ .  $\square$

Since  $\alpha_\Delta = W(v)dv$ , then

$$h_j = (\alpha_\Delta, \chi_j) = iv_j \cdot W_j(v).$$

The estimates on  $W(v)$  easily imply that the series for  $f$  and  $\chi_k f$ ,  $k \geq 1$ , converge. So  $f$  is a solution of (4.2). Let us consider its differential  $df = \nabla_v f(v)dv$ . Here  $\nabla_v f(v) = (\xi_1, \xi_2, \dots)$ , where  $\xi_j = \frac{\partial f}{\partial v_j^+} + i \frac{\partial f}{\partial v_j^-}$  with  $v_j = v_j^+ + iv_j^-$ .

**Lemma 4.3.** *The germ  $v \mapsto Y(v) = \nabla_v f(v)$ ,  $h^m \rightarrow h^{m+\kappa}$ , is n.a. and  $Y(v) = O(v)$ .*

*Proof.* Noting that  $iv_j \cdot W_j = \Phi_j^{\pi/2} v_j \cdot W_j$ , we have  $\nabla_{v_i} f = \sum_j \nabla_{v_i} f_j$ , where

$$\begin{aligned} \nabla_{v_i} f_j &= \int_{\mathbb{T}^j} \theta_j \nabla_{v_i} \left( W_j(\Pi^{\theta_j} v) \cdot (\Phi_j^{\theta_j + \frac{\pi}{2}} v_j) \right) d\theta^j \\ &= \int_{\mathbb{T}^j} \theta_j (\Phi_i^{-\theta_i} \nabla_{v_i}) W_j(\Pi^{\theta_j} v) \cdot (\Phi_j^{\theta_j + \frac{\pi}{2}} v_j) d\theta^j + \delta_{ij} \int_{\mathbb{T}^j} \theta_j (\Phi_j^{-\theta_j - \frac{\pi}{2}} W_j(\Pi^{\theta_j} v)) =: \\ &=: Y_{ij} + \delta_{ij} Z_j. \end{aligned}$$

Here

$$\theta^j = (\theta_1, \dots, \theta_j) \in \mathbb{T}^j, \quad \Pi^{\theta_j} = \Phi_1^{\theta_1} \circ \dots \circ \Phi_j^{\theta_j}, \quad d\theta^j = \frac{d\theta_1}{2\pi} \dots \frac{d\theta_j}{2\pi}.$$

Denote by  $\mathcal{Z}_j^\theta(v)$  the integrand for  $Z_j$ . The germ  $v \mapsto \mathcal{Z}^\theta(v)$ ,  $h^m \rightarrow h^{m+\kappa}$ , is analytic and  $\mathcal{Z}^\theta(|v|) = (\text{diag } \theta_j) \underline{W}(|v|)$ . Hence, the germ  $\mathcal{Z}^\theta$  is n.a. for each  $\theta$ . So the germ  $v \mapsto Z(v)$  also is.

Denote by  $\mathcal{Y}_{ij}^\theta(v)$  the integrand in  $Y_{ij}$ . We have

$$\mathcal{Y}_{ij}^\theta(v) = \theta_j \Phi_i^{-\theta_i} (dW(\Pi^{\theta_j}(v)))_{ij} \Phi_j^{\theta_j + \frac{\pi}{2}} v_j.$$

Using (1.3) we see that  $|\sum_j \mathcal{Y}_{ij}^\theta(|v|)| \leq 2\pi \sum_j (\Phi_W)_{ij}(|v|)|v_j|$ , uniformly in  $\theta$ .

That is, for any  $\theta$  the germ  $h^m \ni v \mapsto \sum_j \mathcal{Y}_{ij}^\theta(v) \in h^{m+\kappa}$  is n.a. So the map  $h^m \ni u \mapsto Y(u) = (Y^1, Y^2, \dots) \in h^{m+\kappa}$  is n.a. It is obvious that  $Y(u) = O(u)$ .  $\square$

For  $0 \leq \tau \leq 1$  we set  $\bar{J}^\tau(v) = (1 - \tau)i + \tau\bar{J}_1(v)$  and  $J^\tau(v) = -(\bar{J}^\tau(v))^{-1}$ . These are well defined operators in  $\mathcal{L}(h^m, h^m)$  antisymmetric with respect to the  $h^0$ -scalar product. Clearly,  $J^\tau(v) = i + \Upsilon^\tau(v)$ , where  $\Upsilon^\tau(v)$  belongs to  $\mathcal{L}(h^m, h^{m+\kappa})$ . Denote by  $\omega^\tau$  the form  $(1 - \tau)\omega_0 + \tau\omega_1 = \bar{J}^\tau(v)dv \wedge dv$ .

We define  $V^\tau(v) = J^\tau(v)(W(v) - Y(v))$ , consider the equation

$$\dot{v}(\tau) = V^\tau(v(\tau)), \quad 0 \leq \tau \leq 1, \quad (4.8)$$

and denote by  $\varphi^\tau$ ,  $0 \leq \tau \leq 1$ , its flow-maps. Since  $V^\tau(v) = O(v)$  and  $V^\tau : h^m \mapsto h^{m+\kappa}$  is n.a. (cf. Lemma 3.5 and its proof), then  $(\varphi^\tau - id) = O(v)$  and  $(\varphi^\tau - id) : h^m \mapsto h^{m+\kappa}$  is n.a. Also

$$\frac{d}{d\tau}(\varphi^\tau)^*\omega^\tau = (\varphi^\tau)^*d(\alpha_\Delta + V^\tau]\omega^\tau) = (\varphi^\tau)^*d(Y(v)dv) = 0$$

since  $Y(v)dv = df$ . So  $(\varphi^\tau)^*\omega^\tau = \text{const}$  and  $(\varphi^0)^*\omega^0 = \omega_0 = (\varphi^1)^*\omega^1$ . That is, the n.a. germ

$$\Psi^+ = (\varphi^1)^{-1} \circ \Psi : h^m \rightarrow h^m$$

is such that  $d\Psi^+(0) = id$  and  $\Psi^{+*}\omega_0 = \omega_0$ . The germ  $(\varphi^1)^{-1} - id : h^m \mapsto h^{m+\kappa}$  is n.a., so  $\Psi_0^+ = \Psi^+ - id : h^m \mapsto h^{m+\kappa}$  is n.a. as well.

Now we show that  $\Psi_0^+ \in \mathfrak{A}_{m,\kappa}$ . Since  $\Psi^+$  is symplectic, then  $d\Psi^+(u)^*i(d\Psi^+(u)) = i$ . Hence,

$$d\Psi_0^+(u)^* = id\Psi_0^+(u)(1 + d\Psi_0^+(u))^{-1}i,$$

and Lemma 3.3 implies that  $\frac{d\Psi_0^+(|u|)^*|v|}{|u|}$  has the required form (1.3).

It remains to check for  $\Psi^+$  properties ii) and a). Let  $v$  be a vector such that  $v_j \neq 0$  for all  $j$ . Since  $\alpha_\Delta = W(v)dv$ , then for each  $j$  we have

$$\begin{aligned} \omega^\tau(V^\tau, iv_j 1_j) &= \langle \bar{J}^\tau(v)V^\tau(v), iv_j 1_j \rangle = \\ &= \langle W(v) - Y(v), iv_j 1_j \rangle = (\alpha_\Delta, \chi_j) - (df, \chi_j) = 0. \end{aligned} \quad (4.9)$$

By (4.4) and a similar relation for the form  $\omega_0$ ,

$$\omega^\tau(\xi_1, \xi_2) = 0 \quad \forall \xi_1, \xi_2 \in i\Sigma_v. \quad (4.10)$$

Here as before  $i\Sigma_v = \text{span}\{iv_j 1_j\}$ . Denote

$$(i\Sigma_v)^\perp = \{\xi \in h^0 \mid \omega^\tau(\xi, \eta) = 0 \ \forall \eta \in i\Sigma_v\}.$$

By (4.10),  $i\Sigma_v \subset (i\Sigma_v)^\perp$ . We claim that

$$i\Sigma_v = (i\Sigma_v)^\perp \quad (4.11)$$

(i.e.,  $i\Sigma_v$  is a Lagrangian subspace for the symplectic form  $\omega^\tau$ ). Indeed, if this is not the case, then we can find a vector  $\xi \in \Sigma_v$ ,  $\|\xi\|_0 = 1$ , such that  $\xi \in (i\Sigma_v)^\perp$ . In particular,  $\omega^\tau(\xi, i\xi) = 0$ . But  $\omega^\tau(0)(\xi, i\xi) = \omega_0(\xi, i\xi) = 1$ . So for small  $v$  we have  $\omega^\tau(\xi, i\xi) > 0$ . Contradiction.

By (4.9), (4.11),  $V^\tau(v) \in (i\Sigma_v)^\perp = i\Sigma_v$ . So solutions  $v(\tau)$  of (4.8) satisfy

$$\frac{1}{2} \frac{d}{d\tau} |v_j(\tau)|^2 = \langle V^\tau(v), v_j 1_j \rangle = 0,$$

and  $I_j(\varphi^\tau(v)) \equiv I_j(v)$  for each  $j$ . By continuity this relation holds for all vectors  $v$  (without assuming that  $v_j \neq 0 \forall j$ ). Hence,  $I_j \circ \Psi \equiv I_j \circ \Psi$  for each  $j$ . This proves ii) and a) for the germ  $\Psi^+$ .

## 5 Proof of Theorem 2.1

The construction of  $\Psi$  that we present below follows the ideas of [Kap91] (also see [Kuk00], pp. 42-44). It relies on the spectral theory of the corresponding Lax operator  $L_u = -\partial_x^2 - u$ .

It will be convenient for us to allow for complex-valued potentials  $u$ :

$$u \in L_2^0(S^1, \mathbb{C}) = \{u \in L_2(S^1, \mathbb{C}), \int_0^{2\pi} u dx = 0\}.$$

We write  $u(x)$  as Fourier series  $u(x) = \frac{1}{2\sqrt{\pi}} \sum_{j \in \mathbb{Z}_0} e^{ijx} w_j$ , where  $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$ , and denote

$$w = (w_j \in \mathbb{C}, j \in \mathbb{Z}_0) = \mathcal{F}(u), \quad u = \mathcal{F}^{-1}(w).$$

Clearly,  $\mathcal{F} : L_2^0(S^1, \mathbb{C}) \rightarrow h^0 = h^0(\mathbb{Z}_0)$  is an isomorphism. Here and below we use the notations

$$h^m = h^m(\mathbb{Z}_0) = \{w : \|w\|_m^2 = \sum_{j \in \mathbb{Z}_0} |j|^{2m} |w_j|^2 < \infty\}.$$

A sequence  $w$  called *real* if  $\mathcal{F}^{-1}(w)$  is a real-valued function. That is, if  $w_j = \overline{w_{-j}}$  for each  $j$ .

We view  $L_u$  as an operator on  $L_2(\mathbb{R}/4\pi\mathbb{Z})$  with the domain  $\mathcal{D}(L_u) = H^2(\mathbb{R}/4\pi\mathbb{Z})$ . The spectrum of  $L_u$  is discrete and for  $u$  real is of the form

$$\sigma(L_u) = \{\lambda_k(u), k \geq 0\}, \quad \lambda_0 < \lambda_1 \leq \lambda_2 < \dots,$$

where  $\lambda_k(u) \rightarrow \infty$  as  $k \rightarrow \infty$ . For  $u$  small  $\sigma(L_u)$  is  $\|u\|_{L_2}$ -close to the spectrum of  $L_0 = -\partial_x^2$ , that is, to the set  $\sigma(L_0) = \{j^2/4, j \geq 0\}$ . More precisely, one has

$$|\lambda_{2j-1} - j^2/4|, |\lambda_{2j} - j^2/4| \leq C\|u\|_{L_2}, \quad j \geq 1,$$

provided  $\|u\|_{L_2} \leq \delta$ , where  $\delta > 0$  sufficiently small.

For  $j \geq 1$  we will denote by  $E_j(u)$  the invariant two-dimensional subspace of  $L_u$ , corresponding to the eigenvalues  $\lambda_{2j-1}(u)$ ,  $\lambda_{2j}(u)$ , and by  $P_j(u)$  the spectral projection on  $E_j(u)$ :

$$P_j(u) = -\frac{1}{2\pi i} \oint_{\gamma_j} (L_u - \lambda)^{-1} d\lambda, \quad \text{Im } P_j(u) = E_j(u), \quad j \geq 1.$$

Here  $\gamma_j$  is a contour in the complex plane which isolates  $\lambda_{2j-1}$  and  $\lambda_{2j}$  from other eigenvalues of  $L_u$ . For the computations that will be performed below we fix the contours as  $\gamma_j = \{\lambda \in \mathbb{C}, |\lambda - j^2/4| = \delta_0 j\}$ ,  $\delta_0 > 0$  small.

Clearly,  $u \mapsto P_j(u)$ ,  $j \geq 1$ , are analytic<sup>6</sup> maps from  $V_\delta = \{u \in L_2^0(S^1, \mathbb{C}), \|u\|_{L_2} \leq \delta\}$  to  $\mathcal{L}(L_2, H^2)$ ,  $L_2 = L_2(\mathbb{R}/4\pi\mathbb{Z})$ ,  $H^2 = H^2(\mathbb{R}/4\pi\mathbb{Z})$ , provided  $\delta$  is sufficiently small. Furthermore, it is not difficult to check that

$$\|P_j(u) - P_{j0}\|_{L_2 \rightarrow L_2} \leq Cj^{-1}\|u\|_{L_2}, \quad \|P_j(u) - P_{j0}\|_{L_2 \rightarrow H^2} \leq Cj\|u\|_{L_2}, \quad (5.1)$$

for  $j \geq 1$  and  $u \in V_\delta$ . Here  $P_{j0}$  is the spectral projection of  $L_0$ , corresponding to a double eigenvalue  $j^2/4$ :

$$\text{Im } P_{j0} = E_{j0}, \quad \text{Ker } P_{j0} = E_{j0}^\perp, \quad E_{j0} = \text{span}\{\cos jx/2, \sin jx/2\}, \quad j \geq 1.$$

Following [Kat66], see also [Kap91], we introduce the transformation operators  $U_j(u)$ ,  $j \geq 1$ :

$$U_j(u) = (I - (P_j(u) - P_{j0})^2)^{-1/2} P_j(u).$$

---

<sup>6</sup>In this section “analytic” means complex analytic.

It follows from (5.1) that the maps  $u \mapsto U_j(u)$  are well defined and analytic on  $V_\delta$ . It turns out (see [Kat66]) that the image of  $U_j(u)$  is  $E_j(u)$  and for  $u$  real one has

$$\|U_j(u)f\|_2 = \|f\|_2, \quad f \in E_{j0}, \quad (5.2)$$

$$\overline{U_j(u)f} = U_j(u)\bar{f}. \quad (5.3)$$

For  $j \in \mathbb{Z}_0$  let us set

$$f_j(u) = U_{|j|}(u)f_{j0} \in E_{|j|}(u), \quad f_{j0} = \frac{1}{\sqrt{2\pi}}e^{-ijx/2}, \quad (5.4)$$

$$z_j(u) = -\sqrt{\pi} \left( (L_u - j^2/4)f_j(u), \overline{f_j(u)} \right). \quad (5.5)$$

Here  $(\cdot, \cdot)$  stands for the standard scalar product in  $L_2([0, 4\pi], \mathbb{C})$ :

$$(f, g) = \int f \bar{g} dx.$$

**Lemma 5.1.** *For  $u$  real, one has*

$$i) \quad \overline{z_j(u)} = z_{-j}(u),$$

$$ii) \quad |z_j(u)|^2 = \pi(\lambda_{2j}(u) - \lambda_{2j-1}(u))^2, \quad j \geq 1.$$

*Proof.* Assertion i) is obvious (see (5.3)). To check ii), consider

$$e_j = \operatorname{Re} f_j = U_j(u)e_{j0}, \quad e_{j0} = \frac{1}{\sqrt{2\pi}} \cos jx/2,$$

$$e_{-j} = \operatorname{Im} f_j = U_j(u)e_{-j0}, \quad e_{-j0} = -\frac{1}{\sqrt{2\pi}} \sin jx/2, \quad j \geq 1.$$

It follows from (5.2), (5.3) that the vectors  $e_j, e_{-j}$  form a real orthonormal basis of  $E_j(u)$ . Let  $M_j(u)$  be the matrix of the self-adjoint operator  $-\sqrt{\pi}(L_u - j^2/4)|_{E_j(u)}$  in this basis:

$$M_j(u) = \begin{pmatrix} a_j^1 & b_j \\ b_j & a_j^2 \end{pmatrix},$$

$$a_j^1 = -\sqrt{\pi} \left( (L_u - \frac{j^2}{4})e_j(u), e_j(u) \right), \quad a_j^2 = -\sqrt{\pi} \left( (L_u - \frac{j^2}{4})e_{-j}(u), e_{-j}(u) \right),$$



$$b_j = -\sqrt{\pi} \left( (L_u - j^2/4)e_j(u), e_{-j}(u) \right) = \frac{1}{2} \operatorname{Im} z_j(u).$$

Consider the deviators  $M_j^D$ ,  $j \geq 1$ , (for a  $2 \times 2$  matrix  $M$  its deviator is the traceless matrix  $M - (\frac{1}{2} \operatorname{tr} M)I$ ):

$$M_j^D(u) = \begin{pmatrix} a_j & b_j \\ b_j & -a_j \end{pmatrix}, \quad a_j = \frac{1}{2}(a_j^1 - a_j^2) = \frac{1}{2} \operatorname{Re} z_j(u).$$

By construction, one has  $|z_j(u)|^2 = 4(a_j^2 + b_j^2) = \pi(\lambda_{2j}(u) - \lambda_{2j-1}(u))^2$ .  $\square$

Functions  $z_j(u)$ ,  $j \in \mathbb{Z}_0$ , are analytic functions of  $u \in V_\delta$ , vanishing at zero. They can be represented by absolutely and uniformly converging Taylor series that we will write in terms of the Fourier coefficients  $w = \mathcal{F}(u)$ .

$$z_j(u) = \sum_{n=1}^{\infty} Z_n^j(w), \quad (5.6)$$

where  $Z_n^j(w)$  are bounded  $n$ - homogeneous functionals on  $h^0(\mathbb{Z}_0)$ :

$$Z_n^j(w) = \sum_{i=(i_1, i_2, \dots, i_n) \in \mathbb{Z}_0^n} \mathcal{K}_n^j(i) w_{i_1} w_{i_2} \dots w_{i_n}, \quad (5.7)$$

$\mathcal{K}_n^j(\cdot)$  being a symmetric function on  $\mathbb{Z}_0^n$ .

Notice that

$$P_j(u) = P_{j0} + (L_0 - j^2/4)^{-1}(I - P_{j0})uP_{j0} + P_{j0}u(L_0 - j^2/4)^{-1}(I - P_{j0}) + O(u^2),$$

$$U_j(u) = P_j(u) + O(u^2).$$

As a consequence,

$$f_j(u) = f_{j0} + (L_0 - j^2/4)^{-1}(I - P_{|j|0})u f_{j0} + O(u^2), \quad j \in \mathbb{Z}_0. \quad (5.8)$$

Substituting (5.8) into (5.5), one gets

$$z_j(u) = \sqrt{\pi}(u f_{j0}, \bar{f}_{j0}) + \sqrt{\pi}(u(L_0 - j^2/4)^{-1}(I - P_{|j|0})u f_{j0}, \bar{f}_{j0}) + O(u^3),$$

which gives that  $Z_1(w) = w$  and

$$Z_2^j(w) = \frac{1}{2\sqrt{\pi}} \sum_{k \in \mathbb{Z}_0, k \neq j} \frac{w_k w_{j-k}}{k(k-j)} \quad (5.9)$$

In a similar way, one can show that

$$Z_3^j(w) = \frac{1}{4\pi} \sum_{\substack{i_1, i_2 \in \mathbb{Z}_0 \\ i_1 \neq j, i_1 + i_2 \neq 0, j}} \frac{w_{i_1} w_{i_2} w_{j-i_1-i_2}}{i_1(i_1-j)(i_1+i_2)(i_1+i_2-j)} - \frac{w_j}{4\pi} \sum_{l \in \mathbb{Z}_0, l \neq j} \frac{w_l w_{-l}}{l^2(l-j)^2}. \quad (5.10)$$

The structure of higher order terms  $Z_n^j(w)$  is described by the following lemma which is the key technical step of our analysis.

**Lemma 5.2.** *One has*

(i)  $\text{supp } \mathcal{K}_n^j(\cdot) \subset \Omega_j^{(n)}$ , where  $\Omega_j^{(n)}$  is the simplex

$$\Omega_j^{(n)} = \{i = (i_1, \dots, i_n) \in \mathbb{Z}_0^n, \sum_{l=1}^n i_l = j\};$$

(ii) for  $n \geq 2$ ,  $\|\mathcal{K}_n^j\|_{l^2(\mathbb{Z}_0^n)} \leq R^n |j|^{1-n}$ ;

(iii) for  $n \geq 3$ ,  $\|\mathcal{B}_n^j\|_{l^2(\mathbb{Z}_0^n)} \leq R^n |j|^{-2}$ , where  $\mathcal{B}_n^j(i_1, \dots, i_n) = \mathcal{K}_n^{i_1}(j, i_2, i_3, \dots, i_n)$ .

Here  $R$  is a positive constant, independent of  $j$  and  $n$ .

Postponing the proof of this lemma till the end of the section, we proceed with the construction of the map  $\Psi$ . Introduce the map  $F$  that associates to  $w \in h^0(\mathbb{Z}_0)$  the sequence  $F(w) = (z_j(u), j \in \mathbb{Z}_0)$ ,  $u = \mathcal{F}^{-1}(w)$ . Since  $Z_1(w) = w$ , we write  $F$  as the sum

$$F = \text{id} + F_2, \quad F_2 = Z_2 + F_3,$$

where  $Z_2(w) = (Z_2^j(w), j \in \mathbb{Z}_0)$ . Notice that, by the construction,

$$dF(0) = \text{id}, \quad F_2(w) = O(w^2), \quad F_3(w) = O(w^3).$$

As a direct consequence of Lemma 5.2 (i), (ii) one gets

**Lemma 5.3.** *For  $j = 2, 3$  the map  $F_j : \mathcal{O}_\delta(h^m(\mathbb{Z}_0)) \rightarrow h^{m+j-1}(\mathbb{Z}_0)$  is analytic and normally analytic.*

*Proof.* It is sufficient to show that for any  $n \geq 2$ ,

$$\|Z_n(v)\|_{m+n-1} \leq R^n \|v\|_m^n, \quad v \in h_R^m(\mathbb{Z}_0).$$

Here  $\underline{Z}_n(v) = (\underline{Z}_n^j(v), j \in \mathbb{Z}_0)$  and

$$\underline{Z}_n^j(v) = \sum_{i=(i_1, i_2, \dots, i_n) \in \mathbb{Z}_0^n} |\mathcal{K}_n^j(i)| v_{i_1} v_{i_2} \dots v_{i_n}.$$

From Lemma 5.2 (i), (ii) and the Cauchy-Schwartz inequality we get

$$\begin{aligned} |\underline{Z}_n^j(v)|^2 &\leq \|\mathcal{K}_n^j\|_{l^2(\mathbb{Z}_0^n)}^2 \left( \sup_{i \in \Omega_j^{(n)}} \Lambda^{-2m}(i) \right) \sum_{i=(i_1, \dots, i_n) \in \Omega_j^{(n)}} \Lambda^{2m}(i) v_{i_1}^2 v_{i_2}^2 \dots v_{i_n}^2 \\ &\leq \frac{R^{2n}}{|j|^{2n-2+2m}} \sum_{i=(i_1, \dots, i_n) \in \Omega_j^{(n)}} \Lambda^{2m}(i) v_{i_1}^2 v_{i_2}^2 \dots v_{i_n}^2. \end{aligned}$$

Here  $\Lambda(i) = |i_1| |i_2| \dots |i_n|$  and at the last step we have used the inequality

$$\sup_{i \in \Omega_j^{(n)}} \Lambda^{-1}(i) \leq R^n |j|^{-1}. \quad (5.11)$$

Summing up with respect to  $j$  one gets that  $\|\underline{Z}_n(v)\|_{m+n-1} \leq R^n \|v\|_m^n$  for  $n \geq 2$ .  $\square$

Consider  $dF_3(w)^t$  – the real transposed of  $dF_3(w)$  with respect to the (standard complex) scalar product in  $h^0(\mathbb{Z}_0)$ :

$$(dF_3(w)h, \bar{g})_{h^0(\mathbb{Z}_0)} = (h, \overline{dF_3(w)^t g})_{h^0(\mathbb{Z}_0)}.$$

Notice that for analytic maps the inequality of Lemma 3.3 becomes an equality:

$$\underline{dF}(|w|)|h| = d\underline{F}(|w|)|h|, \quad \underline{dF}(|w|)^t|h| = d\underline{F}(|w|)^t|h|.$$

**Lemma 5.4.** (i) For  $w \in \mathcal{O}_\delta(h^m)$ ,  $dF_3(w)^t \in \mathcal{L}(h^m, h^{m+2})$  and the map  $w \mapsto dF_3(w)^t$ ,  $\mathcal{O}_\delta(h^m) \rightarrow \mathcal{L}(h^m, h^{m+2})$ , is analytic;

(ii) similarly,  $d\underline{F}_3(v)^t$ ,  $v \in \mathcal{O}_\delta(h_R^m)$ , belongs to  $\mathcal{L}(h_R^m, h_R^{m+2})$  and the map  $v \mapsto d\underline{F}_3(v)^t$ ,  $\mathcal{O}_\delta(h_R^m) \rightarrow \mathcal{L}(h_R^m, h_R^{m+2})$ , is real analytic.

*Proof.* We have

$$(dZ_n(w)^t h)_j = nB_n^j(h, w, \dots, w), \quad (d\underline{Z}_n(v)^t g)_j = n\underline{B}_n^j(g, v, \dots, v)$$

Here  $B_n^j$  and  $\underline{B}_n^j$  are the  $n$ -linear forms with the kernels  $\mathcal{B}_n^j$  and  $|\mathcal{B}_n^j|$  respectively:

$$B_n^j(w^1, \dots, w^n) = \sum_{i=(i_1, \dots, i_n) \in \mathbb{Z}_0^n} \mathcal{B}_n^j(i) w_{i_1}^1 \dots w_{i_n}^n, \quad w^k \in h^m \quad \forall k,$$

$$\underline{B}_n^j(v^1, \dots, v^n) = \sum_{i=(i_1, \dots, i_n) \in \mathbb{Z}_0^n} |\mathcal{B}_n^j(i)| v_{i_1}^1 \dots v_{i_n}^n, \quad v^k \in h_R^m \quad \forall k.$$

To prove the lemma it is sufficient to show that for  $n \geq 3$  the poly-linear map

$$\underline{B}_n = (\underline{B}_n^j, j \in \mathbb{Z}_0) : h_R^m \times \dots \times h_R^m \rightarrow h_R^{m+2}$$

is bounded and verifies:

$$\|\underline{B}_n(v^1, \dots, v^n)\|_{m+2} \leq R^n \prod_{k=1}^n \|v^k\|_m, \quad v^k \in h_R^m, \quad k = 1, \dots, n. \quad (5.12)$$

It follows from Lemma 5.2 (i) that

$$\begin{aligned} |\underline{B}_n^j(v^1, \dots, v^n)|^2 &\leq \|\mathcal{B}_n^j\|_{l^2(\mathbb{Z}_0^n)}^2 \left( \sup_{i \in \Omega_{-j}^{(n)}} \Lambda^{-2m}(i) \right) \\ &\times \sum_{\substack{i=(i_1, \dots, i_n) \in \mathbb{Z}_0^n \\ (-i_1, i_2, i_3, \dots, i_n) \in \Omega_{-j}^{(n)}}} \Lambda^{2m}(i) (v_{i_1}^1 v_{i_2}^2 \dots v_{i_n}^n)^2. \end{aligned}$$

Combining this inequality with (ii) of Lemma 5.2 and using once more (5.11) we get (5.12) for any  $n \geq 3$ .  $\square$

We next denote by  $\tilde{D}$  the operator of multiplication by the diagonal matrix  $\text{diag}(|j|^{1/2}, j \in \mathbb{Z}_0)$ . It defines isomorphisms  $\tilde{D} : h^r \rightarrow h^{r-1/2}$ ,  $r \in \mathbb{R}$ . Let us set  $m' = m + \frac{1}{2} \geq \frac{1}{2}$ . For any analytic germ  $H : h^m \rightarrow h^{m+a}$  we will denote by  $H^{\tilde{D}}$  the germ  $H^{\tilde{D}} = \tilde{D}^{-1} \circ H \circ \tilde{D} : h^{m'} \rightarrow h^{m'+a}$ . In particular,  $F^{\tilde{D}} = \tilde{D}^{-1} \circ F \circ \tilde{D}$ . Due to Lemma 5.3 one has

- (a)  $F^{\tilde{D}} : \mathcal{O}_\delta(h^{m'}) \rightarrow h^{m'}$  is n.a.;
- (b)  $F^{\tilde{D}} - id = F_2^{\tilde{D}} : \mathcal{O}_\delta(h^{m'}) \rightarrow h^{m+1}$  is n.a. and  $F_2^{\tilde{D}}(v) = O(v^2)$ .

Notice that the operations  $F \mapsto F^{\tilde{D}}$  and  $F \mapsto \underline{F}$  commute.

Consider  $(dF_2^{\tilde{D}})^t$ ,  $(d\underline{F}_2^{\tilde{D}})^t$ . We have

**Lemma 5.5.** (i) For  $v \in \mathcal{O}_\delta(h^{m'})$ ,  $dF_2^{\tilde{D}}(v)^t \in \mathcal{L}(h^{m'}, h^{m'+1})$  and the map  $v \mapsto dF_2^{\tilde{D}}(v)^t$ ,  $\mathcal{O}_\delta(h^{m'}) \rightarrow \mathcal{L}(h^{m'}, h^{m'+1})$ , is analytic;

(ii) similarly,  $dF_2^{\tilde{D}}(v)^t$ ,  $v \in \mathcal{O}_\delta(h_R^{m'})$ , belongs to  $\mathcal{L}(h_R^{m'}, h_R^{m'+1})$  and the map  $v \mapsto dF_2^{\tilde{D}}(v)^t$ ,  $\mathcal{O}_\delta(h_R^{m'}) \rightarrow \mathcal{L}(h_R^{m'}, h_R^{m'+1})$ , is real analytic.

*Proof.* As in the proof of Lemma 5.4, it is sufficient to prove the statement, corresponding to  $dF_2^{\tilde{D}}(v)^t$ . We write

$$F_2^{\tilde{D}} = Z_2^{\tilde{D}} + F_3^{\tilde{D}}, \quad \underline{F}_2^{\tilde{D}} = \underline{Z}_2^{\tilde{D}} + \underline{F}_3^{\tilde{D}}.$$

Since  $dF_3^{\tilde{D}}(v)^t = \tilde{D}dF_3^t(\tilde{D}v)\tilde{D}^{-1}$ , Lemma 5.3 implies that the map  $v \mapsto dF_3^{\tilde{D}}(v)^t$ ,  $\mathcal{O}_\delta(h_R^{m'}) \rightarrow \mathcal{L}(h_R^{m'-1}, h_R^{m'+1})$ , is real analytic. Therefore it is also real analytic as a map from  $\mathcal{O}_\delta(h_R^{m'})$  to  $\mathcal{L}(h_R^{m'}, h_R^{m'+1})$ .

Next consider  $\underline{Z}_2^{\tilde{D}} = \tilde{D}^{-1}\underline{Z}_2\tilde{D}$ . Note that  $\underline{Z}_2(w) = \text{const } \tilde{D}^{-2}w * \tilde{D}^{-2}w$ . Accordingly,  $d\underline{Z}_2(w)(f) = \text{const } \tilde{D}^{-2}w * \tilde{D}^{-2}f$ , and

$$d\underline{Z}_2^{\tilde{D}}(v)^t(f) = \text{const } \tilde{D}^{-1}(\tilde{D}^{-1}v * \tilde{D}^{-1}f).$$

Since  $\tilde{D}^{-1} : h^{m'} \rightarrow h^{m'+1/2}$ , where  $m' + 1/2 \geq 1$ , and since the convolution defines a continuous bilinear map  $h^r \times h^r \rightarrow h^r$  if  $r > 1/2$ , then we have

$$\|d\underline{Z}_2^{\tilde{D}}(v)^t(f)\|_{m'+1} \leq C_m \|v\|_{m'} \|f\|_{m'}.$$

So the map  $h^{m'} \ni v \mapsto d\underline{Z}_2^{\tilde{D}}(v)^t \in \mathcal{L}(h_R^{m'}, h_R^{m'+1})$  is bounded, which concludes the proof of Lemma 5.5.  $\square$

We are now in position to finish the proof of Theorem 2.1. Define  $\Psi : \mathcal{O}_\delta(h^{m'}(\mathbb{N})) \rightarrow h^{m'}(\mathbb{N})$  by restricting  $F^{\tilde{D}}$  on the subspace of real sequences  $\{v = (v_j, j \in \mathbb{Z}_0) \in h^{m'}(\mathbb{Z}_0), v_j = \overline{v_{-j}}\}$  (that is on the real potentials  $u$ ) and further projecting it on the positive indices:

$$\Psi = \pi F^{\tilde{D}} \circ \pi^{-1}.$$

Here  $\pi : h^{m'}(\mathbb{Z}_0) \rightarrow h^{m'}(\mathbb{N})$  is the projection:

$$\pi : v = (v_j, j \in \mathbb{Z}_0) \mapsto \pi v = (v_j, j \geq 1),$$

and  $\pi^{-1} : h^{m'}(\mathbb{N}) \rightarrow h^{m'}(\mathbb{Z}_0)$  is the right inverse map:

$$\begin{aligned}\pi^{-1} : v = (v_j, j \geq 1) &\mapsto v' = (v'_j, j \in \mathbb{Z}_0), \\ v'_j &= v_j \text{ for } j \geq 1, \quad v'_j = \overline{v_{-j}} \text{ for } j \leq -1.\end{aligned}$$

Clearly,  $\Psi : \mathcal{O}_\delta(h^{m'}) \rightarrow h^{m'}$ ,  $h^{m'} = h^{m'}(\mathbb{N})$ , is a real analytic map of the form  $\Psi = id + \Psi_2$ , where  $\Psi_2 = \pi F_2^{\tilde{D}} \circ \pi^{-1} = O(v^2)$ , and  $\Psi_2$  is real analytic as a map from  $\mathcal{O}_\delta(h^{m'})$  to  $h^{m'+1}$ . Furthermore, since  $\underline{\Psi} = \pi \underline{F_2^{\tilde{D}}} \circ \pi^{-1}$ ,  $\underline{\Psi}_2 = \pi \underline{F_2^{\tilde{D}}} \circ \pi^{-1}$ , then  $\Psi$  and  $\Psi_2$  are n.a. In addition, one has

**Lemma 5.6.**  $\Psi_2 \in \mathfrak{A}_{m',1}$ .

*Proof.* We already know that  $\Psi_2 : \mathcal{O}_\delta(h^{m'}) \rightarrow h^{m'+1}$  is n.a. and  $\Psi_2(v) = O(v^2)$ . Moreover, since  $d\Psi_2(v)^*g = \pi dF_2^{\tilde{D}}(\pi^{-1}v)^*\pi^{-1}g$ , where  $dF_2^{\tilde{D}}(\pi^{-1}v)^* = \underline{dF_2^{\tilde{D}}(\pi^{-1}v)^t}$ , the map  $v \mapsto d\Psi_2(v)^*$ ,  $\mathcal{O}_\delta(h^{m'}) \rightarrow \mathcal{L}(h^{m'}, h^{m'+1})$  is real analytic by Lemma 5.5. Finally, the representation  $\underline{d\Psi_2(|v|)^*|f|} = \Phi(|v|)|f|$ , where  $\Phi : \mathcal{O}_\delta(h_R^{m'}) \rightarrow \mathcal{L}(h_R^{m'}, h_R^{m'+1})$  is real analytic and  $(\Phi(v))_{jk} \geq 0$  for  $v = |v|$ , required by the definition of  $\mathfrak{A}_{m',1}$ , follows from the identity

$$\underline{d\Psi_2(|v|)^*|f|} = \pi \underline{dF_2^{\tilde{D}}(\pi^{-1}|v|)^t} \pi^{-1}|f|$$

and item (ii) of Lemma 5.5. □

To finish the proof of Theorem 2.1 it remains to note that assertion ii) follows from Lemma 5.1 ii). Indeed, if  $v = \tilde{D}^{-1}\mathcal{F}(u)$ , then

$$|\Psi^j(v)|^2 = j^{-1}|z_j(u)|^2 = \pi|j|^{-1}(\lambda_{2j}(u) - \lambda_{2j-1}(u))^2 = \pi|j|^{-1}\gamma_j^2(u).$$

This concludes the proof of Theorem 2.1.

It remains to prove Lemma 5.2. We will obtain it as a consequence of

**Lemma 5.7.**  $Z_n^j(w)$  (see (5.6)) can be represented as

$$Z_n^j(w) = \sum_{i=(i_1, i_2, \dots, i_n) \in \mathbb{Z}_0^n} \tilde{\mathcal{K}}_n^j(i) w_{i_1} w_{i_2} \dots w_{i_n}, \quad (5.13)$$

where for  $n \geq 2$   $\tilde{\mathcal{K}}_n^j$  satisfies

$$\begin{aligned}|\tilde{\mathcal{K}}_n^j(i)| &\leq R^n \mathcal{A}_n^j(i), \quad \mathcal{A}_n^j(i) = \delta_{j, \sum_{i=1}^n i_i} a_n^j(i_1, \dots, i_{n-1}), \\ a_n^j(i_1, \dots, i_{n-1}) &= \prod_{l=1}^{n-1} \left( \left\langle \sum_{k=1}^l i_k \right\rangle \left\langle \sum_{k=1}^l i_k - j \right\rangle \right)^{-1}\end{aligned} \quad (5.14)$$

Here  $\langle j \rangle = (1 + j^2)^{1/2}$ .

*Remark.* The difference between representation (5.7) and (5.13) is that  $\tilde{\mathcal{K}}_n^j(i)$  are not required to be symmetric functions.

Clearly, item (i) of Lemma 5.2 follows trivially from Lemma 5.7. It is also not difficult to check that (5.14) leads to the following estimates

$$\|\tilde{\mathcal{K}}_n^j\|_{l^2(\mathbb{Z}_0^n)} \leq \frac{R^n}{|j|^{n-1}}, \quad n \geq 2, \quad \sup_{1 \leq l \leq n} \|\mathcal{B}_{n,l}^j\|_{l^2(\mathbb{Z}_0^n)} \leq \frac{R^n}{|j|^2}, \quad n \geq 3,$$

$$\mathcal{B}_{n,l}^j(i_1, \dots, i_n) = \tilde{\mathcal{K}}_n^{i_l}(i_1, \dots, i_{l-1}, j, i_{l+1}, \dots, i_n),$$

which in turn imply (ii), (iii) of Lemma 5.2. So it remains to establish Lemma 5.7.

*Proof of Lemma 5.7.* First notice that for  $n = 2, 3$  the representation (5.13), (5.14) follows directly from the explicit formulas (5.9), (5.10). The general case can be treated as follows. Consider  $z_j(u)$  (see (5.5)) and write it as the sum  $z_j(u) = z_{j,1}(u) + z_{j,2}(u)$ , where

$$z_{j,1}(u) = -\sqrt{\pi}((L_0 - j^2/4)f_j(u), \overline{f_j(u)}), \quad (5.15)$$

$$z_{j,2}(u) = \sqrt{\pi}(uf_j(u), \overline{f_j(u)}), \quad (5.16)$$

The functions  $f_j(u)$ ,  $j \in \mathbb{Z}_0$ , were defined in (5.4). Now it is convenient for us to write them as

$$f_j(u) = (I - P_{|j|1}^2)^{-1/2}(I + P_{|j|1})f_{j0}, \quad P_{|j|1}(u) = P_{|j|}(u) - P_{|j|0}. \quad (5.17)$$

The Taylor expansions for  $z_{j,k}(u)$ ,  $k = 1, 2$ , have the form

$$z_{j,1}(u) = \sum_{n \geq 2} Z_n^{j,1}(w), \quad z_{j,2}(u) = \sum_{n \geq 1} Z_n^{j,2}(w), \quad w = \mathcal{F}(u),$$

where  $Z_n^{j,k}$  are bounded  $n$ -homogeneous functionals on  $h^0(\mathbb{Z}_0)$ , and  $Z_n^j = Z_n^{j,1} + Z_n^{j,2}$ .

We next compute explicitly  $Z_n^{j,k}(w)$ . From (5.17) we have

$$f_j(u) = \sum_{q \geq 0} c_q P_{|j|1}^q(u) f_{j0}, \quad (5.18)$$

where  $c_q = \varphi^{(q)}(0)/q!$  for  $\varphi(x) = (1 - x^2)^{-1/2}(1 + x)$ . Note that  $c_q \geq 0$ .

Further expanding  $P_{|j|1}(u)$  in the power series of  $u$ :

$$P_{|j|1}(u) = -\frac{1}{2\pi i} \sum_{k \geq 1} \oint_{\gamma_{|j|}} T^k(u, \lambda) (L_0 - \lambda)^{-1} d\lambda, \quad T(u, \lambda) = (L_0 - \lambda)^{-1}u,$$

and substituting this expansion into (5.18) we get

$$f_j(u) = f_{j0} + \sum_{n \geq 1} \sum_{1 \leq q \leq n} c_q \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q, \\ |\alpha| = n}} f_{j,q}^\alpha(u),$$

$$f_{j,q}^\alpha(u) =$$

$$\left(\frac{i}{2\pi}\right)^q \oint_{\gamma_{|j|}} \dots \oint_{\gamma_{|j|}} T^{\alpha_1}(u, \lambda_1)(L_0 - \lambda_1)^{-1} \dots T^{\alpha_q}(u, \lambda_q)(L_0 - \lambda_q)^{-1} f_{j0} d\lambda_1 \dots d\lambda_q.$$

Substituting this series into (5.15) and replacing  $u$  by  $\frac{1}{2\sqrt{\pi}} \sum_{j \in \mathbb{Z}_0} w_j e^{ijx}$ , we arrive at the following representation for  $Z_n^{j,1}$ :

$$Z_n^{j,1}(w) = \sum_{i=(i_1, i_2, \dots, i_n) \in \mathbb{Z}_0^n} \tilde{\mathcal{K}}_n^{j,1}(i) w_{i_1} w_{i_2} \dots w_{i_n},$$

$$\tilde{\mathcal{K}}_n^{j,1}(i) = \sum_{\substack{p=(p_1, p_2) \in \mathbb{N}^2, \\ |p| \leq n}} c_{p_1} c_{p_2} \sum_{\substack{\beta=(\beta_1, \dots, \beta_{|p|}) \in \mathbb{N}^{|p|}, \\ |\beta| = n}} \mathcal{S}_{p,\beta}^{j,1}(i), \quad n \geq 2.$$

Here

$$\mathcal{S}_{p,\beta}^{j,1}(i) =$$

$$= - \left(\frac{1}{2\sqrt{\pi}}\right)^{n-1} \left(\frac{i}{2\pi}\right)^{|p|} \oint_{\gamma_{|j|}} \dots \oint_{\gamma_{|j|}} s_{p,\beta}^{j,1}(i, \lambda) d\lambda_1 \dots d\lambda_{|p|}, \quad \lambda = (\lambda_1, \dots, \lambda_{|p|})$$

and

$$s_{p,\beta}^{j,1}(i, \lambda) =$$

$$= \delta_{j, \sum_{l=1}^n i_l} \left( \prod_{m=1}^n \frac{1}{\left(\sum_{l=1}^m i_l - j/2\right)^2 - \mu_m} \right) \left( \prod_{m=1}^{|p|-1} \frac{1}{\left(\sum_{l=1}^{\beta_1 + \dots + \beta_m} i_l - j/2\right)^2 - \lambda_{m+1}} \right)$$

$$\times \frac{\left(\sum_{l=1}^{\beta_1 + \dots + \beta_{p_1}} i_l - j/2\right)^2 - j^2/4}{j^2/4 - \lambda_1}$$
(5.19)

with some  $\mu_m = \mu_m(\lambda; p, \beta) \in \gamma_{|j|}$ .



Since for any  $r$ , we have  $|\lambda_r - j^2/4|, |\mu_r - j^2/4| = \delta_0|j|$ , where  $\delta_0 > 0$  small, one deduces from (5.19) that

$$|s_{p,\beta}^{j,1}(i, \lambda)| \leq \frac{C^n}{\langle j \rangle^{|p|}} \mathcal{A}_n^j(i).$$

As a consequence, one gets  $|\mathcal{S}_{p,\beta}^{j,1}(i)| \leq C^n \mathcal{A}_n^j(i)$ , so that

$$|\tilde{\mathcal{K}}_n^{j,1}(i)| \leq C^n \mathcal{A}_n^j(i) \sum_{\substack{p=(p_1,p_2) \in \mathbb{N}^2, \\ |p| \leq n}} c_{p_1} c_{p_2} \sum_{\substack{\beta=(\beta_1, \dots, \beta_{|p|}) \in \mathbb{N}^{|p|}, \\ |\beta|=n}} 1 \leq C^n \mathcal{A}_n^j(i).$$

Similar computations can be performed for  $Z_n^{j,2}$ ,  $n \geq 2$ . As a result, one gets the representation

$$Z_n^{j,2}(w) = \sum_{i=(i_1, i_2, \dots, i_n) \in \mathbb{Z}_0^n} \tilde{\mathcal{K}}_n^{j,2}(i) w_{i_1} w_{i_2} \dots w_{i_n},$$

with some  $\tilde{\mathcal{K}}_n^{j,2}$ , satisfying the same estimate as  $\tilde{\mathcal{K}}_n^{j,1}$ :  $|\tilde{\mathcal{K}}_n^{j,2}(i)| \leq C^n \mathcal{A}_n^j(i)$ . This concludes the proof of Lemma 5.7.  $\square$

## References

- [Eli84] L. H. Eliasson, *Hamiltonian systems with Poisson commuting integrals*, PhD Thesis, Stockholm University (1984).
- [Eli90] ———, *Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case*, Comment. Math. Helv. **65** (1990), 4–35.
- [Ito89] H. Ito, *Convergence of Birkhoff normal forms for integrable systems*, Comment. Math. Helv. **64** (1989), 412–461.
- [Kap91] T. Kappeler, *Fibration of the phase-space for the Korteweg-de Vries equation*, Ann. Inst. Fourier **41** (1991), 539–575.
- [Kat66] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.
- [KP03] T. Kappeler and J. Pöschel, *KAM & KdV*, Springer, 2003.

- [KP09] S. Kuksin and A. Piatnitski, *Averaging for randomly perturbed KdV: uniqueness of the limit*, Preprint (2009).
- [Kuk00] S. B. Kuksin, *Analysis of Hamiltonian PDEs*, Oxford University Press, Oxford, 2000.
- [Vey78] J. Vey, *Sur certain systèmes dynamiques séparables*, Am. J. Math. **100** (1978), 591–614.
- [Zun05] Nguyen Tien Zung, *Convergence versus integrability in Birkhoff normal form*, Annals of Maths. **161** (2005), 141–156.