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ERGODIC THEOREMS FOR 2D STATISTICAL HYDRODYNAMICS

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We consider the 2D Navier–Stokes system, perturbed by a random force, such that sufficiently many of its Fourier modes are excited (e.g. all of them are). We discuss the results on the existence and uniqueness of a stationary measure for this system, obtained in last years, homogeneity of the measures and some their limiting properties. Next we use these results to prove that solutions of the equations obey the central limit theorem and the strong law of large numbers.

Keywords:

1. Introduction

In this work we interpret the 2D statistical hydrodynamics as a theory of the 2D Navier–Stokes (NS) system, perturbed by a random force:

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \eta(t, x)\,, \quad \text{div } u = 0\,, \tag{1.1}$$

where $u = u(t, x) \in \mathbb{R}^2$ and $p = p(t, x)$. The space-variable x belongs either to a smooth bounded two-dimensional domain and then u vanishes on its boundary, or to the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then we assume that $\int u\,dx \equiv \int \eta\,dx \equiv 0$.[a] In both cases any vector-field $v(x)$ admits a unique decomposition

$$v(x) = w(x) + \nabla p(x)\,, \quad \text{div } w = 0\,,$$

and we denote by Π the projector $v(x) \mapsto w(x)$ (Π is called *Leray's projector*, see [4, 30]). Applying Π to the NS system (1.1) we re-write it as

$$\dot{u}(t) + Lu(t) + B(u(t), u(t)) = \eta(t)\,. \tag{1.2}$$

[a]Physically the most important is the case when x belongs to an unbounded domain (say, $x \in \mathbb{R}^2$) and the velocity-field $u(x)$ is bounded, but has infinite energy $\int |u|^2\,dx$. This case leads to serious complications which we cannot handle.

Here we set $L = -\nu\Pi\Delta$, $B(u,u) = \Pi(u\cdot\nabla)u$ and re-denoted $\Pi\eta = \eta$. In (1.2) we view u and η as curves in the Hilbert space H, formed by square-integrable divergence-free vector-fields. By V we denote the space

$$V = \left\{ u(x) \in H \,\middle|\, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \in L_2 \quad \text{and } u \text{ satisfies the boundary conditions} \right\},$$

given the norm $\|u\| = (\int |\nabla u|^2\, dx)^{1/2}$, and denote by $|\cdot|$ the norm in the space H. It is well known that if $\eta(t)$ is a continuous curve in H, then for any $u_0 \in H$ Eq. (1.2) has a unique solution $u(t)$ (understood in the sense of generalised functions) such that $u(0) = u_0$ and u defines a continuous curve in H, as well as a square-integrable curve in V, see e.g. [4, 30, 12].

Let $\{e_1, e_2, \ldots\}$ be a Hilbert basis of the space H, formed by eigenfunctions of the operator L. In the case of periodic boundary conditions it is formed by the trigonometric vector-fields

$$c_s s^\perp \cos s\cdot x\,, \quad c_s s^\perp \sin s\cdot x\,, \quad s \in \mathcal{S}\,. \tag{1.3}$$

Here $\mathcal{S}$ is a subset of $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$ such that $\mathcal{S} \cap -\mathcal{S} = \oslash$ and $\mathcal{S} \cup -\mathcal{S} = \mathbb{Z}_0^2$, $s^\perp = \binom{-s_2}{s_1}$ for any vector $\binom{s_1}{s_2} \in \mathbb{Z}_0^2$, and $c_s = \frac{\sqrt{2}}{2\pi|s|}$ is the L_2-normalising factor.

By $\{\alpha_j\}$ we denote the eigenvalues, corresponding to the eigenfunctions $\{e_j\}$, and assume that $\alpha_1 \le \alpha_2 \le \cdots$. Note that $\alpha_1 > 0$ and that in the periodic case the eigenfunctions in (1.3) have the eigenvalues $\nu|s|^2$.

Concerning the random force η we assume that either it is a kick-force

$$\eta(t,x) = \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t - Tk)\,, \quad \eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x)\,, \tag{1.4}$$

where $\{b_j \ge 0\}$ are some constants such that $\sum b_j^2 < \infty$ and $\{\xi_{jk}\}$ are independent random variables with k-independent distributions. Or we assume that η is a "white in time force":

$$\eta(t,x) = \frac{d}{dt}\zeta\,, \quad \zeta = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x)\,, \tag{1.5}$$

where $\{\beta_j\}$ are independent standard Wiener processes, defined for $t \in \mathbb{R}$, and the constants $\{b_j \ge 0\}$ are such that

$$\sum \alpha_j b_j^2 < \infty\,. \tag{1.6}$$

In the kick-case (1.4) solutions for (1.2) are normalised to be continuous from the right and can be described as follows. For $t \in [kT, (k+1)T)$, where k is any integer, u is a solution of the free NS system (i.e. it satisfies (1.1) with $\eta = 0$), and at $t = (k+1)T$ it has a jump, equal η_{k+1}, see Fig. 1. Denoting by S the operator of the time-T shift along trajectories of the free NS system, we see that

$$u((k+1)T) = S(u(kT)) + \eta_{k+1}\,. \tag{1.7}$$

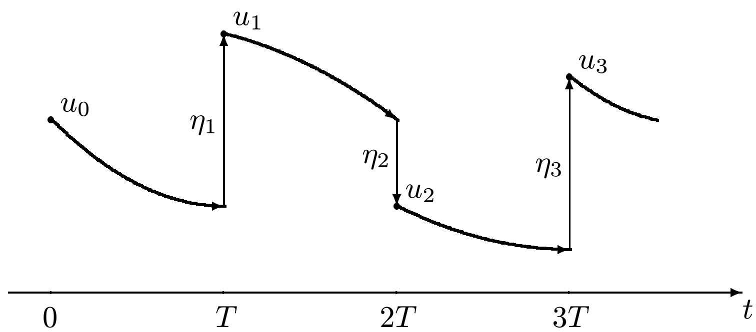

Fig. 1. Solutions of the kicked equation.

This random system defines a Markov chain in the space H. (Here and below all metric spaces are assumed to be provided with the Borel sigma-algebras.) Denoting by $u(t;u_0)$ a solution for (1.2) and (1.4), equal u_0 at $t=0$, we see that the corresponding Markov transition function $P(t,v,\Gamma)$, where $t\in T\mathbb{Z}$, $v\in H$ and Γ is a Borel subset of H, is

$$P(t,v,\Gamma)=\mathbb{P}\{u(t;v)\in\Gamma\}\,. \tag{1.8}$$

This Markov chain defines semigroups $\{S_t\}$ and $\{S_t^*\}$ $(t\in T\mathbb{Z}_+)$ in the space C_b of bounded continuous functions on H and in the space $\mathcal{P}$ of probability (Borel) measures, respectively. Due to (1.8),

$$(S_tf)(v)=\mathbb{E}f(u(t;v))\,,\quad f\in C_b\,,$$
$$(S_t^*\mu)(\Gamma)=\int\mathbb{P}\{u(t;v)\in\Gamma\}\mu(dv)\,,\quad \mu\in\mathcal{P}\,.$$

A measure $\mu\in\mathcal{P}$ is called a *stationary measure* if $S_t^*\mu\equiv\mu$.

Concerning the independent variables ξ_{jk} we assume the following:

(H) Their distributions $\mathcal{D}\xi_{jk}$ have densities against the Lebesgue measure, $\mathcal{D}\xi_{jk}=p_j(s)\,ds$, where p_j's are functions of bounded total variation, supported by the segment $[-1,\,1]$, and such that $\int_{-\varepsilon}^{\varepsilon}p_j(s)\,ds>0$ for all integers j and all $\varepsilon>0$.

That is, we restrict ourselves to the case of bounded random kicks. Concerning equations with unbounded kicks see [19].

Now let us pass to the white-forced NS system (1.2) and (1.5). Integrating (1.2) from 0 to $T>0$ we get:

$$u(T)+\int_0^T(Lu(t)+B(u(t),u(t)))\,dt=u(0)+\zeta(T)-\zeta(0)\,. \tag{1.9}$$

A random vector-field $u(t,x)$, $t\geq0$, which defines an a.s. continuous process $u(t)\in H$ such that its norm $\|u(t)\|$ is square-integrable on every finite time-interval, is called a solution for (1.2) and (1.5) if the relation (1.9) holds a.s., for each $T > 0$. The equality is understood in the sense of generalised functions of x.[b]

It is known (e.g. see [31]) that for any $u_0 \in H$ Eqs. (1.2) and (1.5) has a unique solution, equal u_0 for $t = 0$. Because of that, the white-forced NS system defines a Markov process in H. Its Markov transition function $P(t, v, \Gamma)$ still is defined by the relation (1.8), where now $t \in \mathbb{R}_+$. The corresponding semigroups $\{S_t\}$ and $\{S_t^*\}$ are defined for $t \geq 0$.

2. Stationary Measures

2.1. *Kicked equations*

Armen Shirikyan and the author studied stationary measures for the kicked equation (1.2) and (1.4) in [17, 21, 18, 24, 16] (the last paper is joint with A. Piatnitski). In these works we assume that the force η satisfies the assumption (H) and that

$$b_j \neq 0\,, \quad \forall\, 1 \leq j \leq N_\nu\,, \tag{2.1}$$

where N_ν is a suitable constant, growing to infinity as $\nu \to 0$ (e.g. all b_j are nonzero numbers). In [17] (see also [21]) it is proved that the equation has a unique stationary measure μ and solutions of (1.2) and (1.4) weakly converge to μ in distribution:

$$\mathcal{D}u(t; u_0) \rightharpoonup \mu \quad \text{as} \quad t \to \infty\,, \tag{2.2}$$

for all u_0, where $t \in T\mathbb{Z}_+$.[c] To establish the result we used a Foias–Prodi type reduction of the NS system to an N_ν-dimensional system with delay which is satisfied by the vector, formed by the first N_ν Fourier components of any solution for the NS system (1.2) and (1.4). The new system turned out to be of the Gibbs type (similar systems are studied, say, in [3]). By the assumption (2.1), the noise, which stirs it, is non-degenerate. Therefore due to a Ruelle-type theorem, the new system has a unique invariant ("Gibbs") measure, so the NS system has a unique stationary measure μ.

The condition (2.1) which guarantees the non-degeneracy of the reduced system is crucial for all works on stationary measures for randomly forced PDEs, written after [17] up to now. It is somewhat restrictive since N_ν grows when the viscosity ν decreases, and it is not clear if the uniqueness result remains true under the weaker condition $b_j \neq 0$ for $j \leq N$, where N is an absolute constant. Still, it is shown in [9] that if the system (1.2) is replaced by any its finite-dimensional Galerkin approximation, then the stationary measure is unique provided that (2.1) holds

[b] I.e. if we multiply both parts of (1.9) by any smooth divergence-free vector-field $w(x)$ and next integrate against dx, applying formally integration by parts to the term $(Lu) \cdot w$, we get equal numbers.

[c] The work was done during the year 1999, when its approach was discussed at a number of informal seminars. At the end of that year the first talk on the results obtained was given at a meeting of Moscow Mathematical Society and a preprint of the paper [17] appeared.

with $N_\nu = 3$. This result follows from the Malliavin calculus since it can be checked that the system satisfies the Hörmander non-degeneracy condition.

In [21, 18, 24, 16] we developed a coupling-approach to study the NS system and related kick-forced dissipative nonlinear PDEs.[d] This approach uses not the Foias–Prodi reduction, but the main lemma the reduction is based upon. It gives a shorter proof of the uniqueness (under the assumption (2.1)) and implies that

$$\left|\mathbb{E}f(u(t;u_0)) - \int f(u)\mu(du)\right| \le C(|u_0|)e^{-\sigma t}\,, \quad \forall\, f \in O\,, \tag{2.3}$$

where $t \in T\mathbb{Z}_+$, $\sigma > 0$ is u_0-independent and

$$O = \{f \in C_b|\ |f| \le 1 \text{ and } \operatorname{Lip} f \le 1\}\,. \tag{2.4}$$

That is, for any $u_0 \in H$ the distribution $\mathcal{D}u(t;u_0)$ exponentially fast converge to μ in the Lipschitz-dual norm:

$$\|\mathcal{D}u(t;u_0) - \mu\|_L^* \le C(|u_0|)e^{-\sigma t}\,, \tag{2.5}$$

where for any two measures $\nu_1, \nu_2 \in \mathcal{P}$,

$$\|\nu_1 - \nu_2\|_L^* = \sup_{f\in O}\left|\int f(u)\nu_1(du) - \int f(u)\nu_2(du)\right|\,.$$

This convergence implies the weak convergence (2.2), see [5]. For the final version of the proof see [24], where some ideas of L. Kantorovich are evoked to simplify and clarify the arguments.

Independently a similar coupling-approach to study the problem (1.2), (1.4) and (2.1) was developed by N. Masmoudi and L.-S. Young in [27].

2.2. *White-forced equations*

The first theorem on the uniqueness of a stationary measure for a white-forced NS system is due to Flandoli and Maslovski [11] who considered Eq. (1.2), perturbed by a non-smooth in the space-variable random force $\eta(t,x)$. I.e. by a force (1.5), where $b_j \sim j^{-a}$, $\frac{1}{2} \le a < \frac{3}{8}$. This result is not quite satisfactory since, firstly, the statistical hydrodynamics usually works with forces, smooth in the space-variable, and, secondly, it is unnatural to impose a *lower* bound for the energy of *each* Fourier mode.

After the work [17] on the kick-forced equations, E. Mattingly, Sinai [10] and Bricmont, Kupiainen, Lefevere [2] applied the Foias–Prodi reduction to study the

[d]The short paper [21] was written in December 2000 as the authors' respond to some criticism of the results of [17], made during my lectures at CIMS (New York). In January 2001, when the work was presented at a work-shop in Warwick, Roger Tribe pointed out that the main idea of the work is a form of coupling.

NS system (1.2) with smooth in space white-force (1.5) which satisfies (2.1), reducing it to a finite-dimensional system with delay. Imposing the additional restriction that the sum (1.5) is finite,

$$\eta(t,x) = \frac{d}{dt}\sum_{j=1}^{N'} b_j\beta_j(t)e_j(x)\ , \quad N_\nu \le N' < \infty\,, \tag{2.6}$$

they proved that the system has a unique stationary measure μ and is ergodic. In [10] the convergence (2.2) is not established, but it is shown that for any continuous functional f on H and for μ-a.a. initial data u_0, time-averages for $f(u(t;u_0))$ converge to the ensemble-average $\int f\,d\mu$. Some techniques, which had been developed in [10], were next used in works of other researches (including [2, 20]).

In [2] it is proved that the convergence (2.2) holds for μ-a.a. initial data and is exponentially fast. This is the first work on the randomly forced NS equations, where the exponentially fast convergence to a stationary measure was established. Still, the stipulation "for μ-a.a." restricts applicability of the results of that work. In particular, they cannot be used to derive the central limit theorem for solutions of (1.2) and (1.5), which we discuss in Sec. 4.

We also note that the assumption that the sum (2.6) is finite makes it impossible to use the results of [10] and [2] to study the turbulence-limit $\nu \to 0$ (see Sec. 2.3). Indeed, the restriction $N_\nu \le N'$, where N' is fixed and N_ν grows to infinity as $\nu \to 0$, does not allow to make ν very small.

In [26], J. Mattingly applied a coupling to Eqs. (1.2) and (1.5) which satisfies (2.1) and (2.6), and proved that convergence (2.2) is exponential for all $u(0)$. Unfortunately, we found it very difficult to follow his arguments.

We also mention the papers [7, 14], devoted to studying a class of randomly perturbed parabolic problems with strong nonlinear dissipation, including the Ginzburg–Landau equation.

In [20] Armen Shirikyan and I show that the ideas which we developed earlier to study the kicked equations, apply as well in the white-forced case. Technically the main difference with the kick-case is that now to study distributions of solutions we cannot any more use explicit formulas in terms of iterated integrals, but have to use instead Girsanov's formulas, related to those which were first exploited in [10]. In [20] we prove that Eqs. (1.2), (1.5) and (2.1) has a unique stationary measure μ, and that the convergence (2.3) = (2.5) holds for all u_0, with $C = C_1(1+|u_0|^2)$. Moreover, the convergence holds true if $u(t)$, $t \ge 0$, is a solution such that $u(0)$ is a random variable with a bounded second moment:

$$\|\mathcal{D}u(t) - \mu\|_L^* \le C_1(1+\mathbb{E}|u(0)|^2)e^{-\sigma t}\,, \quad t \ge 0\,. \tag{2.7}$$

Applying the Ito formula to functionals $|u|^{2m}$, $m = 1, 2, \ldots$, and arguing by induction we get that $\mathbb{E}|u(t;u_0)|^{2m} \le C(m,u_0)$, uniformly in $t \ge 0$ (see [10] and [22]). Integrating the Lipschitz functionals $f_{m,M}(u) = |u|^{2m} \wedge M$ against the signed measure $\mathcal{D}u(t;0) - \mu$, using (2.7) and going to limit in t, we find that $\int f_{m,M}\,d\mu \le$

$C(m,0)$ for each $M>0$. Now application of the Beppo–Levi theorem implies that all moments of the measure μ are finite:

$$\int |u|^{2m}\,\mu(du) < \infty\,, \quad \forall\, m \in \mathbb{N}\,. \tag{2.8}$$

If the boundary conditions are periodic and the noise η as a function of x is smoother than we have assumed originally and

$$\sum_{j=1}^{\infty} \alpha_j^l b_j^2 < \infty \quad \text{for some} \quad l \geq 1\,, \tag{2.9}$$

then for $t>0$ any solution $u(t;u_0)$, $u_0 \in H$, a.s. belongs to the space H^l, formed by vector-fields from H which belong to the Sobolev space of order l, and

$$\mathbb{E}\|u(t;u_0)\|_l^r \leq C_{rl}(|u_0|)\,, \quad \forall\, t \geq 1\,, \quad \forall\, r \in \mathbb{N}\,. \tag{2.10}$$

Here $\|\cdot\|_l$ is the norm in H^l, $\|u\|_l = |(-\Delta u)^{l/2} u|$ (in particular, $\|\cdot\|_1 = \|\cdot\|$ and $\|\cdot\|_0 = |\cdot|$). Now the convergence (2.3) holds for locally Lipschitz functionals f on H^{l-1} of polynomial growth:

$$\left|\mathbb{E} f(u(t;u_0)) - \int f(u)\mu(du)\right| \leq C'_f(|u_0|)e^{-\sigma_f t}\,, \quad \forall\, t \geq 1\,, \quad f \in O_{l-1}\,. \tag{2.11}$$

Here $O_{l-1} = \bigcup_{p=1}^{\infty} O_{l-1}^p$, where O_{l-1}^p denotes the set of continuous functionals f on H^{l-1} such that

(i) $|f(u)| \leq 1 + \|u\|_{l-1}^p$,
(ii) $|f(u) - f(v)| \leq \|u-v\|_{l-1}(1 + \|u\|_{l-1}^{p-1} + \|v\|_{l-1}^{p-1})$,

see [22].

In particular, the energy functional $f(u) = |u|^2$ belongs to O_0^2, while the correlation tensor gives rise to the functionals $f(u) = u_i(x)u_j(y)$ (x and y are points in the space-domain, i and j equals 1 or 2), which belong to O_2^2 (and do not belong to O_0^2).

Similar refinement applies to the convergence (2.3) in the kick-forced case.

Let $U(t)$, $t \geq 0$, be a solution for (1.2) and (1.5) such that $\mathcal{D}U(0) = \mu$. Then $\mathcal{D}U(t) \equiv \mu$ and the Ito formula applies to the energy functional $|U(t)|^2$. Taking expectation and differentiating the result in t we find that $2\nu\mathbb{E}\|U(t)\|^2 = \sum b_j^2 =: B_0$. That is,

$$\int \|u\|^2\,\mu(du) = \frac{1}{2\nu}B_0 \tag{2.12}$$

(see [31, 10]). If (1.2) is the NS system with the periodic boundary conditions, then using (2.10) with $l=1$ and applying the Ito formula to the enstrophy functional $|\mathrm{rot}\ u|^2 = \|u\|^2$ we get that

$$\int \|u\|_2^2\,\mu(du) = \int |\Delta u|^2\,\mu(du) = \frac{1}{2\nu}B_1\,, \quad B_1 = \sum \alpha_j b_j^2\,. \tag{2.13}$$

Since the functionals $u \mapsto \|u\|^2$ and $u \mapsto \|u\|_2^2$ belong to O_1 and O_2, respectively, then for any solution $u(t) = u(t; u_0)$ $(u_0 \in H)$ of the NS equation under the periodic boundary conditions we have:

$$\left|\mathbb{E}\|u(t)\|^2 - \frac{1}{2\nu}B_0\right| \le C_0 e^{-\sigma_0 t}\,, \quad \left|\mathbb{E}\|u(t)\|_2^2 - \frac{1}{2\nu}B_1\right| \le C_1 e^{-\sigma_1 t}$$

if $t \ge 1$ and (2.9) holds with $l = 3$.

2.3. *Stationary in space forces and solutions*

In this section we restrict ourselves to the NS system under the periodic boundary conditions. Now the basis $\{e_j\}$ is formed by the vector-fields (1.3), and it is convenient to write it as $\{e_s(x), s \in \mathbb{Z}_0^2\}$, where for any $s \in \mathcal{S}$, $e_s(x)$ is the first vector-field in (1.3) and $e_{-s}(x)$ is the second. Let us consider a white-force (1.5) such that $b_s \equiv b_{-s}$. Then

$$\begin{aligned}\zeta &= \sum_{s\in\mathcal{S}} b_s c_s s^\perp(\beta_s(t)\cos s\cdot x + \beta_{-s}(t)\sin s\cdot x)\\ &= \mathrm{Re}\sum_{s\in\mathcal{S}} b_s' s^\perp(\beta_s - i\beta_{-s})e^{is\cdot x}\,,\end{aligned} \tag{2.14}$$

where $b_s' = b_s c_s$. Now $\alpha_s = \nu|s|^2$ and the assumption (1.6) takes the form $\sum_{s\in\mathcal{S}} {b_s}^2|s|^2 < \infty$. Due to (2.14), for any $y \in \mathbb{R}^2$ we have

$$\begin{aligned}\zeta^y(t,x) := \zeta(t, x+y) &= \mathrm{Re}\sum b_s' s^\perp(\beta_s - i\beta_{-s})(\cos s\cdot y + i\sin s\cdot y)e^{is\cdot x}\\ &= \mathrm{Re}\sum b_s' s^\perp(\beta_s^y - i\beta_{-s}^y)e^{is\cdot x}\,,\end{aligned}$$

where $\beta_s^y = \beta_s^y(t)$, $t \in \mathbb{R}$, and for $s \in \mathcal{S}$ we have $\beta_s^y = \beta_s \cos s\cdot y + \beta_{-s}\sin s\cdot y$, $\beta_{-s}^y = \beta_{-s}\cos s\cdot y - \beta_{-s}\sin s\cdot y$. Since β_s and β_{-s} are independent normal random variables and $\mathcal{D}\beta_{\pm s} = N(0,|t|)$, then β_s^y and β_{-s}^y also are independent and are distributed as $N(0,|t|)$. So the processes $\zeta^y(t)$, $y \in \mathbb{R}^2$, are distributed identically with $\zeta(t)$.

Let μ be the unique stationary measure for Eqs. (1.1), (2.14) and (2.1) and let $U(t) = U(t,x)$, $t \ge 0$, be a solution such that $\mathcal{D}U(t) \equiv \mu$. Then $U^y(t,x) = U(t, x+y)$ is a stationary process in H, satisfying the equation with ζ replaced by ζ^y. Since $\mathcal{D}\zeta^y = \mathcal{D}\zeta$, then $\mathcal{D}U^y(t)$ is a stationary measure for the equation. Therefore, by the uniqueness, $\mathcal{D}U^y(t) \equiv \mu$. That is, the measure $\mathcal{D}U^y(t) \in \mathcal{P}$ is y-independent, and *the process $U(t,x)$ is stationary both in time and space*. Accordingly, *the stationary measure μ is homogeneous*, i.e. it is preserved by transformations of the space H of the form $u(x) \mapsto u(x+y)$ (y is a fixed vector from $\mathbb{R}^2$).

Since $\int U\,dx \equiv 0$, then

$$0 = \mathbb{E}\int U\,dx = \int(\mathbb{E}\,U(t,x))\,dx = (2\pi)^2\mathbb{E}\,U(t,x)\,.$$

That is,

$$\mathbb{E}\, U(t,x) \equiv 0\,, \quad \forall\, t, x\,. \tag{2.15}$$

If (2.9) holds with $l = 3$, i.e. if $\sum {b_s}^2 |s|^6 < \infty$, then due to (2.11) with $f(u) = u(x)$, we have:

$$|\mathbb{E}u(t,x;u_0)| \le C(|u_0|)e^{-\sigma t}\,,$$

for any u_0 and any x.

2.4. *The turbulence-limit*

Due to (2.2) and (2.7), the unique stationary measure μ comprises asymptotic in time stochastic properties of solutions for the NS system, cf. [1, Chap. VI] and [12, Sec. 6.1]. If the force (1.5) is such that $b_j \neq 0$ for all j, then for each positive viscosity ν the equation has a unique stationary measure μ_ν. The limiting properties of this measure as $\nu \to 0$ describe the 2D turbulence. At this moment we do not know much about the turbulence-limit for the NS system, apart from the relations (2.12) and (2.13) (where $\mu = \mu_\nu$) and their immediate consequences.[e] Instead, in the rest of this paper we discuss limiting properties of the stationary measure for the kick-forced equation when the period T between the kicks goes to zero, and asymptotic properties of time-averaged solutions, both in the white-forced and the kicked-forced cases.

For some other PDEs, different from but related to the 2D NS system, some progress in study of the turbulence-limit has been achieved in last years. Namely, in [23] (also see in [24]) a weak form of the Kolmogorov–Obukhov law from the theory of developed turbulence (see [25, Sec. 33]) is obtained for solutions of the randomly forced nonlinear Schrödinger equation with small viscosity. In [8] the small-viscosity 1D Burgers equation is considered. It is proved that when the viscosity goes to zero, the stationary measure weakly converges to a limit, and the limiting measure is studied. In [15] similar analysis is done for the nD Burgers equation. We note that the weak convergence of the stationary measure is a nice specific of the Burgers equation. Most likely, for the NS system this convergence does not hold (so the limiting properties of the stationary measure μ_ν correspond to some "very weak" convergence of measures).

3. Kick-Forces with Short Periods Between the Kicks

Let us consider Eq. (1.2) with the kick-force $\sqrt{\varepsilon}\eta$, where η has the form (1.4) and $T = \varepsilon$, $0 < \varepsilon \ll 1$:

$$\eta = \eta_\varepsilon(t,x) = \sqrt{\varepsilon} \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t - \varepsilon k)\,. \tag{3.1}$$

[e]In particular, these relations imply that the energy-range for the periodic 2D turbulent flow (described by Eq. (1.1)) is bounded uniformly in $\nu \in (0,1]$.

In addition to (H), we assume that $\mathbb{E}\,\xi_{jk} \equiv 0$, $\mathbb{E}\,\xi_{jk}^2 \equiv 1$, (1.6) holds and

$$b_j \neq 0\,, \quad \forall\, j\,. \tag{3.2}$$

Denoting by $u_\varepsilon(t; u_0)$ a solution for (1.2) and (3.1), equal u_0 for $t = 0$, due to the results of Sec. 2.1 for any $\varepsilon > 0$ we have the convergence

$$\mathcal{D}u_\varepsilon(t; u_0) \rightharpoonup \mu_\varepsilon \quad \text{as} \quad t \to \infty\,,$$

where μ_ε is the corresponding stationary measure.

Let us set $X_\varepsilon(t)$ to be equal to $\int_{0+}^{t+0} \eta_\varepsilon(s)\,ds$ for $t \in \mathbb{R}_+ \cap \varepsilon\mathbb{Z}$ and use the linear interpolation to extend X_ε to $\mathbb{R}_+$. Due to the Donsker theorem (see e.g. [29]), we have $\mathcal{D}X_\varepsilon(\cdot) \rightharpoonup \mathcal{D}\zeta(\cdot)$ as $\varepsilon \to 0$, where now $\rightharpoonup$ denotes the weak convergence of measures in the space $C([0, L], H)$, the process ζ is defined in (1.5) and L is any positive number. This convergence underlies the "splitting up method for stochastic PDEs" which in our situation states that

$$\mathcal{D}u_\varepsilon(t; u_0) \rightharpoonup \mathcal{D}u(t; u_0) \quad \text{as} \quad \varepsilon \to 0$$

for each $t \geq 0$, where $u(t; u_0)$ is a solution for (1.2) and (1.5), see in [22]. Using the results of Sec. 2.2 we have

$$\mathcal{D}u(t; u_0) \rightharpoonup \mu \quad \text{as} \quad t \to \infty\,.$$

So altogether we got the convergences

$$\begin{array}{ccc} \mathcal{D}u_\varepsilon(t; u_0) & \xrightarrow[t\to\infty]{} & \mu_\varepsilon \\ \big\downarrow{\scriptstyle \varepsilon\to 0} & & \\ \mathcal{D}u(t; u_0) & \xrightarrow[t\to\infty]{} & \mu \end{array}$$

(the arrows signify weak convergence of measures). In [22] A. Shirikyan and the author of this paper show that $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$. So (3.3) closes up to the commutative diagram:

$$\begin{array}{ccc} \mathcal{D}u_\varepsilon(t; u_0) & \xrightarrow[t\to\infty]{} & \mu_\varepsilon \\ {\scriptstyle \varepsilon\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\to 0} \\ \mathcal{D}u(t; u_0) & \xrightarrow[t\to\infty]{} & \mu \end{array}$$

That is, for distributions of solutions for the NS system, forced by the short-kick force (3.1) and (3.2), the limits $t \to \infty$ and $\varepsilon \to 0$ commute.

4. Ergodic Properties of Solutions

In this section we use the convergences (2.5) and (2.7) and some versions of the classical limiting theorems from the theory of probability to examine ergodic properties of solutions for the randomly forced NS systems.

4.1. *Ergodicity and the strong law of large numbers*

Let us consider Eqs. (1.2) and (1.5) which satisfies (2.1). Let μ be its stationary measure and $U_\tau(t)$, $t \geq \tau$, be a solution such that $\mathcal{D}U_\tau(\tau) = \mu$. Then $\mathcal{D}U_\tau(t) = \mu$ for each t. Passing to the limit as $\tau \to -\infty$, we obtain an a.s. continuous stationary process $U(t) \in H$, $t \in \mathbb{R}$ (defined on a new probability space), such that $\mathcal{D}U(t) \equiv \mu$ and U satisfies (1.2), where $\eta(t)$ is replaced by a new process $\eta'(t)$ with the same distribution (cf. [17, Proposition 1.5]). So U is a Markov process with the same transition function P. Since μ is the unique stationary measure for the equation, than the process U is ergodic, see [28, Theorem 3.2.6]. Therefore the strong law of large numbers applies to the process $h(U)$, where h is any function from $L^2(H, d\mu)$ (see [28, Theorem 3.3.1]):

$$\langle h(U)\rangle_0^T := \frac{1}{T}\int_0^T h(U(t))\,dt \xrightarrow[T\to\infty]{} \int h(u)\mu(du) \quad \text{a.s.} \tag{4.1}$$

If $f \in O$ (see (2.4)), then this convergence holds for any solution:

Theorem 4.1. *If* (2.1) *holds and* $u(t) = u(t; u_0)$ *is a solution of* (1.2) *and* (1.5), *then*

$$\langle f(u)\rangle_0^T \xrightarrow[T\to\infty]{} \int f(u)\,\mu(du) \quad a.s. \tag{4.2}$$

for any $f \in O$ *and* $u_0 \in H$.

Proof. Let P be the measure, defined by the process u in the space of trajectories $\mathcal{H} = C([0,\infty), H)$. Then (4.2) states that the functionals on $\mathcal{H}$, defined by the l.h.s. of (4.2), converge P-a.s. as $T \to \infty$ to the constant $\int f\,d\mu$. So to check (4.2) we can replace $u(t)$ by any weak solution $u^1(t)$ for (1.2) and (1.5), equal u_0 at $t = 0$.[f] Similar, (4.1) remains true if we replace U by a process $u^2(t) \in H$, $t \geq 0$, distributed as U.

For any random variable $T' \geq 0$ which is a.s. finite, we set

$$\langle f(u)\rangle_{T'}^T := I_{T'<T}\frac{1}{T-T'}\int_{T'}^T f(u(t))\,dt\,.$$

Since $|f| \leq 1$, then when checking (4.2) we can replace its l.h.s. by $\langle f(u)\rangle_{T'}^T$.

Due to [20, Proposition 2.6], the weak solutions u^1, u^2 can be chosen in such a way that for a suitable random time T' as above we have

$$|u^1(t) - u^2(t)| \leq Ce^{-\sigma t}\,, \quad \forall\, t \geq T'\,,$$

where C and σ are positive constants. Then

$$|\langle f(u^1) - f(u^2)\rangle_{T'}^T| \leq I_{T'<T}\,\frac{1}{T-T'}\int_{T'}^T |u^1 - u^2|\,dt \leq \frac{CI_{T'<T}}{\sigma(T-T')}\,e^{-\sigma T'}\,. \tag{4.3}$$

[f] That is, by any process u^1 such that $u^1(0) = u_0$ and u^1 satisfies (1.2), where η is replaced by a process $\partial_t\zeta'$ and ζ' has the same distribution as ζ in (1.5). This process defines in $\mathcal{H}$ the same measure P.

Since (4.1) with $h = f$ implies that $\langle f(u^2)\rangle_{T'}^{T} \to \int f\,d\mu$ a.s. and since due to (4.3) $\langle f(u^1)\rangle_{T'}^{T} - \langle f(u^2)\rangle_{T'}^{T} \to 0$ a.s., then (4.2) follows. □

Remark 4.1. (1) If the boundary conditions are periodic and (2.9) holds with some $l \geq 1$, then due to (2.10) and (2.11), the theorem's assertion is valid for any $f \in O_{l-1}$.

(2) An obvious version of the theorem holds for solutions of the kicked equation (1.2) and (1.4).

(3) The arguments used in the proof apply to study other asymptotic properties of the solutions u. In particular, if g is a bounded Lipschitz functional on H and the Law of Iterated Logarithm holds for the stationary process $g(U)$, then it also holds for the processes $g(u)$.

4.2. *The central limit theorem (CLT)*

Let $U(t) \in H$, $t \in \mathbb{R}$, be the stationary weak solution for (1.2), (1.5) and (2.1), constructed above, and $\{\mathcal{F}_{\leq t}, t \in \mathbb{R}\}$, be the corresponding flow of σ-algebras. Let $f \in O$ be a functional such that $\int f(u)\,\mu(du) = 0$. Then, using the Markov property and (2.7), we get:

$$\begin{aligned}\mathbb{E}(\mathbb{E}(f(U_0)|\mathcal{F}_{\leq -t})^2 &= \int (\mathbb{E}f(u(t;u_0)))^2\,\mu(du_0)\\ &\leq C_1^2 e^{-2\sigma t}\int (1+|u_0|^2)^2\,\mu(du_0)\,.\end{aligned}$$

Since the integral in the r.h.s. is bounded due to (2.8), then

$$\mathbb{E}(\mathbb{E}(f(U_0)|\mathcal{F}_{\leq -t}))^2 \leq C_2 e^{-2\sigma t}\,.$$

Let $\mathcal{F}^0_{\leq t}$ be the σ-algebra, generated by the random variables $f(U(s)), s \leq t$. Then $\mathcal{F}^0_{\leq t} \subset \mathcal{F}_{\leq t}$. Denoting $\mathbb{E}(f(U_0)|\mathcal{F}_{\leq -t}) = F$ and using the Jensen inequality we have:

$$\mathbb{E}(\mathbb{E}(F|\mathcal{F}^0_{\leq -t}))^2 \leq \mathbb{E}(\mathbb{E}(F^2|\mathcal{F}_{\leq -t})) = \mathbb{E}F^2\,.$$

So

$$\mathbb{E}(\mathbb{E}(f(U_0)|\mathcal{F}^0_{\leq -t}))^2 \leq C_2 e^{-2\sigma t}\,. \tag{4.4}$$

Let us consider the real-valued stationary process $X(t) = f(U(t)), t \in \mathbb{R}$. Since the process U is ergodic, then X is ergodic as well. Due to the mixing property (4.4), the CLT as in [6, Theorem 7.6],[g] applies to X, and for any positive T we have

$$\mathcal{D}\left(\frac{f(U(0)) + \cdots + f(U((n-1)T))}{\sqrt{n}}\right) \rightharpoonup \hat{\sigma}N(0,1) \quad \text{as} \quad n \to \infty\,, \tag{4.5}$$

[g] This form of the CLT goes back to M. I. Gordin [13].

where

$$\hat\sigma^2 = \hat\sigma^2(T) = \mathbb{E}f(U(0))^2 + 2\sum_{n=1}^{\infty}\mathbb{E}(f(U(0))f(U(nT))) \geq 0\,. \tag{4.6}$$

Due to (4.4), $|\mathbb{E}f(U(0))f(U(nT))| \leq C_2^{1/2}e^{-\sigma nT}$. Therefore $\hat\sigma^2 > 0$ at least if T is sufficiently large.

For any $u_0 \in H$ let us denote $u(t) = u(t; u_0)$. Taking a function $h : \mathbb{R} \to \mathbb{R}$ such that $|h| \leq 1$ and $\operatorname{Lip} h \leq 1$, we wish to estimate

$$\left|\mathbb{E}\, h\left(\frac{f(U(0) + \cdots + f(U((n-1)T)))}{\sqrt{n}}\right)\right. \\ \left. - \mathbb{E}\, h\left(\frac{f(u(0) + \cdots + f(u((n-1)T)))}{\sqrt{n}}\right)\right|. \tag{4.7}$$

Denoting $n^{-1/2}f(U(jT)) = f_U^j$ and $n^{-1/2}f(u(jT)) = f_u^j$, we see that (4.7) is majorized by

$$\sum_{j=0}^{n-1}|\mathbb{E}(h(f_U^0 + \cdots + f_U^j + f_u^{j+1} + \cdots + f_u^{n-1}) \\ - h(f_U^0 + \cdots + f_u^j + f_u^{j+1} + \cdots + f_u^{n-1}))|\,. \tag{4.8}$$

Due to [20, (2.39)], we can replace U and u by weak solutions having the same initial conditions, such that (keeping for the new solutions the same notations) we have

$$\mathbb{P}\{|U(jT) - u(jT)| \geq C_1e^{-\sigma jT}\} \leq C_2(1 + |u_0|^2)e^{-\sigma jT}\,. \tag{4.9}$$

Let us denote the event in the l.h.s. of (4.9) by Q_j, $j = 0, \ldots, n-1$. Since Lip$f$, Lip $h \leq 1$, then everywhere outside Q_j we have

$$|h(\cdots + f_U^j + f_u^{j+1} + \cdots) - h(\cdots + f_u^j + f_u^{j+1} + \cdots)| \leq n^{-1/2}C_1e^{-\sigma jT}\,.$$

As $|h| \leq 1$, then due to (4.9) the jth term in (4.8) is bounded by

$$n^{-1/2}C_1e^{-\sigma jT} + 2C_2(1 + |u_0|^2)e^{-\sigma jT} \leq C_3e^{-\sigma jT}\,,$$

where C_3 depends on $|u_0|$. From other hand, since $|f_u^j|, |f_U^j| \leq n^{-1/2}$, then the jth term is also bounded by $2n^{-1/2}$.

Let us denote by n_T the smallest integer $\geq (\log n)/\sigma T$. Then, majorizing the first n_T terms in (4.8) using the second estimate, and majorizing the rest of them using the first, we get that

$$(4.8) \leq 2n^{-1/2}n_T + C_3\sum_{j=n_T}^{n-1}e^{-\sigma jT} \leq 2n^{-1}n_T + \frac{C_4}{n\sigma T} \leq C_5n^{-1/3}\,.$$

Thus, for each h as above, (4.7) is bounded by $C_5 n^{-1/3}$. Hence,

$$\Bigg\| \mathcal{D}\left(\frac{f(U(0)) + \cdots + f(U((n-1)T))}{\sqrt{n}}\right) - \mathcal{D}\left(\frac{f(u(0)) + \cdots + f(u((n-1)T))}{\sqrt{n}}\right)\Bigg\|_L^* = O(n^{-1/3}), \qquad (4.10)$$

where now $\|\cdot\|_L^*$ stands for the Lipschitz-dual norm for measures on the real line. Since the convergence in this norm is equivalent to the weak convergence of measures [5], then (4.5) and (4.10) imply the CLT for the process $f(u(t))$:

Theorem 4.2. *If* (2.1) *holds,* μ *is the unique stationary measure for* (1.2) *and* (1.5) *and* $u = u(t; u_0)$ *is any solution, then for any functional* $f \in O$ *such that* $\int f\, d\mu = 0$, *we have the convergence*

$$\mathcal{D}\left(\frac{f(u(0)) + \cdots + f(u((n-1)T))}{\sqrt{n}}\right) \rightharpoonup \hat{\sigma} N(0,1) \quad \text{as} \quad n \to \infty\,, \qquad (4.11)$$

where $\hat{\sigma}$ *is defined in* (4.6).

If the boundary conditions are periodic and (2.9) holds for some $l \geq 1$, then f can be taken from the space O_{l-1}. In particular, if $l = 3$, then we can take $f(u) = u^i(x)$, or $f(u) = u^i(x)u^j(y)$, where x and y are fixed points from the space-domain and $i, j \in \{1, 2\}$.

The same result is true for solutions of the kicked equation (1.2), (1.4) and (2.1) (in this case we choose T to be equal to the interval between the kicks). Proof remains the same.

As an example, let us consider the NS equation under the periodic boundary conditions, perturbed by the space-stationary force $\partial_t \zeta$, where the process ζ is defined in (2.14). Assuming that (2.9) holds with $l \geq 3$ (i.e. that $\sum {b_s}^2 |s|^6 < \infty$) and using (2.15) we get that

$$\mathcal{D}\left(\frac{u(0,x) + \cdots + u((n-1)T, x)}{\sqrt{n}}\right) \rightharpoonup \hat{\sigma} N(0,1) \quad \text{as} \quad n \to \infty\,. \qquad (4.12)$$

The dispersion $\hat{\sigma}^2$ is x-independent since it is defined in terms of the process U and the latter is stationary in time and in space, see in Sec. 2.3.

We claim (giving no proof) that the integral version of (4.12) also is true:

$$\mathcal{D}\left(T^{-1/2} \int_0^T u(s,x)\, ds\right) \rightharpoonup \sigma' N(0,1) \quad \text{as} \quad T \to \infty\,. \qquad (4.13)$$

This convergence is related to the following physical effect. Let us imagine that we are trying to measure the velocity u of the fluid at a point x, using some device. Then what we really measure will be the averaged quantity const $\cdot \int_0^T u(s,x)\, ds$. If our device is unsophisticated, then $T \gg 1$ and due to (4.13) we shall get the false impression that $u(t,x)$ is a Gaussian random variable. The normal distribution of the velocity u was "observed" in a number of experiments in the first half of the last century (some of them are discussed in [1, Sec. 8.1]). We believe that the arguments above explain this phenomenon.

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