Stochastic 3D Navier-Stokes equations in a thin domain and its α-approximation

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Abstract

In the thin domain $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is a two-dimensional torus, we consider the 3D Navier-Stokes equations, perturbed by a white in time random force, and the Leray α-approximation for this system. We study ergodic properties of these models and their connection with the corresponding $2D$ models in the limit $\varepsilon \to 0$. In particular, under natural conditions concerning the noise we show that in some rigorous sense the 2D stationary measure μ comprises asymptotical in time statistical properties of solutions for the 3D Navier-Stokes equations in $\mathcal{O}_\varepsilon$, when $\varepsilon \ll 1$.

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1 Introduction

In this paper we study the stochastic Navier-Stokes equations (NSE) in a thin three-dimensional domain $O_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is the torus $\mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$. That is, in O_ε we consider the 3D NSE, perturbed by a random force, which is smooth as a function of the space-variable x, while as a function of time t it is a white noise. Using the Leray projection Π_ε we write the equation as

$$u' + \nu A_\varepsilon u + B_\varepsilon(u, u) = f_\varepsilon + \dot{W}_\varepsilon \,. \tag{1.1}$$

Here A_ε is the Stokes operator $-\Pi_\varepsilon \Delta$, $B_\varepsilon(u,\, u) = \Pi_\varepsilon((u \cdot \nabla)u)$, $f_\varepsilon(x)$ is a deterministic part of the force and $\dot{W}_\varepsilon(t,\, x)$ is the time-derivative of a Wiener process $W_\varepsilon(t,\, x)$ in an appropriate function space. The equation is supplemented with the free boundary conditions in the thin direction (see (2.4) below).

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This boundary value problem for the 3D NSE describes a special case of anisotropic 3D turbulence, important for the meteorology (see, e.g., [9]). The natural related question is to find out *up to what extend this anisotropic 3D turbulence can be approximated by 2D turbulence.* In our work we continue the rigorous study of this problem, initiated in [6].

The study of global existence of strong solutions for the deterministic Navier-Stokes equations in thin three-dimensional domains began with the papers of Raugel and Sell [24, 25], who proved global existence of strong solutions for large initial data and forcing terms in the case of periodic or mixed boundary conditions. After these initial results, a series of papers by different authors followed, in which the results of Raugel and Sell were sharpened and generalised in various ways, see [1, 27, 22, 14, 23, 15, 7]. In the quoted works it was also shown that for $\varepsilon \ll 1$ solutions of the 3D NSE in an ε-thin domain becomes close to solutions of the corresponding 2D NSE.

From other hand, the 2D NSE perturbed by a random force were intensively studied in recent years by many authors, see [12, 19, 2, 11, 18, 13] and references therein. Under some mild restriction on the random force it was proved that the equation has a unique stationary measure, which governs stochastic properties of its solutions as time goes to infinity; we refer to the survey [18] for details.

In [6] the authors of this work considered the 3D NSE in $\mathcal{O}_\varepsilon$, perturbed by a random kick-force. That is, we considered the equation (1.1), when the r.h.s. $f_\varepsilon + \dot{W}_\varepsilon$ is replaced by a random kick-force. Assuming that the force is not too big and is genuinely random we proved that the equation, regarded as a random dynamical system in the H^1-space of a divergence-force vector fields, has a unique stationary measure; that all solutions converge to this measure in distribution, and that the two-dimensional part of the stationary measure (defined below) converges, when ε goes to zero, to a stationary measure for the 2D NSE on the torus $\mathbb{T}^2$. It is shown in [6] that the results obtained apply to study asymptotical properties of various physically relevant characteristics of the flow, described by the Navier-Stokes equations in the 3D domain $\mathcal{O}_\varepsilon$.

Our goal in this work is to extend the results of [6] to the stochastic NSE (1.1). This tasks complicates by the well known difficulty: no matter how small ε is, almost every solution for the stochastic NSE (1.1) exists only finite time.[1] So we cannot study its asymptotical in time properties directly. To resolve this difficulty we apply a trick, often used in physics: we regularise the equation, study its limiting properties and next remove the regularisation. For the regularised equation we take the α-model, introduced by J. Leray in [21] for an analytical study of the NSE. Namely, we replace the nonlinearity $(u \cdot \nabla)u$ by $(G_\alpha u \cdot \nabla)\, u$, where $G_\alpha = (1 + \alpha A_\varepsilon)^{-1}$, and write thus regularised equation as

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u,\, u) = f_\varepsilon + \dot{W}_\varepsilon\,, \tag{1.2}$$

see Section 2.5. Analytical properties of eq. (1.2) are as good as those of the

[1] More precisely, when time grows, the (strong) solution inevitably becomes very large, so due to the well known lack of a corresponding result on the 3D NSE we cannot guarantee that it keeps existing.

2D NSE (in fact, they are even better). [2] In particular, for any initial data the equation has a unique solution, existing for all t, and the techniques, developed for the stochastic 2D NSE allow to show that the equation has a stationary measure μ_ε^α, which is unique if the force $\dot W_\varepsilon$ is nondegenerate. In the latter case every solution $u(t,\,x)$ of (1.2) converges to this measure in distribution:

$$\mathcal{D}\,u(t) \rightharpoonup \mu_\varepsilon^\alpha \quad \text{as} \quad t \to \infty\,,$$

see Theorem 2.5. Our goal is to study behaviour of the measure μ_ε^α when $\varepsilon \to 0$ and $\alpha \to 0$.

To describe the results, we consider the operator M_ε of averaging in the thin direction x_3, which maps 3D velocity fields on $\mathcal{O}_\varepsilon$ to 2D fields on $\mathbb{T}^2$ by the formula

$$(M_\varepsilon u)(x') = \left(\frac{1}{\varepsilon}\int_0^\varepsilon u_1(x',x_3)\,dx_3,\ \frac{1}{\varepsilon}\int_0^\varepsilon u_2(x',x_3)\,dx_3\right),\ x' = (x_1,x_2) \in \mathbb{T}^2. \tag{1.3}$$

As in the previous works on the NSE in thin 3D domains, we compare $M_\varepsilon u(t)$, where $u(t)$ satisfies (1.2), with solutions for the 2D equation

$$v' + A_0 v + B_0(v,\,v) = \widetilde f(x') + \dot{\widetilde W}(t,\,x'), \quad x' \in \mathbb{T}^2\,, \tag{1.4}$$

where A_0 and B_0 are the corresponding 2D Stokes operator and the bilinear operator, and $\widetilde f$ and $\widetilde W$ are limits of $M_\varepsilon f_\varepsilon$ and $M_\varepsilon W_\varepsilon$ as $\varepsilon \to 0$ (below we assume that these limits exist). Under a mild nondegeneracy assumption on the noise $\dot{\widetilde W}$, the equation has a unique stationary measure μ (which is a Borel measure in the L_2-space of divergence-free vector fields on $\mathbb{T}^2$), see Theorem 2.2. We also consider the α-approximation for equation (1.4) by putting $B_0((1+\alpha A_0)^{-1}v, v)$ in (1.4) instead of $B_0(v,v)$. Under the same nondegeneracy assumptions it also has a unique stationary measure μ^α.

Our main results are presented in Theorem 3.1 and in Corollary 3.2. In addition to some nondegeneracy conditions on the random forces they require that (i) the correlation operator of the Wiener process $W_\varepsilon(t)$ in the L_2-space of vector-functions on $\mathcal{O}_\varepsilon$ with respect to the normalised measure $\varepsilon^{-1}\,dx_1\,dx_2\,dx_3$ has a finite trace, bounded uniformly in ε; (ii) the L_2-norms of functions $f_\varepsilon(x)$ are bounded uniformly in ε, and (iii) the correlation operator K_0 of 2D Wiener process $\widetilde W(t,\,x')$ satisfies the condition $\operatorname{tr} A_0 K_0 < \infty$.

Theorem 3.1 states that the projection $M_\varepsilon \mu_\varepsilon^\alpha$ of a stationary measure μ_ε^α for (1.2) weakly converges as $\varepsilon \to 0$ to a unique stationary measure μ^α of the 2D Leray approximation which, in its turn, converges as $\alpha \to 0$ to the unique stationary measure μ of 2D NSE (1.4). Moreover, we also show that $M_\varepsilon \mu_\varepsilon^\alpha$ converges to μ as $\varepsilon \to 0$ and $\alpha \to 0$ simultaneously. In particular, if the 3D

[2] Because of that the Leray α-model (with deterministic forces) and the closely related Camassa–Holm equations were intensively used in the last decade for analytical study of 3D fluid flows. E.g., see [4, 3] and references in this articles.

noise $\dot{W}_\varepsilon$ is nondegenerate, then the stationary measure μ_ε^α is unique, and for any solution $u_\varepsilon^\alpha(t)$ of (1.2) we have

$$M_\varepsilon \mathcal{D}\, u_\varepsilon^\alpha(t) \xrightarrow{t\to\infty} M_\varepsilon \mu_\varepsilon^\alpha \xrightarrow{\varepsilon,\alpha\to 0} \mu\,,$$

where the arrows indicates the weak convergence of measures.

Theorem 3.1 and also some compactness argument make it possible to deal with the limit 'first $\alpha \to 0$, next $\varepsilon \to 0$'. Indeed, one can see that the set of measures $\{\mu_\varepsilon^\alpha, 0 < \alpha \le 1\}$ is tight in the space of Borel measures in the corresponding L_2-space (see, e.g., Theorem 2.7 below). Let us denote by $\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ the set of all its limiting points as $\alpha \to 0$. By Theorem 2.7 this set consists of (weakly) stationary measures for the 3D NSE (1.1). As a corollary from Theorem 3.1 we get that the set $M_\varepsilon(\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha)$ weakly converges to the 2D stationary measure μ as $\varepsilon \to 0$. This means that choosing for each $\varepsilon > 0$ any $\mu_\varepsilon \in \mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ we have $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$.

Theorem 3.1 and Corollary 3.2 jointly show that

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0} M_\varepsilon \mu_\varepsilon^\alpha = \lim_{\alpha,\varepsilon\to 0} M_\varepsilon \mu_\varepsilon^\alpha = \lim_{\varepsilon\to 0}\mathop{\mathrm{Lim}}_{\alpha\to 0} M_\varepsilon \mu_\varepsilon^\alpha = \mu. \tag{1.5}$$

That is, in some rigorous sense the anisotropic 3D turbulence, described by eq. (1.1) with a force, having bounded normalised intensity, may be approximated by the 2D turbulence, described by eq. (1.4). In particular, the averaged normalised energy of the 3D flow is close to that of a corresponding 2D flow (as well as averaging of any functional of the flow, which is continuous in the L_2-norm), see at the end of Section 3. In the same time, we cannot prove that averaged enstrophy or enstrophy production of the 3D flow converges to that of the 2D flow, see a discussion in Section 3.

The ideas which we use in the proof of Theorem 3.1 make it also possible to show that M_ε-component of solution $u_\varepsilon^\alpha(t)$ for (1.2) converges in distribution as $\varepsilon, \alpha \to 0$ to a solution $v(t)$ of (1.4), if $M_\varepsilon u_\varepsilon^\alpha(0) \to v(0)$. See Proposition 3.3 for exact statement.

Our results may be generalised to the case when $\mathcal{O}_\varepsilon = \Gamma \times (0, \varepsilon)$, where Γ is any Riemann surface, e.g. the 2D sphere. Since the Earth atmosphere occupies a thin layer around the Earth, then this generalisation may be relevant for meteorology.

The paper is organised as follows. In Section 2 we describe the models under the consideration, quote several known results concerning statistical solutions and stationary measures and give some preliminary results on dependence of statistical characteristics on ε as $\varepsilon \to 0$. This section also contains Theorem 2.5 and Theorem 2.7 on the existence and limiting properties of stationary measures μ_ε^α and corresponding statistical solutions for fixed ε, which, as we believe, are of independent interest. In Section 3 we formulate our main results (Theorem 3.1 and Corollary 3.2). The proofs defer to Section 4. In this section we also prove Proposition 2.6 which makes an auxiliary step in the proof of the uniqueness of the stationary measure μ_ε^α for (1.2) in Theorem 2.5.

Notations. We denote by $\mathcal{D}(\cdot)$ the distribution of a random variable, denote by the symbol $\rightharpoonup$ the weak convergence of measures and denote by $|\cdot|_{\mathcal{L}(H)}$ the operator-norm for operators in a Hilbert space H.

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2 Models

2.1 3D Navier-Stokes equations in a thin domain

Let $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is the torus $\mathbb{T}^2 = \mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$, $l_1, l_2 > 0$, and $\varepsilon \in (0, 1]$. Let $x = (x', x_3) = (x_1, x_2, x_3) \in \mathcal{O}_\varepsilon$, and let

$$u(x) = (u_1(x), u_2(x), u_3(x)), \quad x \in \mathcal{O}_\varepsilon,$$

stand for a vector function on $\mathcal{O}_\varepsilon$. On the domain $\mathcal{O}_\varepsilon$ we consider the Navier-Stokes equations (NSE) perturbed by a white noise

$$\partial_t u - \nu \triangle u + \sum_{j=1}^{3} u_j \partial_j u + \nabla p = f_\varepsilon + \dot{W}_\varepsilon \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.2}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon. \tag{2.3}$$

We supplement the equations with the free boundary conditions in the thin direction. Thus we impose the following boundary conditions:

$$\begin{cases} x' \in \mathbb{T}^2 \quad \text{(i.e., } u \text{ is } (l_1, l_2)\text{-periodic with respect to } (x_1, x_2)\text{),} \\ \text{and} \\ u_3|_{x_3=\varepsilon} = 0, \quad \partial_3 u_j|_{x_3=\varepsilon} = 0, \quad j = 1, 2, \\ u_3|_{x_3=0} = 0, \quad \partial_3 u_j|_{x_3=0} = 0, \quad j = 1, 2. \end{cases} \tag{2.4}$$

Here above $f_\varepsilon = f_\varepsilon(x)$ is a deterministic time-independent force, and $\dot{W}_\varepsilon(t)$ is generalised derivative of a Wiener process with values in appropriate function space (see Section 2.2 below).

Let $\mathcal{W}_\varepsilon$ be the space, formed by divergence-free vector fields $u = (u_j)_{j=1,2,3}$ on $\mathcal{O}_\varepsilon$ such that

$$u \in \left[H^2(\mathcal{O}_\varepsilon)\right]^3, \quad \int_{\mathcal{O}_\varepsilon} u_j dx = 0, \ j = 1, 2,$$

and condition (2.4) is satisfied. Let V_ε (respectively, H_ε) be the closure of $\mathcal{W}_\varepsilon$ in $\left[H^1(\mathcal{O}_\varepsilon)\right]^3$ (respectively, in $\left[L^2(\mathcal{O}_\varepsilon)\right]^3$). In the space H_ε we introduce the normalised inner product

$$(u, v)_\varepsilon = \frac{1}{\varepsilon} \int_{\mathcal{O}_\varepsilon} uv \, dx, \quad u, v \in H_\varepsilon,$$

and denote by $|\cdot|_\varepsilon$ the corresponding norm. The norm in the space V_ε is given by

$$\|u\|_\varepsilon \equiv |\nabla u|_\varepsilon = \varepsilon^{-1/2}\left[a_\varepsilon(u,u)\right]^{1/2}.$$

Here and below

$$a_\varepsilon(u,v) = \sum_{j=1}^{3} \int_{\mathcal{O}_\varepsilon} \nabla u_j \cdot \nabla v_j \, dx.$$

We denote by A_ε the Stokes operator defined as an isomorphism from V_ε onto the dual V'_ε by

$$(A_\varepsilon u, v)_{V_\varepsilon, V'_\varepsilon} = \varepsilon^{-1} a_\varepsilon(u,v), \quad u, v \in V_\varepsilon\,,$$

where $(\cdot,\cdot)_{V_\varepsilon,V'_\varepsilon}$ is the duality between V_ε and V'_ε generated by the inner product $(\cdot,\cdot)_\varepsilon$ in H_ε. The operator is extended to H_ε as a linear unbounded self-adjoint operator with a domain $\mathrm{D}(A_\varepsilon) = \mathcal{W}_\varepsilon$. Let Π_ε be the Leray projector on H_ε in $(L^2(\mathcal{O}_\varepsilon))^3$. Then

$$(A_\varepsilon u)(x) = (-\Pi_\varepsilon \Delta u)(x), \quad \text{for almost all } x \in \mathcal{O}_\varepsilon$$

for every $u \in \mathrm{D}(A_\varepsilon)$.

Now we consider the trilinear form

$$b_\varepsilon(u,v,w) = \sum_{j,l=1}^{3} \int_{\mathcal{O}_\varepsilon} u_j \, \partial_j v_l \, w_l \, dx, \quad u, v \in \mathrm{D}(A_\varepsilon),\ w \in (L^2(\mathcal{O}_\varepsilon))^3.$$

It defines a bilinear operator B_ε by the formula

$$(B_\varepsilon(u,v), w)_{V_\varepsilon, V'_\varepsilon} = \varepsilon^{-1} b_\varepsilon(u,v,w), \quad u, v, w \in V_\varepsilon\,,$$

and the system (2.1)–(2.4) can be written in the Leray form (1.1).

It is proved in the works on deterministic equations, mentioned in Introduction (see, e.g., [15, 27]), that if the random component $\dot{W}_\varepsilon$ of the force vanishes, while $f_\varepsilon \in H_\varepsilon$ and $u_0 \in V_\varepsilon$ are bounded in certain sense, then for $\varepsilon \ll 1$ the problem (1.1) has a unique strong solution. In [6] a similar result has been obtained for the 3D NSE, perturbed by a random kick-force. In this work we are concerned with forces, having non-trivial white component $\dot{W}_\varepsilon$. We begin their study with discussion of basic properties of the white forces and statistical (weak) solutions.

2.2 Noise

We assume that the Wiener process W_ε has the form

$$W_\varepsilon(t,x) = \sum_j b_j^\varepsilon \beta_j(t)\, e_{\lambda_j}(x) + \sum_j \hat{b}_j^\varepsilon \, \hat{\beta}_j(t) e_{\Lambda_j^\varepsilon}(x)\,. \tag{2.5}$$

Here b^ε_j, $\hat b^\varepsilon_j$ are real numbers such that

$$B^\varepsilon_0 = \sum_j (b^\varepsilon_j)^2 < \infty\,, \quad \hat B^\varepsilon_0 = \sum_j (\hat b^\varepsilon_j)^2 < \infty\,, \tag{2.6}$$

and $\beta_j(t)$, $\hat\beta_j(t)$ are standard independent Wiener process, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ (so $W_\varepsilon(0) = 0$). The system of vectors $\{e_{\lambda_j}, e_{\Lambda^\varepsilon_j}; j \geq 1\}$ is the orthonormal basis of H_ε, formed by eigenfunctions of the Stokes operator, corresponding to eigenvalues $\{\lambda_j, \Lambda^\varepsilon_j\}$ (see Appendix). We also note that the eigenfunctions e_{λ_j} have the structure $e_{\lambda_j} = (\widetilde e_{\lambda_j}; 0)$, where $\widetilde e_{\lambda_j}$ are the eigenfunctions of the $2D$ Stokes operator on $\mathbb{T}^2$ which correspond to the same eigenvalues.

For any vectors $f = \sum_j f_j\, e_{\lambda_j} + \sum_j \hat f_j e_{\Lambda^\varepsilon_j}$ and $h = \sum_j h_j\, e_{\lambda_j} + \sum_j \hat h_j e_{\Lambda^\varepsilon_j}$ from H_ε, we have $\mathbf{E}(W_\varepsilon(t), f)_\varepsilon = 0$ and

$$\mathbf{E}(W_\varepsilon(t), f)_\varepsilon (W_\varepsilon(s), h))_\varepsilon = (t \wedge s)\,(K_\varepsilon f, h)_\varepsilon,$$

where the correlation operator K_ε is diagonal in the basis $\{e_{\lambda_j}, e_{\Lambda^\varepsilon_j},\, j \geq 1\}$:

$$K_\varepsilon e_{\lambda_j} = [b^\varepsilon_j]^2 e_{\lambda_j}, \quad K_\varepsilon e_{\Lambda^\varepsilon_j} = [\hat b^\varepsilon_j]^2 e_{\Lambda^\varepsilon_j}, \quad j = 1, 2, \ldots \tag{2.7}$$

The relations above imply that

$$\mathbf{E}|W_\varepsilon(t)|^2_\varepsilon = t \cdot \left[\sum_j [b^\varepsilon_j]^2 + \sum_j [\hat b^\varepsilon_j]^2\right] \equiv t \cdot \operatorname{tr} K_\varepsilon < \infty.$$

It is well known that for a.e. ω the corresponding realisation of the process W_ε defines a continuous curve $W_\varepsilon(t) \in H_\varepsilon$, see [10].

2.3 3D statistical solutions

We recall now some results from [28] (see also [29]) concerning statistical solutions of problem (2.1)-(2.4).

Let us denote by $\mathcal{W}^{-s}_\varepsilon$ the completion of the space H_ε with respect to the norm $|A^{-s}_\varepsilon \cdot|_\varepsilon$ with some $s > 5/4$, and for any $T > 0$ let $\mathcal{Z}^\varepsilon_T$ be the space of functions $u(x,t)$ in $C(0,T; \mathcal{W}^{-s}_\varepsilon)$ such that

$$|\, u\, |_{\mathcal{Z}^\varepsilon_T} \equiv \sup_{0\leq t\leq T} |A^{-s}_\varepsilon(u(t))|_\varepsilon + \Big(\int_0^T |u(\tau)|^2_\varepsilon d\tau\Big)^{1/2} < \infty\,.$$

We also set

$$\mathcal{Z}^\varepsilon = \{u \in C(0,\infty; \mathcal{W}^{-s}_\varepsilon) : \, u_T \in \mathcal{Z}_T \;\text{ for any }\; T > 0\}\,,$$

where $u_T := u\,|_{[0,T]}$. This is a complete metric space with respect to the distance

$$\operatorname{dist}_{\mathcal{Z}^\varepsilon}(u,v) = \sum_{n=1}^\infty 2^{-n} \frac{|(u-v)_n|_{\mathcal{Z}^\varepsilon_n}}{1 + |(u-v)_n|_{\mathcal{Z}^\varepsilon_n}}\,.$$

It is proved in [28] that if $f_\varepsilon \in H_\varepsilon$ and $u_0(x)$ is a random variable, independent from the force $\dot{W}_\varepsilon$ and satisfying $\mathbf{E}|u_0|_\varepsilon^{2+\eta} < \infty$ for some $\eta > 0$, then the problem (2.1)-(2.4) has a *statistical solution* which is a Borel probability measure P_ε in $\mathcal{Z}^\varepsilon$ supported by the set of functions $u(x,t)$ in $C(\mathbb{R}_+; \mathcal{W}_\varepsilon^{-s})$ such that

$$| u |_T \equiv \sup_{0 \le s,t \le T} \frac{|A_\varepsilon^{-s}(u(t)-u(s))|_\varepsilon}{|t-s|^\delta} + \Big(\int_0^T \|u(\tau)\|_\varepsilon^2 d\tau \Big)^{1/2} < \infty \tag{2.8}$$

for all T (δ is any fixed number from the interval (0,1/2)). It means that there exists a new probability space and on this space there exist processes $\hat{u}(t) \in H_\varepsilon$, $t \ge 0$, and $\widehat{W}_\varepsilon(t) \in H_\varepsilon$, $t \ge 0$, such that $\mathcal{D}(\hat{u}(\cdot)) = P_\varepsilon$ and

- $\widehat{W}_\varepsilon$ is a Wiener process, distributed as the process W_ε;
- $\mathcal{D}\hat{u}(0) = \mathcal{D}u_0$ and the random variable $\hat{u}(0)$ is independent from the process $\widehat{W}_\varepsilon$;
- the process $\hat{u}(t)$, $t \ge 0$, satisfies eq. (1.1) with W_ε replaced by $\widehat{W}_\varepsilon$. That is,

$$\hat{u}(t) - \hat{u}(0) + \int_0^t \big(\nu A_\varepsilon \hat{u} + B_\varepsilon(\hat{u},\hat{u}) - f_\varepsilon \big)\, ds = \widehat{W}_\varepsilon(t) \quad \forall\, t \ge 0\,, \tag{2.9}$$

  almost surely (the equality (2.9) is understood in the usual sense: it holds true after we multiply it in H_ε by any function $\varphi \in \mathcal{W}_\varepsilon \cap C^\infty(\mathcal{O}_\varepsilon)$ and replace $(B_\varepsilon(\hat{u},\hat{u}),\varphi)_\varepsilon$ by $-(B_\varepsilon(\hat{u},\varphi),\hat{u})_\varepsilon$).

We note that in [28, 29] the statistical solutions are defined in terms of Kolmogorov's equation. That definition is equivalent to the one above.

It is also proved in [28, 29] that eq. (1.1) has a *stationary statistical solution* which is a statistical solution, defined by a stationary Borel measure P_ε. That is, the measure P_ε is invariant under the translations

$$\mathcal{Z}^\varepsilon \to \mathcal{Z}^\varepsilon, \quad u(\cdot) \mapsto u(\tau + \cdot),\ \tau \ge 0\,.$$

The trace-measure of the measure P_ε, i.e., its image under the mapping $u(\cdot) \mapsto u(0)$, is a measure on V_ε, called a *weakly stationary measure* for eq. (1.1).

Statistical solutions for the 2D NSE and the α-approximation for the 3D NSE which we consider later in this work are defined similarly. In Theorem 2.7 below we construct stationary statistical solutions P_ε for (1.1) as limits (when $\alpha \to 0$) of the statistical solutions to the corresponding α-approximations (1.2). Due to lack of the uniqueness statement these 3D solutions P_ε may be different from the solutions constructed in [28] by the Galerkin method.

2.4 Corresponding 2D Navier-Stokes equations

Our goal is to study solutions for (2.1)–(2.4) when $\varepsilon \to 0$. Under this limit problem (2.1)–(2.4) is closely related to the 2D NSE on $\mathbb{T}^2$ (see Introduction).

To describe this relation we first define the space

$$\widetilde{V}=\left\{u\in H^1(\mathbb{T}^2;\mathbb{R}^2):\,\mathrm{div}'\,u=0,\ \int_{\mathbb{T}^2}u\,dx'=0\right\},$$

where the prime in $div'u$ indicates that we consider the differential operation with respect to the variable $x'=(x_1,x_2)$ (in contrast with $x=(x_1,x_2,x_3)\equiv(x',x_3)$). Next we define the space $\widetilde{H}$ as a closure of $\widetilde{V}$ in $\left[L_2(\mathbb{T}^2)\right]^2$. We denote by $|\cdot|_{\mathbb{T}^2}$ and $(\cdot,\cdot)_{\mathbb{T}^2}$ the (standard) L_2-norm and L_2-inner product in $\widetilde{H}$, and denote by $\|\cdot\|_{\mathbb{T}^2}=|\nabla\cdot|_{\mathbb{T}^2}$ the norm in the space $\widetilde{V}$. The subscripts in $|\cdot|_{\mathbb{T}^2}$, $(\cdot,\cdot)_{\mathbb{T}^2}$ and $\|\cdot\|_{\mathbb{T}^2}$ will be often omitted when apparent from the context.

One can see that the averaging operator M_ε given by (1.3) maps the spaces H_ε and V_ε in $\widetilde{H}$ and $\widetilde{V}$ respectively. The operator

$$\left(M_\varepsilon^* v\right)(x)=u(x)\quad\text{with}\quad u_j(x)=v_j(x')\text{ for }j=1,2\text{ and }u_3=0$$

defines isometric embeddings $M_\varepsilon^*:\widetilde{V}\to(V_\varepsilon,\|\cdot\|_\varepsilon)$ and $M_\varepsilon^*:\widetilde{H}\to(H_\varepsilon,|\cdot|_\varepsilon)$. This operator is a right inverse to M_ε, i.e. $M_\varepsilon\circ M_\varepsilon^*=\mathrm{id}$, and is adjoint to the operator $M_\varepsilon:(H_\varepsilon,|\cdot|_\varepsilon)\to\widetilde{H}$. We also define the operator $\hat{M}_\varepsilon$ in H_ε (resp. in V_ε) by the formula

$$\hat{M}_\varepsilon u=(M_\varepsilon u;0)=M_\varepsilon^* M_\varepsilon u,\quad u\in H_\varepsilon\ (\text{resp. }u\in V_\varepsilon). \tag{2.10}$$

The operator $\hat{M}_\varepsilon$ defines an orthogonal projector in H_ε and in V_ε. So

$$V_\varepsilon=\hat{M}_\varepsilon V_\varepsilon\oplus\hat{N}_\varepsilon V_\varepsilon\,,\quad\text{where}\quad\hat{N}_\varepsilon=I-\hat{M}_\varepsilon\,. \tag{2.11}$$

By an analogy with the deterministic NSE (see, e.g., [27]) and the equation, perturbed by a kick-force [6], we can *conjecture* that if the limits

$$\widetilde{f}=\lim_{\varepsilon\to0}M_\varepsilon f\in\widetilde{H}\quad\text{and}\quad b_j^0=\lim_{\varepsilon\to0}b_j^\varepsilon,\quad j\ge1\,,$$

exist, then the M_ε-projections of solutions to (2.1)–(2.4) should be close (when $\varepsilon\ll1$) to solutions of the following 2D NSE on $\mathbb{T}^2$

$$\partial_t v-\nu\triangle' v+\sum_{j=1}^2 v_j\partial_j v+\nabla' p=\widetilde{f}+\dot{\widetilde{W}}\quad\text{in}\quad\mathbb{T}^2\times(0,+\infty), \tag{2.12}$$

$$\int_{\mathbb{T}^2}v(t,x')\,dx'=0\,;\quad\mathrm{div}\,'v=0\quad\text{in}\quad\mathbb{T}^2\times(0,+\infty), \tag{2.13}$$

$$v(x',0)=\widetilde{v}_0(x')\quad\text{in}\quad\mathbb{T}^2. \tag{2.14}$$

Here $\widetilde{W}(t,x')=\widetilde{W}(t)$ is the Wiener process in $\widetilde{H}$ of the form

$$\widetilde{W}(t)=\sum_j b_j^0\beta_j(t)\,\widetilde{e}_{\lambda_j},\quad B_0=\sum_j(b_j^0)^2<\infty\,,$$

where $\widetilde{e}_{\lambda_j} \equiv M_\varepsilon e_{\lambda_j}$ are eigenfunctions of the 2D Stokes operator (see Appendix). So a.a. realisation of $\widetilde{W}$ defines a continuous trajectory in $\tilde{H}$, and the correlation operator $\tilde{K}_0$ of the process is the diagonal operator

$$K_0 \widetilde{e}_{\lambda_j} = [b_j^0]^2 \widetilde{e}_{\lambda_j}, \quad j = 1, 2, \dots ,$$

cf. Section 2.2.

In the abstract form problem (2.12)–(2.14) can be written as

$$u' + \nu A_0 u + B_0(u, u) - \widetilde{f} = \dot{\widetilde{W}}, \quad u(0) = u_0, \tag{2.15}$$

where A_0 and B_0 are the corresponding two dimensional Stokes operator and bilinear operator.

A random field $u(t, x)$ is called a (strong) solution of the problem (2.12)-(2.14) (written in the form (2.15)) on a segment $[0, T]$ if for a.a. ω it defines a curve in $C([0, T], \widetilde{H}) \cap L_2([0, T], \widetilde{V})$, satisfying

$$u(t) - u_0 + \int_0^t \left(\nu A_0 u(\tau) + B_0(u(\tau), u(\tau)) - \widetilde{f}\right) d\tau = \widetilde{W}(t), \tag{2.16}$$

for all $0 \le t \le T$. A random field $u(t, x)$ which defines a random process with trajectories in the space

$$\mathcal{Z} = C(0, \infty; \widetilde{H}) \cap L_{2\,loc}(0, \infty; \widetilde{V}) , \tag{2.17}$$

is a solution of (2.15) for $t \in [0, \infty)$ if it is a solution on any finite segment $[0, T]$.

We also consider the process $\widetilde{W}_\varepsilon(t) = M_\varepsilon W_\varepsilon(t)$. It has the form above with the correlation operator $M_\varepsilon K_\varepsilon M_\varepsilon^*$, which is the diagonal operator in $\widetilde{H}$ with the eigenvalues $(b_i^\varepsilon)^2$. Below we also deal with the 2D NSE (2.15) with $\widetilde{W} = \widetilde{W}^\varepsilon$ and $f = \widetilde{f}_\varepsilon = M_\varepsilon f_\varepsilon$:

$$u' + \nu A_0 u + B_0(u, u) = \widetilde{f}_\varepsilon + \dot{\widetilde{W}}_\varepsilon, \quad u(0) = u_0^\varepsilon \, . \tag{2.18}$$

The stochastic evolution equation (2.15) was studied by many authors (see, e.g., [10, 28, 29, 17, 18] and the references therein). Here we will recall basic result on the existence and uniqueness of its solutions from [17].

Theorem 2.1 *If $\widetilde{f} \in \widetilde{H}$ and $u_0 = u_0^\omega$ is a random variable in $\widetilde{H}$ independent from the process $\widetilde{W}(t)$ and such that $\mathbf{E}|u_0|^2 < \infty$, then eq.* (2.15) *has a unique (up to equivalence) solution $u(t)$, $t \ge 0$. If, in addition, $tr(A_0 K_0) = \sum \lambda_j (b_j^0)^2 < \infty$ and $\mathbf{E}\exp(\beta_0 \parallel u_0 \parallel^2) < \infty$ for some $\beta_0 \in \left(0, \nu |K_0|^{-1}_{\mathcal{L}(\widetilde{H})}\right)$, then for every $\beta_1 \le \frac{1}{2} \cdot \beta_0 \left(\nu - \beta_0 |K_0|_{\mathcal{L}(\widetilde{H})}\right)$ we have*

$$\mathbf{E}\exp\left(\beta_0 \|u(t)\|^2 + \beta_1 \int_0^t \mid A_0 u(\tau) \mid^2 d\tau\right) \le e^{\gamma t} \cdot \mathbf{E}\exp(\beta_0 \parallel u_0 \parallel^2) \tag{2.19}$$

for all $t \geq 0$, where $\gamma = \frac{\beta_0}{2\nu}\Big(|\,\widetilde{f}\,|^2 + \nu\, tr(A_0K_0)\Big)$. Furthermore for any positive λ there exists a constant $D_{\beta_0,\lambda} > 0$ such that

$$\mathbf{E}\exp(\beta_0\|u(t)\|^2) \leq D_{\beta_0,\lambda} + e^{-\lambda(t-s)} \cdot \mathbf{E}\exp(\beta_0\|u(s)\|^2), \quad t > s \geq 0. \quad (2.20)$$

Proof. The proof of the first part can be found in [17]. For the proof of relations (2.19) and (2.20) we refer to [5]. □

The solution u, constructed in this theorem, will be denoted $u(t;u_0) = u(t,x;u_0)$. It defines a Markov process in the space $\widetilde{H}$ with the transition function $P_t(v,\cdot) = \mathcal{D}u(t;v)$; see [17, 28, 10, 18].

Clearly Theorem 2.1 remains true for problem (2.18) with the noise $\widetilde{W}_\varepsilon = M_\varepsilon W_\varepsilon$ and the force $\widetilde{f}_\varepsilon$ depending on ε. The corresponding constants β_0, β_1 and γ in Theorem 2.1 can be chosen independent of ε provided that

$$|\widetilde{f}_\varepsilon|_{\mathbb{T}^2} \leq c_1 \quad \text{and} \quad \mathrm{tr}\left(A_0 M_\varepsilon K M_\varepsilon^*\right) \equiv \sum_j \lambda_j [b_j^\varepsilon]^2 \leq c_2,$$

where the constants c_1 and c_2 do not depend on ε.

A Borel measure μ on $\widetilde{H}$ is said to be a *stationary measure for eq.* (2.15) if it is a stationary measure for the Markov process which the equation defines in the space $\widetilde{H}$. This means that

$$\int_{\widetilde{H}} \mathbf{E}g(u(t;u_0))\mu(du_0) = \int_{\widetilde{H}} g(u_0)\mu(du_0) \quad \text{for any } g \in C_b(\widetilde{H}), \text{ any } \ t \geq 0.$$

Let us write the force $\widetilde{f}(x)$ as $\widetilde{f} = \sum_j \tilde{f}_j \widetilde{e}_{\lambda_j}$.

Theorem 2.2 *Assume that $tr(A_0K_0) < \infty$, that $b_j^0 \neq 0$ if $\tilde{f}_j \neq 0$ ($j = 1, 2, \dots$), and that $b_j^0 \neq 0$, $j = 1, \dots, N$ for N large enough. Then there exists a unique stationary measure μ for* (2.15)*, and every solution of* (2.15) *given by Theorem 2.1 converges to μ in distribution when $t \to \infty$. This measure satisfies*

$$\int_{\widetilde{H}} \exp\left\{\beta_0 \parallel u \parallel^2\right\} \mu(du) < \infty \quad \textit{for any } \beta_0 < \nu|K_0|^{-1}_{\mathcal{L}(\widetilde{H})}. \quad (2.21)$$

Proof. The existence of the stationary measure is well-known (it follows from the standard Krylov-Bogolyubov procedure, e.g. see [17, 28]). Concerning the uniqueness of the measure under the imposed assumptions see [26, 18]. The claimed estimate (2.21) follows from (2.20) and the Fatou lemma in the standard way (see [28, 29] for similar arguments). □

Remarks. 1) If we replace the l.h.s. of (2.21) by $\int \exp\left\{\beta_0|u|^2\right\}\mu(du)$, then the assertion of Theorem 2.2 would remain true for the 2D NSE on any compact Riemann surface Γ. Since the arguments of this work use only basic properties of the operators A_ε and B_ε in (1.1), then its main results, stated in Section 3, remain true for domain $\mathcal{O}_\varepsilon = \Gamma \times (0,\varepsilon)$.

2) The hypotheses concerning diffusion coefficients b_j^0 in Theorem 2.2 are not optimal and can be relaxed in one way or another. In particular, the recent paper [13] suggests a rather general geometrical characterization of noises for which the 2D NSE (2.12), (2.13) in the vorticity formulation is ergodic on the square torus $\mathbb{T}^2 = \mathbb{R}^2/(l\mathbb{Z} \times l\mathbb{Z})$ with $\widetilde{f} \equiv 0$.

The uniqueness of the measure μ implies the same property for statistical solutions. More precisely, we have the following assertion.

Corollary 2.3 *Let the hypotheses of Theorem 2.2 be in force. Then problem* (2.15) *has a unique stationary statistical solution (in the sense of definitions in Section 2.3) which is a Borel probability measure on the space $\mathcal{Z}$ given by* (2.17).

Proof. Let $u(t), t \geq 0$, be a solution of (2.15), such that $\mathcal{D}u(0) = \mu$. Its distribution is a Borel measure $\tilde{P}$ in the space $\mathcal{Z}$. This is stationary statistical solution of the equation (cf. Section 2.3). Let P' be another stationary statistical solution. Then $P' = \mathcal{D}v(\cdot)$, where $v(t)$ is a solution of (2.15) with $\widetilde{W}$ replaced by another process, distributed as $\widetilde{W}$. So $\vartheta = \mathcal{D}v(0)$ is a stationary measure for the Markov process, defined by the equation, and $\vartheta = \mu$ by the theorem above. Accordingly, P' is the distribution of trajectories of the Markov process with the initial measure μ. So $P' = \tilde{P}$; that is, the stationary statistical solution $\tilde{P}$ for the 2D NSE is unique. $\square$

2.5 Leray α-approximation of stochastic 3D Navier-Stokes equations

It is unknown if the 3D NSE (2.1)-(2.4) has a unique solution. So to make a progress in its study we replace the equation by its *Leray α-approximation* [21], in order later to send α to zero. That is, we consider the equations

$$\partial_t u - \nu \triangle u + \sum_{j=1}^{3} v_j \partial_j u + \nabla p = f_\varepsilon + \dot{W}_\varepsilon \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.22}$$

$$\text{div } u = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty),$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon\,,$$

where the forces f_ε and $\dot{W}_\varepsilon$ are the same as in (2.1). These equations are supplemented with boundary conditions (2.4), and the vector field $v = (v_1, v_2, v_3)$ solves the elliptic problem

$$v - \alpha \triangle v = u, \;\; \text{div } v = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{2.23}$$

and satisfies the same boundary conditions.

In the Leray representation the problem above takes the form

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u) = f_\varepsilon + \dot{W}_\varepsilon, \quad u(0) = u_0, \tag{2.24}$$

where $G_\alpha = (I + \alpha A_\varepsilon)^{-1}$, $\alpha \geq 0$, is the Green operator for problem (2.23) with boundary conditions (2.4). The nonlinear term $B_\varepsilon(G_\alpha u, u)$ in (2.24) possesses the properties:

$$(B_\varepsilon(G_\alpha v, u), u) = 0$$

and

$$|B_\varepsilon(G_\alpha u_1, u_1) - B_\varepsilon(G_\alpha u_2, u_2)|_\varepsilon \leq C_{\alpha,\varepsilon} \left(\|u_1\|_\varepsilon + \|u_2\|_\varepsilon\right) \|u_1 - u_2\|_\varepsilon. \quad (2.25)$$

This allows to obtain for the α-model results, similar to those in the $2D$ case.

Theorem 2.4 *Assume that $W_\varepsilon(t)$ is the Wiener process in H_ε of the form* (2.5) *and relations* (2.6) *holds. Let $f \in H_\varepsilon$. Then there exists a unique (strong) solution $u(t)$ to* (2.24) *for any initial data u_0 which is independent from the noise and satisfies $\mathbf{E}|u_0|_\varepsilon^2 < \infty$. Moreover, for any $n \geq 1$ we have*

$$\mathbf{E}|u(t)|_\varepsilon^{2n} + \frac{\nu n}{2} \int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)} \|u\|_\varepsilon^2 d\tau \leq \mathbf{E}|u(0)|_\varepsilon^{2n} + b_n t \leq \infty\,, \quad (2.26)$$

where $b_n = C_n \nu^{1-n} \sigma_\varepsilon^n$ with $\sigma_\varepsilon = tr\, K_\varepsilon + \frac{\lambda_1}{\nu} |f_\varepsilon|_\varepsilon^2$ and K_ε is given by (2.7).

Proof. Due to the regularity in (2.25) the existence and uniqueness of strong solutions can be obtained by the same argument as for 2D NSE (see, e.g., [17, 18] or [10]). Now we prove (2.26). Here our arguments are formal. To make them rigorous one should consider the Galerkin approximations for the problem.

Let us consider the functional $F(u(t)) = |u(t)|_\varepsilon^{2n}$. Using the Ito formula we have

$$dF = 2n|u|_\varepsilon^{2(n-1)}(u, du)_\varepsilon + n\left\{|u|_\varepsilon^{2(n-1)} tr\, K_\varepsilon + 2(n-1)|u|_\varepsilon^{2(n-2)}(K_\varepsilon u, u)_\varepsilon\right\} dt.$$

Since $(u, du)_\varepsilon = -\nu\|u\|_\varepsilon^2 dt + (u, f_\varepsilon dt + dW_\varepsilon)_\varepsilon$, then integrating the equality above we obtain

$$\begin{aligned} &\mathbf{E}|u(t)|_\varepsilon^{2n} + 2\nu n \int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)} \|u\|_\varepsilon^2 d\tau \\ &= \mathbf{E}|u(0)|_\varepsilon^{2n} + 2n \int_0^t \mathbf{E}\left(|u|_\varepsilon^{2(n-1)}(u, f_\varepsilon)_\varepsilon\right) d\tau \\ &\quad + n \int_0^t \mathbf{E}\left(|u|_\varepsilon^{2(n-1)} tr\, K_\varepsilon + 2(n-1)|u|_\varepsilon^{2(n-2)}(K_\varepsilon u, u)_\varepsilon\right) d\tau. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathbf{E}|u(t)|_\varepsilon^{2n} + \nu n \int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)} \|u\|_\varepsilon^2 d\tau \\ &\leq \mathbf{E}|u(0)|_\varepsilon^{2n} + C_n \left(tr\, K_\varepsilon + \frac{\lambda_1}{\nu}|f_\varepsilon|_\varepsilon^2 + |K_\varepsilon|_{\mathcal{L}(H_\varepsilon)}\right) \int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)} d\tau. \end{aligned}$$

Since $|K_\varepsilon|_{\mathcal{L}(H_\varepsilon)} \le tr\, K_\varepsilon$, then the second term in the r.h.s. is bounded by

$$C_n' \sigma_\varepsilon \int_0^t \mathbf{E}|u|_\varepsilon^{2(n-1)} d\tau \le C_n' \sigma_\varepsilon \left(\int_0^t \mathbf{E}|u|_\varepsilon^{2n} d\tau \right)^{(n-1)/n} t^{1/n}$$
$$\le \frac{\nu n}{2} \lambda_1^n \int_0^t \mathbf{E}|u|_\varepsilon^{2n} d\tau + t C_n''(\sigma_\varepsilon)^n \nu^{1-n} .$$

Since $\lambda_1 |u|_\varepsilon^2 \le \|u\|_\varepsilon^2$, this implies the required estimate. □

Under the condition $\sigma_\varepsilon \le C$ for all $0 \le \varepsilon \le \varepsilon_0$ the constant b_n in (2.26) is independent from α and ε. Therefore if $\mathbf{E}|u(0)|_\varepsilon^{2n} \le C$ for all $0 \le \varepsilon \le \varepsilon_0$, then the theorem above provides a priori estimates for solutions of (2.24), uniform in α and ε. This observation is important in the limit transitions below.

Let us decompose the force $f_\varepsilon = f_\varepsilon(x)$ in (2.22) in the basis of H_ε:

$$f_\varepsilon(x) = \sum f_j e_{\lambda_j}(x) + \sum \hat{f}_j e_{\Lambda_j^\varepsilon}(x)$$

Theorem 2.5 *1) Eq.* (2.24) *has a stationary measure μ_ε^α in H_ε, satisfying*

$$\int_{H_\varepsilon} |u|_\varepsilon^{2(n-1)} \|u\|_\varepsilon^2 \mu_\varepsilon^\alpha(du) \le C_n, \quad n = 1, 2, \dots , \tag{2.27}$$

where the constants C_n are increasing functions of $\sigma_\varepsilon = \operatorname{tr} K_\varepsilon + \frac{\lambda_1}{\nu}|f_\varepsilon|_\varepsilon^2$, independent from α.

2) There are constants $n(\varepsilon, \alpha)$ and $\hat{n}(\varepsilon, \alpha)$ such that if $b_j^\varepsilon \ne 0$ for $j \le n$ and $\hat{b}_j^\varepsilon \ne 0$ for $j \le \hat{n}$, and if

$$b_j^\varepsilon \ne 0 \ \text{ if } \ f_j \ne 0 \quad \text{ and } \quad \hat{b}_j^\varepsilon \ne 0 \ \text{ if } \ \hat{f}_j \ne 0, \qquad \forall j , \tag{2.28}$$

then a stationary measure is unique, and every solution of (2.24) *converges to it in distribution (in the space of Borel measures in H_ε) as time goes to infinity.*

3) Assume that $tr K_\varepsilon \le c_0$ and $|f_\varepsilon|_\varepsilon \le c_1$ for all ε. Then $n(\varepsilon, \alpha)$ may be chosen independent from ε, and the assumption $\hat{b}_j^\varepsilon \ne 0$ for $j \le \hat{n}$ may be dropped if $\varepsilon \le \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(\alpha) > 0$.

Note that the assumption in 2) holds trivially if the noise $\dot{W}_\varepsilon$ is nondegenerate, i.e. all coefficients b_j^ε and $\hat{b}_j^\varepsilon$ are non-zero. We do not know whether the nondegeneracy noise hypotheses in Theorem 2.5 can be relaxed significantly (the characterization suggested in [13] is not applied here because, in contrast with [13], we deal with a 3D hydrodynamical problem).

Proof. The first assertion follows from the Bogolyubov-Krylov arguments and the Fatou lemma in the standard way, cf. [18], Section 4.4.

The second assertion follows from the techniques, developed in recent works on the randomly forced 2D NSE, discussed in Introduction. More specifically, in [20, 26] the 2D NSE is written in the abstract form as

$$u' + Lu + B(u, u) = f + \dot{W} \tag{2.29}$$

(in [20] $f = 0$, but it is shown in [26] that the arguments of that work apply to equations with non-zero f). The proof in [20, 26] uses only basic properties of the linear operator L and the quadratic operator B. It is straightforward that the operators νA_ε and B_ε in (2.24) satisfy these properties, if we choose for the basic function space the space $(H_\varepsilon, |\cdot|_\varepsilon)$. So the main theorems in the references above apply and imply the uniqueness of a stationary measure and the assertion about the convergence.

The proof in [20, 26] uses in a critical way the 'squeezing property', stating that asymptotical in time behaviour of a solution for (2.29) with a deterministic r.h.s. is determined by its finite-dimensional part, formed by first few Fourier harmonics of the solution. The validity of this property for the α-model can be checked by literal repeating of the classical arguments due to Foias-Prodi, exploited in [20, 26] (see Proposition A.1 in [20]). The finite-dimensional part corresponds to the subspace of H_ε, spanned by the vectors $e_{\lambda_j}, j \le n(\varepsilon, \alpha)$, and $e_{\Lambda_j^\varepsilon}, j \le \tilde{n}(\varepsilon, \alpha)$, where the corresponding eigenvalues λ_j and Λ_j^ε contain all eigenvalues of the Stokes operator A_ε, smaller than a suitable threshold Λ.

To prove the last assertion of the theorem we have to estimate how the numbers n and $\tilde{n}$ grow when $\varepsilon \to 0$. Let us write the spectrum of the Stokes operator A_ε, formed by the two branches $\{\lambda_j\}$ and $\{\Lambda_j^\varepsilon\}$ (see Appendix) as $\mu_1 \le \mu_2 \le \dots$, and denote by $\{b_{\mu_j}\}$ the corresponding coefficients in the decomposition of the Wiener process W_ε. By [20, 26] a stationary measure is unique if (2.28) holds and $b_{\mu_j} \neq 0$ for $j \le N_\mu$. The constant $N = N_\mu$ should be so big that the assumptions (A.1) and (A.2) of Proposition A.1 in [20] imply the estimate (A.3). This can be achieved with help of the following proposition (which is an analog of Proposition A.1 [20] for the case considered).

Proposition 2.6 *Let u_i, $i = 1, 2$ be solutions to the (deterministic) problems*

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u) = \eta_i(t), \quad i = 1, 2.$$

Assume that

$$\int_s^t \|u_1(\tau)\|_\varepsilon^2 \, d\tau \le \rho + K(t-s), \quad 0 \le s \le t \le s + T, \tag{2.30}$$

where ρ, K and T are nonnegative constants. Let $P \equiv P_{n,\hat{n}}$ be the spectral orthoprojector on the subspace [3]

$$Span \left\{ e_{\lambda_j}, e_{\Lambda_i^\varepsilon} \,:\, 1 \le j \le n, \ 1 \le i \le \hat{n} \right\}, \quad n \ge 1, \ \hat{n} \ge 0,$$

and $Q = 1 - P$. If $Pu_1(t) = Pu_2(t)$ and $Q\eta_1(t) = Q\eta_2(t)$ for all $t \in [s, s+T]$, then

$$|u_1(t) - u_2(t)|_\varepsilon \le e^{-m_{n,\hat{n}}(t-s) + \rho_n} |u_1(s) - u_2(s)|_\varepsilon, \quad t \in [s, s+T], \tag{2.31}$$

where

$$m_{n,\hat{n}} = \frac{\nu}{2} \min\left\{\lambda_{n+1}, \Lambda_{\hat{n}+1}^\varepsilon\right\} - \frac{c_0 K}{\alpha}\left(\varepsilon^2 + \alpha^{1/2}\lambda_{n+1}^{-1/2}\right)$$

[3] If $\hat{n} = 0$, then this subspace equals Span $\{e_{\lambda_j} : 1 \le j \le n\}$.

and

$$\rho_n = \frac{c_0\rho}{\alpha}\big(\varepsilon^2 + \alpha^{1/2}\lambda_{n+1}^{-1/2}\big).$$

We prove this proposition in Section 4.3.

The structure of the constant $m_{n,\hat{n}}$ and the fact that $\Lambda_k^\varepsilon \geq \varepsilon^{-2}$ for all k imply the third assertion of the theorem by the same argument as in [20]. □

Let $u_\varepsilon^\alpha(t),\ t \geq 0$, be a stationary solution of (2.24), corresponding to the stationary measure μ_ε^α. Then $P_\varepsilon^\alpha = \mathcal{D}u_\varepsilon^\alpha(\cdot)$ is a stationary statistical solution of (2.24) in the space $\mathcal{Z}^\varepsilon$. For the same reason as in the 2D case (see Corollary 2.3), under the assumptions of item 2) of the theorem this equation has a unique stationary statistical solution. Other properties of P_ε^α are collected in the following assertion.

Theorem 2.7 *Assume that*

$$tr\, K_\varepsilon \leq c_0, \quad |f_\varepsilon|_\varepsilon^2 \leq c_1 \tag{2.32}$$

for some ε-independent constants c_0, c_1 and denote $P_\varepsilon^\alpha = \mathcal{D}u_\varepsilon^\alpha(\cdot)$. Then

1) for any fixed $\varepsilon > 0$ the set of measures $\{P_\varepsilon^\alpha, 0 < \alpha \leq 1\}$ is tight in the space of Borel measures on $\mathcal{Z}^\varepsilon$ and the corresponding trace-measures μ_ε^α are tight in H_ε.

2) Let P_ε be any limiting measure for this family as $\alpha \to 0$.[4] *Then the measure P_ε is a stationary statistical solution of the 3D NSE* (1.1) *in the space $\mathcal{Z}^\varepsilon$. Its trace-measure μ_ε (i) satisfies estimates* (2.27)*, (ii) is a limiting point for μ_ε^α in H_ε as $\alpha \to 0$, and (iii) is a weakly stationary measure for* (1.1) *(see Section 2.3 for the corresponding definitions).*

Proof. 1) The tightness of the set $\{P_\varepsilon^\alpha, 0 < \alpha \leq 1\}$ follows by repeating the argument from [28, Chap. IV]. Moreover, in the same way as in [28, Chap. IV] we can derive from (2.27) the estimate

$$\int_{\mathcal{Z}^\varepsilon} |u|_T^{1+\kappa} P_\varepsilon^\alpha(du) \leq C_T \tag{2.33}$$

for every $T > 0$ and for some $\kappa > 0$, where $|\cdot|_T$ is given by (2.8) and the constant C_T does not depend on α. The tightness of μ_ε^α follows from (2.27) with $n = 1$.

2) By the Skorokhod representation theorem (see [16]), there exists a new probability space and on this space there are random processes $\hat{u}_\varepsilon^{\alpha_j}(t) \in H_\varepsilon$, $t \geq 0$, and $\hat{u}_\varepsilon(t) \in H_\varepsilon$, $t \geq 0$, such that $\mathcal{D}(\hat{u}_\varepsilon^{\alpha_j}) = P_\varepsilon^{\alpha_l}$, $\mathcal{D}(\hat{u}_\varepsilon) = P_\varepsilon$ and

$$\hat{u}_\varepsilon^{\alpha_j} \to \hat{u}_\varepsilon \quad \text{in} \quad \mathcal{Z}^\varepsilon \tag{2.34}$$

almost surely. Since $P_\varepsilon^{\alpha_j}$ is a statistical solution, then a.s. $u = \hat{u}_\varepsilon^{\alpha_j}$ satisfies

$$u(t) - u(0) + \int_0^t \big(\nu A_\varepsilon u + B_\varepsilon(G_{\alpha_j} u, u) - f\big)ds = W_\varepsilon^{\alpha_j}(t), \quad \forall t \geq 0\,, \tag{2.35}$$

[4] this means that $P_\varepsilon^{\alpha_j} \rightharpoonup P_\varepsilon$ in the space of Borel measures on $\mathcal{Z}^\varepsilon$ for some sequence $\alpha_j \to 0$.

where $W_\varepsilon^{\alpha_j}(t)$ is a Wiener process, distributed as $W_\varepsilon(t)$. The validity of equation (2.35) is understood in the same way as that of (2.9).

Let e be any basis vector e_{λ_k} or $e_{\Lambda_k^\varepsilon}$, and b_e be the corresponding coefficient b_k^ε or $\hat{b}_k^\varepsilon$. Let us denote by $\xi_e^{\alpha_j}(t)$ the H_ε-scalar product of the l.h.s. of (2.35) with e, where $u = \hat{u}_\varepsilon^{\alpha_j}$ and we replaced $b_\varepsilon(G_{\alpha_j}u, u, e)$ with $-b_\varepsilon(G_{\alpha_j}u, e, u)$. Then (2.35) implies that $\xi_\varepsilon^{\alpha_j}(t)$ is a scalar Wiener process with the dispersion $\mathbf{E}(\xi_\varepsilon^{\alpha_j}(t))^2 = b_e^2 t$. The convergence (2.34) and estimate (2.33) imply that

$$\xi_e^{\alpha_j}(t) \to \xi_e^0(t) \quad \text{a.s.},$$

uniformly for t in finite segments, where the process $\xi_\varepsilon^0(t)$ is obtained by replacing $\hat{u}_\varepsilon^{\alpha_j}(t)$ by $\hat{u}_\varepsilon(t)$. Therefore $\xi_\varepsilon^0(t)$ also is a Wiener process with the dispersion $b_\varepsilon^2 t$.

Now let us take any two basis vectors $e' \neq e''$. Since the Wiener processes $\xi_{e'}^{\alpha_j}(t)$ and $\xi_{e''}^{\alpha_j}(t)$ are independent, then the limiting processes $\xi_{e'}^0(t)$ and $\xi_{e''}^0(t)$ are independent as well. Therefore we see that the process $\hat{u}_\varepsilon(t)$ satisfies (2.35) with $\alpha_j := 0$, where the Wiener process $W_\varepsilon^{\alpha_j}(t)$ is replaced by an equidistributed process $W_\varepsilon^0(t)$. So P_ε is a statistical solution of the 3D NSE.

The estimates on the measure μ_ε follows from the estimates (2.27) on the measures μ_ε^α, the convergence (2.34) and the Fatou lemma. The theorem is proved. □

2.6 Leray α-approximation of stochastic 2D NSE

We also consider the α-approximation for the 2D NSE (2.15):

$$u' + \nu A_0 u + B_0(G_\alpha^0 u, u) = \widetilde{f} + \dot{\widetilde{W}}, \quad u(0) = u_0, \tag{2.36}$$

where $G_\alpha^0 = (1 + \alpha A_0)^{-1}$, $\alpha > 0$. This equation possesses the same basic properties as the 2D NSE: given a suitable initial condition it has a unique solution, and under the assumptions of Theorem 2.2 it has a unique stationary measure in the space $\widetilde{H}$. However when $\alpha > 0$ we cannot guarantee bounds (2.19), (2.20) and (2.21) for the exponential moments. Indeed, the proof of these estimates for solutions of (2.15) uses essentially the orthogonality relation $b_0(u, u, A_0 u) = 0$. Since $b_0(G_\alpha^0 u, u, A_0 u)$ does not vanish identically, then the corresponding arguments do not apply to eq. (2.36).

In the case $\alpha > 0$ we can only use the orthogonality relation $b_0(G_\alpha^0 u, u, u) = 0$. It implies bounds for exponential moments of the L_2-norms of solutions:

Theorem 2.8 *If $\widetilde{f} \in \widetilde{H}$ and $u_0 = u_0^\omega$ is a random variable in $\widetilde{H}$ independent from the process $\widetilde{W}(t)$ and such that $\mathbf{E}|u_0|^2 < \infty$, then eq.* (2.36) *has a unique (up to equivalence) solution $u(t)$, $t \geq 0$. If, in addition, $\mathbf{E}\exp(\beta_0|u_0|^2) < \infty$ for some $\beta_0 \in \left(0, \nu|K_0|^{-1}_{\mathcal{L}(\widetilde{H})}\lambda_1^{-1}\right)$, where $\lambda_1 > 0$ is the first eigenvalue of the 2D Stokes operator, then for every $\beta_1 \leq \frac{1}{2} \cdot \beta_0 \left(\nu - \beta_0\lambda_1|K_0|_{\mathcal{L}(\widetilde{H})}\right)$ we have*

$$\mathbf{E}\exp\left(\beta_0|u(t)|^2 + \beta_1\int_0^t \|u(\tau)\|^2 d\tau\right) \leq e^{\gamma t} \cdot \mathbf{E}\exp(\beta_0|u_0|^2)$$

for all $t \geq 0$, where $\gamma = \frac{\beta_0}{2\nu\lambda_1}\Big(|\,\widetilde{f}\,|^2 + \nu\lambda_1\, trK_0\Big)$. Furthermore for any positive λ there exists a constant $D_{\beta_0,\lambda} > 0$ such that

$$\mathbf{E}\exp(\beta_0|u(t)|^2) \leq D_{\beta_0,\lambda} + e^{-\lambda(t-s)} \cdot \mathbf{E}\exp(\beta_0|u(s)|^2), \quad t > s \geq 0.$$

Proof. This is a slight modification of argument given in [17] and [5]. To obtain the required inequalities we first apply the Ito formula to the corresponding processes. □

This theorem implies an analogy of Theorem 2.2 for eq. (2.36):

Theorem 2.9 *Assume that the hypotheses of Theorem 2.2 are in force. Then there exists a unique stationary measure μ^α for (2.36). This measure satisfies $\int_{\widetilde{H}} \exp\left\{\beta_0|u|^2\right\} \mu^\alpha(du) < \infty$ for any $\beta_0 < \nu|K_0|^{-1}_{\mathcal{L}(\widetilde{H})}\lambda_1^{-1}$. Every solution of (2.36) given by Theorem 2.8 converges to μ^α in distribution when $t \to \infty$, in the weak topology of the space of Borel measures in $\widetilde{H}$.*

3 Main results

In the theorem below $M_\varepsilon\mu$ stands for the image of a measure μ under the map M_ε defined in (1.3).

Theorem 3.1 *Assume that*

- *the assumptions of Theorem 2.2 hold;*
- *there exist constants c_0, c_1 independent of ε such that (2.32) holds;*
- *we have that $\lim_{\varepsilon\to 0}\sum_j [b_j^\varepsilon - b_j^0]^2 = 0$ and $\lim_{\varepsilon\to 0}|\widetilde{f} - M_\varepsilon f_\varepsilon| = 0$.*

Let μ_ε^α be a stationary measure for eq. (2.24), given by the first item of Theorem 2.5. Then

- *$M_\varepsilon\mu_\varepsilon^\alpha \rightharpoonup \mu^\alpha$ in $\widetilde{H}$ as $\varepsilon \to 0$, where μ^α is a unique stationary measure for the corresponding 2D α-model (2.36), and $\mu^\alpha \rightharpoonup \mu$ as $\alpha \to 0$, where μ is a unique stationary measure for the 2D NSE (2.15).*
- *$M_\varepsilon\mu_\varepsilon^\alpha \rightharpoonup \mu$ in $\widetilde{H}$ as $\varepsilon \to 0$ and $\alpha \to 0$ simultaneously.*

We note that in Theorem 3.1 we do not assume that the stationary measure μ_ε^α is unique for $\alpha, \varepsilon > 0$. However under condition (2.32) we can use the third statement of Theorem 2.5 to achieve the uniqueness for $\varepsilon \leq \varepsilon_0(\alpha)$ by increasing the parameter N in the hypotheses of Theorem 2.2.

As we will see in the proof given below an assertion similar to Theorem 3.1 can be easily established for stationary statistical solutions P_ε^α.

We recall that under the assumptions (2.32) the set $\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha$ of Borel measures in H_ε, limiting for the family $\{\mu_\varepsilon^\alpha\}$ as $\alpha \to 0$, is formed by weakly stationary measures for (1.1) (see Theorem 2.7). The next result dealing with the iterated limit 'first $\alpha \to 0$, then $\varepsilon \to 0$' follows immediately from Theorem 3.1 by the standard commutation limits argument.

Corollary 3.2 *Under the assumptions of Theorem 3.1 the only limiting point of the family of measures* $M_\varepsilon\,(\mathrm{Lim}_{\,\alpha\to 0}\mu_\varepsilon^\alpha)$ *as* $\varepsilon \to 0$ *is the unique stationary measure* μ *of* (2.15).

Under the conditions of Theorem 3.1 we obviously have relation (1.5) claimed in the Introduction. Moreover, using (2.27) one can see that

$$\lim_{\varepsilon\to 0}\int_{H_\varepsilon} |\hat{N}_\varepsilon u|_\varepsilon^2 \mu_\varepsilon^\alpha(du) = 0 \quad \text{uniformly in } \alpha,$$

where $\hat{N}_\varepsilon$ is the 'vertical' projection defined in Section 2.4. Accordingly, the averaged normalised energy

$$E(\mu_\varepsilon^\alpha) = \frac{1}{2}\int_{H_\varepsilon} |u|_\varepsilon^2 \mu_\varepsilon^\alpha(du) = \int_{H_\varepsilon}\Big(\frac{1}{2\varepsilon}\int_{\mathcal{O}_\varepsilon} |u(x)|^2\,dx\Big)\mu_\varepsilon^\alpha(du)$$

possesses the property

$$\lim_{\alpha\to 0}\lim_{\varepsilon\to 0} E(\mu_\varepsilon^\alpha) = E(\mu) \equiv \frac{1}{2}\int_{\widetilde{H}} |u|_{\mathbb{T}^2}^2 \mu(du).$$

In contrast with the kick model considered in [6] we are not able to establish similar convergence of averaged enstrophy and enstrophy production. However for a *fixed* $\alpha > 0$ the convergence properties of the measures μ_ε^α can be improved and the hypotheses concerning f_ε and W_ε can be relaxed. This happens since using additional regularity provided by α-approximation, in addition to relation (2.26), we can estimate the projections $\hat{M}_\varepsilon u$ and $\hat{N}_\varepsilon u$ separately. We do not discuss this issue in details.

We also note that literally repeating the proof of Theorem 3.1 we get a similar result for solutions of the initial-value problems:

Proposition 3.3 *Let* $u_{0\varepsilon} \in H_\varepsilon$, $0 < \varepsilon \le 1$, *be a non-random vector such that* $|u_{0\varepsilon}|_\varepsilon \le C$ *and* $M_\varepsilon u_{0\varepsilon} \to u_0$ *in* $\widetilde{H}$ *as* $\varepsilon \to 0$. *Let* $u_\varepsilon^\alpha(t)$, $t \ge 0$, *be a solution of* (2.24) *such that* $u_\varepsilon^\alpha(0) = u_{0\varepsilon}$ *and* $u(t)$, $t \ge 0$, *be a solution of* (2.15), *equal* u_0 *at* $t = 0$. *Then under the assumptions of Theorem 3.1 we have*

$$\mathcal{D}M_\varepsilon u_\varepsilon^\alpha(\cdot) \rightharpoonup \mathcal{D}u(\cdot) \;\; \text{as} \;\; \varepsilon, \alpha \to 0$$

in the space of Borel measures in $\tilde{\mathcal{Z}} = L_2^{loc}(\mathbb{R}_+; \widetilde{H}) \cap C(\mathbb{R}_+; \widetilde{\mathcal{W}}^{-1})$.

4 Proofs

In this section we prove Theorem 3.1 and Proposition 2.6 (needed to complete the proof of Theorem 2.5).

4.1 Preliminaries

We define the operators $\hat{M}_\varepsilon$ and $\hat{N}_\varepsilon = I - \hat{M}_\varepsilon$ as in Section 2.4. The most important property of these operators is that they are spectral (orthogonal) projectors for the Stokes operator A_ε. In particular, they map the space V_ε to itself, are orthogonal in both spaces H_ε and V_ε, and commute with A_ε. Other properties of the operators $\hat{M}_\varepsilon$ and $\hat{N}_\varepsilon$ which we use in the further considerations are listed below (we refer to [27] for the proofs):

(i) $\hat{M}_\varepsilon \partial_{x_i} = \partial_{x_i} \hat{M}_\varepsilon$ and $\hat{N}_\varepsilon \partial_{x_i} = \partial_{x_i} \hat{N}_\varepsilon$, $i = 1, 2$.

(ii) If one of the vector fields u, w, v lies in $\hat{N}_\varepsilon V_\varepsilon$ and two others belong to $\hat{M}_\varepsilon V_\varepsilon$, then $b_\varepsilon(u, w, v) = 0$. In particular, for all $u, w, v \in V_\varepsilon$ we have

$$b_\varepsilon(u, w, \hat{M}_\varepsilon v) = b_\varepsilon(\hat{M}_\varepsilon u, \hat{M}_\varepsilon w, \hat{M}_\varepsilon v) + b_\varepsilon(\hat{N}_\varepsilon u, \hat{N}_\varepsilon w, \hat{M}_\varepsilon v). \tag{4.1}$$

(iii) For any $\varepsilon \in (0, 1)$ we have that

$$|\hat{N}_\varepsilon u|_\varepsilon \leq \varepsilon |\partial_3 \hat{N}_\varepsilon u|_\varepsilon \quad \text{for all } u \in V_\varepsilon. \tag{4.2}$$

4.2 Proof of Theorem 3.1

For definiteness we consider the second case when both ε and α tend to 0 (the case of a fixed $\alpha > 0$ is simpler).

Let P_ε^α be a stationary statistical solution of problem (2.24). Then μ_ϵ^α is a trace-measure for P_ε^α. Since estimates (2.27) are uniform in α and ε, then the families $\{M_\varepsilon \mu_\varepsilon^\alpha\}_{0 \leq \varepsilon, \alpha \leq 1}$ and$\{M_\varepsilon P_\varepsilon^\alpha\}_{0 \leq \varepsilon, \alpha \leq 1}$ are tight in the spaces $\widetilde{H}$ and $\tilde{\mathcal{Z}} = L_2^{loc}(\mathbb{R}_+; \widetilde{H}) \cap C(\mathbb{R}_+; \widetilde{\mathcal{W}}^{-1})$ respectively, see [28] or the proof of Theorem 2.7. Denote by $\bar{P}$ any limiting measure for the family $\{M_\varepsilon P_\varepsilon^\alpha\}$ as $\varepsilon, \alpha \to 0$. Then for suitable sequences $\varepsilon_j \to 0$ and $\alpha_j \to 0$ we have

$$M_{\varepsilon_j} P_{\varepsilon_j}^{\alpha_j} \rightharpoonup \bar{P} \text{ in } \widetilde{Z}, \quad M_{\varepsilon_j} \mu_{\varepsilon_j}^{\alpha_j} \rightharpoonup \bar{\mu} \text{ in } \widetilde{H},$$

where $\bar{\mu}$ is a trace-measure for $\bar{P}$. Using the Skorokhod representation theorem (see [16]), we construct on a new probability space random processes $\hat{u}_{\varepsilon_j}^{\alpha_j}(t) \in H_{\varepsilon_j}$ and $\hat{v}(t) \in \tilde{H}$, $t \geq 0$, such that $\mathcal{D}\hat{u}_{\varepsilon_j}^{\alpha_j}(\cdot) = P_{\varepsilon_j}^{\alpha_j}$, $\mathcal{D}\hat{v}(\cdot) = \hat{P}$, and

$$M_{\varepsilon_j} \hat{u}_{\varepsilon_j}^{\alpha_j} \to \hat{v} \quad \text{in} \quad \tilde{\mathcal{Z}}, \text{ a.s.} \tag{4.3}$$

(cf. Lemma 5.9 in [6]). Since $P_{\varepsilon_j}^{\alpha_j}$ is a stationary statistical solution, then $\hat{u}_{\varepsilon_j}^{\alpha_j}(t)$ is a stationary process, satisfying the corresponding α-approximation of (2.9) with a suitable Wiener process $\widehat{W}_\varepsilon = \widehat{W}_j$. The process $\hat{v}(t)$ also is stationary. Let us denote $M_{\varepsilon_j} \hat{u}_{\varepsilon_j}^{\alpha_j}(t) = \hat{v}_{\varepsilon_j}^{\alpha_j}(t)$. Then $v = \hat{v}_{\varepsilon_j}^{\alpha_j}$ a.s. satisfies

$$v(t) - v(0) + \int_0^t \big(\nu A_0 v + M_{\varepsilon_j} B_{\varepsilon_j}(G_{\alpha_j} \hat{u}_{\varepsilon_j}^{\alpha_j}, \hat{u}_{\varepsilon_j}^{\alpha_j}) - M_{\varepsilon_j} f_{\varepsilon_j}\big) ds = M_{\varepsilon_j} \widehat{W}_j(t),$$

for any $t \geq 0$. Let us take any vector $e = \widetilde{e}_{\lambda_k}$ (see Appendix), multiply the last equation by e in $\tilde{H}$ and denote the l.h.s. by $\xi_e^j(t)$. Consider the 3-linear term

$$b_j(t) := (M_{\varepsilon_j} B_{\varepsilon_j}(G_{\alpha_j}\hat{u}, \hat{u}), e)_{\mathbb{T}^2} = \varepsilon^{-1} b_{\varepsilon_j}(G_{\alpha_j}\hat{u}, \hat{u}, M^*_{\varepsilon_j} e),$$

where we abbreviated $\hat{u} = \hat{u}^{\alpha_j}_{\varepsilon_j}$. Due to (4.1),

$$b_j(t) = \varepsilon_j^{-1} b_{\varepsilon_j}(\hat{M}_{\varepsilon_j} G_{\alpha_j}\hat{u}, \hat{M}_{\varepsilon_j}\hat{u}, M^*_{\varepsilon_j} e) + \varepsilon_j^{-1} b_{\varepsilon_j}(\hat{N}_{\varepsilon_j} G_{\alpha_j}\hat{u}, \hat{N}_{\varepsilon_j}\hat{u}, M^*_{\varepsilon_j} e).$$

Since $M_\varepsilon G_\alpha = G^0_\alpha M_\varepsilon$, then we obtain

$$b_j(t) = b_0(G^0_{\alpha_j} v, v, e) + \varepsilon_j^{-1} b_{\varepsilon_j}(\hat{N}_{\varepsilon_j} G_{\alpha_j}\hat{u}, \hat{N}_{\varepsilon_j}\hat{u}, M^*_{\varepsilon_j} e).$$

Using (4.2) we have that

$$\varepsilon^{-1} b_\varepsilon(\hat{N}_\varepsilon G_\alpha \hat{u}, \hat{N}_\varepsilon \hat{u}, M^*_\varepsilon e) \leq C \max_{x' \in \mathbb{T}^2} \{|\nabla e(x')|\} |N_\varepsilon \hat{u}|_\varepsilon |N_\varepsilon G_\alpha \hat{u}|_\varepsilon \leq C_e \varepsilon^2 \|N_\varepsilon \hat{u}\|^2_\varepsilon.$$

Since

$$|A_0^{-1/2} \left[I - G^0_\alpha\right] |_{\mathcal{L}(\widetilde{V})} \leq \max_{\lambda > 0} \frac{\alpha \lambda^{1/2}}{1 + \alpha\lambda} \leq \alpha^{1/2},$$

we also have that

$$|b_0\big((I - G_{\alpha_j})v, v, e\big)| \leq C_e \alpha_j^{1/2} \|v\|_{\mathbb{T}^2} |v|_{\mathbb{T}^2} \leq C_e \alpha_j^{1/2} \|\hat{u}_{\varepsilon_j}(t)\|^2_{\varepsilon_j}.$$

Therefore

$$|b_j(t) - b_0(\hat{v}_{\varepsilon_j}, \hat{v}_{\varepsilon_j}, e)| \leq C_e \left[\varepsilon_j^2 + \alpha_j^{1/2}\right] \|\hat{u}_{\varepsilon_j}(t)\|^2_{\varepsilon_j}. \tag{4.4}$$

Now let us denote by $\xi_e(t)$ the scalar product in $\tilde{H}$ of the vector e with the l.h.s. of eq. (2.16), where u is replaced by $\hat{v}$. The convergence (4.3) and the estimates $(2.27)_{n=1}$, (4.4) imply that for any $T > 0$ we have

$$\xi_e^j(\cdot) \to \xi_e(\cdot) \;\text{ in }\; C[0,T] \;\text{ as }\; j \to \infty$$

almost surely (if necessary, we replace the sequence $j = 1, 2, \dots$ by a suitable subsequence). Moreover, since

$$|b_j(t)| \leq C|\hat{u}^{\alpha_j}_{\varepsilon_j}(t)|^2 \;\text{ for }\; 0 \leq t \leq T,$$

where $C = C(e,T)$, then by (2.27) all moments of the random variables $|\xi_e^j(t)|$ are bounded uniformly in j and in $t \in [0,T]$. Now arguing as when proving Theorem 2.7, we get that $\bar{P} = \mathcal{D}(\hat{v})$ is a stationary statistical solution of the 2D NSE. Accordingly, $\bar{\mu}$ is a stationary measure and $\bar{\mu} = \mu$ by the uniqueness. This completes the proof of Theorem 3.1.

4.3 Proof of Proposition 2.6

Now we complete the proof of Theorem 2.5 on the uniqueness of the stationary measure for the 3D α-approximation (2.24). To do this we need to establish Proposition 2.6 only.

For $u = u_1 - u_2$ we have that

$$u' + \nu A_\varepsilon u + B_\varepsilon(G_\alpha u, u_1) + B_\varepsilon(G_\alpha u_2, u) = \eta_1 - \eta_2.$$

Since $u = Qu$, we obtain that

$$\frac{1}{2}\frac{d}{dt}|Qu|_\varepsilon^2 + \nu|A_\varepsilon^{1/2}Qu|_\varepsilon^2 + \varepsilon^{-1}b_\varepsilon(G_\alpha u, u_1, Qu) = 0. \tag{4.5}$$

The projector P can be written in the form $P = P_n^1 + P_{\hat{n}}^2$, where P_n^1 and $P_{\hat{n}}^2$ are the spectral orthoprojectors on the subspaces Span $\left\{e_{\lambda_j} \,:\, 1 \le j \le n\right\}$ and Span $\left\{e_{\Lambda_i^\varepsilon} \,:\, 1 \le i \le \hat{n}\right\}$. Therefore

$$|A_\varepsilon^{1/2}Qu|_\varepsilon^2 \ge \min\left\{\lambda_{n+1}, \Lambda_{\hat{n}+1}^\varepsilon\right\}|Qu|_\varepsilon^2. \tag{4.6}$$

Now we estimate the nonlinear term in (4.5). Since

$$u = Qu = (1 - P_n^1)\hat{M}_\varepsilon u + (1 - P_{\hat{n}}^2)\hat{N}_\varepsilon u \equiv Q_n^1\hat{M}_\varepsilon u + Q_{\hat{n}}^2\hat{N}_\varepsilon u,$$

we have that

$$b_\varepsilon(G_\alpha u, u_1, Qu) = b_\varepsilon(\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u, u_1, Qu) + b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu). \tag{4.7}$$

To estimate the first term in r.h.s. of (4.7) we use the relation

$$|b_\varepsilon(\hat{N}_\varepsilon u, w, v)| \le c\varepsilon^2|A_\varepsilon\hat{N}_\varepsilon u|_\varepsilon \cdot \|w\|_\varepsilon \cdot |v|_\varepsilon,$$

see [27] and Lemma 6.3 in [6]. Since $|A_\varepsilon^{1/2}G_\alpha|_{\mathcal{L}(H_\varepsilon)} \le \alpha^{-1/2}$, this inequality implies that

$$\begin{aligned} |b_\varepsilon(\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u, u_1, Qu)| &\le C\varepsilon^2|A_\varepsilon\hat{N}_\varepsilon G_\alpha Q_{\hat{n}}^2 u|_\varepsilon\|u_1\|_\varepsilon|Qu|_\varepsilon \\ &\le C\varepsilon^2|A_\varepsilon^{1/2}G_\alpha|_{\mathcal{L}(H_\varepsilon)}\|Q_{\hat{n}}^2 u\|_\varepsilon\|u_1\|_\varepsilon|Qu|_\varepsilon \\ &\le \delta\varepsilon\|Qu\|_\varepsilon^2 + \frac{C_\delta\varepsilon^3}{\alpha}\|u_1\|_\varepsilon^2|Qu|_\varepsilon^2 \end{aligned}$$

for any $\delta > 0$. To estimate the second term in r.h.s. of (4.7) we note that

$$|b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu)| \le C\varepsilon \max_{x'\in\mathbb{T}^2}\left\{|(M_\varepsilon G_\alpha Q_n^1 u)(x')|\right\}\|u_1\|_\varepsilon|Qu|_\varepsilon,$$

where

$$\max_{x'\in\mathbb{T}^2}\left\{|(M_\varepsilon G_\alpha Q_n^1 u)(x')|\right\} \le C|M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2}^{1/2}|A_0 M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2}^{1/2}. \tag{4.8}$$

Since $|M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2} \le C\lambda_{n+1}^{-1/2}|A_0^{1/2}Q_n^1 u|_{\mathbb{T}^2}$ and

$$|A_0 M_\varepsilon G_\alpha Q_n^1 u|_{\mathbb{T}^2} \le |A_0^{1/2}G_\alpha^0|_{\mathcal{L}(\widetilde{H})} \cdot |A_0^{1/2}Q_n^1 u|_{\mathbb{T}^2},$$

then the r.h.s. in (4.8) is $\le C\big(\lambda_{n+1}\alpha\big)^{-1/4}\|Q_n^1 u\|_\varepsilon$. Therefore

$$|b_\varepsilon(\hat{M}_\varepsilon G_\alpha Q_n^1 u, u_1, Qu)| \le \delta\varepsilon\|Qu\|_\varepsilon^2 + \frac{C_\delta \varepsilon}{\sqrt{\alpha\lambda_{n+1}}}\|u_1\|_\varepsilon^2 |Qu|_\varepsilon^2$$

for every $\delta > 0$. Thus

$$|b_\varepsilon(G_\alpha u, u_1, Qu)| \le \frac{\nu}{2}\varepsilon|A_\varepsilon^{1/2}Qu|_\varepsilon^2 + c_0\left(\frac{\varepsilon^3}{\alpha} + \frac{\varepsilon}{\sqrt{\alpha\lambda_{n+1}}}\right)\|u_1\|_{0,\varepsilon}^2|Qu|_\varepsilon^2$$

and we get from (4.5) an (4.6) that

$$\frac{d}{dt}|Qu|_\varepsilon^2 + \nu\min\left\{\lambda_{n+1}, \Lambda_{\hat{n}+1}^\varepsilon\right\}|Qu|_\varepsilon^2 \le c_0\left(\frac{\varepsilon^2}{\alpha} + \frac{1}{\sqrt{\alpha\lambda_{n+1}}}\right)\|u_1\|_\varepsilon^2|Qu|_\varepsilon^2.$$

Now the desired relation in (2.31) follows from Gronwall's lemma.

5 Appendix: spectral problem for the Stokes operator

The spectral boundary value problem which corresponds to operator A_ε has the form

$$\begin{cases} -\Delta w = \lambda w, \quad \operatorname{div} w = 0 \quad \text{in } \mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0,\varepsilon), \\ w(x', x_3) \text{ is } (l_1, l_2)\text{-periodic with respect to } x', \\ w_3|_{x_3=\varepsilon} = 0, \quad \partial_3 w_j|_{x_3=\varepsilon} = 0, \quad j = 1,2, \\ w_3|_{x_3=0} = 0, \quad \partial_3 w_j|_{x_3=0} = 0, \quad j = 1,2. \\ \int_{\mathcal{O}_\varepsilon} w_j dx = 0, \ j = 1,2. \end{cases}$$

Using the spectral decomposition (2.11) one can see that the spectrum consists of two branches. Recalling estimate (4.2) we find that these branches are: (i) the spectrum of the 2D Stokes operator A_0, $0 < \lambda_1 \le \lambda_2 \le \ldots$, and (ii) series of eigenvalues $0 < \Lambda_1^\varepsilon \le \Lambda_2^\varepsilon \le \ldots$, depending on ε and greater than ε^{-2} . We denote the corresponding eigenfunctions e_{λ_j} and $e_{\Lambda_j^\varepsilon}$. We have

$$\hat{M}_\varepsilon e_{\lambda_j} = e_{\lambda_j}, \quad \hat{M}_\varepsilon e_{\Lambda_j^\varepsilon} = 0,$$

where the (spectral) projector $\hat{M}_\varepsilon$ is defined by (2.10). One can also see that $e_{\lambda_j} = (\widetilde{e}_{\lambda_j}; 0)$, where $\widetilde{e}_{\lambda_j} \equiv M_\varepsilon e_{\lambda_j}$ is the eigenfunction for the 2D Stokes operator on $\mathbb{T}^2$ which correspond to the eigenvalue λ_j. The eigenvalues λ_j are properly ordered numbers $\left(s_1\frac{2\pi}{l_1}\right)^2 + \left(s_2\frac{2\pi}{l_2}\right)^2$, $s = (s_1, s_2) \in \mathbb{Z}^2 \setminus \{0\}$, so that $C^{-1}j \le \lambda_j \le Cj$ for all j, with some $C > 1$ (see, e.g., [8]). We normalise the eigenfunctions as follows:

$$|e_{\lambda_j}|_\varepsilon = |e_{\Lambda_j^\varepsilon}|_\varepsilon = 1 \quad \forall\, j\, .$$

It is also obvious that $|\widetilde{e}_{\lambda_j}|_{\mathbb{T}^2} = 1$ and $\|\widetilde{e}_{\lambda_j}\|_{\mathbb{T}^2} = \sqrt{\lambda_j}$ for all j.

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