Hölder estimates for solutions of parabolic SPDEs

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1 Introduction

This paper is devoted to L^∞ and Hölder estimates for solutions of second order parabolic stochastic partial differential equations (SPDEs) in bounded domains, under the minimal regularity assumptions on the coefficients. Recent years several works were focused at the L^p and $W^{l,p}$ theory for parabolic SPDEs. The $W^{l,p}$-estimates were usually obtained under rather strong regularity assumptions on the coefficients. On the other hand, in various qualitative and applied problems the estimates in terms of L^∞- or even L^p-norms of the coefficients are very important. Big progress was achieved by N.V.Krylov in the important work [1], where the regularity assumptions were reduced essentially for equations in the whole R^d. In our work we propose another techniques which allow to reduce farther the regularity assumptions, to treat initial-boundary problems and to obtain more specific results for solutions of SPDEs with additive noise (this class of equations is smaller than that in [1] but still it is sufficiently large and important). Our approach differs essentially from [1] and gives rise to a shorter proof.

In the present paper we study SPDEs of the form

$$
\begin{gathered}
du = Au\,dt + \sum_{j=1}^{N} f^j(x,t)dw_t^j, \quad (x,t) \in Q \times (0,\infty), \\
u|_{t=0} = 0, \qquad u|_{\partial Q} = 0,
\end{gathered} \tag{1}
$$

where Q is a smooth bounded domain in R^n, w_t^j are standard independent one-dimensional Brownian motions and $f^j(x,t)$ are progressively measurable L^∞- or L^p-functions, in the latter case p is sufficiently large. Regarding the operator A we assume that either A is a generic second order uniformly elliptic operator with sufficiently smooth coefficients, or A is a divergence form uniformly elliptic operator with measurable bounded coefficients.

In this paper we prove that a solution of (1) is a Hölder-continuous function a.s. and that corresponding L^∞- and Hölder norms have finite momenta of any order.

For a stochastically perturbed heat equation (i.e. for equation (1) with $A = \Delta$) under periodic boundary conditions these estimates were proved by one of the authors in the Appendix to [2].

It should be noted that results of the paper can be generalised to equations involving infinite-dimensional stochastic term $\sum_{j=1}^{\infty} f^j(x,t)dw_t^j$ instead of finite-dimensional one; the L^∞-convergence (or L^p-convergence with sufficiently large p) of the sum $\sum_{j=1}^{\infty} |f^j(x,t)|$ is then required.

The L_∞-estimates we derive in this paper can be used to prove solvability on some nonlinear SPDEs and examine regularity of their solutions (see [2]). In corresponding arguments the L_∞-estimates play a role, similar to one which the classical maximum principle plays in deterministic equations.

In the first part of the paper we obtain several estimates for Hölder and L^∞-norms of solutions for deterministic parabolic equations. These estimates are used in the following sections to derive pointwise bounds for increments of the solution of (1) and bounds for the corresponding function norms.

2 Setting of the problem. Technical estimates.

Let w_t^j be standard independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ be a corresponding filtration. We assume that $f^j(x,t)$ are $\mathcal{F} \times \mathcal{B}$-measurable, $\mathcal{F}_t$-adapted functions, where $\mathcal{B}$ stands for the Borel σ-algebra, and that at least one of the following two assumptions holds:

$\mathbf{B}^\infty$ $\quad |f(x,t)| \le 1$ almost surely (a.s.), for all x and t,

$\mathbf{B}^p$ $\quad \|f(\cdot,t)\|_{L^p(Q)} \le 1$ a.s., for every t.

In the statements below A is a uniformly elliptic second order differential operator of the form

$$A = a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

with smooth coefficients.

Theorem 1. *Let $\mathbf{B}^\infty$ hold. Then, under the conditions above,*

- *for any $k \ge 0$ and any $T \ge 0$, a solution of problem (1) satisfies the estimate*

$$\mathbf{E}\big|u(\cdot,t)\big|^k_{L^\infty(Q_T)} \le C_k \exp(\bar{c}T), \tag{2}$$

where $Q_T = Q \times [T, T+1]$ and $\bar{c} = \max(0, \max_Q c(x))$;

- *the solution $u(x,t)$ is Hölder-continuous a.s., and for any $T \ge 0$, $k \ge 0$ and $\theta < 1/2$ the upper bound*

$$\mathbf{E}\big|u\big|^k_{C^\theta(Q_T)} \le C(k,\theta)\exp(\bar{c}T) \tag{3}$$

holds.

In (2), (3) and everywhere below we write $|u|_{L^\infty(Q_T)}$ instead of $\big|u\,|_{Q_T}\big|_{L^\infty(Q_T)}$, etc.

Theorem 2. *If $\mathbf{B}^p$ holds for some $p > n$ then*

- *for any $k \geq 0$ and any $T \geq 0$, a solution of problem (1) satisfies the estimate (2);*
- *the solution $u(x,t)$ is Hölder-continuous a.s., and for any $T \geq 0$, $k \geq 0$ and $\theta < 1/2 - n/2p$ the upper bound (3) holds.*

Remark 1. *If zero order term $c(x)$ of the operator A is non-positive, then the exponential factor disappears in all the estimates (2)-(3) above. On the other hand, a simple change of the unknown functions $U = \exp(-\alpha t)u$ allows to reduce, under an appropriate choice of α, problem (1) with arbitrary A to a problem involving the operator A with non-positive zero order term.*

Later on we always assume without loss of generality that

$$c(x) \leq 0.$$

In the case of a uniformly elliptic operator A of the divergence form with bounded measurable coefficients one has:

Theorem 3. *Let A be a uniformly elliptic divergence form operator with bounded coefficients. Then, under the assumption $\mathbf{B}^\infty$,*

- *for any $k \geq 0$ and any $T \geq 0$, the solution of problem (1) satisfies the estimate*
$$\mathbf{E}\big|u(\cdot,t)\big|^k_{L^\infty(Q_T)} \leq C_k;$$
- *the solution $u(x,t)$ is Hölder-continuous a.s., and there is $\alpha > 0$ depending only on lower and upper ellipticity constants of A and on geometry of the domain Q, such that for any $T \geq 0$, $k \geq 0$ and $\theta < \alpha$ the upper bound*
$$\mathbf{E}\big|u\big|^k_{C^\theta(Q_T)} \leq C(k,\theta) \tag{4}$$
holds.

Under the assumption $\mathbf{B}^p$ with $p > n$,

- *there exists $\alpha_0 > 0$ such that the estimate (4) holds true for every $T \geq 0$, $k \geq 0$ and $\theta \leq \alpha_0$.*

The proof of the theorems relies on several auxiliary statements. Let us consider the following initial-boundary problem:

$$\frac{\partial v}{\partial t} - Av = 0, \qquad v\big|_{t=0} = F(x), \;\; v\big|_{\partial Q} = 0, \tag{5}$$

and denote by S_t the corresponding semigroup:

$$v(\cdot, t) = (S_t F)(\cdot), \quad F = F(\cdot).$$

The next statement deals with a "smooth" operator A.

Lemma 1. *Let $|F(x)| < M$. Then, for any $\theta < 1$, the following estimates hold with some $c > 0$:*

$$\|v(\cdot, t)\|_{C^\theta(Q)} \le c(\theta) M t^{-\theta/2} \exp(-ct), \tag{6}$$

$$|v(x, t + \delta) - v(x, t)| \le c(\theta) M t^{-\theta} \delta^\theta \exp(-ct) \tag{7}$$

Moreover, if $\|F\|_{L^q(Q)} \le M$ and $q > 1$, then

$$\|v(\cdot, t)\|_{C^\theta(Q)} \le c(\theta) M t^{-\theta/2 - d/(2q)} \exp(-ct), \tag{8}$$

$$|v(x, t + \delta) - v(x, t)| \le c(\theta) M t^{-\theta - d/(2q)} \delta^\theta \exp(-ct). \tag{9}$$

Proof. Denoting by $Q(x, y, t)$ the Green function associated with problem (5), we have that

$$\|G(x, \cdot, t)\|_{L^p(Q)} \le c(p) t^{-d/(2q)}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{10}$$

(see, for example, [3]). This implies that

$$\|v(\cdot, t)\|_{L^\infty(R^d)} \le c(p) t^{-d/(2q)} \|F\|_{L^q(R^d)} \tag{11}$$

for any $F \in L^q(R^d)$.

Now, in addition to problem (5) we consider the initial-boundary problems in the domains $Q_\gamma = \gamma Q = \{\gamma x \, : \, x \in Q\}$, $\gamma \ge 1$:

$$\begin{gathered} \frac{\partial V}{\partial t} - A_\gamma V = 0, \quad x \in Q_\gamma, \\ V\big|_{t=0} = F(\gamma x), \quad V\big|_{\partial Q_\gamma} = 0, \end{gathered} \tag{12}$$

where

$$A_\gamma = a_{ij}(\gamma x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \gamma^{-1}b_i(\gamma x)\frac{\partial}{\partial x_i} + \gamma^{-2}c(\gamma x).$$

Due to the smoothness of the coefficients of A and of the domain Q, by the standard elliptic estimates one has:

$$|\frac{\partial}{\partial t}V(x,1)| + |\nabla_x V(x,1)| \le c\|V(\cdot,\frac{1}{2})\|_{L^\infty(Q_\gamma)} \tag{13}$$

uniformly in $x \in Q_\gamma$, where the constant c does not depend on γ (see [6], [5]). After an appropriate rescaling, the last estimate reads

$$t\,|\frac{\partial}{\partial t}v(x,t)| + t^{1/2}|\nabla_x v(x,t)| \le c\|v(\cdot,\frac{t}{2})\|_{L^\infty(Q)} \tag{14}$$

for all t, $0 < t \le 1$. From (11) and (14) it follows that

$$|\frac{\partial}{\partial t}v(x,t)| \le ct^{-1-d/(2q)}\|F\|_{L^q(Q)}, \tag{15}$$

$$|\frac{\partial}{\partial x}v(x,t)| \le ct^{-(1/2)-d/(2q)}\|F\|_{L^q(Q)}, \tag{16}$$

for all $t \le 1$. Finally, we use the interpolation inequality [1]

$$\|v\|_{C^\theta} \le 2\|v\|_{L^\infty}^{1-\theta}\,\|v\|_{C^1}^{\theta}$$

to derive (6), (7) and (8), (9) from (11), (15) and (16). For large t the required estimates are the consequences of the exponential decay of the Green function. This completes the proof of the lemma. □

In the case of an operator A with measurable coefficients we have:

Lemma 2. *Let A be a divergence form uniformly elliptic operator with bounded coefficients. Then there exists $\alpha > 0$ depending only on the ellipticity constants of A and on the domain Q, such that for any $\theta < \alpha$ a solution v for the problem (5) admits all the estimates of Lemma 1.*

[1] Denoting $\|v\|_{L^\infty}$ by γ_0 and $\|v\|_{C^1}$ by γ_1, we have $|v(x+y)-v(x)| \le 2\gamma_0$ and $|v(x+y)-v(x)| \le \gamma_1|y|$. Raising these inequalities to the degrees $(1-\theta)$ and θ respectively and multiplying the new ones, we get that $|v(x+y)-v(x)| \le 2^{1-\theta}\gamma_0^{1-\theta}\gamma_1^{\theta}|y|^{\theta}$. This relation implies the inequality we use.

Proof. The proof is rather similar to that of preceding Lemma and we outline it briefly. According to [4] the inequality (10) (hence, also (11)) is still valid. The bound (13) is not true any more but instead we can use the Nash-type estimates for the Hölder norms of the solution $V(x,t)$ (see [5], [6]) to get that

$$\|V\|_{C^{\alpha}(Q_\gamma\times[1,2])} \le c\|V(\cdot,\frac{1}{2})\|_{L^\infty(Q)}$$

for some $\alpha > 0$ and c independent of γ. To complete the proof one can follow the above arguments. $\square$

Now we proceed with problem (1) in a "smooth" case. Its solution does exist, is unique and can be written in the form (see [7], Chapter 5; [1])

$$u(t) = \int_0^t \sum_{j=1}^N S_{t-\tau} f^j(x,\tau)\, dw_\tau.$$

The last formula is, in fact, the definition of a "mild solution". Clearly it is sufficient to consider one element of the sum above. In what follows the summation sign and the index j will be omitted. We begin by pointwise estimating an increment of the solution in x variables. For this aim we denote $[S_{t-\tau}f(x_1,\tau) - S_{t-\tau}f(x_2,\tau)]$ by $g(t,\tau)$, and introduce a random variable

$$U = u(t,x_1) - u(t,x_2) = \int_0^t g(t,\tau)dw_\tau$$

The quadratic characteristics of the stochastic integral is given by $X(t) = \int_0^t g^2(t,\tau)d\tau$. From Lemma 1 we derive the estimate

$$X(t) \le M^2 c(\theta)|x_1 - x_2|^{2\theta} \int_0^t t^{-\theta}\exp(-2ct)dt \le M^2 c'(\theta)|x_1-x_2|^{2\theta},$$

valid for any $\theta < 1$. Thus, by the Burkholder-Devis-Handy inequality we have

$$\mathbf{E}|U|^p \le c_1(p)\mathbf{E}(X^{p/2}) \le c(p,\theta)M^p|x_1-x_2|^{p\theta}, \quad \theta < 1. \tag{17}$$

Proceeding similar with an increment in time we define

$$R = u(x, t+\delta) - u(x,t) = \int\limits_t^{t+\delta} S_{t+\delta-\tau} f(x,\tau) dw_\tau +$$

$$+ \int\limits_0^t (S_{t+\delta-\tau} f(x,\tau) - S_{t-\tau} f(x,\tau)) dw_\tau = I_1 + I_2.$$

The first integral I_1 can be estimated as follows:

$$\mathbf{E}|I_1|^p \leq c\delta^{p/2} M^p. \tag{18}$$

Indeed, if we denote $S_{t+\delta-\tau} f(x,\tau)$ by $h_1(t,\tau)$, then

$$\int\limits_t^{t+\delta} h_1^2(t,\tau) d\tau \leq M^2 \delta,$$

and required estimate (18) is a direct consequence of the Burkholder-Davis-Handy inequality.

In order to estimate I_2 we define

$$h_2(t,\tau) = S_{t+\delta-\tau} f(x,\tau) - S_{t-\tau} f(x,\tau).$$

By Lemma 1 we have

$$|h_2(t,\tau)| \leq c(\theta) M (t-\tau)^{-\theta} \delta^\theta \exp(-c(t-\tau))$$

for any $\theta < 1/2$. Thus,

$$\int\limits_0^t h_2^2(t,\tau) d\tau \leq M^2 \delta^{2\theta} \int\limits_0^t (t-\tau)^{-2\theta} \exp(-2c(t-\tau)) d\tau \leq c(\theta) M^2 \delta^{2\theta}$$

and by the Burkholder-Davis-Handy inequality we have

$$\mathbf{E}|I_2|^p \leq c(\theta) \delta^{\theta p} M^p, \quad \theta < 1/2. \tag{19}$$

Combining (17), (18) and (19), we obtain

$$\mathbf{E}|u(x_1,t_1) - u(x_2,t_2)|^p \leq c(\theta)(|t_1 - t_2| + |x_1 - x_2|)^{\theta p} \tag{20}$$

for any $p > 1$ and any $\theta < 1/2$.

3 Boundedness and Hölder-continuity of the solution, higher momenta

This section deals with estimation of $L^\infty-$ and C^θ-norms of the solution of problem (1). The first statement here is given by the following

Lemma 3. *Let the functions $f^j(x,t)$ satisfy condition $\mathbf{B}^\infty$ of the preceding section. Then*

- *There is a.s. continuous version of the solution $u(x,t)$ of problem (1);*
- *if $u(x,t)$ is as above, then, for any $T \geq 0$ and any $p \geq 1$,*

$$\mathbf{E}\|u\|^p_{L^\infty(Q\times[T,T+1]} \leq c(p) < \infty.$$

Proof. The first statement follows immediately from (20) by the Kolmogorov theorem; it just suffices to choose a sufficiently large p.

To prove the second one we consider a cylinder $Q^T = Q \times [T, T+1]$ and define a family of sets:

$$\mathcal{K}_n = \{k \in Z^{d+1} \,|\, \frac{k}{2^n} \in Q^T\}, \quad n = 1, 2, \ldots .$$

For arbitrary vector $e \in Z^{d+1}$ such that $|e|_1 = \max\limits_{1\leq j\leq d} |e_j| = 1$ we then set $\xi_k^{n,e} = |u((k+e)2^{-n}) - u(k2^{-n})|$. By Lemma 1 we have

$$\mathbf{E}|\xi_k^{n,e}|^p \leq c(\theta)2^{-np\theta}, \quad 0 < \theta < 1/2, \tag{21}$$

for any $p \geq 1$. Let us introduce the events

$$\mathcal{A}^{n,e}_{k,q} = \{\omega \in \Omega \,|\, \xi_k^{n,e} \geq Kq^n\}.$$

From (21) by the Chebyshev inequality we get

$$\mathbf{P}\big(\mathcal{A}^{n,e}_{k,q}\big) \leq \frac{\mathbf{E}|\xi_k^{n,e}|^p}{K^p q^{np}} \leq c(\theta)\frac{2^{-np\theta}}{K^p q^{np}}.$$

For each n the total number of the events $\mathcal{A}^{n,e}_{k,q}$ is not greater than $2^{\bar{c}(d,Q)n}$. Thus, for the union $\mathcal{A}^n_q = \bigcup\limits_{k\in\mathcal{K}} \left(\bigcup\limits_{|e|_1=1} \mathcal{A}^{n,e}_{k,q} \right)$ we obtain

$$\mathbf{P}(\mathcal{A}^n_q) \leq c(\theta)K^{-p}q^{-pn}2^{\bar{c}(d,Q)n-\theta pn} = c(\theta)K^{-p}\alpha^n,$$

where $\alpha = 2^{\bar{c}(d,Q)}/(2^\theta q)^p$. If we put $\theta = 1/3$, $q = 2^{-1/6}$, then $\alpha = 2^{\bar{c}(d,Q)}/2^{p/6}$. Finally, taking $p > 6\bar{c}(d,Q) + 1$ we have $\alpha < 1/2$. Hence, for the event

$$\mathcal{A} = \bigcup_{n>1} \mathcal{A}_q^n$$

the following estimate holds true:

$$\mathbf{P}(\mathcal{A}) \le c(\theta) K^{-p}. \tag{22}$$

From now on we assume without loss of generality that Q is the d-dimensional cube $(0,1)^d$. Indeed, to reduce the case of a general domain to that of cube it suffices to extend the function $u(x,t)$ in the exterior of Q as zero and rescale if necessary the arguments.

Any point of Q can be represented in the form

$$x = \sum_{j=1}^{\infty} e(j) 2^{-j}, \quad |e(j)|_1 \le 1.$$

Let us denote $x(m) = \sum\limits_{j=1}^{m} e(j) 2^{-j}$ (as usual, $x(0) = 0$). Clearly, $u(x(0),t) = 0$. Then, by the definition of the set $\mathcal{A}_{k,q}^{n,e}$, for any $\omega \notin \mathcal{A}$ we have

$$|u(x(m),t) - u(x(m+1),t)| \le K q^m = K 2^{-m/6}.$$

Therefore,

$$|u(x,t)| \le K \sum_{m=1}^{\infty} 2^{-m/6} = K 2^{1/6} (2^{1/6} - 1). \tag{23}$$

It now follows from (22) and (23) that

$$\mathbf{P}\{\|u\|_{L^\infty(Q^T)} \ge K\} \; \le \; c_p (K+1)^{-p}$$

for any $p > \bar{c}(d,Q)$. Finally, denoting $U_T = \|u\|_{L^\infty(Q^T)}$, we find that

$$\mathbf{E} U_T^s = \int_0^\infty x^s d\mathbf{P}\{U_t \le x\} =$$

$$= s \int_0^\infty x^{s-1} \mathbf{P}\{U_t \le x\} dx \le s c_p \int_0^\infty x^{s-1} (x+1)^{-p} dx < \infty$$

if $p > s+1$. The Lemma is proved. $\square$

In order to estimate Hölder norms of $u(x,t)$, let us first formulate the following simple assertion.

Lemma 4. *Let a function $g(y)$ satisfy the estimate*

$$\operatorname*{osc}_J \leq \gamma_n \tag{24}$$

for any small cube J which is a mesh of the grid $2^{-n}Z^{d+1}$, i.e. $J = 2^{-n}j + [0, 2^{-n}]^{d+1}$, where $j \in Z^{d+1}$. Then, for any $\Delta \in R^{d+1}$, one has

$$|g(y+\Delta) - g(y)| \leq 2\gamma_{[\log_2(1/|\Delta|)]},$$

where $[\cdot]$ stands for the integer part.

Proof. For $n = [\log_2(1/|\Delta|)]$ let us consider the family of closed cubes defined above. Clearly, either both y and $y+\Delta$ belong to the same cube or they are situated in adjacent cubes having at least one point in common. So the statement of Lemma follows. ☐

Next, for a solution of (1) and for any $\omega \notin \mathcal{A}$ the inequality (24) holds with $\gamma_n = Kq^n$. Thus,

$$|u((x,t)+\Delta) - u(x,t)| \leq 2Kq^{[\log_2(1/|\Delta|)]} \leq 2Kq^{\log_2(1/|\Delta|)-1} =$$

$$= 2Kq^{-1}2^{\log_2 q \, \log_2(1/|\Delta|)} = 2Kq^{-1}|\Delta|^{\log_2(1/q)}.$$

If we put $q = 2^{-\theta_1}$ with $\theta_1 < \theta < 1/2$, then $\log_2(1/q) = \theta_1$, and the above inequality implies

$$|u((x,t)+\Delta) - u(x,t)| \leq 2K2^{\theta_1}\Delta^{\theta_1} \leq 4K\Delta^{\theta_1}.$$

Therefore, for all $\omega \notin \mathcal{A}$:

$$\|u\|_{C^{\theta_1}(Q^T)} \leq 4K.$$

Taking into account (22) we derive from the last estimate that

$$\mathbf{P}\{\|u\|_{C^{\theta_1}(Q^T)} > K\} \leq c(p,\theta_1)K^{-p}$$

for all $p > 1$. Finally, for any $\theta_1 < 1/2$ and any $m \geq 1$ the bound

$$\mathbf{E}\|u\|^m_{C^{\theta_1}(Q^T)} \leq c(m,\theta_1)$$

follows in a usual way, and we arrive at the assertion:

Lemma 5. *For any $\theta < 1/2$ a solution $u(x,t)$ of problem (1) is a C^θ-function a.s. Moreover, for any cylinder Q^T and any $m \geq 1$, we have* $\mathbf{E}\|u\|^m_{C^\theta(Q^T)} \leq c(m,\theta)$.

Now, to complete the **proof of Theorem 1** it is merely sufficient to refer to Lemma 3 and 5.

Other theorems can be proven exactly in the same way.

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