DAMPED-DRIVEN KDV AND EFFECTIVE EQUATIONS FOR LONG-TIME BEHAVIOUR OF ITS SOLUTIONS

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Abstract. For the damped-driven KdV equation

$$\dot u - \nu u_{xx} + u_{xxx} - 6uu_x = \sqrt{\nu}\,\eta(t,x)\,,\ x \in S^1,\ \int u\,dx \equiv \int \eta\,dx \equiv 0\,,$$

with $0 < \nu \le 1$ and smooth in x white in t random force η, we study the limiting long-time behaviour of the KdV integrals of motions $(I_1, I_2, \dots)$, evaluated along a solution $u^\nu(t,x)$, as $\nu \to 0$. We prove that for $0 \le \tau := \nu t \lesssim 1$ the vector $I^\nu(\tau) = (I_1(u^\nu(\tau,\cdot)), I_2(u^\nu(\tau,\cdot)), \dots)$, converges in distribution to a limiting process $I^0(\tau) = (I_1^0, I_2^0, \dots)$. The j-th component I_j^0 equals $\frac{1}{2}(v_j(\tau)^2 + v_{-j}(\tau)^2)$, where the vector $v(\tau) = (v_1(\tau), v_{-1}(\tau), v_2(\tau), \dots)$ is a solution of a system of *effective equations* for the damped-driven KdV. These new equations are a quasilinear stochastic heat equation with a non-local nonlinearity, written in the Fourier coefficients. They are well posed.

0 Introduction

In this work we continue the study of randomly perturbed and damped KdV equation, commenced in [KuPi]. Namely, we consider the equation

$$u_t - \nu u_{xx} + u_{xxx} - 6uu_x = \sqrt{\nu}\,\eta(t,x)\,, \tag{0.1}$$

where $x \in S^1 \stackrel{\text{def}}{=} \mathbb{R}/2\pi\mathbb{Z}$, $\int_{S^1} u\,dx = 0$, and $\nu > 0$ is a small positive parameter. The random stationary force $\eta = \eta(t,x)$ is

$$\eta = \frac{d}{dt}\Big(\sum_{s\in\mathbb{Z}_0} b_s\beta_s(t)e_s(x)\Big)\,.$$

Here $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, β_s are standard independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $\{e_s,\ s \in \mathbb{Z}_0\}$ is the usual trigonometric basis

$$e_s(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\cos(sx)\,, & s > 0\,,\\ -\frac{1}{\sqrt{\pi}}\sin(sx)\,, & s < 0\,. \end{cases}$$

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The coefficients ν and $\sqrt{\nu}$ in (0.1) are balanced in such a way that solutions of the equation stays of order one as $t \to \infty$ and $\nu \to 0$, see [KuPi]. The coefficients b_s are non-zero and are even in s, i.e.

$$b_s = b_{-s} \neq 0 \quad \forall\, s \geq 1\,.$$

When $|s| \to \infty$ they decay faster than any negative power of $|s|$: for any $m \in \mathbb{Z}^+$ there is $C_m > 0$ such that

$$|b_s| \leq C_m |s|^{-m} \quad \forall\, s\,.$$

This implies that the force $\eta(t,x)$ is smooth in x for any t. We study behaviour of solutions for (0.1) with given smooth initial data

$$u(0,x) = u_0(x) \in C^\infty(S^1) \tag{0.2}$$

for

$$0 \leq t \leq \nu^{-1} T\,, \quad 0 < \nu \ll 1\,. \tag{0.3}$$

Here T is any fixed positive constant.

The KdV equation $(0.1)_{\nu=0}$ is integrable. That is to say, the function space $\{u(x) : \int u\,dx = 0\}$ admits analytic symplectic coordinates $v = (\mathbf{v}_1, \mathbf{v}_2, \dots) = \Psi(u(\cdot))$, where $\mathbf{v}_j = (v_j, v_{-j})^t \in \mathbb{R}^2$, such that the quantities $I_j = \frac{1}{2}|\mathbf{v}_j|^2$, $j \geq 1$, are actions (integrals of motion), while $\varphi_j = \operatorname{Arg} \mathbf{v}_j$, $j \geq 1$, are angles. In the (I,φ)-variables KdV takes the integrable form

$$\dot I = 0\,, \quad \dot\varphi = W(I)\,, \tag{0.4}$$

where $W(I) \in \mathbb{R}^\infty$ is the *frequency vector*, see section 1.2. (The actions I and the angles q were constructed first (before the Cartesian coordinates v), starting with the pioneer works by Novikov and Lax in 1970's. See in [MT], [ZMNP], [Ku1], [KP].) The integrating map Ψ is called the *nonlinear Fourier transform.* (The reason is that an analogy of Ψ, a map which integrates the linearised KdV equation $\dot u + u_{xxx} = 0$, is the usual Fourier transform.)

We are mostly concerned with behaviour of actions $I(u(t)) \in \mathbb{R}^\infty$ of solutions for the perturbed KdV equation (0.1) for t, satisfying (0.3). To this end, let us write equations for $I(v)$ and $\varphi(v)$, using the slow time $\tau = \nu t \in [0,T]$:

$$dI(\tau) = F(I,\varphi)d\tau + \sigma(I,\varphi)d\beta(\tau)\,, \quad d\varphi = \nu^{-1} W(I) d\tau + \dots\,, \tag{0.5}$$

where the dots stand for terms of order one, $\beta = (\beta_1, \beta_2, \dots)^t$ and $\sigma(I,\varphi)$ is an infinite matrix. For finite-dimensional stochastic systems of the form (0.5) under certain non-degeneracy assumptions, for the I-component of solutions for (0.5) the averaging principle holds. That is, when $\nu \to 0$ the I-component of a solution converges in distribution to a solution of the averaged equation

$$dI = \langle F\rangle(I)d\tau + \langle\sigma\rangle(I)d\beta(\tau)\,. \tag{0.6}$$

Here $\langle F\rangle$ is the averaged drift, $\langle F\rangle = \int F(I,\varphi)d\varphi$, and the dispersion matrix $\langle\sigma\rangle$ is a square root of the averaged diffusion $\int \sigma(I,\varphi)\sigma^t(I,\varphi)d\varphi$. This result was claimed in [Kh] and was first proved in [FW]; see [Ki] for recent development. In [KuPi] we established "half" of this result for solutions of e.g. (0.6) which corresponds to (0.1).

Namely, we have shown that for solutions $u_\nu(\tau,x)$ of (0.1), (0.2), where $t=\nu^{-1}\tau$ and $0<\tau\le T$,

(i) the set of laws of actions $\{\mathcal{D}I(u_\nu(\tau))\}$ is tight in the space of continuous trajectories $I(\tau)\in h_I^p$, $0\le\tau\le T$, where the space h_I^p is given the norm $|I|_{h_I^p}=2\sum_{j=1}^\infty j^{1+2p}|I_j|$ and p is any number ≥ 3;

(ii) any limiting measure $\lim_{\nu_j\to 0}\mathcal{D}I(u_{\nu_j}(\,\cdot\,))$ is a law of a weak solution $I^0(\tau)$ of e.g. (0.6) with the initial condition

$$I(0)=I_0:=I(u_0)\,. \tag{0.7}$$

The solutions $I^0(\tau)$ are *regular* in the sense that all moments of the random variables $\sup_{0\le\tau\le T}|I^0(\tau)|_{h_I^r}$, $r\ge 0$, are finite.

Similar results are obtained in [KuPi] for limits (as $\nu_j\to 0$) of stationary in time solutions for e.g. (0.1).

If we knew that (0.6), (0.7) has a unique solution $I^0(\tau)$, then (ii) would imply that

$$\mathcal{D}I\big(u_\nu(\,\cdot\,)\big)\rightharpoonup\mathcal{D}I^0(\,\cdot\,)\quad\text{as }\nu\to 0\,, \tag{0.8}$$

as in the finite-dimensional case. But the uniqueness is far from obvious since (0.6) is a bad equation in the bad phase-space $\mathbb{R}_+^\infty$: the dispersion $\langle\sigma\rangle$ is not Lipschitz in I, and the drift $\langle F\rangle(I)$ is an unbounded operator. In this paper we show that still the convergence (0.8) holds true:

Theorem A. *The problem* (0.6), (0.7) *has a solution* $I^0(\tau)$ *such that the convergence* (0.8) *holds.*

The proof of this result, given in section 4, Theorem 4.5, relies on a new construction, crucial for this work. Namely, it turns out that the 'bad' equation (0.6) may be lifted to a system of 'good' *effective equations* on the variable $v=(\mathbf{v}_1,\mathbf{v}_2,\dots)$, $\mathbf{v}_j\in\mathbb{R}^2$, which transforms to (0.6) under the mapping

$$\pi_I: v\mapsto I,\qquad I_j=\tfrac12|\mathbf{v}_j|^2.$$

To derive the effective equations we evoke the mapping Ψ to transform e.g. (0.1), written in the slow time τ, to a system of stochastic equations on the vector $v(\tau)$

$$d\mathbf{v}_k(\tau)=\nu^{-1}d\Psi_k(v)V(u)d\tau+P_k(v)d\tau+\sum_{j\ge1}B_{kj}(v)d\boldsymbol{\beta}_j(\tau)\,,\quad k\ge 1\,. \tag{0.9}$$

Here $V(u)=-u_{xxx}+6uu_x$ is the vector-field of KdV, $P_k\,d\tau+\sum B_{kj}\,d\boldsymbol{\beta}_j$ is the perturbation $u_{xx}+\eta(\tau,x)$, written in the v-variables, and $\boldsymbol{\beta}_j$'s are standard Wiener processes in $\mathbb{R}^2$ (so B_{kj}'s are 2×2-blocks). We will refer to the system (0.9) as the *v-equations.* This system becomes singular as $\nu\to 0$.

The *effective equations* for (0.9) as $\nu\to 0$ is a system of regular stochastic equations

$$d\mathbf{v}_k(\tau)=\langle P\rangle_k\,d\tau+\sum_j\langle\langle B\rangle\rangle_{kj}(v)d\beta_j(\tau)\,,\quad k\ge 1\,. \tag{0.10}$$

To define the *effective drift* $\langle P\rangle$ and the *effective dispersion* $\langle\langle B\rangle\rangle$, for any $\theta\in\mathbb{T}^\infty$ let us denote by Φ_θ the linear operator in the space of sequences $v=(\mathbf{v}_1,\mathbf{v}_2,\dots)$ which rotates each two-vector $\mathbf{v}_j$ by the angle θ_j. The rotations Φ_θ act on vector-fields on the v-space, and $\langle P\rangle$ is the result of the action of Φ_θ on P, averaged in θ:

$$\langle P\rangle(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta} P(\Phi_\theta v) d\theta \tag{0.11}$$

($d\theta$ is the Haar measure on $\mathbb{T}^\infty$).

Consider the diffusion operator $BB^t(v)$ for the v-equations (0.9). It defines a (1,1)-tensor on the linear space of vectors v. The averaging of this tensor with respect to the transformations Φ_θ is a tensor, corresponding to the operator

$$\langle BB^t\rangle(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta} \cdot \big((BB^t)(\Phi_\theta v)\big) \cdot \Phi_\theta \, d\theta \,. \tag{0.12}$$

This is the *averaged diffusion operator*. The effective dispersion operator $\langle\langle B\rangle\rangle(v)$ is its non-symmetric square root:

$$\langle\langle B\rangle\rangle(v) \cdot \langle\langle B\rangle\rangle^t(v) = \langle BB^t\rangle(v) \,. \tag{0.13}$$

Such a square root is non-unique. The one chosen in this work is given by an explicit construction and is analytic in v (while the *symmetric* square root of $\langle BB^t\rangle(v)$ is only a Hölder-$\frac{1}{2}$ continuous function of v). The effective equations are weakly invariant under the action of the group $\mathbb{T}^\infty$: if $v(\tau)$ is a weak solution, then for each $\theta \in \mathbb{T}^\infty$ the curve $\Phi_\theta v(\tau)$ is a weak solution as well. See sections 1.5 and 2.

Let us provide the space of vectors v with the norms $|\cdot|_r$, $r \geq 0$, where $|v|_r^2 = \sum_j |\mathbf{v}_j|^2 j^{1+2r}$. A solution of e.g. (0.10) is called *regular* if all moments of all random variables $\sup_{0\leq\tau\leq T} |v(\tau)|_r$, $r \geq 0$, are finite.

Theorem B. *System* (0.10) *has at most one regular strong solution* $v(\tau)$ *such that* $v(0) = \Psi(u_0)$.

This result is proved in section 4, where we show that system (0.10) is a quasilinear stochastic heat equation, written in Fourier coefficients.

The effective system (0.10) is useful to study e.g. (0.1) since this is a lifting of the averaged equations (0.6). The corresponding result, stated below, is proved in section 3:

Theorem C. *For every weak solution* $I^0(\tau)$ *of* (0.6) *as in assertion* (ii) *there exists a regular weak solution* $v(\tau)$ *of* (0.10) *such that* $v(0) = \Psi(u_0)$ *and* $\mathcal{D}(\pi_I(v(\,\cdot\,))) = \mathcal{D}(I^0(\,\cdot\,))$. *The other way round, if* $v(\tau)$ *is a regular weak solution of* (0.11), *then* $I(\tau) = \pi_I(v(\tau))$ *is a weak solution of* (0.6).

We do not know if a regular weak solution of problem (0.6), (0.7) is unique. But from Theorem B we know that a regular weak solution of the Cauchy problem for the effective equations (0.10) is unique, and through Theorem C it implies uniqueness of a solution for (0.6), (0.7) as in item (ii). This proves Theorem A.

In section 5 we evoke some intermediate results from [KuPi] to show that, after averaging in τ, distribution of the actions of a solution u^ν for (0.1) becomes asymptotically (as $\nu \to 0$) independent of distribution of the angles, and the angles become uniformly distributed on the torus $\mathbb{T}^\infty$. In particular, for any continuous function

$f \geq 0$ such that $\int f = 1$, we have

$$\int_0^T f(\tau)\mathcal{D}\varphi\big(u^\nu(\tau)\big)d\tau \rightharpoonup d\theta \quad \text{as } \nu \to 0\,.$$

This convergence justifies the *random phase approximation* for solutions of (0.1) with $0 < \nu \ll 1$. The approximation is often claimed in modern physics for various nonlinear PDE, but never was rigorously proved. (Usually physicists claim the random phase approximation for solutions of deterministic nonlinear PDE. That is a much more complicated assertion.)

The recipe (0.11) allows us to construct effective equations for other perturbations of KdV, with or without randomness. These are non-local nonlinear equations with interesting properties. In particular, if the perturbation is given by a Hamiltonian nonlinearity $\nu(\partial/\partial x)f(u,x)$, then the effective system is Hamiltonian and integrable (its Hamiltonian depends only on the actions I).

The effective equations (0.10) are instrumental in the study of other problems, related to e.g. (0.1). In particular, they may be used to prove the convergence (0.8) when $u_\nu(\tau)$ are stationary solutions of (0.1) and $I^0(\tau)$ is a stationary solution for (0.6). See [Ku3] for a discussion of these and some related results; the proof will be published elsewhere. Moreover, we are certain that corresponding effective equations may be used to study other perturbations of KdV, including the damped equation $(0.1)_{\eta=0}$.

The damped-driven KdV (0.1) may be cautiously regarded as a model for the 2d Navier–Stokes equations with small viscosity and small random force, under periodic boundary conditions (those equations are responsible for the space-periodic 2d turbulence). See the Introduction to [KuPi], and see [Ku2] for some results on the 2d Navier–Stokes, related to the problem which we address in this work.

Our results are also related to Whitham averaging for perturbed KdV, see the Appendix.

Agreements. Analyticity of maps $B_1 \to B_2$ between Banach spaces B_1 and B_2, which are the real parts of complex spaces B_1^c and B_2^c, is understood in the sense of Fréchet. All analytic maps which we consider possess the following additional property: for any R a map analytically extends to a complex $(\delta_R > 0)$–neighbourhood of the ball $\{|u|_{B_1} < R\}$ in B_1^c. Such maps are Lipschitz on bounded subsets of B_1. When a property of a random variable holds almost sure, we often drop the specification "a.s.". All metric spaces are provided with the Borel sigma-algebras. All sigma-algebras which we consider in this work are assumed to be completed with respect to the corresponding probabilities.

Notation. χ_A stands for the indicator function of a set A (equalling 1 in A and vanishing outside it). By $\mathcal{D}\xi$ we denote the distribution (i.e. the law) of a random variable ξ. For a measurable set $Q \subset \mathbb{R}^n$ we denote by $|Q|$ its Lebesgue measure.

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the constructions in section 1.4, and for critical remarks on a preliminary version of this work.

1 Preliminaries

Solutions of problem (0.1), (0.2) satisfy a priori estimates which are uniform in t and ν (see [KuPi]):

$$\mathbb{E}\big\{\exp\big(\sigma\|u(t)\|_0^2\big)\big\} \le c_0\,, \quad \mathbb{E}\big(\|u(t)\|_m^k\big) \le c_{m,k}\,, \tag{1.1}$$

for any $m, k \ge 0$ and any $\sigma \le (2\max b_s^2)^{-1}$. Here $\|\cdot\|_m$ stands for the norm in the Sobolev space $H^m = \big\{u \in H^m(S^1) : \int u\,dx = 0\big\}$, $\|u\|_m^2 = \int(\partial^m u/\partial x^m)^2\,dx$. To study further properties of solutions for (0.1) with small ν we need the nonlinear Fourier transform Ψ which integrates the KdV equation.

1.1 Nonlinear Fourier transform for KdV. For $s \ge 0$, denote by h^s the Hilbert space formed by the vectors $v = (v_1, v_{-1}, v_2, v_{-2}, \dots)$ and provided with the weighted l_2-norm $|\cdot|_s$,

$$|v|_s^2 = \sum_{j=1}^{\infty} j^{1+2s}(v_j^2 + v_{-j}^2)\,.$$

We set $\mathbf{v}_j = \binom{v_j}{v_{-j}}$, $j \in \mathbb{Z}^+ = \{j \ge 1\}$, and will also write vectors v as $v = (\mathbf{v}_1, \mathbf{v}_2, \dots)$. For any $v \in h^s$ we define the vector of actions $I(v) = (I_1, I_2, \dots)$, $I_j = \frac{1}{2}|\mathbf{v}_j|^2$. Clearly $I \in h^s_{I+} \subset h^s_I$. Here h^s_I is the weighted l^1-space,

$$h^s_I = \left\{ I : |I|_{h^s_I} = 2\sum_{j=1}^{\infty} j^{1+2s}|I_j| < \infty \right\},$$

and h^s_{I+} is the positive octant $h^s_{I+} = \{h \in h^s_I : I_j \ge 0\ \forall\, j\}$.

Theorem 1.1. *There exists an analytic diffeomorphism $\Psi : H^0 \mapsto h^0$ and an analytic functional K on h^0 of the form $K(v) = \widetilde{K}(I(v))$, where the function $\widetilde{K}(I)$ is analytic in a suitable neighbourhood of the octant h^0_{I+} in h^0_I, with the following properties:*

1. *The mapping Ψ defines, for any $m \in \mathbb{Z}^+$, an analytic diffeomorphism $\Psi : H^m \to h^m$. This is a symplectomorphism if the space H^m is given a symplectic structure by means of the two-form Ω_2, $\Omega_2[\xi(x), \eta(x)] = -\int(\partial/\partial x)^{-1}\xi(x) \wedge \eta(x)dx$, and the space h^m, by the two-form $\omega_2 = \sum dv_k \wedge dv_{-k}$.*
2. *The map $d\Psi(0)$ takes the form $\sum u_s e_s \mapsto v$, $v_s = |s|^{-1/2}u_s$.*
3. *A curve $u \in C^1(0, T; H^0)$ is a solution of the KdV equation $(0.1)_{\nu=0}$ if and only if $v(t) = \Psi(u(t))$ satisfies the equation*

$$\dot{\mathbf{v}}_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \widetilde{K}}{\partial I_j}(I)\mathbf{v}_j\,, \quad j \in \mathbb{Z}^+\,. \tag{1.2}$$

4. *For $m = 0, 1, 2, \dots$ there are polynomials P_m and Q_m such that*

$$\big|d^j\Psi(u)\big|_m \le P_m\big(\|u\|_m\big)\,, \quad \big\|d^j(\Psi^{-1}(v))\big\|_m \le Q_m\big(|v|_m\big)\,, \quad j = 0, 1, 2\,,$$

for all u and v and all $m \ge 0$.

See [KP] for items 1-3 and [KuPi] for item 4. The coordinates $v=\Psi(u)$ are called the *Birkhoff coordinates*, and the form (1.2) of KdV, its *Birkhoff normal form.*

The analysis in section 4 requires the following amplification of Theorem 1.1, stating that the nonlinear Fourier transform Ψ “is quasilinear”:

PROPOSITION 1.2. *For any $m\geq 0$ the map $\Psi-d\Psi(0)$ defines an analytic mapping from H^m to h^{m+1}.*

A local version of the last statement which deals with the germ of Ψ at the origin, is established in [KuP]. Proof of the proposition in the general case, based on the spectral theory of Hill operators, will be given in a separate publication.

1.2 Equation (0.1) in Birkhoff coordinates. Applying the Itô formula to the nonlinear Fourier transform Ψ, we see that for $u(t)$, satisfying (0.1), the function $v(\tau)=\Psi(u(\tau))$, where $\tau=\nu t$, is a solution of the system

$$d\mathbf{v}_k=\nu^{-1}d\Psi_k(u)V(u)d\tau+P_k^1(v)d\tau+P_k^2(v)d\tau+\sum_{j\geq1}B_{kj}(v)d\boldsymbol{\beta}_j(\tau)\,,\quad k\geq1\,. \tag{1.3}$$

Here $\boldsymbol{\beta}_j=\binom{\beta_j}{\beta_{-j}}\in\mathbb{R}^2$, $V(u)=-u_{xxx}+6uu_x$ is the vector field of KdV, $P^1(v)=d\Psi(u)u_{xx}$ and $P^2(v)d\tau$ is the Itô term,

$$P_k^2(v)=\frac12\sum_{j\geq1}b_j^2\big[d^2\Psi_k(u)\,(e_j,e_j)+d^2\Psi_k(u)\,(e_{-j},e_{-j})\big]\in\mathbb{R}^2.$$

Finally, the dispersion matrix B is formed by 2×2-blocks B_{kj}, $k,j\geq1$, where

$$B_{kj}(u)=b_j(d\Psi(u))_{kj}.$$

Due to the analyticity of the map Ψ, Proposition 1.2 and the fast decay of the coefficients b_j, for any $m\geq0$, the matrix-functions $B_{kj}(u)$, $k,j\geq1$, analytically extend to a complex neighbourhood O^m of H^m in the complex Sobolev space $H^m\otimes\mathbb{C}$, where

$$\big\|B_{kj}(u)\big\|\leq C_N\big(\|u\|_m\big)j^{-N}k^{-m-3/2}, \tag{1.4}$$

for any N with a suitable positive continuous function $C_N(\,\cdot\,)$.

We will call (1.3) the *system of v-equations.*

The v-equations imply the following relations for the actions vector I:

$$dI_k=\mathbf{v}_k^tP_k^1(v)d\tau+\mathbf{v}_k^tP_k^2(v)d\tau+\frac12\sum_{j\geq1}\|B_{kj}\|_{HS}^2d\tau+\sum_{j\geq1}\mathbf{v}_k^tB_{kj}(v)d\boldsymbol{\beta}_j(\tau)\,, \tag{1.5}$$

$k\geq1$. Here $\|B_{kj}\|_{HS}^2$ is the squared Hilbert–Schmidt norm of the 2×2 matrix B_{kj}, i.e. the sum of squares of all its four elements.

Estimates (1.1) and e.g. (1.5) imply that

$$\mathbf{E}\sup_{0\leq\tau\leq T}|I(\tau)|_{h_I^m}^k\leq C_{m,k}\quad\forall\,m,k\geq0\,. \tag{1.6}$$

See in [KuPi].

1.3 Averaged equations. For a vector $v = (\mathbf{v}_1, \mathbf{v}_2, \dots)$ denote by $\varphi(v) = (\varphi_1, \varphi_2, \dots)$ the vector of angles. That is, φ_j is the argument of the vector $\mathbf{v}_j \in \mathbb{R}^2$, $\varphi_j = \arctan(v_{-j}/v_j)$ (if $\mathbf{v}_j = 0$, we set $\varphi_j = 0$). The vector $\varphi(v)$ belongs to the infinite-dimensional torus $\mathbb{T}^\infty$. We provide the latter with the Tikhonov topology (so it becomes a compact metric space) and the Haar measure $d\theta = \prod(d\theta_j/2\pi)$. We will identify a vector v with the pair (I, φ) and write $v = (I, \varphi)$.

The torus $\mathbb{T}^\infty$ acts on each space h^m by the linear rotations Φ_θ, $\theta \in \mathbb{T}^\infty$, where $\Phi_\theta : (I, \varphi) \mapsto (I, \varphi + \theta)$. For any continuous function f on h^m we denote by $\langle f \rangle$ its angular average,

$$\langle f \rangle(v) = \int_{\mathbb{T}^\infty} f(\Phi_\theta v) d\theta \,.$$

The function $\langle f \rangle(v)$ is as smooth as $f(v)$ and depends only on I. Furthermore, if $f(v)$ is analytic on h^m, then $\langle f \rangle(I)$ is analytic on h^m_I; for the proof see [KuPi].

Averaging equations (1.5) using formally the rules of stochastic averaging (see [Kh], [FW]), we get the averaged system

$$\begin{aligned} dI_k(\tau) = \langle \mathbf{v}_k^t P_k^1 \rangle(I) d\tau + \langle \mathbf{v}_k^t P_k^2 \rangle(I) d\tau \\ + \frac{1}{2} \Big\langle \sum_{j \ge 1} \|B_{kj}\|^2_{HS} \Big\rangle (I) d\tau + \sum_{j \ge 1} K_{kj}(I) d\beta_j(\tau) \,, \quad k \ge 1 \,, \end{aligned} \tag{1.7}$$

with the initial condition

$$I(0) = I_0 = I\big(\Psi(u_0)\big) \,. \tag{1.8}$$

Here the dispersion matrix K is a square root of the averaged diffusion matrix S,

$$S_{km}(I) \stackrel{\text{def}}{=} \Big\langle \sum_{l \ge 1} \mathbf{v}_k^t B_{kl} \mathbf{v}_m^t B_{ml} \Big\rangle (I) \,, \tag{1.9}$$

not necessary symmetric. That is,

$$\sum_{l \ge 1} K_{kl}(I) K_{ml}(I) = S_{km}(I) \tag{1.10}$$

(we abuse the language since the l.h.s. is not K^2 but KK^t, i.e. it is $|K|^2$). If in (1.7) we replace K by another square root of S, we will get a new equation which has the same set of weak solutions, see [Y].

Note that system (1.7) is very irregular: its drift operator $\langle G_k^1 \rangle$ is unbounded and the dispersion matrix $K(I)$ is not Lipschitz continuous in I.

1.4 Averaging principle. Let us fix any $p \ge 3$ and denote

$$\mathcal{H}_I = C\big([0,T], h^p_{I+}\big) \,, \quad \mathcal{H}_v = C\big([0,T], h^p\big) \,. \tag{1.11}$$

In [KuPi] we have proved the following results: given any $T > 0$, for the process $I^\nu(\tau) = \{I(v^\nu(\tau)) : 0 \le \tau \le T\}$ the following theorem holds:

Theorem 1.3. *Let $u^\nu(t)$, $0 < \nu \le 1$, be a solution of* (0.1), (0.2) *and $v^\nu(\tau) = \Psi(u^\nu(\tau))$, $\tau = \nu t$, $\tau \in [0,T]$. Then the family of measures $\mathcal{D}(I^\nu(\,\cdot\,))$ is tight in the space of (Borel) measures in $\mathcal{H}_I$. Any limit point of this family, as $\nu \to 0$, is* *the distribution of a weak solution $I^0(\tau)$ of the averaged equations* (1.7), (1.8). *It satisfies the estimates*

$$\mathbb{E} \sup_{0\le\tau\le T} \left|I^0(\tau)\right|^N_{h^m_I} < \infty \quad \forall\, m, N \in \mathbb{N}\,, \tag{1.12}$$

and

$$\mathbf{E} \int_0^T \chi_{\{I^0_k(\tau)\le\delta\}}(\tau)d\tau \to 0 \quad \textit{as} \quad \delta \to 0\,, \tag{1.13}$$

for each k.

Remarks. (1) The convergence (1.13) is proved in Lemma 4.3 of [KuPi]. There is a flaw in the *statement* of Lemma 4.3: the convergence (1.13) is claimed there for any fixed τ (without integrating in $d\tau$). This is true only for the case of stationary solutions, cf. the next remark. The proof of the main results in [KuPi] uses exactly (1.13), cf. there estimate (5.7). See the Appendix below, where the proof of Lemma 4.3 is rewritten for purposes of this work.

(2) A similar result holds when $u^\nu(t) = u^\nu_{\rm st}(t)$, $t \ge 0$, is a stationary solution of (0.1), see [KuPi].

1.5 Dispersion matrix *K*. The matrix $S(I)$ is symmetric and positive but its spectrum contains 0. Consequently, its symmetric square root $\sqrt{S}(I)$ has low regularity in I at points of the set

$$\partial\, h^p_{I+} = \{I \in h^p_{I+} : I_j = 0 \text{ for some } j\}$$

(which is dense in h^p_I). [Matrix elements of $\sqrt{S}(I)$ are Lipschitz functions of the arguments $\sqrt{I_1}, \sqrt{I_2}, \dots$. Cf. [IW, Prop. IV.6.2].] Now we construct a 'regular' square root K (i.e. a dispersion matrix) which is an analytic function of v, where $I(v) = I$. This regularity will be sufficient for our purposes.

We will obtain a dispersion matrix $K = \{K_{lm}\}(v)$, $I(v) = I$, as the matrix of a dispersion operator $\mathbf{K} : Z \to l_2$, where Z is an auxiliary separable Hilbert space and the operator depends on the parameter v, $\mathbf{K} = \mathbf{K}(v)$. The matrix K is written with respect to some orthonormal basis in Z and the standard basis $\{f_j, j \ge 1\}$ of l_2. Below for a space Z we take a suitable L^2-space $Z = L^2(X, \mu(dx))$. For any Schwartz kernel $\mathcal{M} = \mathcal{M}(j,x)$ we denote by $\mathrm{Op}(\mathcal{M})$ the corresponding integral operator from $L^2(X)$ to l_2:

$$\mathrm{Op}(\mathcal{M})\, g(\,\cdot\,) = \sum_j f_j \int \mathcal{M}(j,x) g(x)\mu(dx)\,.$$

We will define the dispersion operator $\mathbf{K}(v)$ by its Schwartz kernel $\mathcal{K}(j,x)(v)$, $\mathbf{K}(v) = \mathrm{Op}(\mathcal{K}(v))$. For any choice of the orthonormal basis in Z the Parseval identity holds:

$$\sum_{l\ge1} K_{kl}(v) K_{ml}(v) = \int_X \mathcal{K}(k,x)(v)\mathcal{K}(m,x)(v)\mu(dx) \quad \forall\, k, m\,. \tag{1.14}$$

Since a law of a zero-meanvalue Gaussian process is defined by its correlations, then, due to (1.14), the law of the process $\sum_{l\ge1} f_l \sum_{m\ge1} K_{lm}\beta_m(\tau) \in l_2$ does not depend on the choice of the orthonormal basis in Z; it depends only on the correlation operator $\mathbf{K}$ (i.e. on its kernel $\mathcal{K}$) and not on a matrix K. Accordingly, we will *formally* denote the differential of this process as

$$\sum_{l\geq 1} f_l \sum_{m\geq 1} K_{lm}\, d\beta_m(\tau) = \sum_{l\geq 1} f_l \int_X \mathcal{K}(l,x) d\beta_x(\tau)\mu(dx)\,, \tag{1.15}$$

where $\beta_x(\tau)$, $x\in X$, are standard independent Wiener processes on some probability space. (We cannot find continuum independent copies of a random variable on a standard probability space. So indeed this is just a notation.) We will call the differential in the l.h.s. (its integral) a *physical realisation* of the formal differential in the r.h.s. (of the corresponding formal stochastic integral). Naturally, if in a stochastic equation the diffusion is written as the r.h.s. in (1.15), then only weak solutions of the equation are well defined. A stochastic equation, where a formal diffusion is replaced by its physical realisation, is called a *physical realisation* of the equation.

Notation (1.15) agrees well with the Itô formula. Indeed, denote the differential in (1.15) by $d\eta$ and let $f(\eta)$ be a C^2-smooth function. Then due to (1.14)

$$\begin{aligned} df(\eta) &= \Big(\frac{1}{2}\sum_{k,r}\frac{\partial^2 f}{\partial\eta_k\partial\eta_r}\sum_m K_{km}K_{rm}\Big)d\tau + \sum_{k,m}\frac{\partial f}{\partial \eta_k}K_{km}\,d\beta_m(\tau)\\ &= \Big(\frac{1}{2}\sum_{k,r}\frac{\partial^2 f}{\partial\eta_k\partial\eta_r}\int_X \mathcal{K}(k,x)\mathcal{K}(r,x)\mu(dx)\Big)d\tau \\ &\quad + \sum_k \frac{\partial f}{\partial\eta_k}\int_X \mathcal{K}(k,x)d\beta_x(\tau)\mu(dx)\,. \end{aligned} \tag{1.16}$$

Due to (1.14) the matrix $K(v)$ satisfies equation (1.10) if

$$\begin{aligned} \int_X \mathcal{K}(k,x)(v)\mathcal{K}(m,x)(v)\mu(dx) &= \sum_{l\geq 1} K_{kl}(v)K_{ml}(v)\\ &= S_{km}(I) = \sum_{l\geq 1}\big\langle (\mathbf{v}_k^t B_{kl}(v))(\mathbf{v}_m^t B_{ml}(v))\big\rangle\,. \end{aligned} \tag{1.17}$$

The element $S_{km}(I)$ of the matrix $S(I)$ in the right-hand side of (1.17) equals

$$\begin{aligned} &\sum_{l\geq 1}\int_{\mathbb{T}^\infty}\big((\mathbf{v}_k^t B_{kl})(\Phi_\theta v)\big)\big((\mathbf{v}_m^t B_{ml})(\Phi_\theta v)\big)\,d\theta\\ &\quad = \sum_{l\geq 1}\int_{\mathbb{T}^\infty}\big((\mathfrak{L}_{\theta_k}\mathbf{v}_k)^t B_{kl}(\Phi_\theta v)\big)\big((\mathfrak{L}_{\theta_m}\mathbf{v}_m)^t B_{ml})(\Phi_\theta v)\big)\,d\theta\\ &\quad = \mathbf{v}_k^t\mathbf{v}_m^t\sum_{l\geq 0}\int_{\mathbb{T}^\infty}\big(\mathfrak{L}_{-\theta_k}\cdot B_{kl}(\Phi_\theta v)\big)\big(\mathfrak{L}_{-\theta_m}\cdot B_{ml}(\Phi_\theta v)\big)d\theta\,. \end{aligned}$$

Here and below $\mathfrak{L}_\theta$ stands for the rotation of the plane $\mathbb{R}^2$ by the angle θ, or the matrix of this linear operator (so that $\mathfrak{L}_{-\theta_k}\cdot B_{kl}$ and $\mathfrak{L}_{-\theta_m}\cdot B_{ml}$ are multiplication of the 2×2-matrices).

Let us choose for X the space $X = \mathbb{Z}^+ \times \mathbb{T}^\infty = \{(l,\theta)\}$ and equip it with the measure $\mu(dx) = dl \times d\theta$, where dl is the counting measure in $\mathbb{Z}^+$ and $d\theta$ is the Haar measure in $\mathbb{T}^\infty$. Consider the following Schwartz kernel $\mathcal{K}$:

$$\mathcal{K}(k;l,\theta)(v) = \mathbf{v}_k^t \mathcal{R}(k;l,\theta)(v)\,, \quad \mathcal{R}(k;l,\theta)(v) = (\mathfrak{L}^k_{-\theta} B_{kl})\big(\Phi_\theta(v)\big)\,. \tag{1.18}$$

Then (1.17) is fulfilled. So

$$\begin{aligned}&\text{for any choice of the basis in } L_2(\mathbb{Z}^+ \times \mathbb{T}^\infty)\\ &\text{the matrix } K(v) \text{ of } \mathrm{Op}(\mathcal{K}(v)) \text{ satisfies (1.10) with } I = I(v)\,.\end{aligned} \tag{1.19}$$

The differential (1.15) where $\mathcal{K} = \mathcal{K}(k;l,\theta)(v)$ (and $x = (l,\theta)$), depends on v, but its law depends only on $I(v)$. Due to (1.4), for any $m \ge 0$ there exists a complex neighbourhood Q_m of h^m in $h^m \otimes \mathbb{C}$ such that for every $k,l \ge 1$ and $\theta \in \mathbb{T}^\infty$ the matrix-function $\mathcal{R}(k,l,\theta)(v)$ analytically in v extends to Q_m and there satisfies the estimates

$$\big\|\mathcal{R}(k,l,\theta)(v)\big\| \le C_N\big(\|v\|_m\big) k^{-m-3/2} l^{-N}, \quad \forall\, N \in \mathbb{N}\,. \tag{1.20}$$

We formally write the averaged equations (1.7) with the constructed above dispersion operator $\mathrm{Op}(\mathcal{K}(v))$, $I(v) = I$, as

$$\begin{aligned} dI_k(\tau) = \langle \mathbf{v}_k^t P_k^1\rangle(I)d\tau + \langle \mathbf{v}_k^t P_k^2\rangle(I)d\tau + \frac{1}{2}\Big\langle \sum_{j\ge1} \|B_{kj}\|^2_{HS}\Big\rangle(I)d\tau \\ + \sum_{l\ge1}\int_{\mathbb{T}^\infty} \mathbf{v}_k^t \mathcal{R}(k,l,\theta)(v)\, d\boldsymbol{\beta}_{l,\theta}(\tau)d\theta\,. \end{aligned} \tag{1.21}$$

Let us fix a basis in the space $L_2(\mathbb{Z}^+ \times \mathbb{T}^\infty)$, Wiener processes $\{\beta_m(\,\cdot\,),\, m \ge 1\}$, and consider the corresponding physical realisation of this equation. Let $\xi \in h^p$ be a random variable, independent of the processes $\{\beta_m(\tau)\}$.

Definition 1.4. (I) A pair of processes $I(\tau) \in h^p_I, v(\tau) \in h^p$, $0 \le \tau \le T$, such that $I(v(\tau)) \equiv I(\tau)$, $v(0) = \xi$ and

$$\mathbf{E} \sup_{0\le\tau\le T} |v(\tau)|^N_m < \infty \quad \forall\, m, N\,, \tag{1.22}$$

is called a regular strong solution of (1.21) in the space $h^p_I \times h^p$, corresponding to the basis and the Wiener processes above, if

(i) I and v are adapted to the filtration, generated by ξ and the processes $\{\beta_m(\tau)\}$;
(ii) the integrated in τ version of the physical realisation of (1.21) holds a.s.

(II) A pair of processes (I,v) is called a regular weak solution of (1.21) if it is a regular strong solution for some choice of the basis and the Wiener processes $\{\beta_m\}$, defined on a suitable extension of the original probability space (see in [KaS]).

Lemma 1.5. *If $(I(\tau), v(\tau))$, $0 \le \tau \le T$, is a regular weak solution of e.g. (1.21), then $I(\tau)$ is a weak solution of (1.7), where the matrix $\{K_{km}(I)\}$ is the symmetric square root of the matrix $\{S_{km}(I)\}$.*

Proof. Clearly the process $I(\tau)$ is a solution to the (local) martingale problem, associated with e.g. (1.7) (see [KaS, Prop. 4.2 & Prob. 4.3]). So $I(\tau)$ is a weak solution of (1.7), see [Y] and Corollary 6.5 in [KuPi]. □

The representation of the averaged equations (1.7) in the form (1.21) is crucial for this work. It is related to the construction of non-selfadjoint dispersion operators in the work [DIPP] and is inspired by it. We are thankful to A. Piatnitski for corresponding discussion.

2 Effective Equations

The goal of this section is to lift the averaged equations (1.7) to equations for the vector $v(\tau)$ which transform to (1.7) under the mapping $v \mapsto I(v)$. Using Lemma 1.5 we instead lift equations (1.21). We start the lifting with the last two terms in the right-hand side of (1.21). They define the Itô differential

$$\frac{1}{2}\Big\langle \sum_{j\geq 1} \|B_{kj}\|^2_{\mathrm{HS}} \Big\rangle(I)d\tau + \sum_{l\geq 1}\int_{\mathbb{T}^\infty} \mathbf{v}_k^t \mathcal{R}(k;l,\theta)(v)d\boldsymbol{\beta}_{l,\theta}(\tau)d\theta\,. \tag{2.1}$$

Consider the differential $d\mathbf{v}_k = \sum_{l\geq 1}\int_{\mathbb{T}^\infty} \mathcal{R}(k;l,\theta)(v)d\boldsymbol{\beta}_{l,\theta}(\tau)d\theta$. Due to (1.16), for $J_k = \frac{1}{2}|\mathbf{v}_k|^2$ we have

$$dJ_k = \frac{1}{2}\bigg(\sum_{l\geq 1}\int_{\mathbb{T}^\infty} \big\|\mathcal{R}(k;l,\theta)\big\|^2_{\mathrm{HS}} d\theta\bigg)d\tau + \sum_{l\geq 1}\int_{\mathbb{T}^\infty} \mathbf{v}_k^t \mathcal{R}(k;l,\theta)(v)d\boldsymbol{\beta}_{l,\theta}(\tau)d\theta\,.$$

Notice that the diffusion term in the last formula coincides with that in (2.1). The drift terms also are the same since $\|\mathfrak{L}_{\theta'}B_{kl}\|^2_{\mathrm{HS}} = \|B_{kl}\|^2_{\mathrm{HS}}$ for any rotation $\mathfrak{L}_{\theta'}$.

Now consider the first part of the differential on the right-hand side of (1.7),

$$\langle \mathbf{v}_k^t P_k^1\rangle(I)d\tau + \langle \mathbf{v}_k^t P_k^2\rangle(I)d\tau\,. \tag{2.2}$$

Recall that $P^1 = d\Psi(u)u_{xx}$ with $u = \Psi^{-1}(v)$ and that $P^2(v)$ is the Itô term. We have

$$\begin{aligned}\langle \mathbf{v}_k^t P_k^1\rangle(I) &= \int_{\mathbb{T}^\infty} (\mathbf{v}_k^t P_k^1)(\Phi_\theta v)d\theta = \int_{\mathbb{T}^\infty} \mathbf{v}_k^t \left(\mathfrak{L}_{-\theta_k} d\Psi_k(\Pi_\theta u)\frac{\partial^2}{\partial x^2}(\Pi_\theta u)\right)d\theta\\ &= \mathbf{v}_k^t R_k^1(v)\,,\quad u = \Psi^{-1}(v)\,,\end{aligned}$$

where $R_k^1(v) = \int_{\mathbb{T}^\infty} \mathfrak{L}_{-\theta_k} d\Psi_k(\Pi_\theta u)\frac{\partial^2}{\partial x^2}(\Pi_\theta u)d\theta$, and the operators Π_θ are defined by the relation $\Pi_\theta u = \Psi^{-1}(\Phi_\theta v)$. Similarly,

$$\langle \mathbf{v}_k^t P_k^2\rangle(I) = \int_{\mathbb{T}^\infty} (\mathbf{v}_k^t P_k^2)(\Phi_\theta v)d\theta = \mathbf{v}_k^t \int_{\mathbb{T}^\infty} \mathfrak{L}_{-\theta_k} P_k^2(\Phi_\theta v)d\theta =: \mathbf{v}_k^t R_k^2(v)\,.$$

Consider the differential $d\mathbf{v}_k = R_k^1(v)d\tau + R_k^2(v)d\tau$. Then $d\big(\frac{1}{2}|\mathbf{v}_k|^2\big) = (2.2)$.

Now consider the system of equations,

$$d\mathbf{v}_k(\tau) = R_k^1(v)d\tau + R_k^2(v)d\tau + \sum_{l\geq 1}\int_{\mathbb{T}^\infty} \mathcal{R}(k;l,\theta)(v)d\boldsymbol{\beta}_{l,\theta}(\tau)d\theta\,,\quad k\geq 1\,. \tag{2.3}$$

The arguments above prove that if $v(\tau)$ satisfies (1.21), then $I(v(\tau))$ satisfies (1.7). Using Lemma 1.5 we get

PROPOSITION 2.1. *If $v(\tau)$ is a regular weak solution of equations* (2.3), *then $(I(v(\tau)), v(\tau))$ is a regular weak solution of* (1.7).

Here a *regular weak solution* of (2.3) is a weak solution, satisfying (1.22).

Equations (2.3) are called the system of *effective equations* for (0.1). They are obtained from the v-equations by removing the KdV vector field and averaging the result. These equations are weakly invariant under the rotations Φ_θ:

- The drift $R^1(v) + R^2(v)$ in the effective equations (2.3) is an averaging of the vector-field $P(v) = P^1(v) + P^2(v)$, see (0.11).
- The kernel $\mathcal{R}(k;l,\theta)(v)$ defines a linear operator $\mathbf{R}(v) := \mathrm{Op}(\mathcal{R}(v))$ from the space $L_2 := L_2(\mathbb{Z}^+ \times \mathbb{T}^\infty)$ to the space $h := h^{-1/2}$, see section 1.5 (recall that the space h is given the l_2-scalar product). The operator $\mathbf{R}(v)\mathbf{R}(v)^t : h \to h$ has the matrix $X(v)$, formed by 2×2-blocks
$$X_{kj}(v) = \sum_l \int_{\mathbb{T}^\infty} \mathcal{R}(k;l,\theta)(v)\mathcal{R}(j;l,\theta)(v)d\theta\,.$$
Due to (1.18) this is the matrix of the averaged diffusion operator (0.12). Consider any physical realisation $\sum_j \langle\langle B\rangle\rangle_{kj}(v)d\beta_j(\tau)$ of the diffusion in (2.3). Then also $\langle\langle B\rangle\rangle(v)\langle\langle B\rangle\rangle^t(v) = X(v)$, see (1.14). So the dispersion operator in (2.3) is a non-symmetric square root of the averaged diffusion operator in the v-equations. Cf. relation (0.13) and its discussion.
- If $v(\tau)$ is a regular weak solution of (2.3), then $\Phi_\theta v(\tau)$ is a regular weak solution for each θ.

System (1.21) has locally Lipschitz coefficients and does not have a singularity at $\partial\, h^I_{p+}$, but its dispersion operator depends on v. Now we construct an equivalent system of equations on I which is v-independent, but has weak singularities at $\partial\, h^I_{p+}$.

The dispersion kernel in equations (1.21) is $\mathbf{v}_k^t\mathcal{R}(k;l,\theta)(v)$. Let us abbreviate it as $\mathcal{K}_k(l,\theta)(v)$. Then $\mathcal{K}_k(l,\theta)(v) = \mathbf{v}_k^t B_{kl}(v)|_{v:=\Phi_\theta v}$. Clearly

$$\mathcal{K}_k(l,\theta)(\Phi_\phi v) = \mathcal{K}_k(l,\theta+\phi)(v) \quad \forall\, \phi \in \mathbb{T}^\infty. \tag{2.4}$$

Denoting, as before, by $\mathrm{Op}(\mathcal{K}(v))$ the linear operator $L_2(\mathbb{N}\times\mathbb{T}^\infty) \to l_2$ with the kernel $\mathcal{K}(v) = \mathcal{K}_k(l,\theta)(v)$, $v = (I,\varphi)$, we have

$$\mathrm{Op}\big(\mathcal{K}(I,\varphi_1+\varphi_2)\big) = \mathrm{Op}\big(\mathcal{K}(I,\varphi_1)\big) \circ U(\varphi_2)\,. \tag{2.5}$$

Here $U(\varphi)$ is the unitary operator in the space $L = L_2(\mathbb{N}\times\mathbb{T}^\infty)$, corresponding to the rotation of $\mathbb{T}^\infty$ by an angle $\varphi \in \mathbb{T}^\infty$.

Let us provide $L_2(\mathbb{T}^1, dx/2\pi)$ with the basis $\xi_j(\theta)$, $j \in \mathbb{Z}$, where $\xi_0 = 1$, $\xi_j = \sqrt{2}\cos jx$ if $j \ge 1$ and $\xi_j = \sqrt{2}\sin jx$ if $j \le -1$. For $i \in \mathbb{N}$ and $s = (s_1, s_2, \dots) \in \mathbb{Z}^{\mathbb{N}}$, $|s| < \infty$, define
$$E_{i,s}(l,\theta) = \delta_{l-i} \prod_{j\in\mathbb{Z}} \xi_{s_j}(\theta_j)$$
(the infinite product is well defined since a.a. factors equal 1). These functions define a basis in L. For $n \ge 1$ denote by L^n the subspace of L, spanned by the vectors $E_{i,s}$ such that $i \le n$, $|s| \le n$ and $s_j = 0$ if $|j| > n$. It is easy to see that the operators $U(\varphi)$ leave all the spaces L^n invariant. Let $(E_r, r \in \mathbb{N})$, be the

functions $E_{i,s}$, reparameterised by the natural parameter in such a way that each space L^n is generated by first few functions E_r:

$$L^n = \operatorname{span}\{E_1, \dots, E_{M(n)}\}\,. \tag{2.6}$$

For any $v = (I, \varphi)$ the matrix $\mathcal{K}(v)$ with the elements

$$\mathcal{K}_{kr}(v) = \big(\mathcal{K}_k(l,\theta)(v), E_r(l,\theta)\big)_L = \int_{\mathbb{Z}^+\times\mathbb{T}^\infty} \mathcal{K}_k(l,\theta)(v)E_r(l,\theta)(dl\times d\theta)$$

is the matrix of the operator $\mathrm{Op}(\mathcal{K}(v))$ with respect to the basis $\{E_r\}$. Due to (2.5) for $v = (I, \varphi)$ the operator $\mathrm{Op}(\mathcal{K}(I, \varphi))$ equals $\mathrm{Op}(\mathcal{K}(I, 0)) \circ U(\varphi)$. So its matrix is

$$\mathcal{K}_{kr}(I,\varphi) = \sum_m M_{km}(I)U_{mr}(\varphi)\,,$$

where the matrix $M_{km}(I)$ corresponds to the kernel $\mathcal{K}_k(l,\theta)(I,0)$ and $U_{mr}(\varphi)$ is the matrix of the operator $U(\varphi)$ (both matrices are formed by 2×2-blocks). Clearly $\|K(I,\varphi)\|_{HS} = \|M(I)\|_{HS}$ for each (I,φ). Taking into account the form of the functions $E_{i,s}(l,\theta)$ we see that any $U_{mr}(\varphi)$ is a smooth function of each argument φ_j and is independent of φ_k with k large enough. In particular,

$$\text{any matrix element } U_{mr}(\varphi) \text{ is a Lipschitz function of } \varphi \in \mathbb{T}^\infty. \tag{2.7}$$

Note that the Lipschitz constant of U_{mr} depends on m and r.

Let us denote the drift in the system (1.21) by $F_k(I)d\tau$. Then the physical realisation of (1.21), corresponding to the basis $\{E_r\}$, is the system

$$dI_k(\tau) = F_k(I)d\tau + \sum_{m,r} M_{km}(I)U_{mr}(\varphi)d\beta_r\,, \quad k \ge 1\,. \tag{2.8}$$

Consider the processes $\tilde\beta_m(\tau)$, $m \ge 1$,

$$d\tilde\beta_m(\tau) = \sum_r U_{mr}(\varphi(\tau))d\beta_r(\tau)\,, \quad \tilde\beta_m(0) = 0\,. \tag{2.9}$$

Since U is an unitary operator, then $\tilde\beta_m(\tau)$, $m \ge 1$, are standard independent Wiener processes. So we may write (2.8) as

$$dI_k(\tau) = F_k(I)d\tau + \sum_m M_{km}(I)d\tilde\beta_m(\tau)\,, \quad k \ge 1\,. \tag{2.10}$$

Note that each weak solution of (2.10) is a weak solution of (2.8) and vice versa. Due to (1.19) the matrix M satisfies (1.10). So equations (2.10) have the same weak solutions as equations (1.7).

Now consider system (2.3) for $v(\tau)$. Denote by $\mathcal{R}_{km}(v)$ the matrix, corresponding to the kernel $\mathcal{R}(k;l,\theta)(v)$ in the basis $\{E_k\}$. Denoting $R^1_k + R^2_k = R_k$ we write (2.3) as follows:

$$\begin{aligned} d\mathbf{v}_k &= R_k(v)d\tau + \sum_r \mathcal{R}_{kr}(v)d\beta_r(\tau) \qquad (2.11)\\ &= R_k(v)d\tau + \sum_{m,l,r} \mathcal{R}_{kl}(v)U_{ml}(\varphi)U_{mr}(\varphi)d\beta_r(\tau)\,. \end{aligned}$$

So

$$d\mathbf{v}_k = R_k(v)d\tau + \sum_m \tilde{\mathcal{R}}_{km}(v)d\tilde{\beta}_m(\tau)\,, \quad k \geq 1\,, \tag{2.12}$$

where

$$\tilde{\mathcal{R}}_{km}(v) = \sum_l \mathcal{R}_{kl}(v)U_{ml}(\varphi)\,,$$

so that $\|\tilde{\mathcal{R}}(v)\|_{HS} = \|\mathcal{R}(v)\|_{HS} < \infty$ for each v. As before, equations (2.3) and (2.12) have the same sets of weak solutions.

We have established

LEMMA 2.2. *Equations* (2.12) *have the same set of regular weak solutions as equations* (2.11)*, and equations* (2.10) *as equations* (1.7)*. The Wiener processes* $\{\beta_r(\tau),\, r \geq 1\}$ *and* $\{\tilde{\beta}_m(\tau),\, m \geq 1\}$ *are related by formula* (2.9)*, where* $v(\tau) = (I(\tau), \varphi(\tau))$ *and the unitary matrix* $U(\varphi)$ *satisfies* (2.7)*.*

We also note that if a process $v(\tau)$ satisfies only one equation (2.12), then it also satisfies the corresponding equation (2.11).

3 Lifting of Solutions

3.1 The theorem. In this section we prove an assertion which in some sense is inverse to that of Proposition 2.1. For any $\vartheta \in \mathbb{T}^\infty$ and any vector $I \in h_I^p$ we set

$$V_\vartheta(I) = (\mathbf{V}_{\vartheta\,1}, \mathbf{V}_{\vartheta\,2}, \dots) \in h^p, \quad \mathbf{V}_{\vartheta\,r} = \mathbf{V}_{\vartheta_r}(I_r)\,, \quad \text{where}$$
$$\mathbf{V}_\alpha(J) = (\sqrt{2J}\cos\alpha, \sqrt{2J}\sin\alpha)^t \in \mathbb{R}^2.$$

Then $\varphi_j(V_\vartheta(I)) = \vartheta_j \ \forall j$ and for every ϑ the map $I \mapsto V_\vartheta(I)$ is right-inverse to the map $v \mapsto I(v)$. For $N \geq 1$ and any vector I we denote

$$I^{>N} = (I_{N+1}, I_{N+2}, \dots)\,, \quad V_\vartheta^{>N}(I) = (\mathbf{V}_{\vartheta\,N+1}(I), \mathbf{V}_{\vartheta\,N+2}(I), \dots)\,.$$

Theorem 3.1 (Lifting). *Let* $I^0(\tau) = (I_k^0(\tau),\, k \geq 1,\, 0 \leq \tau \leq T)$*, be a weak solution of system* (1.7)*, constructed in Theorem 1.3. Then, for any vector* $\vartheta \in \mathbb{T}^\infty$*, there is a regular weak solution* $v(\tau)$ *of system* (2.3) *such that*

(i) *the law of* $I(v(\,\cdot\,))$ *in the space* $\mathcal{H}_I$ *(see* (1.11)*) coincides with that of* $I^0(\,\cdot\,)$*;*
(ii) $v(0) = V_\vartheta(I_0)$ *a.s.*

Proof of the theorem is based on a simple idea, which is obscured by serious technical complications. We start by sketching the idea and explaining the technical difficulties. The complete proof is given in the next subsection.

Sketch of the proof. Let us rewrite the averaged system for $I(\tau)$ in the form (2.10) and the effective v-equation in the form (2.12). It suffices to show that they have weak solutions $I(\tau)$ and $v(\tau)$, corresponding to the same Wiener processes $\tilde{\beta}_m(\tau)$, such that the law of $I(\tau)$ equals to the law of $I^0\tau)$, $I(v(\tau)) \equiv I(\tau)$ and (ii) holds. Abusing notation a bit, we will denote a solution for (2.10), distributed as I^0, by I^0. We regard (2.10), (2.12) as a system of stochastic equations on $(I, v)(\tau)$ and approximate it by a system which is finite-dimensional in v:

$$dI_k(\tau) = F_k(I)d\tau + \sum_{m\ge1} M_{km}(I)d\tilde\beta_m(\tau)\,, \quad k\ge 1\,, \tag{3.1}$$

$$d\mathbf{v}_k(\tau) = R_k(v)d\tau + \sum_{m\ge1} \tilde{\mathcal{R}}_{km}(v)d\tilde\beta_m(\tau)\,, \quad k\le N\,. \tag{3.2}$$

0n the r.h.s. of e.g. (3.2) we replace $v(\tau)$ by $v=(v^N(\tau), V_\vartheta^{>N}(I(\tau))$, where $v^N(\tau) = (\mathbf{v}_1,\dots,\mathbf{v}_N)(\tau)\in\mathbb{R}^{2N}$. So (3.1), (3.2) is a system of equations on the vector (I, v^N). We are looking for solutions such that

$$I(\mathbf{v}_k(\tau)) \equiv I_k(\tau) \quad \forall\, k\le N\,. \tag{3.3}$$

For the I-component of a solution we take $I=I^0$. Then it remains to find $v^N(\tau)$, satisfying (3.2), (3.3).

Assume that for each N we constructed a solution (I,v^N) and denote by ${}^Nv(\tau)$ the process ${}^Nv(\tau) = (v^N(\tau), V_\vartheta^{>N}(I(\tau))$. Due to (3.3) and (1.12) the family of measures $\mathcal{D}(I(\,\cdot\,), {}^Nv(\,\cdot\,))$ is tight in the space $\mathcal{H}_I\times\mathcal{H}_v$. Any limiting measure is a weak solution for (2.10), (2.12) and the v-component of this solutions satisfies (i) and (ii).

It remains for each $N\ge1$ to find a solution for (3.1)–(3.3). To do this, for any fixed $k\le N$ consider the equation for $\varphi_k(\tau)=\varphi(\mathbf{v}_k(\tau))$ which follows from (3.2) by the Ito formula,

$$d\varphi_k(\tau) = R_k^\varphi(v)d\tau + \sum_{m\ge1}\tilde{\mathcal{R}}^\varphi_{km}(v)d\tilde\beta_m(\tau)\,, \tag{3.4}$$

where

$$R_k^\varphi(v) = \left(\nabla_{\mathbf{v}_k}\arctan\left(\frac{v_k}{v_{-k}}\right)\right)\cdot R_k(v) + \frac12\sum_{m\ge1}\left(\nabla^2_{\mathbf{v}_k}\arctan\left(\frac{v_k}{v_{-k}}\right)\right)\tilde{\mathcal{R}}_{km}\cdot\tilde{\mathcal{R}}_{km}\,,$$

$$\tilde{\mathcal{R}}^\varphi_{km}(v) = \left(\nabla_{\mathbf{v}_k}\arctan(\frac{v_k}{v_{-k}})\right)\cdot\tilde{\mathcal{R}}_{km}(v)\,.$$

Here $\cdot$ stands for the inner product in $\mathbb{R}^2$. If $\mathbf{v}_k(\tau)\ne0$, then near τ equation (3.2) is equivalent to (3.1)+(3.4). Let us denote

$$[I] = \min_{1\le j\le N}\{I_j\}\,. \tag{3.5}$$

For any $\delta>0$ we can cover $[0,T]$ by a countable union of random closed intervals $\Lambda_j, j\ge1$, and $\Delta_j, j\ge1$, such that

(i) $\Lambda_0\le\Delta_1\le\Lambda_1\le\Delta_2\le\dots$ (any two neighboring intervals intersect each other by a point),

(ii) $[I]\ge\delta$ on each Λ_j, and $[I]\le2\delta$ on each Δ_j.

We construct a solution v^N on these intervals iteratively. Assume that we know it at the left end point of some Λ_j. To construct $v^N(\tau)$ on Λ_j we note that since every $|\mathbf{v}_k(\tau)|^2 = 2I_k\ge2\delta$ on Λ_j, then we can replace each pair of equations (3.1), (3.2) with $k \le N$ by the pair (3.1), (3.4). That is, replace the system (3.1)−(3.3) by the system (3.1), (3.4) (in the r.h.s. of (3.4) we replace $\mathbf{v}_k$ by $\mathbf{V}_{\varphi_k}(I_k)$ if $k \le N$ and replace it by $\mathbf{V}_{\vartheta_k}(I_k)$ if $k > N$). Since $I(\tau) = I^0(\tau)$ is known, it remains to solve (3.4), regarded as a stochastic equation with progressively measurable coefficients for the vector $\varphi^N(\tau) = (\varphi_1, \dots, \varphi_N)(\tau)$. Such a solution φ^N exists. It defines $v^N(\tau)$ for $\tau \in \Lambda_j$.

On each interval Δ_j we have $[I] \le 2\delta$. By (1.13) the union of these intervals becomes small with δ. So if we can extend $v^N(\tau)$ to these intervals "in a controllable way", keeping property (3.3), we may next go to a limit as $\delta \to 0$ to construct a required weak solution of (3.1)−(3.3). The task of extending $v^N(\tau)$ to the intervals Δ_j turned out to be surprisingly complicated. Indeed, by (3.3) we have $v_k^+(\tau) = \sqrt{2I_k(\tau)}\cos\varphi_k(\tau)$, $v_k^-(\tau) = \sqrt{2I_k(\tau)}\sin\varphi_k(\tau)$ with an unknown phase φ_k. So a priori $|\dot{\mathbf{v}}_k| \sim \delta^{-1/2}$ and the restriction of v^N to $\cup\Delta_j$ becomes non-negligible as $\delta \to 0$. To construct a "right" lifting $\mathbf{v}_k(\tau)$ of $I_k(\tau)$ when I_k is small, we use the fact that the process $I(\tau)$ is a limit of the processes $I^\nu(\tau) = I(v^\nu(\tau))$, where v^ν solves (1.3). The process $v^\nu(\tau)$ is a lifting of $I^\nu(\tau)$ which is singular (i.e. very fast) as $\nu \to 0$. In [KuPi] we suggest a rather involved construction (repeated below in the Appendix) which modifies any process $\mathbf{v}_k^\nu(\tau)$, $k \ge 1$, to a process $\tilde{\mathbf{v}}_k^\nu(\tau)$ such that $I(\tilde{\mathbf{v}}_k^\nu) = I(\mathbf{v}_k^\nu)$ and $\left|\frac{d}{d\tau}\tilde{\mathbf{v}}_k^\nu\right| \sim 1$ as $\nu \to 0$. A limit in the distribution of the processes $\tilde{\mathbf{v}}_k^\nu(\tau)$ as $\nu = \nu_j \to 0$ gives us a 'right' lifting of I_k. Taking $k = 1, \dots, N$ we get an extension of the process $v^N(\tau)$ to Δ_j, keeping the property (3.3), and such that $|\dot{v}^N| \sim 1$ as $\delta \to 0$.

Iterating these two constructions we get a process $(I, v^N)(\tau)$, $0 \le \tau \le T$, which solves (3.1)−(3.3) for $\tau \in \cup\Lambda_j$ and satisfies a good estimate on the remaining small set $\cup\Delta_j$. In Lemma 3.4 we show that any limit distribution of the processes (I, v^N) as $\delta \to 0$ is a weak solution for (3.1)−(3.3). The lemma's proof is straightforward but long; it occupies subsection 3.3. Considering the processes ${}^N v = (v^N(\tau), V_\vartheta^{>N}(I^0(\tau))$ and sending $N \to \infty$ we get a required weak solution $v(\tau)$.

The construction, explained above, becomes complicates when the norm of the process $I(\tau)$ is large. So, in fact, we begin the proof by introducing the stopping times $\tau_P = \inf_\tau\{|I(\tau)|_{h_I^p} = P\}$, and replacing $I^0(\tau)$ by a trivial modification for $\tau \ge \tau_P$. We construct a weak solution $v_P(\tau)$, corresponding to the modified process $I^0(\tau)$, and next go to the limit as $P \to \infty$ to get a real solution $v(\tau)$.

3.2 Proof of Theorem 3.1.

Step 1. *Redefining the equations for large amplitudes.*

For any $P \in \mathbb{N}$ consider the stopping time

$$\tau_P = \inf\left\{\tau \in [0,T] \;\middle|\; |v(\tau)|_p^2 \equiv |I(v(\tau))|_{h_I^p} = P\right\}$$

(here and in similar situations below $\tau_P = T$ if the set is empty). For $\tau \ge \tau_P$ and each $\nu > 0$ we redefine equations (1.3) to the trivial system

$$d\mathbf{v}_k = b_k d\boldsymbol{\beta}_k(\tau)\,, \quad k \ge 1\,, \tag{3.6}$$

and redefine accordingly equations (1.5) and (1.7). We will denote the new equations as $(1.3)_P$, $(1.5)_P$ and $(1.7)_P$. If $v_P^\nu(\tau)$ is a solution of $(1.3)_P$, then $I_P^\nu(\tau) = I(v_P^\nu(\tau))$

satisfies $(1.5)_P$. That is, for $\tau \le \tau_P$ it satisfies (1.5), while for $\tau \ge \tau_P$ it is a solution of the Itô equations

$$dI_k = \tfrac{1}{2} b_k^2\, d\tau + b_k (v_k\, d\beta_k + v_{-k}\, d\beta_{-k}) = \tfrac{1}{2} b_k^2\, d\tau + b_k \sqrt{2I_k}\, dw_k(\tau)\,, \quad k \ge 1\,, \tag{3.7}$$

where $w_k(\tau)$ is the Wiener process $\int^\tau (\cos\varphi_k\, d\beta_k + \sin\varphi_k\, d\beta_{-k})$. So $(1.5)_P$ is the system of equations

$$dI_k = \chi_{\tau\le\tau_P} \cdot \big\langle \text{r.h.s. of } (1.5)\big\rangle + \chi_{\tau\ge\tau_P} \Big(\tfrac{1}{2} b_k^2\, d\tau + b_k \sqrt{2I_k}\, dw_k(\tau)\Big)\,, \quad k \ge 1\,. \tag{3.8}$$

Accordingly, the averaged system $(1.7)_P$ may be written as

$$dI_k = \chi_{\tau\le\tau_P} \Big(F_k(I) d\tau + \sum_j K_{kj}(I) d\beta_j(\tau)\Big) + \chi_{\tau\ge\tau_P} \Big(\tfrac{1}{2} b_k^2\, d\tau + b_k \sqrt{2I_k}\, d\beta_k(\tau)\Big)\,, \tag{3.9}$$

$k \ge 1$. Here (as in (2.8)) $F_k\, d\tau$ abbreviates the drift in e.g. (1.7), and for $\tau \ge \tau_P$ we replaced the Wiener process w_k by the process β_k; this does not change weak solutions of the system.

Similarly to v^ν and I^ν (see Lemma 4.1 in [KuPi]), the processes v_P^ν and I_P^ν meet the estimates

$$\mathbf{E} \sup_{0\le\tau\le T} |I(\tau)|^M_{h_I^m} = \mathbf{E} \sup_{0\le\tau\le T} |v(\tau)|^{2M}_{h_I^m} \le C(M,m,T)\,, \tag{3.10}$$

uniformly in $\nu \in (0,1]$.

Due to Theorem 1.3 for a sequence $\nu_j \to 0$ we have $\mathcal{D}(I^{\nu_j}(\,\cdot\,)) \rightharpoonup \mathcal{D}(I^0(\,\cdot\,))$. Choosing a suitable subsequence we achieve that also $\mathcal{D}(I_P^{\nu_j}(\,\cdot\,)) \rightharpoonup \mathcal{D}(I_P(\,\cdot\,))$ for some process $I_P(\tau)$, for each $P \in \mathbb{N}$. Clearly $I_P(\tau)$ satisfies estimates (3.10).

LEMMA 3.2. *For any $P \in \mathbb{N}$, $I_P(\tau)$ is a weak solution of $(1.7)_P$=(3.9) such that $\mathcal{D}(I_P) = \mathcal{D}(I^0)$ for $\tau \le \tau_P$ (that is, images of the two measures under the mapping $I(\tau) \mapsto I(\tau \wedge \tau_P)$ are equal) and $\mathcal{D}(I_P(\,\cdot\,)) \rightharpoonup \mathcal{D}(I^0(\,\cdot\,))$ as $P \to \infty$.*

Proof. The process $I_P^\nu(\tau)$ satisfies the system of Itô equations $(1.5)_P$=(3.8) which we now abbreviate as

$$dI_{Pk}^\nu = \mathfrak{F}_k\big(\tau, v_P^\nu(\tau)\big) d\tau + \sum_j \mathcal{S}_{kj}\big(\tau, v_P^\nu(\tau)\big) d\beta_j(\tau)\,, \quad k \ge 1\,. \tag{3.11}$$

Denote by $\langle \mathfrak{F}\rangle_k(\tau, I)$ and $\langle \mathcal{S}\mathcal{S}^t\rangle_{km}(\tau, I)$ the averaged drift and diffusion. Then

$$\langle \mathfrak{F}\rangle_k = \chi_{\tau\le\tau_P} F_k(I) + \chi_{\tau\ge\tau_P} \tfrac{1}{2}\, b_k, \quad \langle \mathcal{S}\mathcal{S}^t\rangle_{km} = \chi_{\tau\le\tau_P} S_{km}(I) + \chi_{\tau\ge\tau_P} \delta_{km} b_k^2 2I_k$$

(cf. (2.8) and (1.9)). We claim that

$$\Upsilon_\nu^q := \mathbf{E} \sup_{0\le\tau\le T} \left| \int_0^\tau \Big(\mathfrak{F}_k(s, v_P^\nu(s)) - \langle \mathfrak{F}\rangle_k (s, I_P^\nu(s)) ds\Big) \right|^q \to 0 \quad \text{as } \nu \to 0\,, \tag{3.12}$$

for $q = 1$ and 4. Indeed, since $\mathfrak{F}_k = \langle \mathfrak{F}\rangle_k$ for $\tau \ge \tau_P$ and $v_P^\nu = v^\nu$, $I_P^\nu = I^\nu$ for $\tau \le \tau_P$, then

$$\Upsilon_\nu^q \le \mathbf{E} \sup_{0\le\tau\le T} \left| \int_0^\tau \mathfrak{F}_k\big(s, v^\nu(s)\big) - F_k\big(I^\nu(s)\big) ds \right|^q .$$

But the r.h.s. goes to zero with ν, see in [KuPi, Prop. 5.2 & relation (6.17)]. So (3.12) holds true.

Relations (3.11) and (3.12) with $q = 1$ imply that for each k the process $Z_k(\tau) = I_k(\tau) - \int_0^\tau \langle \mathfrak{F}_k \rangle \, ds$, regarded as the natural process on the space $\mathcal{H}_I$, given the natural filtration and the measure $\mathcal{D}(I_P)$, is a square integrable martingale, cf. Proposition 6.3 in [KuPi]. Using the same arguments and (3.12) with $q = 4$ we see that for any k and m the process $Z_k(\tau)Z_m(\tau) - \int_0^\tau \langle \mathcal{S}\mathcal{S}^t \rangle_{km} \, ds$ also is a $\mathcal{D}(I_P)$-martingale. This means that the measure $\mathcal{D}(I_P)$ is a solution of the martingale problem for e.g. $(1.7)_P$=(3.9). That is, $I_P(\tau)$ is a weak solution of $(1.7)_P$.

Since $\mathcal{D}(I_P^\nu) = \mathcal{D}(I^\nu) =: \mathbf{P}^\nu$ for $\tau \le \tau_P$, then passing to the limit as $\nu_j \to 0$ we get the second assertion of the lemma. As $\mathbf{P}^\nu\{\tau_P < T\} \le CP^{-1}$ uniformly in ν (cf. (3.10)), then the last assertion also follows. □

Step 2. *Equations for* v^N.

By Lemma 2.2 the process $I^0(\tau)$ satisfies (2.10). For any $N \in \mathbb{N}$ we consider the Galerkin-like approximation (3.1), (3.2) for equations (2.12), coupled with e.g. (2.10). As at Step 1 we redefine the I-equations (3.1) after τ_P to equations (3.7) and the v-equations to (3.6). We denote the system thus obtained by $\mathbf{S}_v$. By Lemma 3.2 the process $I_P(\tau)$ satisfies the new I-equations, and we will take $I_P(\tau)$ for the I-component of a solution for $\mathbf{S}_v$. To solve $\mathbf{S}_v$ for $0 \le \tau \le T$ we first solve (3.1)+(3.2) till time τ_P and next solve the trivial system (3.6) for $\tau \in [\tau_P, T]$. The second step is obvious. So we will mostly analyse the first step.

Denote

$$\hat{\Omega} = \Omega_I \times \Omega_N = C(0, T; h_I^p) \times C(0, T; \mathbb{R}^{2N}),$$

and denote by π_I, π_N the natural projections $\pi_I : \hat{\Omega} \to \Omega_I$, $\pi_N : \hat{\Omega} \to \Omega_N$. Provide the Banach spaces $\hat{\Omega}, \Omega_I$ and Ω_N with the Borel sigma-algebras and the natural filtrations of sigma-algebras. Let $\{\tilde{\mathcal{F}}_t,\ 0 \le t \le T\}$, be the filtration for $\tilde{\Omega}$.

Our goal is to construct a weak solution of system $\mathbf{S}_v$ such that its distribution $\mathbf{P} = \mathbf{P}_P^N = \mathcal{D}(I, v^N)$ satisfies $\pi_I \circ \mathbf{P} = \mathcal{D}(I_P(\,\cdot\,))$ and $I(v^N(\,\cdot\,)) = I^N(\,\cdot\,)$ $\mathbf{P}$-a.s. After that we will go to a limit as $P \to \infty$ and $N \to \infty$ to get a required weak solution v of (2.3).

We will construct $\mathbf{P}_P^N$ as the limit when $\delta \to 0$ of measures $\mathbf{P}_\delta = \mathcal{D}(I(\,\cdot\,), v^N(\,\cdot\,))$, where the process $(I(\tau), v^N(\tau)$ "solves $\mathbf{S}_v$ for τ outside the (small) random set, where $[I(\tau)] \lesssim \delta$" (see (3.5)).

Step 3. *Construction of a measure* $\mathbf{P}_\delta$.

Fix any positive δ. For the process $I(\tau) = I^0(\tau)$ we define stopping times $\theta_j^\pm \le T$ such that $\dots < \theta_j^- < \theta_j^+ < \theta_{j+1}^- < \dots$ as follows:

- If $[I(0)] \le \delta$, then $\theta_1^- = 0$; otherwise $\theta_0^+ = 0$.
- If θ_j^- is defined, then θ_j^+ is the first moment after θ_j^- when $[I(\tau)] \ge 2\delta$ (if this never happens, then we set $\theta_j^+ = T$; similar in the item below).
- If θ_j^+ is defined, then θ_{j+1}^- is the first moment after θ_j^+ when $[I(\tau)] \le \delta$.

We denote $\Delta_j = [\theta_j^-, \theta_j^+]$, $\Lambda_j = [\theta_j^+, \theta_{j+1}^-]$ and set

$$\Delta = \Delta^\delta = \cup \Delta_j \,, \quad \Lambda = \Lambda^\delta = \cup \Lambda_j \,.$$

For segments $[0, \theta_j^-]$ and $[0, \theta_j^+]$, which we denote below $[0, \theta_j^\pm]$, we will iteratively construct processes $(I, v^N)(\tau) = (I, v^N)^{j,\pm}(\tau)$ such that $\mathcal{D}(I(\cdot)) = \mathcal{D}(I_P(\cdot))$, $v^N(\tau) = v^N(\tau \wedge \theta_j^\pm)$ and $\mathcal{D}(I^N(\tau)) = \mathcal{D}(I(v^N(\tau))$ for $\tau \le \theta_j^\pm$. Moreover, on each segment $\Lambda_r \subset [0, \theta_j^\pm]$ the process (I, v^N) will be a weak solution of $\mathbf{S}_v$. Next we will obtain a desirable measure $\mathbf{P}_\delta$ as a limit of the laws of these processes as $j \to \infty$.

For the sake of definiteness assume that $0 = \theta_0^+$.

a) $\tau \in \Lambda_0$. We wish to construct a weak solution (I, v^N) of system $\mathbf{S}_v$ for $0 \le \tau \le \theta_1^-$ such that, as before, $\mathcal{D}(I) = \mathcal{D}(I_P)$. We will only show how to do this on the segment $[0, \theta_1^- \wedge \tau_P]$ since construction of a solution for $\tau \ge \tau_P$ is trivial.

Let $(I(\tau), v^N(\tau))$ be the natural process on $\tilde\Omega$ (corresponding to some measure on $\tilde\Omega$). Assume that for some fixed $k \le N$ its component $\mathbf{v}_k(\tau) \in \mathbb{R}^2$ satisfies equation $(3.2)_k$. Then $I_k(\tau) = I(\mathbf{v}_k(\tau))$ satisfies $(3.1)_k$, and for $0 \le \tau \le \theta_1^-$ the angle $\varphi_k(\tau) = \varphi(\mathbf{v}_k(\tau))$ satisfies equation (3.4). Clearly on the domain $\{v^N \in \mathbb{R}^{2N} \mid [I(v^N)] \ge \delta,\ |v^N|_p \le P\}$ the factors $\nabla_{\mathbf{v}_k} \arctan(v_k/v_{-k})$ and $\nabla^2_{\mathbf{v}_k} \arctan(v_k/v_{-k})$ are smooth, Lipschitz and bounded.

Since the mapping $\mathbf{v} \mapsto (I, \varphi)$ is a diffeomorphism of the domains $\{|\mathbf{v}| > \delta\}$ and $\left(\frac12 \delta^2, \infty\right) \times S^1$, then, on the contrary,

$$\begin{aligned}&\text{if for } \tau \le \theta_1^- \text{ the process } I_k(\tau) \text{ satisfies } (3.1)_k \text{ and } \varphi_k(\tau) \text{ satisfies}\\ &(3.4), \text{ where } \mathbf{v}_k(\tau) = \mathbf{V}_{\varphi_k(\tau)}(I_k(\tau)), \text{ then } \mathbf{v}_k(\tau) \text{ satisfies } (3.2)_k\,.\end{aligned} \tag{3.13}$$

Similar assertions hold for the equations modified after the stopping time τ_P,

LEMMA 3.3. *For any positive δ and for ϑ as in Theorem 3.1, for $0 \le \tau \le \vartheta_1^-$, the system $\mathbf{S}_v$ has a weak solution (I, v^N) such that $\mathcal{D}(I(\cdot)) = \mathcal{D}(I_P(\cdot))$ and $\frac12 |\mathbf{v}_k|^2(\tau) \equiv I_k(\tau)$, $\mathbf{v}_k(0) = \mathbf{V}_{\vartheta\, k}(I_0)$ for $k \le N$.*

Proof. Denote by $\mathbf{S}_\varphi$ the system $(3.1)_P$+$(3.4)_{P,\,1\le k\le N}$. Its solution is a process $(I(\tau), \varphi^N(\tau))$ and in the φ-equations we substitute $\mathbf{v}_k = \mathbf{V}_{\varphi_k}(I_k(\tau))$, $1 \le k \le N$.

For any integer $M \ge N$ of the form $M = M(n)$, $n \ge 1$ (see (2.6)), we call the *M-truncation of system* $\mathbf{S}_\varphi$ a system, obtained from $\mathbf{S}_\varphi$ by omitting in equations (3.4) with $1 \le k \le N$ the terms $\tilde{\mathcal{R}}^\varphi_{km}(v) d\tilde\beta_m$ with $m > M$. The dispersion matrix for the modified φ-equations is $\tilde{\mathcal{R}}^{\varphi M} = \{\tilde{\mathcal{R}}^\varphi_{km}, 1 \le k \le N, 1 \le m \le M\}$. Since $U(\varphi) L^n = L^n$, then for $m \le M = M(n)$ we have $U_{jm}(\varphi) = 0$ if $j > M$. So for $m \le M$

$$\tilde{\mathcal{R}}_{km}(v) = \sum_{l=1}^M \mathcal{R}_{kl}(v) U_{lm}(\varphi)\,.$$

This relation and (2.7) imply that for $0 \le \tau \le \theta_1^- \wedge \tau_P$ the diffusion matrix $\tilde{\mathcal{R}}^{\varphi M}$ is Lipschitz continuous in φ. In the M-truncation of the system $\mathbf{S}_\varphi$ the I-component is known (it equals I^0) and the φ-equations form a system with progressively measurable coefficients, Lipschitz in φ^N. So it has a unique strong solution $\varphi^{N,M}$, e.g. see in [Kry2]. This gives us a solution $(I^0, \varphi^{N,M})(\tau)$, $0 \le \tau \le \theta_1^-$, for the M-truncated system. We first extend this solution to the segment $[\theta_1^- \wedge \tau_P, \theta_1^-]$, and next to $[0,T]$ by setting $\varphi^{N,M}(\tau) = \varphi^{N,M}(\tau \wedge \theta_1^-)$.

For any N and for $0 \le \tau \le \theta_1^- \wedge \tau_P$ we have

$$\big\|\tilde{\mathcal{R}}^{\varphi M}(v)\big\|_{HS} \le C(\delta,P)\big\|\tilde{\mathcal{R}}(v)\big\|_{HS} = C(\delta,P)\|\mathcal{R}(v)\|_{HS}\,.$$

Since all moments of the random variable $\sup_\tau \|\mathcal{R}(v(\tau))\|_{HS}$ are finite, then the family of processes $(I_P, \varphi^{N,M}) \in h_P^I \times \mathbb{T}^N$, $M \ge 1$, is tight. Any limiting as $M \to \infty$ measure solves for $0 \le \tau \le \theta_1^-$ the martingale problem, corresponding to system $\mathbf{S}_\varphi$. So this is a law of a weak solution (I_P, φ^N) of that system (i.e. $(I_P,\varphi^N)(\tau)$ satisfies the system with suitably chosen Wiener processes $\tilde{\beta}_m$). Due to (3.13) the process (I_P, v^N) with $\mathbf{v}_j(\tau) = \mathbf{V}_{\varphi_j(\tau)}(I_{Pj}(\tau))$, $j \le N$, is a weak solution of $\mathbf{S}_v$. We have constructed a desirable weak solution $(I, v^N)(\tau)$, $0 \le \tau \le \theta_1^-$. □

We denote by $\mathbf{P}_1^-$ the law of the constructed solution (I, v^N). This is a measure in $\hat{\Omega}$, supported by trajectories (I, v^N) such that $v^N(\tau)$ is stopped at $\tau = \theta_1^-$.

b) Now we extend $\mathbf{P}_1^-$ to a measure $\mathbf{P}_1^+$ on $\hat{\Omega}$, supported by trajectories (I, v^N), where v^N is stopped at time θ_1^+.

Let us denote by $\Theta = \Theta^{\theta_1^-}$ the operator which stops any continuous trajectory $\eta(\tau)$ at time $\tau = \theta_1^-$. That is, replaces it by $\eta(\tau \wedge \theta_1^-)$.

Since $\mathcal{D}(I_P^\nu(\,\cdot\,)) \rightharpoonup \mathcal{D}(I_P(\,\cdot\,))$ as $\nu = \nu_j \to 0$, then we can represent the laws $\mathbf{P}_1^-$ and $\mathcal{D}(v_P^\nu))$ by distributions of processes $(I_P'(\tau), {v_P'}^N(\tau))$ and ${v_P'}^\nu(\tau)$, defined on a new probability space Ω, such that

$$\big({v_P'}^\nu(\,\cdot\,)\big) \to I_P'(\,\cdot\,) \quad \text{as } \nu = \nu_j \to 0 \text{ in } \mathcal{H}_I \text{ a.s.}\,,$$

and

$$I({v_P'}^N) \equiv {I_P'}^N \quad \text{for } \tau \le \theta_1^-\,.$$

Since ${v_P'}^\nu(\tau,\omega)$, $0 \le \tau \le T$, is a diffusion process, we may replace it by a continuous process $w_P^\nu(\tau;\omega,\omega_1)$ on an extended probability space $\Omega \times \Omega_1$ such that

1. $\mathcal{D}\, w_P^\nu = \mathcal{D}\, {v_P'}^\nu$;
2. for $\tau \le \theta_1^- = \theta_1^-(\omega)$ we have $w_P^\nu = {v_P'}^\nu$ (in particular, then w_P^ν is independent of ω_1);
3. for $\tau \ge \theta_1^-$ the process w_P^ν depends on ω only through the initial data $w_P^\nu(\theta_1^-,\omega,\omega_1) = {v_P'}^\nu(\theta_1^-,\omega)$. For a fixed ω it satisfies $(1.3)_P$ with suitable Wiener processes β_j's, defined on the space Ω_1.

Using a construction from [KuPi], presented in the Appendix, for each ω we construct a continuous process $(\bar{w}^\nu, \widetilde{w}^{\nu N})(\tau;\omega,\omega_1) \in h^p \times \mathbb{R}^{2N}$, $\tau \ge \theta_1^-$, $\omega_1 \in \Omega_1$, such that for each ω we have

(i) the law of the process $\bar{w}^\nu(\tau;\omega,\omega_1)$, $\tau \ge \theta_1^-$, $\omega_1 \in \Omega_1$, is the same as of the process $w_P^\nu(\tau;\omega,\omega_1)$;
(ii) $I(\widetilde{w}^{\nu N}) = I^N(\bar{w}^\nu)$ for $\tau \ge \theta_1^-$, and $\varphi\big(\widetilde{w}^{\nu N}(\theta_1^-)\big) = \varphi({v_P'}^N(\theta_1^-))$ a.s. in Ω_1;
(iii) the law of the process $\widetilde{w}^{\nu N}(\tau)$, $\tau \ge \theta_1^-$, is that of an Itô process

$$dv^N = B^N(\tau)d\tau + a^N(\tau)dw(\tau)\,, \tag{3.14}$$

where for every τ the vector $B^N(\tau)$ and the matrix $a^N(\tau)$ satisfy ν-independent estimates

$$\left|B^N(\tau)\right| \le C\,,\quad C^{-1}I \le a^N(a^N)^t(\tau) \le CI \quad \text{a.s.}\,, \tag{3.15}$$

with some $C = C(P,M)$.

Next for $\nu = \nu_j$ consider the process

$$\xi_P^\nu(\tau) = \left(I_P^\nu(\tau) = I(\bar w^\nu(\tau))\,,\ \chi_{\tau\le\theta_1^-} {v'_P}^N + \chi_{\tau>\theta_1^-}\widetilde{w}^{\nu N}\right),\quad 0\le\tau\le T\,.$$

Due to (3.10) and (iii) the family of laws $\{\mathcal{D}(\xi_P^{\nu_j}),\, j\ge 1\}$, is tight in the space $C(0,T;h_p^I\times\mathbb{R}^{2N})$. Consider any limiting measure Π (corresponding to a suitable subsequence $\nu'_j\to 0$) and represent it by a process $\tilde\xi_P(\tau) = (\tilde I_P(\tau), \tilde v_P^N(\tau))$, i.e. $\mathcal{D}\tilde\xi_P = \Pi$. Clearly,

(iv) $\mathcal{D}(\tilde\xi_P)\,|_{\tau\le\theta_1^-} = \mathbf{P}_1^-$,

(v) $\mathcal{D}(\tilde I_P) = \mathcal{D}(I_P)$.

Since any measure $\mathcal{D}(\xi_P^\nu)$ is supported by the closed set, formed by all trajectories $(I(\tau), v^N(\tau))$ satisfying $I^N\equiv I(v^N)$, then the limiting measure Π also is supported by it. That is, the process $\tilde\xi_P$ satisfies

(vi) $I(\tilde v_P^N(\tau)) \equiv \tilde I_P^N(\tau)$ a.s.

Moreover, for the same reasons as in the Appendix, the law of the limiting process $\tilde v_P^N(\tau)$, $\tau\ge\theta_1^-$, is that of an Itô process (3.14), (3.15). (Note that for $\tau\ge\theta_1^-$ the process $\tilde v_P^N$ *is not* a solution of (3.2)).

Now we set

$$\mathbf{P}_1^+ = \Theta^{\theta_1^+}\circ\mathcal{D}(\tilde\xi_P)\,.$$

c) The constructed measure $\mathbf{P}_1^+$ gives us the distribution of a process $(I(\tau), v^N(\tau))$ for $\tau\le\theta_1^+$. Next we solve system $\mathbf{S}_v$ on the interval $\Lambda_1 = [\theta_1^+, \theta_2^-]$ with the initial data $(I(\theta_1^+), v^N(\theta_1^+))$ and iterate the construction.

It is easy to see that a.s. the sequence $\theta_j^\pm$ stabilises at $\tau = T$ after a finite (random) number of steps. Accordingly, the sequence of measures $\mathbf{P}_j^\pm$ converges to a limiting measure $\mathbf{P}_\delta$ on $\hat\Omega$.

d) On the space $\tilde\Omega$, given the measure $\mathbf{P}_\delta$, consider the natural process which we denote $\xi_\delta(\tau) = (I_\delta(\tau), v_\delta^N(\tau))$. We have

1. $\mathcal{D}(I_\delta(\cdot)) = \mathcal{D}(I_P)$,
2. $I(v_\delta^N(\cdot)) \equiv I_\delta^N$ a.s.,
3. for $\tau\in\Lambda^\delta$ the process ξ_δ is a weak solution of $\mathbf{S}_v$, while for $\tau\in\Delta^\delta$ the process $v_\delta^N(\tau)$ is distributed as an Itô process (3.14).

Step 4. *Limit $\delta\to 0$.*

Due to 1–3 the set of measures $\{\mathbf{P}_\delta,\, 0<\delta\le 1\}$ is tight. Let $\mathbf{P}_P$ be any limiting measure as $\delta\to 0$. Clearly it meets 1 and 2 above.

LEMMA 3.4. *The measure $\mathbf{P}_P$ is a solution of the martingale problem for system $\mathbf{S}_v$.*

The lemma is proved in the next subsection.

Step 5. *Limit* $P \to \infty$.

Due to 1, 2 above, relations (3.10) and Lemma 3.4 the set of measures $\mathbf{P}_P$, $P \to \infty$, is tight. Consider any liming measure $\mathbf{P}^N$ for this family. Repeating in a simpler way the proof of Lemma 3.4, we find that $\mathbf{P}^N$ solves the martingale problem (3.1) + (3.2). It still satisfies 1 and 2 (see Step 3d)). Let $(I(\tau), v^N(\tau))$ be a weak solution for (3.1) + (3.2) such that its law equals $\mathbf{P}^N$. Denote by ${}^N v(\tau)$ the process $(v^N(\tau), V^{>N}(\tau))$ and denote by μ^N its law in the space $\mathcal{H}_v$ (see (1.11)).

Step 6. *Limit* $N \to \infty$.

Due to (1.12) the family of measures $\{\mu^N\}$ is tight in $\mathcal{H}_v$. Let $N_j \to \infty$ be a sequence such that $\mu^{N_j} \rightharpoonup \mu$.

The process ${}^N v(\tau)$ satisfies equations $(2.12)_{1\le k\le N}$ with suitable standard independent Wiener processes $\tilde\beta_m(\tau)$. Due to Lemma 2.2 and a remark made after it, the process also satisfies equations $(2.11)_{1\le k\le N}$. Repeating again the proof of Lemma 3.4 we see that μ is a martingale solution of the system $(2.11)_{1\le k\le N}$ for any $N \ge 1$. Hence, μ is a martingale solution of (2.11) and of (2.3). Let $v(\tau)$ be a corresponding weak solution of (2.11), $\mathcal{D}(v(\,\cdot\,)) = \mu$. As $\mu^{N_j} \rightharpoonup \mu$, then the process v satisfies assertions (i) and (ii) in Theorem 3.1 and the theorem is proved.

3.3 Proof of Lemma 3.4. Consider the space $\hat\Omega$ with the natural filtration $\tilde{\mathcal{F}}_\tau$, provide it with a measure $\mathbf{P}_\delta$ and, as usual, complete the sigma-algebras $\tilde{\mathcal{F}}_\tau$ with respect to this measure. As before we denote by $\xi_\delta(\tau) = (I_\delta(\tau), v_\delta^N(\tau), 0 \le \tau \le T)$, the natural process on $\hat\Omega$.

(i) For $k \ge 1$ consider the process $I_{\delta k}(\tau)$. It satisfies the I_k-equation in $\mathbf{S}_v$:

$$dI_k = F_k^P(\tau, I)d\tau + \sum M_{km}^P(\tau, I)d\tilde\beta_m(\tau)\,. \tag{3.16}$$

Here F_k^P equals F_k for $\tau \le \tau_P$ and equals $\frac12 b_k^2\ \tau > \tau_P$, while M_{km}^P equals M_{km} for $\tau \le \tau_P$ and equals $b_k\sqrt{2I_k}$ for $\tau > \tau_P$, cf. (3.9). For each $\delta > 0$ and any k the process $\chi_k^I(\tau) = I_k(\tau) - \int_0^\tau F_k^P(s, I(s))ds$ is an $\mathbf{P}_\delta$-martingale. Due to (1.12) the L_2-norm of these martingales are bounded uniformly in τ and δ. Since $\mathbf{P}_\delta \rightharpoonup \mathbf{P}_P$ and the laws of the processes χ_k^I, corresponding to $\delta \in (0,1]$ are tight in $C[0,T]$, then $\chi_k^I(\tau)$ also is an $\mathbf{P}_P$-martingale.

(ii) Consider a process $\mathbf{v}_{\delta k}$, $1 \le k \le N$. It satisfies $\mathbf{S}_v$ for $\tau \in \Lambda^\delta$ and satisfies the k-th equation in (3.14) for $\tau \in \Delta^\delta$, where the vector $B^N(\tau)$ and the operator $a^N(\tau)$, $\tau \in \Delta^\delta$, meet the estimates (3.15). So $\mathbf{v}_{\delta k}$ satisfies the Itô equation

$$\begin{aligned} d\mathbf{v}_k(\tau) &= \big(\chi_{\tau\in\Lambda^\delta} R_k^P(\tau, v) + \chi_{\tau\in\Delta^\delta} B_k^N(\tau)\big)d\tau \\ &\qquad + \chi_{\tau\in\Lambda^\delta} \sum_m \tilde{\mathcal{R}}_{km}^P(\tau, v)d\tilde\beta_m(\tau) + \chi_{\tau\in\Delta^\delta} \sum_r a_{kr}^N(\tau)dw_r(\tau) \\ &=: A_k^\delta(\tau)d\tau + \sum_{m\ge1} G_{km}^\delta(\tau, v)d\tilde\beta_m(\tau) + \sum_{r=1}^{2N} C_{kr}^\delta(\tau)dw_r(\tau)\,. \end{aligned} \tag{3.17}$$

Note that the random dispersion matrices $G^\delta(\tau)$ and $C^\delta(\tau)$ are supported by non-intersecting random time-sets.

For any $\delta > 0$ the process $\chi_k^\delta(\tau) = \mathbf{v}_k(\tau) - \int_0^\tau A_k^\delta(s)ds \in \mathbb{R}^2$ is an $\mathbf{P}_\delta$-martingale. Let us compare $\int A_k^\delta\, ds$ with the corresponding term in $\mathbf{S}_v$. For this end we consider the quantity

$$\mathbf{E} \sup_{0\le\tau\le T} \left| \int_0^\tau A_k^\delta(s)ds - \int_0^\tau R_k^P\big(s, v(s)\big)ds \right| \\ \le \mathbf{E} \int_{\Delta^\delta} \big|R_k^P(s, v(s))\big| ds + \mathbf{E} \int_{\Delta^\delta} \big|B_k^N(s)\big| ds =: \Upsilon_1 + \Upsilon_2\,. \quad (3.18)$$

By (3.10) and(1.13),

$$\Upsilon_1^2 \le \mathbf{E} \int_0^T |R_k^P|^2\, ds \cdot \mathbf{E} \int_0^T \chi_{\Delta^\delta}(s) ds \le C(P) o_\delta(1)\,.$$

Similar $\Upsilon_2 \le C(P)\, o_\delta(1)$. So (3.18) goes to zero with δ. Since the L_2-norms of the martingales χ_k^δ are uniformly bounded and their laws are tight in $C(0,T;\mathbb{R}^2)$, then $\chi_k^0(\tau) = \mathbf{v}_k(\tau) - \int_0^\tau R_k^P(s)ds$ is an $\mathbf{P}_P$-martingale. Indeed, let us take any $0 \le \tau_1 \le \tau_2 \le T$ and let $\Phi \in C_b(\hat{\Omega})$ be any function such that $\Phi(\xi(\,\cdot\,))$ depends only on $\xi(\tau)_{\,0\le\tau\le\tau_1}$. We have to show that

$$\mathbf{E}^{\mathbf{P}_P}\big((\chi_k^0(\tau_2) - \chi_k^0(\tau_1))\Phi(\xi)\big) = 0\,. \quad (3.19)$$

The l.h.s. equals

$$\lim_{\delta\to 0} \mathbf{E}^{\mathbf{P}_\delta}\big((\chi_k^0(\tau_2) - \chi_k^0(\tau_1))\Phi(\xi)\big) \\ = \lim_{\delta\to 0} \mathbf{E}^{\mathbf{P}_\delta}\bigg(\Phi(\xi)\bigg(\mathbf{v}_k(\tau_2) - \mathbf{v}_k(\tau_1) - \int_{\tau_1}^{\tau_2} R_k^P(s)ds\bigg)\bigg) \\ = \lim_{\delta\to 0} \mathbf{E}^{\mathbf{P}_\delta}\bigg(\Phi(\xi) \int_{\tau_1}^{\tau_2} \big(A_k^\delta(s) - R_k^P(s)\big)ds\bigg)$$

(we use that χ_k^δ is a $\mathbf{P}_\delta$-martingale). The r.h.s. is

$$\le C \lim_{\delta\to 0} \mathbf{E}^{\mathbf{P}_\delta} \sup_\tau \left| \int_0^\tau \big(A_k^\delta(s) - R_k^P(s)\big)ds \right| \le C \lim_{\delta\to 0} (\Upsilon_1 - \Upsilon_2) = 0\,.$$

So (3.19) is established.

(iii) For the same reasons as in (i), for each k and l the process

$$\chi_k^I(\tau)\chi_l^I(\tau) - \frac{1}{2}\int_0^\tau \sum_m M_{km}^P\big(s, I(s)\big) M_{lm}^P\big(s, I(s)\big) ds$$

is an $\mathbf{P}_P$-martingale.

(iv) Due to (3.17), for any δ and any $k, l \le N$ the process

$$\chi_k^\delta(\tau)\chi_l^\delta(\tau) - \frac{1}{2}\int_0^\tau \Big(\sum_m G_{km}^\delta G_{lm}^\delta + C_{km}^\delta C_{lm}^\delta\Big) ds \\ =: \chi_k^\delta(\tau)\chi_l^\delta(\tau) - \frac{1}{2}\int_0^\tau \big(X_{kl}(s) + Y_{kl}(s)\big) ds$$

is a $\mathbf{P}_\delta$-martingale. We compare it with the corresponding expression for system $\mathbf{S}_v$. To do this we first consider the expression

$$\mathbf{E}\sup_{0\le\tau\le T}\left|\frac{1}{2}\int_0^\tau\Big(\sum_m \tilde{\mathcal{R}}^P_{km}\tilde{\mathcal{R}}^P_{lm}-X_{kl}-Y_{kl}\Big)ds\right|$$
$$\le \mathbf{E}\,\frac{1}{2}\int_0^{\mathbb{T}}\Big|\sum_m \tilde{\mathcal{R}}^P_{km}\tilde{\mathcal{R}}^P_{lm}\Big|\chi_{s\in\Delta^\delta}ds+\mathbf{E}\,\frac{1}{2}\int_0^T\Big|\sum_m a^N_{km}a^N_{lm}\Big|\chi_{s\in\Delta^\delta}ds\,. \quad (3.20)$$

As in (ii), the r.h.s. goes to zero with δ. Hence, $\chi^0_k(\tau)\chi^0_l(\tau)-\frac{1}{2}\int_0^\tau\tilde{\mathcal{R}}^P_{km}\tilde{\mathcal{R}}^P_{lm}ds$ is an $\mathbf{P}_P$-martingale by the same arguments that prove (3.19).

(v) Finally consider the I,v-correlation. For $k\ge1$ and $1\le l\le N$ the process

$$\mathbb{R}^2\ni\chi^I_k(\tau)\chi^\delta_l(\tau)-\frac{1}{2}\int_0^\tau\sum_m M^P_{km}G^\delta_{lm}ds-\frac{1}{2}\int_0^\tau\sum_{m\ge1}\sum_{r=1}^{2N}M^P_{km}C^\delta_{lr}d[\tilde\beta_m,w_r](s)$$
$$=:\chi^I_k(\tau)\chi^\delta_l(\tau)-\frac{1}{2}\int_0^\tau\Xi^\delta_{kl}(s)ds$$

is an $\mathbf{P}_\delta$ martingale. We know that

1. the matrix $\frac{d}{ds}[\tilde\beta_m,w_r](s)$ is constant in s and is such that l_2-norms of all its columns and rows are bounded by one;
2. $\|M^P\|_{HS},\ \|C^\delta\|_{HS}\le C(P)$ for all δ.

Therefore,

$$\Big|\sum_{m\ge1}\sum_{r=1}^{2N}M^P_{km}C^\delta_{lr}\frac{d}{ds}[\tilde\beta_m,w_r](s)\Big|\le C_1(P)\,.$$

Now repeating once again the arguments in (ii) we find that

$$\mathbf{E}\sup_{0\le\tau\le T}\frac{1}{2}\left|\int_0^\tau\Big(\sum_m M^P_{km}\tilde{\mathcal{R}}^P_{lm}-\Xi^\delta_{kl}\Big)ds\right|\to0$$

as $\delta\to0$. Therefore the process $\chi^I_k(\tau)\chi^\delta_l(\tau)-\frac{1}{2}\int_0^\tau\sum_m M^P_{km}\tilde{\mathcal{R}}^P_{lm}ds$ is an $\mathbf{P}_P$-martingale.

Due to (i)–(v) the measure $\mathbf{P}_P$ is a martingale solution for e.g. $(3.1)_P$. □

4 Uniqueness of Solution

In this section we will show that a regular solution of the effective equation (2.3) (i.e. a solution that satisfies estimates (1.12)) is unique. Namely, we will prove the following result:

Theorem 4.1. *If $v^1(\tau)$ and $v^2(\tau)$ are strong regular solutions of some physical realisation of* e.g. (2.3) *such that $v^1(0)=v^2(0)$ a.s., then $v^1(\cdot)=v^2(\cdot)$ a.s.*

Using the Yamada–Watanabe arguments (see, for instance, [KaS]), we conclude that uniqueness of a strong regular solution for (2.3) implies uniqueness of a regular weak solution. So we get

Corollary 4.2. *If v^1 and v^2 are regular weak solutions of equations* (2.3) *such that $\mathcal{D}(v^1(0)) = \mathcal{D}(v^2(0))$, then $\mathcal{D}(v^1(\,\cdot\,)) = \mathcal{D}(v^2(\,\cdot\,))$.*

Corollary 4.3. *Under the assumptions of Theorem 3.1 the law of a lifting $v(\tau)$ is defined in a unique way.*

Evoking Theorem 3.1 we obtain

Corollary 4.4. *Let $I^1(\tau)$ and $I^2(\tau)$ be weak regular solutions of* (1.7), (1.8) *as in Theorem 1.3 (i.e. these are two limiting points of the family of measures $\mathcal{D}(I^\nu(\,\cdot\,))$). Then their laws coincide.*

These results and Theorem 1.3 jointly imply

Theorem 4.5. *The action vector $I^\nu(\,\cdot\,)$ converges in the law in the space $\mathcal{H}_I$ to a regular weak solution $I^0(\,\cdot\,)$ of* (1.7), (1.8)*. Moreover, the law of I^0 equals $I \circ \mathcal{D}(v(\,\cdot\,))$, where $v(\tau)$ is a unique regular weak solution of* (2.3) *such that $v(0) = \mathbf{V}_\vartheta(I_0)$. Here ϑ is any fixed vector from the torus $\mathbb{T}^\infty$.*

Proof of Theorem 4.1. Denote by $(\,\cdot\,,\cdot\,)_0$ the inner product in h^0. For a fixed $\kappa > 0$ we introduce the stopping time Θ:

$$\Theta = \min\big\{\tau \le T : |v^1(\tau)|_{h^2} \vee |v^2(\tau)|_{h^2} = \kappa\big\}$$

(if the set is empty we set $\Theta = T$). Due to (1.22)

$$\mathbf{P}\{\Theta < T\} \le c\kappa^{-1}.$$

Denote

$$v^j_\kappa(\tau) = v^j(\tau \wedge \Theta)\,, \quad w(\tau) = v^1_k(\tau) - v^2_k(\tau)\,.$$

To prove the theorem it suffices to show that $w(\tau) = 0$ a.s., for each $\kappa > 0$.

We have

$$dw_k(\tau) = \chi_{\tau<\Theta}\Big\{\big[R^1_k(v^1_\kappa) - R^1_k(v^2_\kappa)\big]d\tau - \big[R^2_k(v^1_\kappa) - R^2_k(v^2_\kappa)\big]d\tau \\ + \sum_{l\ge1}\int_{\mathbb{T}^\infty}\big[\mathcal{R}(k;l,\theta)(v^1_\kappa) - \mathcal{R}(k;l,\theta)(v^2_\kappa)\big]\,d\boldsymbol{\beta}_{l,\theta}d\theta\Big\}\,.$$

Application of the Itô formula yields

$$\begin{aligned}\mathbf{E}|w(\tau)|_0^2 = \mathbf{E}\int_0^{\tau\wedge\Theta} &\big(w(s), [R^1(v^1_\kappa) - R^1(v^2_\kappa)]\big)_0 ds \\ +\mathbf{E}\int_0^{\tau\wedge\Theta} &\big(w(s), [R^2(v^1_\kappa) - R^2(v^2_\kappa)]\big)_0 ds \\ +\frac12\mathbf{E}\int_0^{\tau\wedge\Theta} &\sum_{l\ge1}\int_{\mathbb{T}^\infty}\big|\mathcal{R}(\,\cdot\,,l,\theta)(v^1_\kappa) - \mathcal{R}(\,\cdot\,,l,\theta)(v^2_\kappa)\big|_0^2\,d\theta ds \\ &\equiv \Xi_1 + \Xi_2 + \Xi_3\,.\end{aligned}$$

We will estimate the three terms in the r.h.s. and start with the term Ξ_3. By (1.20) and the Cauchy inequality we have

$$\big|\mathcal{R}(\,\cdot\,,l,\theta)(v^1_\kappa(s)) - \mathcal{R}(\cdot,l,\theta)(v^2_\kappa(s))\big|_0^2 \le C(N,\kappa)l^{-N}|w(s)|_0^2$$

for any $N \in \mathbb{Z}^+$. Therefore,

$$\Xi_3 \le C(\kappa)\mathbf{E}\int_0^{\tau\wedge\Theta} |w(s)|_0^2 ds\,.$$

For similar reasons $\Xi_2 \le C(\kappa)\mathbf{E}\int_0^{\tau\wedge\Theta} |w(s)|_0^2 ds$.

Estimating the term Ξ_1 is more complicated since the map $v \mapsto R^1(v)$ is unbounded in every space h^p. We recall that $\mathcal{L}^{-1} := d\Psi(0)$ is the diagonal operator

$$\mathcal{L}^{-1}\Big(\sum_s u_s f_s\Big) = v\,, \quad v_s = |s|^{-1/2}u_s \ \forall\ s \in \mathbb{Z}_0\,,$$

and introduce $\Psi_0(u) = \Psi(u) - \mathcal{L}^{-1}u$. According to Proposition (1.2), Ψ_0 defines analytic maps $H^m \mapsto h^{m+1}$, $m \ge 0$. We denote by G the inverse map $G = \Psi^{-1}$. Then $G(v) = \mathcal{L}(v) + G_0(v)$, where $G_0 : h^m \to H^{m+1}$ is analytic for any $m \ge 0$. Finally, denote $R^1(v) - \widehat{\Delta}v = R^0(v)$, where $\widehat{\Delta}$ is the Fourier-image of the Laplacian: $\widehat{\Delta}v = v'$, where $\mathbf{v}'_j = -j^2\mathbf{v}_j$, $\forall\, j$.

Lemma 4.6. *For any $m \ge 0$ the map $R^0 : h^m \to h^{m-1}$ is analytic.*

So the effective equation (2.3) is a quasilinear stochastic heat equation, written in Fourier coefficients.

Proof. We have

$$R^1(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta}\mathcal{L}^{-1}\Delta(G\Phi_\theta v)d\,\theta + \int_{\mathbb{T}^\infty} \Phi_{-\theta}d\Psi_0(G\Phi_\theta v)\Delta(G\Phi_\theta v)\,d\,\theta\,.$$

The first integrand equals

$$\Phi_{-\theta}\mathcal{L}^{-1}\Delta\mathcal{L}\Phi_\theta v + \Phi_{-\theta}\mathcal{L}^{-1}\Delta(G_0\Phi_\theta v) = \widehat{\Delta}v + \Phi_{-\theta}\mathcal{L}^{-1}\Delta(G_0\Phi_\theta v)$$

since $\mathcal{L}^{-1}\Delta\mathcal{L}\Phi_\theta = \widehat{\Delta}$ and $\widehat{\Delta}$ commutes with the operators Φ_θ.

So

$$R^0(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta}\mathcal{L}^{-1}\Delta(G_0\Phi_\theta v)d\,\theta + \int_{\mathbb{T}^\infty} \Phi_{-\theta}d\Psi_0(G\Phi_\theta v)\Delta(G\Phi_\theta v)\,d\,\theta\,.$$

Clearly for any θ the first integrand defines an analytic map from h^m to h^{m-1}. We have $d\Psi_0(u_\theta) : H^m \to h^{m+1}$. Since the map Ψ is symplectic, then also $d\Psi_0(u_\theta) : H^r \to h^{r+1}$ for $-m-2 \le r \le m$ (cf. Proposition 1.4 in [Ku1]). So for any θ the second integrand also defines an analytic map $h^m \to h^{m-1}$. Now the assertion follows. □

By this lemma with $m = 1$

$$\begin{aligned}\Xi_1 &= \mathbf{E}\int_0^{\tau\wedge\Theta} \big(-|w(s)|_1^2 + (w(s), R^0(v_\kappa^1) - R^0(v_\kappa^2))_0\big)ds\\ &\le \mathbf{E}\int_0^{\tau\wedge\Theta} \big(-|w(s)|_1^2 + C_\kappa|w(s)|_0|w(s)|_1\big)ds \le \mathbf{E}C'_\kappa\int_0^{\tau\wedge\Theta} |w(s)|_0^2 ds\,.\end{aligned}$$

Combining the obtained estimates for Ξ_1, Ξ_2 and Ξ_3, we arrive at the inequality

$$\mathbf{E}|w(\tau)|_0^2 \le C_\kappa^1\int_0^\tau \mathbf{E}|w(s)|_0^2 ds\,.$$

Since $\mathbf{E}|w(0)|_0^2 = 0$, then $\mathbf{E}|w(\tau)|_0^2 = 0$ for all τ. This completes the proof of Theorem 4.1. □

5 Limiting Joint Distribution of Action-Angles

For a solution $u^\nu(t)$ of (0.1), (0.2) we denote by $I^\nu(\tau) = I(v^\nu(\tau))$ and $\varphi^\nu(\tau) = \varphi(v^\nu(\tau))$ its actions and angles, written in the slow time τ. Theorem 4.5 describes limiting behaviour of $\mathcal{D}I^\nu$ as $\nu \to 0$. In this section we study joint distribution of $I^\nu(\tau)$ and $\varphi^\nu(\tau)$, mollified in τ. That is, we study the measures $\mu_f^\nu = \int_0^T f(s)\mathcal{D}\Big(I^\nu(s), \varphi^\nu(s)\Big)ds$ on the space $h_I^p \times \mathbb{T}^\infty$, where $f \geq 0$ is a continuous function such that $\int_0^T f = 1$.

Theorem 5.1. *As $\nu \to 0$,*

$$\mu_f^\nu \rightharpoonup \left(\int_0^T f(s)\mathcal{D}(I^0(s))ds\right) \times d\varphi\,. \tag{5.1}$$

In particular, $\int_0^T f(s)\mathcal{D}(\varphi^\nu(s))ds \rightharpoonup d\varphi$.

Proof. Let us first replace $f(\tau)$ with a characteristic function

$$\bar f(\tau) = \frac{1}{T_2 - T_1}\chi_{\{T_1 \leq \tau \leq T_2\}}\,, \quad 0 \leq T_1 < T_2 \leq T\,.$$

Due to (1.6) the family of measures $\{\mu_{\bar f}^\nu, \nu > 0\}$ is tight in $h_I^p \times \mathbb{T}^\infty$. Consider any limiting measure $\mu_{\bar f}^{\nu_j} \rightharpoonup \mu_{\bar f}$.

Let $F(I, \varphi) = F^0(I^m, \varphi^m)$, where F^0 is a bounded Lipschitz function on $\mathbb{R}_+^m \times \mathbb{T}^m$. We claim that

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathbf{E}F\big(I^\nu(s), \varphi^\nu(s)\big)ds \to \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathbf{E}\langle F\rangle\big(I^0(s)\big)ds \quad \text{as} \quad \nu \to 0\,. \tag{5.2}$$

Indeed, due to Theorem 4.5 we have

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathbf{E}\langle F\rangle\big(I^\nu(s)\big)ds \to \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathbf{E}\langle F\rangle\big(I^0(s)\big)ds \quad \text{as } \nu \to 0\,.$$

So (5.2) would follow if we prove the convergence

$$\mathbf{E}\left|\int_0^\tau F\big(I^\nu(s), \varphi^\nu(s)\big) - \langle F\rangle\big(I^\nu(s)\big)\right|ds \to 0\,, \quad \text{as } \nu \to 0\,, \tag{5.3}$$

for any τ. But (5.3) is established in [KuPi] (see there (6.9) and below) for $F^0(I^m,\varphi^m) = F_k(I^m, 0; \varphi^m, 0))$, where $F_k(I, \varphi)$ is the drift in e.g. (1.5). The arguments in [KuPi] are general and apply to any bounded Lipschitz function F^0.

Relation (5.2) implies that $\mu_{\bar f} = ((T_2 - T_1)^{-1} \int_{T_1}^{T_2} \mathcal{D}(I^0(s))ds) \times d\varphi$. So (5.1) is established for characteristic functions. Accordingly, (5.1) holds, firstly, for piecewise constant functions $f(\tau)$ with finitely many discontinuities and, secondly, for continuous functions. □

6 Appendix

6.1 Whitham averaging. The n-gap solutions of the KdV equation under the zero-meanvalue periodic boundary condition have the form (0.4), where $0 = I_{n+1} = I_{n+2} = \dots$. They form a subset of the bigger family of space-quasiperiodic n-gap solutions, discovered in the 1970s by Novikov and Lax. These quasiperiodic solutions may be written as $\Theta^n(Kx+Wt+\varphi; w)$, were the parameter w has dimension $2n+1$, Θ^n is an analytic function on $\mathbb{T}^n \times \mathbb{R}^{2n+1}$ and the vectors $K, W \in \mathbb{R}^n$ depend on w. See in [ZMNP], [DuN], [LLV], [Ku1].

Denote by $X = \nu x$ and $T = \nu t$ slow space- and time-variables. We want to solve either the KdV itself, or some its ν-perturbation (say, e.g. $(0.1)_{\eta=0}$) in the space of functions, bounded as $|x| \to \infty$ (not necessarily periodic in x). We are looking for solutions with the initial data

$$u_0(x) = \Theta^n\big(Kx + \varphi_0; w_0(X)\big)\,,$$

where $w_0(X) \in \mathbb{R}^{2n+1}$ is a given vector-function. Assuming that a solution $u(t,x)$ exists, decomposes in asymptotical series in ν and that the leading term may be written as

$$u^0(t,x) = \Theta^n\big(Kx + Wt + \varphi_0; w(T,X)\big)\,, \tag{6.1}$$

Whitham shows that $w(T,X)$ has to satisfy a nonlinear hyperbolic system, known now as the *Whitham equations*. In the last 40 years much attention was given to the Whitham equations and Whitham averaging (i.e. to the claim that an exact solution $u(t,x)$ may be written as $u = u^0(t,x) + o(1)$, where u^0 has the form (6.1)). Many results were obtained for the Whitham equations for KdV and for other integrable systems, e.g. see [ZMNP], [Kr], [DuN] (we note that in the last section of [DuN] the authors discuss the damped equation $(0.1)_{\eta=0}$). In these works the Whitham equations are postulated as a first principle, without precise statements on their connection with the original problem. Rigorous results on this connection, i.e. results on Whitham averaging, are very few, and these are examples rather than general theorems since they apply to *some* initial data and hold in *some* domains in the space-time $\mathbb{R}^2$, see in [LLV]. (Also see [S] for some related problems and results.)

In the spirit of the Whitham theory our results may be cast in the following way. Consider a perturbed KdV equation

$$\dot u + u_{xxx} - 6uu_x = \nu f(u, u_x, u_{xx})\,, \tag{6.2}$$

and take initial condition $u_0(x)$ of the form above with arbitrary n, where w_0 is an x-independent constant such that $u_0(x)$ is 2π-periodic with zero meanvalue. Let us write u_0 as a periodic ∞-gap potential $u_0(x) = \Theta^\infty(Kx + \varphi_0; I_0)$, where $\Theta^\infty : \mathbb{T}^\infty \times \mathbb{R}^\infty_+ \to \mathbb{R}$ and now $K \in \mathbb{Z}^\infty$, $\varphi_0 \in \mathbb{T}^\infty$ (see [MT] for a theory of ∞-gap potentials). We may write a solution of (6.2) as $u^\nu(t,x) = \Theta^\infty(Kx + \varphi^\nu(\tau); I^\nu(\tau))$, $\tau = \nu t$, with unknown phases $\varphi^\nu \in \mathbb{T}^\infty$ and actions $I^\nu \in \mathbb{R}^\infty_+$. The main task is to recover the actions. To do this we write the effective equations for $I(\tau)$, corresponding to (6.2). Namely, we rewrite (6.2) using the non-linear Fourier transform Ψ, pass to the slow time τ, delete from the obtained v-equations the KdV vector-field

$d\Psi\circ V$ and apply to the rest the averaging (0.11). We claim that for some classes of perturbed KdV equations the vector $I^0(\tau)=\pi_I(v(\tau))$, where v solves the effective equations, approximates $I^\nu(\tau)$ well with small ν. Our work justifies this claim for the damped-driven perturbations (0.1) in the sense that the convergence (0.8) holds.

This special case of Whitham averaging deals with perturbations of solutions for KdV which oscillate fast in time (since we write them using the slow time τ), while the general case treats solutions which oscillate fast both in the slow time T and slow space X. The effective equations serve to find approximately the action vector $I^\nu(\tau)\in\mathbb{R}^\infty_+$ which represents a space-periodic solution for (6.2) as an infinite-gap potential $\Theta^\infty(Kx+\varphi^\nu(\tau);I^\nu(\tau))$. They play a role, similar to that of the Whitham equations, serving to find the parameter $w(T,X)\in\mathbb{R}^{2n+1}$, describing n-gap potentials (6.1) which approximate (non-periodic in space) solutions.

6.2 Lemma 4.3 from [KuPi]. Below we present a construction from [KuPi], used essentially in section 3.

For $\tau\ge\theta'\ge 0$ consider a solution $v(\tau)=v^\nu_P(\tau)$ of equation $(1.3)_P$. For any $N\in\mathbb{N}$ we will construct a process $(\bar v,\tilde v^N)(\tau)\in h^p\times\mathbb{R}^{2N}$, $\tau\ge\theta'$, such that

1. $\mathcal{D}(\bar v(\,\cdot\,))=\mathcal{D}(v(\,\cdot\,))$;
2. $I(\tilde v^N(\tau))\equiv I^N(v(\tau))$, a.s.;
3. $\varphi(\tilde v^N(\theta'))=\varphi^0$, where φ^0 is a given vector in $\mathbb{T}^N$;
4. the process $\tilde v^N(\tau)$ satisfies certain estimates uniformly in ν.

For $\eta_1,\eta_2\in\mathbb{R}^2\setminus\{0\}$ we denote by $U(\eta_1,\eta_2)$ the operator in $SO(2)$ such that $U(\eta_1,\eta_2)\frac{\eta_1}{|\eta_1|}=\frac{\eta_2}{|\eta_2|}$. If $\eta_1=0$ or $\eta_2=0$, we set $U(\eta_1,\eta_2)=\text{id}$.

Let us abbreviate in e.g. $(1.3)_P$ $\;(P^1_k(v)+P^2_k(v))^P=A^P_k(v)$. Then the equation takes the form

$$d\tilde{\mathbf v}_k=\big(\nu^{-1}d\Psi_k(u)V(u)\big)^P d\tau+A^P_k(v)d\tau+\sum_{j\ge1}B^P_{kj}(v)d\boldsymbol\beta_j(\tau)\,,\quad 1\le k\le N\,. \tag{6.3}$$

For $1\le k\le N$ we introduce the functions

$$\tilde A_k(\tilde{\mathbf v}_k,v)=U(\tilde{\mathbf v}_k,\mathbf v_k)A^P_k(v)\,,\quad \tilde B_{kj}(\tilde{\mathbf v}_k,v)=U(\tilde{\mathbf v}_k,\mathbf v_k)B^P_{kj}(v)\,,$$

and define an additional stochastic system for a vector $\tilde v^N=(\tilde{\mathbf v}_1,\dots,\tilde{\mathbf v}_N)\in\mathbb{R}^{2N}$:

$$d\tilde{\mathbf v}_k=\tilde A_k(\tilde{\mathbf v}_k,v)d\tau+\sum_{j\ge1}\tilde B_{kj}(\tilde{\mathbf v}_k,v)d\boldsymbol\beta_j(\tau)\,,\quad 1\le k\le N\,. \tag{6.4}$$

Consider the system of equations (6.3), (6.4), where $\tau\ge\theta'$, with the initial condition

$$\tilde v^N(\theta')=V^N_{\varphi^0}\big(I(v^N(\theta'))\big) \tag{6.5}$$

and with the given $v(\theta')$. It has a unique strong solution, defined while

$$|\mathbf v_k|,|\tilde{\mathbf v}_k|\ge c>0\quad\forall\,k\le N\,,$$

for any fixed $c>0$.

Denote $[(v,\tilde v)]=\big(\min_{1\le j\le N}\frac12|\mathbf v_j|^2\big)\wedge\big(\min_{1\le j\le N}\frac12|\tilde{\mathbf v}_j|^2\big)$. Fix any $\gamma\in(0,1/4]$ and define stopping times $\tau^\pm_j\in[\theta',T],\dots,\ \tau^-_j<\tau^+_j<\tau^-_{j+1}<\dots$, as in Step 3 in section 3.1. Namely,

- If $[(\mathbf{v}_0, \mathbf{v}_0)] \le \gamma$, then $\tau_1^- = 0$. Otherwise $\tau_0^+ = 0$.
- If τ_j^- is defined, then τ_j^+ is the first moment after τ_j^- when $[(v(\tau), \tilde v(\tau))] \ge 2\gamma$ (if this never happens, then $\tau_j^+ = T$).
- If τ_j^+ is defined, then τ_{j+1}^- is the first moment after τ_j^+ when $[(v, \tilde v)] \le \gamma$.

Next for $0 < \gamma \le 1/4$ we construct a continuous process $(v(\tau), \tilde v^{\gamma N}(\tau))$, $\tau \ge \theta'$, where $v(\tau) \equiv v_P^\nu(\tau)$, $\tilde v^N(\theta')$ is given (see (6.5)), and for $\tau > \theta'$ the process $\tilde v^{\gamma N}$ is defined as follows:

(i) If $\tilde v^{\gamma N}(\tau_j^+)$ is known, then we extend $\tilde v^{\gamma N}$ to the segment $\Delta_j := [\tau_j^+, \tau_{j+1}^-]$ in such a way that $(v(\tau), \tilde v^{\gamma N}(\tau))$ satisfies (6.3), (6.4).

(ii) If $\tilde v^{\gamma N}(\tau_j^-)$ is known, then on the segment $\Lambda_j = [\tau_j^-, \tau_j^+]$ we define $\tilde v^{\gamma N}$ as

$$\tilde v^{\gamma N}(\tau) = U\big(\tilde{\mathbf{v}}_k(\tau_j^-), \mathbf{v}_k(\tau_j^-)\big)\mathbf{v}_k(\tau)\,, \quad k \le N\,.$$

By applying Itô's formula to the functional $J(\tau) = (I_k(v(\tau)) - I_k(\tilde v^{\gamma N}(\tau))^2$ we derive that if $J(\tau_j^+) = 0$, then $J(\tau) = 0$ for all $\tau \in \Delta_j$ (see Lemma 7.1 in [KuPi]). Hence, the process $\tilde v^{\gamma N}(\tau)$ is well defined for $\tau \in [\theta', T]$ and

$$I_k(v(\tau)) \equiv I_k\big(\tilde v^{\gamma N}(\tau)\big)\,, \quad k \le N\,. \tag{6.6}$$

Let us abbreviate $U_k^j = (U(\tilde{\mathbf{v}}_k(\tau_j^-), \mathbf{v}_k(\tau_j^-))$. Then on an interval Λ_j the process $\tilde v^{\gamma N}$ satisfies the equation

$$d\tilde{\mathbf{v}}_k^\gamma = U_k^j\big((\nu^{-1} d\Psi_k(u)V(u))^P + A_k^P(v)\big)d\tau + \sum_l U_k^j \circ B_{kl}^P(v)d\boldsymbol{\beta}_l(\tau)\,. \tag{6.7}$$

Formally letting $|\tilde{\mathbf{v}}_k|/|\mathbf{v}_k| = 1$ if $\mathbf{v}_k = 0$, we make the function $|\tilde{\mathbf{v}}_k^\gamma|/|\mathbf{v}_k| \equiv 1$ along all trajectories.

Due to (6.4) and (6.7), $\tilde v^{\gamma N}$ is an Itô process

$$d\tilde{\mathbf{v}}_k^\gamma = \hat A_k(\tau)d\tau + \sum \hat B_{kj}(\tau)d\boldsymbol{\beta}_j(\tau)\,, \quad 1 \le k \le N\,. \tag{6.8}$$

The coefficients $\hat A_k = \hat A_k^\gamma$ and $\hat B_{kj} = \hat B_{kj}^\gamma$ a.s. satisfy the estimates

$$\big|\hat A^\gamma(\tau)\big| \le \nu^{-1} C\,, \quad C^{-1}E \le \hat B^\gamma (\hat B^\gamma)^t \le CE \tag{6.9}$$

for all τ, where C depends only on N and P and we regard $\hat B^\gamma$ as an $2N \times 2N$-matrix.

Let us set

$$\mathcal{A}_k^\gamma(\tau) = \tilde{\mathbf{v}}_k(\theta') + \int_{\theta'}^\tau \hat A_k^\gamma(s)ds\,, \quad \mathcal{M}_k^\gamma(\tau) = \sum_j \int_{\theta'}^\tau \hat B_{kj}^\gamma\, d\beta_j(\tau)$$

(cf. (6.5)) and consider the process

$$\xi^\gamma(\tau) = \big(v^\gamma(\tau), \mathcal{A}^\gamma(\tau), \mathcal{M}^\gamma(\tau)\big) \in h^p \times \mathbb{R}^{2N} \times \mathbb{R}^{2N}, \quad \tau \ge \theta'.$$

Then $\tilde v^{\gamma N} = \mathcal{A}^\gamma(\tau) + \mathcal{M}^\gamma(\tau)$ and due to (6.9) the family of laws of the processes ξ^γ is tight in the space $C(\theta', T; h^p) \times C(\theta', T; \mathbb{R}^{2N}) \times C(\theta', T; \mathbb{R}^{2N})$. Consider any limiting (as $\gamma_j \to 0$) law $\mathcal{D}^0$ and find any process $(\bar v(\tau), \mathcal{A}^0(\tau), \mathcal{M}^0(\tau))$, distributed as $\mathcal{D}^0$. Denote $\tilde v^N(\tau) = \mathcal{A}^0(\tau) + \mathcal{M}^0(\tau)$ and consider the process $(\bar v(\tau), \tilde v^N(\tau)) \in h^p \times \mathbb{R}^{2N}$.

It is easy to see that it satisfies 1–3. In [KuPi] we show that estimates (6.9) imply that

$$\mathcal{A}^0(\tau) = \int_{\theta'}^{\tau} B^N(s)ds\,, \quad \mathcal{M}^0(\tau) = \int_{\theta'}^{\tau} a^N(s)dw(s)\,,$$

where $w(s) \in \mathbb{R}^{2N}$ is a standard Wiener process, while B^N and a^N meet (3.15). That is, $\tilde{v}^N(\tau)$ is an Itô process

$$d\tilde{v}^N(\tau) = B^N(\tau)d\tau + a^N(\tau)dw(\tau)\,, \tag{6.10}$$

where

$$\big|\hat{B}(\tau)\big| \le C\,, \quad C^{-1}E \le a^N(a^N)^t(\tau) \le CE \quad \forall\,\tau\,,\ a.s. \tag{6.11}$$

These are the estimates, mentioned in item 4 above.

Now by (6.9) and Theorem 4 from section 2.2 in [Kry1], applied to the Itô process $\tilde{\mathbf{v}}_k$, we have

$$\mathbf{E}\int_{\theta'}^{T} \chi_{\{I_k(v_P^\nu(\tau))\le\delta\}}d\tau \le C\delta\,, \quad \forall\,k \le N\,, \tag{6.12}$$

where $C = C(N,P)$.

Taking $\theta' = 0$ and passing to a limit as $\nu \to 0$ we see that the process $I_{Pk}(\tau)$ also meets (6.12). Since $\mathcal{D}(I_P(\,\cdot\,)) \rightharpoonup \mathcal{D}(I(\,\cdot\,))$ as $P \to \infty$, then we get estimate (1.13).

For any $\nu > 0$ the processes I_P^ν and I^ν coincide on the event $\{\sup_\tau |I^\nu(\tau)|_{h_p^I} \le P\}$. Due to (1.6) probability of this event goes to 1 as $P \to \infty$, uniformly in ν. So (6.12) also implies that

$$\mathbf{E}\int_0^T \chi_{\{I_k^\nu(\tau)\le\delta\}} \to 0 \quad \text{as } \delta \to 0\,, \tag{6.13}$$

uniformly in ν.

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