Amer. Math. Soc. Transl.
(2) Vol. **206**, 2002

On Exponential Convergence to a Stationary Measure for Nonlinear PDEs Perturbed by Random Kick-Forces, and the Turbulence Limit

Sergei B. Kuksin

Dedicated to M. I. Vishik on the occasion of his 80th birthday.

ABSTRACT. For a class of random dynamical systems which describe dissipative nonlinear PDEs perturbed by a bounded random kick-force, we propose a "direct proof" of the uniqueness of the stationary measure and exponential convergence of solutions to this measure, by showing that the transfer operator acting in the space of probability measures given the Kantorovich metric, defines a contraction of this space. Next we use results of [**Kuk97, Kuk99**] to study properties of this measure in the turbulence limit (as the viscosity goes to zero), for some nonlinear PDEs.

0. Introduction

In [**KS00**][1] (see also [**KS02**]) A. Shirikyan and the author of this paper considered a class of nonlinear dissipative PDEs perturbed by smooth in space random forces. We proved that these equations, treated as random dynamical systems in a function space, have unique stationary measures. The forces considered in [**KS00**] have the form of bounded random kicks and satisfy some nondegeneracy assumption; the class of equations includes the 2D Navier–Stokes equations:

$$\text{(NS)}\qquad \begin{aligned} &\dot u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \eta(t,x)\,, \quad x\in\mathbb{T}^2;\\ &\operatorname{div} u = 0\,, \qquad \int u\,dx = \int \eta\,dx = 0. \end{aligned}$$

Here η is a kick-force; see Section 1. In particular, the results of [**KS00**] imply that if η "contains noise in each Fourier mode", then (for any positive ν) solutions of (NS), treated as random processes in a function space H of divergence-free vector fields, converge in distribution to a unique measure μ_ν on H. This measure comprises

2000 *Mathematics Subject Classification.* Primary 37H99, 35Q30.

Key words and phrases. Coupling, kick-force, Navier–Stokes equations, random dynamical system, stationary measure.

The paper is a part of a research supported by EPSRC, grant GR/N63055/01.

[1]The postscript files of this paper as well as of other related publications by the author can be downloaded from his web-page.

asymptotic in time properties of solutions. For $\nu \ll 1$ it describes the 2D turbulence; see [**VF88, Gal01**] and the Introduction in [**KS00**].

The proof in [**KS00**] is based on a reduction of the original infinite-dimensional random dynamical system (defined by a PDE we consider) to a 1D Gibbs system with a finite-dimensional phase space. Later, E. Mattingly, Sinai [**EMS01**] and Bricmont, Kupiainen, Lefevere [**BKL00**] used similar approaches to show that the (NS) system perturbed by a white (in time) force η also has a unique stationary measure. In [**KS01b**] the (NS) equation with an unbounded kick-force is studied and the scheme of [**KS00**] is used to prove the uniqueness and ergodicity of a stationary measure.

J.-P. Eckmann and M. Hairer have recently considered another class of randomly forced nonlinear PDEs and obtained for them similar results; see [**EH01**].

In [**KS01a, KPS01**] the author of this paper and his collaborators developed a coupling approach to study the systems under discussion. This approach gives a shorter proof of the uniqueness and implies that any solution of the system converges in distribution exponentially fast to the stationary measure.[2] Independently a similar coupling approach to the study (NS) was proposed by N. Masmoudi and L.-S. Young in [**MY01**].

The main result of this work is Theorem 1.2, where we present a "direct proof" of the uniqueness and exponential convergence by showing that the transfer-operator, corresponding to a random dynamical system as above and acting in the space of probability measures, given the Kantorovich(–Wasserstein) metric, defines a contraction of this space.

In Section 5 we consider complex Ginzburg–Landau equations and evoke results of [**Kuk97, Kuk99**] to study some properties of their stationary measures in the turbulence-limit (as the coefficient of the Laplacian goes to zero). The results of this section do not apply to the equations (NS).

The proof of Theorem 1.2 presented in this work can be treated as reinterpreting of the arguments from [**KS01a, KPS01**]: it is based on the coupling approach and uses essentially Lemma 3.2 of [**KS01a**] (which is the heart of the proof in [**KS01a**]). In addition to the coupling techniques, we now use some ideas originated in the works by Kantorovich on the mass-transfer problem in the 1940's; see [**KA77, Dud89**].

Notation. We denote by $(\Omega, \mathcal{F}, \mathsf{P})$ and $(\Omega', \mathcal{F}', \mathsf{P}')$ different probability spaces, and abbreviate them to Ω and Ω', respectively. All metric spaces are given Borel sigma-algebras. $\mathcal{D}(\cdot)$ signifies the distribution of a random variable.

A Hilbert space H with a norm $\|\cdot\|$ is fixed in this work. We use the following notation for objects related to H:

$\mathcal{B} = \mathcal{B}(H)$, the sigma-algebra of Borel subsets of H;

C_b, the space of bounded continuous functions on H, given the sup-norm;

$\mathcal{P}$, the space of Borel probability measures on H;

$\mathcal{P}(A)$, measures from $\mathcal{P}$ supported by a subset $A \subset H$;

$B(R)$, the closed ball of radius R in H centered at the origin.

---

[2] In [**BKL00**] the exponential convergence is proven for almost all (with respect to stationary measure) initial conditions.

Acknowledgements. A preliminary short version [**Kuk01**] of this paper was written and typed during my visit to IHES in June 2001, and I sincerely thank the institute for hospitality.

1. A class of random dynamical systems

Let H be a Hilbert space with a norm $\|\cdot\|$ and an orthonormal basis $\{e_j\}$, and let $S: H \to H$ be a continuous map such that $S(0)=0$.

Let $\{\eta_k, k \in \mathsf{Z}\}$ be a sequence of i.i.d. random variables $\Omega \to H$ of the form

$$\eta_k = \eta_k^\omega = \sum_{j=1}^{\infty} b_j \, \xi_{jk} \, e_j \,, \tag{1.1}$$

where $b_j \ge 0$ are constants and $\sum b_j^2 < \infty$. It is assumed that $\{\xi_{jk} = \xi_{jk}^\omega\}$ are independent random variables such that $|\xi_{jk}| \le 1$ for all j, k, ω and

$$\mathcal{D}(\xi_{jk}) = \; p_j(r)\, dr \qquad \forall\, j, k \,. \tag{1.2}$$

Here $p_1, p_2, \dots$ are functions of bounded variation, supported by the segment $[-1,1]$, and

$$\int_{-\varepsilon}^{\varepsilon} p_j(r)\, dr > 0 \qquad \forall\, j \ge 1 \,,\ \varepsilon > 0 \,. \tag{1.3}$$

We consider the following *random dynamical system* (RDS) in H:

$$u(k) = S\left(u(k-1)\right) + \eta_k =: F_k^\omega \left(u(k-1)\right),\ k \ge 1 \,. \tag{1.4}$$

This RDS defines a family of Markov chains in H with the transition function

$$P\left(k, v, \Gamma\right) = \mathsf{P}\left\{u(k) \in \Gamma\right\}, \quad \Gamma \in \mathcal{B}(H)\,,$$

where $u(\cdot) = \ u(\cdot; v)$ is a solution for (1.4) such that $u(0) = v$. Let $\{\mathfrak{S}_k\}$ and $\{\mathfrak{S}_k^*\}$ be the corresponding Markov semigroups, acting in the space C_b of bounded continuous functions on H, and in the space $\mathcal{P}$ of Borel probability measures, respectively:

$$\mathfrak{S}_k f(v) = \mathsf{E}\, f\left(u(k; v)\right),\ f \in C_b \,,$$

$$\mathfrak{S}_k^* \mu(\Gamma) = \int_H \mathsf{P}\left\{u(k; v) \in \Gamma\right\} \mu(dv)\,,\ \mu \in \mathcal{P}\,,$$

where u is the solution for (1.4) as above.

For any $v \in H$ and $k = 0, 1, \dots$ we abbreviate

$$\mu_v(k) = P\left(k, v, \cdot\right) = \mathcal{D}\left(u(k; v)\right).$$

Now we impose some assumptions on the map S. The "right" ones are given in [**KS01a**]; see conditions A–C there. In this work we replace them by shorter and stronger conditions A′) and B′). The new conditions hold for the RDS which corresponds to the 2D Navier–Stokes equations (see the example below). Our proof of the Main Theorem works under the conditions A–C, but it becomes somewhat longer, and the notation is more cumbersome.

A′) The map S is Lipschitz uniformly on bounded subsets of H, and there exists a positive constant $\gamma_0 < 1$ such that

$$\|S\left(u\right)\| \le \gamma_0 \|u\| \qquad \forall\, u \in H \,. \tag{1.5}$$

B′) For any $R > 0$ there is a sequence $\gamma_N(R) > 0$ $(N \ge 1)$ which converges to zero as $N \to \infty$, such that

$$\|Q_N(S(u_1) - S(u_2))\| \le \gamma_N(R)\,\|u_1 - u_2\| \quad \text{for all} \quad u_1, u_2 \in B(R)\,.$$

Here Q_N stands for the orthogonal projector $H \to \overline{\mathrm{span}}\,\{e_{N+1}, e_{N+2}, \dots\}$.

EXAMPLE 1.1. Let us consider the 2D Navier–Stokes equations perturbed by a random kick-force η:

$$\dot u - \nu\,\Delta u + (u\cdot\nabla)u + \nabla p = \eta(t,x) \equiv \sum_{k\in\mathsf{Z}} \eta_k(x)\,\delta(t-k)\,, \tag{1.6}$$

$$\operatorname{div} u = 0\,, \int u\,dx \equiv \int \eta\,dx \equiv 0\,; \qquad x \in \mathsf{T}^2.$$

Let H be the L^2-space of divergence-free vector fields on T^2 with zero space-average, and let $\{e_j\}$ be the usual trigonometric basis of H. Let us assume that the kicks η_k are random variables in H having the form (1.1) and satisfying (1.3). By normalizing solutions $u(t) \in H$ of (1.6) to be continuous from the right, we observe that the equation can be written in the form (1.4), where $u(k) = u(k,\cdot) \in H$, $k \in \mathsf{Z}$, and the operator S is the time-one shift along trajectories of the free Navier–Stokes system. The condition A′) obviously holds with $\gamma_0 = e^{-\lambda}$, where λ is the minimum eigenvalue of $-\nu\Delta$ in H. It is also well known that S satisfies B′); see e.g. [**KS00**].

A measure $\mu \in \mathcal{P}$ is called a *stationary measure* for the RDS (1.4) if $\mathfrak{S}_k^*\mu = \mu \quad \forall\, k$. The goal of this work is to prove the following result:

THEOREM 1.2. *There exists a constant $N \ge 1$ such that if*

$$b_j \ne 0 \qquad \forall\, j \le N\,, \tag{1.7}$$

then the RDS (1.4) has a unique stationary measure μ. Moreover, there exists a constant $\kappa \in (0,1)$ such that

$$|(\mu_u(t), f) - (\mu, f)| \le C\,\kappa^t \quad \textit{for} \quad t = 1, 2, \dots, \tag{1.8}$$

for every Lipschitz function f on H such that $|f| \le 1$ and $\operatorname{Lip} f \le 1$. *The constant C depends only on $\|u\|$.*

The theorem applies to the 2D NS equation (1.6). Moreover, it is known that if the sequence $\{b_j\}$ decays faster than every negative degree of j, then the corresponding measure μ is concentrated on the set of smooth functions:

$$\mu(H \cap C^\infty) = 1. \tag{1.9}$$

This property immediately follows from the Chapman–Kolmogorov relation since the corresponding map S sends the space H to $H \cap C^\infty$ (see in [**KS00**]).

2. Preliminaries

2.1. Estimates for solutions. Since $|\xi_{jk}| \le 1$, we have

$$\|\eta_k^\omega\| \le K_1 = (b_1^2 + b_2^2 + \cdots)^{1/2} < \infty \quad \text{for all } k \text{ and } \omega. \tag{2.1}$$

So

$$\|F_k^\omega(u)\| \le \gamma_0\|u\| + K_1\,,$$

and any ball $B(R)$ with $R \geq K_1/(1-\gamma_0)$ is invariant for the RDS (1.4).[3] The same estimate above implies that

$$\|u(k;v)\| \leq \gamma_0^k \|v\| + K_1(1+\cdots+\gamma_0^{k-1}) \leq \gamma_0^k \|v\| + \frac{K_1}{1-\gamma_0}\,, \tag{2.2}$$

for all $k \geq 0$, $v \in H$ and all ω.

2.2. The coupling. Let $\mu_1, \mu_2 \in \mathcal{P}$.

Definition 2.1. A pair of random variables ξ_1, ξ_2, defined on the same probability space and valued in H, is called a *coupling* for (μ_1, μ_2) if $\mathcal{D}\,\xi_1 = \mu_1$ and $\mathcal{D}\,\xi_2 = \mu_2$.

For basic results on the coupling see [**Lin92**] and the Appendix in [**KS01a**].

The following Lemma 2.2 claims that the measures $\mu_{u_1}(1)$, $\mu_{u_2}(1)$ admit a coupling which possesses some special properties if $\|u_1 - u_2\| \ll 1$. The lemma was first proved in [**KS01a**]. For the reader's convenience we repeat its proof here.

Let us take any positive R.

Lemma 2.2. *There is a probability space* $(\Omega, \mathcal{F}, \mathsf{P})$*, an integer* $N = N(R) \geq 1$ *and a constant* $C_* = C_*(R) > 0$ *such that if (1.7) holds, then for any* $u_1, u_2 \in B\,(R)$ *the measures* $\mu_{u_1}(1), \mu_{u_2}(1)$ *admit a coupling* (V_1, V_2)*, where* $V_j = V_j(u_1, u_2; \omega)$, $j = 1, 2$*, possess the following properties:*

(i) *the maps* $V_1, V_2 : B(R)^2 \times \Omega \to H$ *are measurable;*

(ii) *denoting* $d = \|u_1 - u_2\|$*, we have*

$$\mathsf{P}\,\{\|V_1 - V_2\| \geq d/2\} \leq C_*\,d\,. \tag{2.3}$$

Proof. Below, $\|\mu - \nu\|_{\mathrm{var}}$ signifies variational distance between measures μ and ν (see [**Dud89, KA77**]). We recall that if the measures have densities $p_\mu(x)$ and $p_\nu(x)$ against a measure $dm(x)$, then

$$\|\mu - \nu\|_{\mathrm{var}} = \frac{1}{2} \int |p_\mu(x) - p_\nu(x)|\, dm(x). \tag{2.4}$$

By P_N we denote the orthogonal projector

$$P_N : H \to \mathrm{span}\,\{e_1, \ldots, e_N\} =: H_N,$$

and recall that Q_N is the orthogonal projector to $\overline{\mathrm{span}}\,\{e_{N+1}, e_{N+2}, \ldots\}$.

We abbreviate pairs of the form V_1, V_2 to $V_{1,2}$.

Let $(\Omega_1, \mathcal{F}_1, \mathsf{P}_1)$ be the probability space on which the random variables $\{\eta_k\}$ are defined, and let $(\Omega_2, \mathcal{F}_2, \mathsf{P}_2)$ be the probability space, where a coupling is defined for the measures ν_1, ν_2, specified below. We shall show that the set $\Omega = \Omega_1 \times \Omega_2$ endowed with the σ-algebra and the probability of direct product is the required probability space.

The random variables V_1, V_2 are sought in the form

$$V_1 = S(u_1) + \xi_1, \quad V_2 = S(u_2) + \xi_2,$$

where $\xi_{1,2}$ are some random variables on Ω such that $\mathcal{D}(\xi_1) = \mathcal{D}(\xi_2) = \mathcal{D}(\eta_1)$. It is clear that $\mathcal{D}(V_{1,2}) = \mu_{u_{1,2}}(1)$. To define the random variables $\xi_{1,2}$, we specify their projections $P_N\xi_{1,2}$ and $Q_N\xi_{1,2}$, where $N \geq 1$ is a sufficiently large integer to be chosen below.

[3] A set $A \subset \mathcal{B}(H)$ is said to be *invariant* for the RDS (1.4) if $v \in A$ implies that $u(k;v) \in A$ a.s. for all k. That is, $P\,(k, u, A) = 1$ for $k \geq 0$ and $u \in A$.

We set

$$Q_N\xi_1 = Q_N\xi_2 = Q_N\tilde\eta_1,$$

where $\tilde\eta_1$ is the natural extension of η_1 to Ω, i.e., $\tilde\eta_1(\omega) = \eta_1(\omega_1)$ for $\omega = (\omega_1, \omega_2) \in \Omega$. To define $P_N\xi_{1,2}$, let us write $\nu_{1,2} := P_N\mu_{u_{1,2}}(1)$ and assume that we have proved the inequality

$$\|\nu_1 - \nu_2\|_{\mathrm{var}} \le C_* d, \tag{2.5}$$

where $C_* > 0$ is a constant not depending on $u_{1,2} \in B(R)$. Then the measures $\nu_{1,2}$ admit a coupling $\Xi_{1,2}(\omega_2)$ depending on the parameter (u_1, u_2) (i.e., $\Xi_{1,2} = \Xi_{1,2}(\omega_2; u_1, u_2)$), such that[4]

$$\mathsf{P}\{\Xi_1 \ne \Xi_2\} = \|\nu_1 - \nu_2\|_{\mathrm{var}} \le C_* d. \tag{2.6}$$

Moreover, the maps $\Xi_{1,2}$ are measurable with respect to $(\omega_2, u_1, u_2) \in \Omega_2 \times B^2$; see [**Lin92**] and Theorem 4.2 in [**KS01a**].

Retaining the same notation for the natural extensions of Ξ_1 and Ξ_2 to Ω, we now set

$$P_N\xi_{1,2} = \Xi_{1,2} - P_N S(u_{1,2})$$

and note that $\mathsf{P}_N V_1 \ne \mathsf{P}_N V_2$ if and only if $\Xi_1 \ne \Xi_2$. Let $N \ge 1$ be large enough that $\gamma_N(R) \le 1/2$ (see condition B$'$)). In this case, if $P_N V_1 = P_N V_2$, then

$$\|V_1 - V_2\| = \big\|Q_N(V_1 - V_2)\big\| = \big\|Q_N(S(u_1) - S(u_2))\big\| \le \|u_1 - u_2\|/2 = d/2.$$

Inequality (2.3) now follows from (2.6). Clearly, the coupling $V_{1,2}$ satisfies (i).

Thus, it remains to establish (2.5). To this end, we set $v_{1,2} = \mathsf{P}_N S(u_{1,2})$ and note that, in view of condition A$'$),

$$\|v_1 - v_2\| \le C(R)d. \tag{2.7}$$

Since $b_j \ne 0$ for $1 \le j \le N$, (1.2) implies that $\mathcal{D}(P_N\eta_1) = p(x)\,dx$, where dx is Lebesgue measure on the finite-dimensional space H_N and

$$p(x) = \prod_{j=1}^N q_j(x_j), \quad q_j(x_j) = b_j^{-1} p_j(x_j/b_j), \quad x = (x_1, \dots, x_N) \in H_N,$$

is a bounded function with compact support. It follows that

$$\nu_{1,2} = \mathcal{D}(v_{1,2} + P_N\eta_1) = p(x - v_{1,2})\,dx.$$

Therefore, by (2.4),

$$\|\nu_1 - \nu_2\|_{\mathrm{var}} = \frac{1}{2}\int_{H_N} |p(x - v_1) - p(x - v_2)|\,dx.$$

We claim that

$$\int_{H_N} |p(x - v_1) - p(x - v_2)|\,dx \le |v_1 - v_2| \sum_{j=1}^N b_j^{-1}\,\mathrm{Var}(p_j), \tag{2.8}$$

where $\mathrm{Var}(p_j)$ stands for the total variation of p_j. The desired inequality (2.5) follows immediately from (2.7) and (2.8).

[4]A coupling that satisfies the first relation in (2.6) is called *maximal*.

To prove (2.8), we first assume that the p_j are C^1-smooth functions. In this case, we have

$$\begin{aligned}
&\int_{H_N} |p(x-v_1)-p(x-v_2)|\,dx \\
&\quad \le |v_1-v_2| \int_{H_N}\int_0^1 \big|(\nabla p)(x-\theta v_1-(1-\theta)v_2)\big|\,d\theta dx \\
&\quad = |v_1-v_2| \int_{H_N} \big|(\nabla p)(x)\big|\,dx \le |v_1-v_2| \sum_{j=1}^N \int_{\mathbb{R}} \big|\partial_{x_j} q_j(x_j)\big|\,dx_j \\
&\quad = |v_1-v_2| \sum_{j=1}^N \mathrm{Var}(q_j).
\end{aligned}$$

It remains to note that $\mathrm{Var}(q_j)=b_j^{-1}\,\mathrm{Var}(p_j)$.

Inequality (2.8) in the general case can be easily derived by a standard approximation procedure; we omit the corresponding arguments. □

2.3. A metric on the space $\mathcal{P}$. Let us take any number

$$R' > K_1/(1-\gamma_0).$$

We fix it from now on and abbreviate $B(R')=B$. Due to the results of Section 2.1, the ball B is invariant for the RDS (1.4). Next we take any $\gamma_1\in(\gamma_0,1)$ and any positive d_0 such that

$$d_0 \le \min\left\{\frac{1}{4\,C_*}, \frac{1-\gamma_1}{2\,C_*}, 1\right\}, \tag{2.9}$$

where the constant $C_*=C_*(R')$ is as in Lemma 2.2. For $k\in\mathbb{Z}$ we set

$$d_k = \gamma_1^k\, d_0\,.$$

We may assume that d_0 and R' are chosen so that

$$\tfrac12\, d_{-L} = R'$$

for some $L\ge 1$. Below we consider only the numbers d_k with $k\ge -L$.

Let us introduce an equivalent metric d in the space H:

$$d\,(u_1,u_2) = \|u_1-u_2\|\wedge d_0\,,$$

and consider the set $\mathcal{O}\subset C_b$ formed by all functions f such that

$$|f(u_1)-f(u_2)| \le d\,(u_1,u_2) \quad \text{for all} \quad u_1,u_2\,.$$

Clearly,

$$\tfrac12\, d_0 f\in\mathcal{O} \qquad \text{if} \quad |f|\le 1 \quad \text{and} \quad \mathrm{Lip}\, f\le 1\,. \tag{2.10}$$

For any two measures $\mu_1,\mu_2\in\mathcal{P}$ we define the *Kantorovich distance* $d_K(\mu_1,\mu_2)$ as

$$d_K(\mu_1,\mu_2) = \sup_{g\in\mathcal{O}}\{(\mu_1-\mu_2,g)\}\,. \tag{2.11}$$

It is known that the space $\mathcal{P}$ is complete with respect to this distance (see [**KA77**], [**Dud89**]), and it is easy to see that $\mathcal{P}(B)$ is a closed subset of $\mathcal{P}$.

LEMMA 2.3. *Suppose that there exists a sequence $\zeta_k \to 0$ such that for $k \geq 1$ and $u, v \in B$ we have $d_K(\mu_u(k), \mu_v(k)) \leq \zeta_k$. Then there exists a unique measure $\mu \in \mathcal{P}(B)$ such that*

$$d_K(\mu_u(k), \mu) \leq \zeta_k \qquad \text{for} \quad k \geq 1, \ u \in B\,. \tag{2.12}$$

PROOF. Let us take any function $f \in \mathcal{O}$. Since the set B is invariant for (1.3), we can use the Chapman–Kolmogorov relation and the assumption of the lemma to get, for $\ell \geq k \geq 0$ and $u, v \in B$, that

$$\begin{aligned}(\mu_v(\ell) - \mu_u(k), f) &= \int_B P(\ell - k, v, dz) \int_B (P(k, z, dw) - P(k, u, dw)) f(w) \\ &\leq \zeta_k \int_B P(\ell - k, v, dz) = \zeta_k\,.\end{aligned}$$

Hence

$$d_K(\mu_v(\ell), \mu_u(k)) \leq \zeta_k. \tag{2.13}$$

Since the space $(\mathcal{P}, d_K)$ is complete, there exists a unique measure $\mu \in \mathcal{P}$ such that $d_K(\mu_u(k), \mu) \to 0$ as $k \to \infty$, for every $u \in B$. Passing to the limit in (2.13) as $\ell \to \infty$ we recover (2.12). It is clear that $\operatorname{supp} \mu \subset B$. Thus $\mu \in \mathcal{P}(B)$ and the lemma is proved. □

3. A Kantorovich-type functional

First we shall construct a special bounded measurable function f_K on $B \times B$, vanishing on the diagonal. To define the function, we consider a partition of $B \times B$ into sets Q_ℓ, $-L \leq \ell \leq \infty$. Here Q_∞ is the diagonal of $B \times B$,

$$Q_r = \{(u_1, u_2) \in B \times B \mid d_{r+1} < \|u_1 - u_2\| \leq d_r\}$$

if $0 \leq r < \infty$, and

$$Q_r = \left\{(u_1, u_2) \in B \times B \mid \|u_1 - u_2\| > d_0, \tfrac{1}{2}\gamma_1\, d_r < \|u_1\| \vee \|u_2\| \leq \tfrac{1}{2}\, d_r\right\}$$

if $-L \leq r < 0$.

Now we define the function f_K:

$$f_K(u_1, u_2) = \begin{cases} d_r & \text{if } (u_1, u_2) \in Q_r, \quad 0 \leq r \leq \infty\,, \\ \widetilde{d}_\ell & \text{if } (u_1, u_2) \in Q_\ell, \quad \ell < 0\,, \end{cases}$$

where $d_\infty = 0$ and the numbers $\{\widetilde{d}_\ell\}$ such that

$$d_0 \leq \widetilde{d}_{-1} \leq \cdots \leq \widetilde{d}_{-L} \tag{3.1}$$

are constructed below. Clearly,

$$\tilde{d}_{-L} \geq f_K(u_1, u_2) \geq d\,(u_1, u_2) \tag{3.2}$$

for all u_1, u_2.

For any pair of measures $\mu_1, \mu_2 \in \mathcal{P}(B)$ we define a Kantorovich-type functional $\mathcal{K}(\mu_1, \mu_2)$ as follows:

$$\mathcal{K}(\mu_1, \mu_2) = \inf \{\mathsf{E}\, f_K(U_1, U_2)\}\,, \tag{3.3}$$

where the infimum is taken over all couplings (U_1, U_2) for (μ_1, μ_2).

Everywhere below (and also in Theorem 1.2), $N = N(R')$ is the constant of Lemma 2.2.

THEOREM 3.1. *Let us assume that the assumption (1.7) holds. Then there exists $\kappa < 1$ such that*

$$\mathcal{K}(\mathfrak{S}_1^*(\mu_1), \mathfrak{S}_1^*(\mu_2)) \leq \kappa\, \mathcal{K}(\mu_1, \mu_2) \tag{3.4}$$

for all $\mu_1, \mu_2 \in \mathcal{P}(B)$ (provided that the numbers $\widetilde{d}_{-1}, \ldots, \widetilde{d}_{-L}$ are chosen accordingly).

The theorem is proved in the next section. Now we continue to study the RDS (1.4), taking the theorem for granted.

Let (U_1, U_2) be a coupling for (μ_1, μ_2). Using (3.2), for any $g \in \mathcal{O}$ we get

$$(\mu_1 - \mu_2, g) = \mathsf{E}\,(g\,(U_1) - g\,(U_2)) \leq \mathsf{E}\; d\,(U_1, U_2) \leq \mathsf{E}\, f_K(U_1, U_2)\,.$$

Taking supremum in $g \in \mathcal{O}$ and using (2.11), next taking infimum in (U_1, U_2) and using (3.3) we find that[5]

$$d_K(\mu_1, \mu_2) \leq \mathcal{K}(\mu_1, \mu_2)\,. \tag{3.5}$$

Let us take any $u_1, u_2 \in B$. Then $\mu_{u_1}(k), \mu_{u_2}(k) \in \mathcal{P}(B)$ for all $k \geq 0$. Iterating (3.4) and using (3.5) together with the first inequality in (3.2), we obtain

$$\begin{aligned} d_K(\mu_{u_1}(k), \mu_{u_2}(k)) &\leq \mathcal{K}(\mu_{u_1}(k), \mu_{u_2}(k)) \\ &\leq \kappa^k \mathcal{K}(\mu_{u_1}(0), \mu_{u_2}(0)) \\ &= \kappa^k f_K(u_1, u_2) \leq \kappa^k\, \widetilde{d}_{-L}\,. \end{aligned} \tag{3.6}$$

Applying Lemma 2.3 we get that there exists a unique measure $\mu \in \mathcal{P}(B)$ such that

$$d_K(\mu_u(k), \mu) \leq \kappa^k\, \widetilde{d}_{-L} \qquad \forall\, k \geq 0,\ u \in B.$$

Let us take a measure $\nu \in \mathcal{P}(B)$. For a function $f \in \mathcal{O}$ we have

$$(\mathfrak{S}_k^*(\nu) - \mu, f) = \int (\mu_u(k) - \mu, f)\, d\nu(u) \leq \kappa^k\, \widetilde{d}_{-L}\,.$$

Hence,

$$d_K(\mathfrak{S}_k^*(\nu), \mu) \leq \kappa^k\, \widetilde{d}_{-L} \qquad \forall\, k \geq 0\,, \quad \nu \in \mathcal{P}(B)\,. \tag{3.7}$$

Now let us take any $u \in H$. Due to (2.2) there exists $\ell = \ell\,(\|u\|)$ such that $\mu_u(\ell) \in \mathcal{P}(B)$. Since $\mu_u(k+\ell) = \mathfrak{S}_k^*\, \mu_u(\ell)$, we denote $k + \ell = t$ and get from (3.7) that

$$d_K(\mu_u(t), \mu) \leq \kappa^{t-\ell}\, \widetilde{d}_{-L}\,, \tag{3.8}$$

for any $u \in H$, where $\ell = \ell\,(\|u\|)$. Due to (2.10) and (2.11) with $g = \frac{d_0}{2} f$, (3.8) implies (1.8) with $C = \frac{2}{d_0} \widetilde{d}_{-L}\, \kappa^{-\ell}$.

The estimate (1.8) easily implies that μ is the unique stationary measure. Indeed, if $\widetilde{\mu}$ is another one, then for any function f as in (1.8) we have

$$\begin{aligned} |(\widetilde{\mu}, f) - (\mu, f)| &= \left| \int (\mu_u(k), f)\, \widetilde{\mu}(du) - \int (\mu, f)\, \widetilde{\mu}(du) \right| \\ &\leq \int |(\mu_u(k) - \mu, f)|\, \widetilde{\mu}(du)\,. \end{aligned}$$

[5]A celebrated theorem of Kantorovich says that the inequality (3.5) transforms to the equality if in (3.3) we replace $f(U_1, U_2)$ by $d(U_1, U_2)$. See [**Dud89, KA77**].

The integrand is bounded by 2 and goes to zero as $k \to \infty$ due to (1.8). Thus, the integral goes to zero as $k \to \infty$ as well and $(\widetilde{\mu}, f) = (\mu, f)$ for all functions as above. Hence, $\mu = \widetilde{\mu}$.

Theorem 1.2 is proved.

4. Proof of Theorem 3.1

Let us take any $A' > \mathcal{K}(\mu_1, \mu_2)$. Then there exists a coupling (U_1', U_2') for (μ_1, μ_2) such that $\mathsf{E}\, f_K(U_1', U_2') \le A'$. The random variables U_1', U_2' are defined on some probability space $(\Omega', \mathcal{F}', \mathsf{P}')$. Since the supports of μ_1, μ_2 belong to B, we may assume that $U_1', U_2' \in B$ for all ω'.

Applying Lemma 2.2 with $R = R'$, we find measurable maps $V_1, V_2 : B^2 \times \Omega \to H$ which satisfy (2.3) and

$$\mathcal{D}\left(V_j(u_1, u_2, \cdot)\right) = \mu_{u_j}(1) = P\left(1, u_j, \cdot\right) \tag{4.1}$$

for $j = 1, 2$. Consider the following random variables U_1, U_2, defined on the probability space $\Omega \times \Omega'$:

$$U_j(\omega, \omega') = V_j(U_1'(\omega'), U_2'(\omega'); \omega)\,, \quad j = 1, 2\,.$$

We take any $f \in C_b$. Using (4.1) and the fact that $\mathcal{D}(U_1') = \mu_1$, we get

$$\begin{aligned} \mathsf{E}^{\omega,\omega'} f(U_1) &= \mathsf{E}^{\omega'}[\mathsf{E}^{\omega} f(V_1(U_1'(\omega'), U_2'(\omega'); \omega)] \\ &= \mathsf{E}^{\omega'} \int P(1, U_1'(\omega'), du)\, f(u) \\ &= \int \mu_1(dv) \int P(1, v, du)\, f(u) \\ &= (\mathfrak{S}_1^*(\mu_1), f)\,. \end{aligned}$$

Therefore, $\mathcal{D}(U_1) = \mathfrak{S}_1^*(\mu_1)$. Similarly, $\mathcal{D}(U_2) = \mathfrak{S}_1^*(\mu_2)$, so (U_1, U_2) is a coupling for $(\mathfrak{S}_1^*(\mu_1), \mathfrak{S}_1^*(\mu_2))$.

If we can prove that

$$\mathsf{E}^{\omega} f_K(V_1(u_1, u_2; \omega), V_2(u_1, u_2; \omega)) \le \kappa\, f_K(u_1, u_2) \tag{4.2}$$

for all $u_1, u_2 \in B$, then

$$\begin{aligned} \mathsf{E}\, f_K(U_1, U_2) &= \mathsf{E}^{\omega'}[\mathsf{E}^{\omega} f_K(V_1(U_1', U_2'; \omega), V_2(U_1', U_2'; \omega))] \\ &\le \kappa\, \mathsf{E}^{\omega'} f_K(U_1', U_2') \le \kappa\, A'. \end{aligned}$$

Thus, $\mathcal{K}(\mathfrak{S}_1^*(\mu_1), \mathfrak{S}_1^*(\mu_2)) \le \kappa\, A'$ and (3.4) would follow since A' is an arbitrary number bigger than $\mathcal{K}(\mu_1, \mu_2)$. It remains to check (4.2).

Let us find $k \in [-L, \infty]$ such that $(u_1, u_2) \in Q_k$. If $k = \infty$, then $u_1 = u_2$, so $V_1 = V_2$ and (4.2) holds trivially. Now let $0 \le k < \infty$. Then, due to (2.3),

$$\mathsf{P}\left\{(V_1, V_2) \in \bigcup_{r \ge k+1} Q_r\right\} \ge 1 - C_*\, d_k\,.$$

Since $f_K \le d_{k+1}$ for $(V_1, V_2) \in \cup_{r \ge k+1} Q_r$ and $f_K \le \sup f_K = \widetilde{d}_{-L}$ for all (V_1, V_2), it follows that

$$\mathsf{E}\, f_K(V_1, V_2) \le d_{k+1}(1 - C_*\, d_k) + \widetilde{d}_{-L}\, C_*\, d_k\,.$$

As $f_K(u_1, u_2) = d_k$, then in this case

$$\frac{\mathsf{E}\, f_K(V_1, V_2)}{f_K(u_1, u_2)} \le \gamma_1(1 - C_*\, d_k) + C_*\, \widetilde{d}_{-L}\,.$$

Therefore, (4.2) holds with some k-independent $\kappa < 1$ if

$$C_* \widetilde{d}_{-L} < 1 - \gamma_1 \,. \tag{4.3}$$

If $-L \le k \le -1$, then $\|u_1\|, \|u_2\| \le \frac{1}{2} d_k$ and $\|S(u_j)\| \le \gamma_0 \frac{1}{2} d_k$ for $j = 1, 2$. As $d_k > d_0, \gamma_0 < \gamma_1$ and the random variable η is smaller, with a positive probability, than any fixed positive constant (see (1.3)), we have

$$\mathsf{P}\,\{\|V_1\|, \|V_2\| \le \frac{1}{2} d_{k+1}\} \ge \theta > 0 \,. \tag{4.4}$$

If $k \le -2$, then this means that

$$\mathsf{P}\Big\{(V_1, V_2) \in \bigcup_{r \ge k+1} Q_r\Big\} \ge \theta \,.$$

Since $f \le \widetilde{d}_{-L}$, we then have

$$\mathsf{E}\, f_K(V_1, V_2) \le \theta\, \widetilde{d}_{k+1} + (1-\theta)\, \widetilde{d}_{-L} \,. \tag{4.5}$$

As $f_K(u_1, u_2) = \widetilde{d}_k$, then (4.2) holds for $-L \le k \le -2$ if

$$\theta\, \widetilde{d}_{k+1} + (1-\theta)\, \widetilde{d}_{-L} = \kappa\, \widetilde{d}_k \,. \tag{4.6}$$

If $k = -1$, then for any ω from the event on the left-hand side of (4.4) we have $\|V_1\|, \|V_2\| \le \frac{1}{2} d_0$. Therefore $\|V_1 - V_2\| \le d_0$ and $(V_1, V_2) \in \bigcup_{r \ge 0} Q_r$. Thus, the relation (4.5) still holds for $k = -1$ if we denote

$$\widetilde{d}_0 = d_0 \,.$$

With this choice of $\widetilde{d}_0$, (4.2) holds for all negative k if so does (4.6).

The relations (4.6) are equivalent to

$$\widetilde{d}_{-L+1} = \frac{\kappa + \theta - 1}{\theta}\, \widetilde{d}_{-L}$$

and

$$\widetilde{d}_{-L+r} = \frac{1}{\theta}\, (\kappa\, \widetilde{d}_{-L+r-1} - (1-\theta)\, \widetilde{d}_{-L})$$

for $r \ge 2$. That is,

$$\widetilde{d}_{-L+r} = \frac{\widetilde{d}_{-L}}{\theta} \left[\left(\frac{\kappa}{\theta}\right)^{r-1} \left(\kappa + \theta - 1 - \frac{\theta(1-\theta)}{\kappa - \theta}\right) + \frac{\theta(1-\theta)}{\kappa - \theta} \right]$$

for $1 \le r \le L - 1$.

Let us assume that $\kappa = 1 - \varepsilon$, where $0 < \varepsilon \ll 1$. Then

$$\widetilde{d}_{-L+r} = \frac{\widetilde{d}_{-L}}{\theta} \left[\left(-\left(\frac{1}{\theta}\right)^{r-1} \frac{\varepsilon}{1-\theta} + O\,(\varepsilon^2) \right) + \frac{\theta(1-\theta)}{1-\theta-\varepsilon} \right] , \tag{4.7}$$

where $O\,(\varepsilon^2)$ depends on $r \le L$. Choosing $\varepsilon = \varepsilon_L$ sufficiently small, we see that the numbers $\widetilde{d}_{-L+r}$ $(0 \le r \le L)$ decay when r grow from 0 to L; so they satisfy all relations in (3.1) (if $\widetilde{d}_0 = d_0$).

We have seen that a function f_K, constructed using the numbers $\{\widetilde{d}_\ell\}$ as above, satisfies (4.2) and (3.1) if it satisfies (4.3) and if $\widetilde{d}_0 = d_0$. Due to (4.7), $\widetilde{d}_{-L} = \widetilde{d}_0\,(1 + O\,(\varepsilon))$. Taking $\widetilde{d}_0 = d_0$, we have $\widetilde{d}_{-L} = d_0\,(1 + O\,(\varepsilon))$. Due to (2.9), $d_0 \le (1-\gamma_1)/2\, C_*$. Thus, (4.3) is satisfied if ε is sufficiently small.

We have constructed constants $\widetilde{d}_k$ such that the corresponding function f_K satisfies (3.4) with some $\kappa = 1 - \varepsilon < 1$. The theorem is proved.

5. A kick-forced CGL equation

Let us consider the following *complex Ginzburg–Landau* (CGL) equation:

$$-i\dot{u} + \nu\Delta u - |u|^2 u = \eta(t,x) \equiv \sum_k \eta_k(x)\delta(t-k), \quad 0 < \nu \le 1, \tag{5.1}$$

which we shall study under the odd 2-periodic boundary conditions:

$$u(t,x) = u(t,x_1,\dots,x_j+2,\dots,x_n) = -u(t,x_1,\dots,-x_j,\dots,x_n) \tag{5.2}$$

for each j. For simplicity we restrict ourselves to the case

$$n \le 3.$$

Clearly, any function satisfying (5.2) vanished at the boundary of the cube K^n of half-periods,

$$K^n = \{0 \le x_j \le 1 \quad \forall\, j\}.$$

Since the equation (5.1) has no nonlinear dissipation, it is sometimes called a *nonlinear Schrödinger equation.*

We denote by $\{e_j\}$ the L^2-normalized trigonometric basis (over reals) of the space of odd 2–periodic complex functions. It is assumed that the force η has the form (1.1), (1.2), where the densities p_j satisfy the assumptions of Section 1, and

$$0 < b_j \le C'_m j^{-m} \quad \forall\, j, m, \tag{5.3}$$

with some positive constants C'_m. These relations imply that

$$|\eta_k(x)| \le C_* \quad \text{for all} \quad k, x, \omega,$$

and

$$\|\eta_k\|_m \le K_m \quad \text{for all} \quad k, \omega, \tag{5.4}$$

for each $m \in \mathbb{N}$. Here C_* and K_m are finite constants and

$$\|u(x)\|_m^2 = \int_{K^n} (-\Delta)^m u\,\bar{u}\,dx.$$

Let us normalize the solutions for (5.1), (5.2) to be continuous from the right. Then for any $m \ge 2$ this equation defines a RDS (1.4) in the space $H = H^m$, where H^m stands for the space of odd 2-periodic complex Sobolev functions with L^2-integrable derivatives up to order m. This RDS satisfies the conditions A–C of [**KS00**], mentioned in Section 1. Accordingly, the RDS has a unique stationary measure $\mu = \mu_\nu$. This measure is supported by smooth functions (cf. (1.9)), so it is m-independent by the uniqueness. Moreover, if $u(t,x)$ is a solution for (5.1) such that $u(0,x) \in H^m$ is a nonrandom function, then

$$\mathsf{E} f(u(j,\cdot)) \to \int_{H^m} f(u)\mu_\delta(du) \tag{5.5}$$

exponentially fast, where f is a bounded Lipschitz function on H^m.

Writing a solution u for (5.1), (5.2) in the polar coordinate form $u = re^{i\phi}$, we see that outside the zero-set $\{u = 0\}$, r satisfies the equation

$$\dot{r} = \nu\Delta r - \nu r|\nabla\phi|^2 + \mathrm{Re}\,(\eta e^{-i\phi}).$$

Therefore,

$$\dot{r} \le \nu\Delta r + C_* \sum_k \delta(t-k). \tag{5.6}$$

Our formal derivation of the differential inequality (5.6) can be easily justified by approximating the measure $\sum_k \delta(t-k)$ in the formula for η by regular functions $f_\epsilon(t)$ supported by the union of small intervals centered at integers (then the approximating function $\eta_\epsilon(t,x)$ equals $f_\epsilon(t)\eta_k(x)$ if t is close to some integer k, and vanishes if t is distant from the set of integers).

Let us consider a solution u for the equation (5.1), (5.2) supplemented by zero initial conditions

$$u(0,x)=0. \tag{5.7}$$

Applying the Maximum Principle [**Lan97**] to (5.6) we get that

$$|u(j+t)|_\infty \le 2^{n/2}|u(j)|_\infty e^{-\nu tc}, \quad 0\le t<1,$$

and

$$|u(j)|_\infty \le C\sum_{l=1}^{j} e^{-\nu lc} < \frac{C}{1-e^{-\nu c}} \le C_1\nu^{-1},$$

for $j=0,1,\dots$. Here $C=2^{n/2}C_*$, $c=n\pi^2/4$ and $|u(t)|_\infty = |u(t,\cdot)|_{L^\infty}$; see [**Kuk99**] for details. In particular,

$$|u(t)|_\infty \le C_2\delta^{-1} \quad \forall\, t\ge 0.$$

For $j=0,1,\dots$, the solution u satisfies equation (5.1) with $\eta=0$ in the interval $[j,j+1)$. Therefore, for $t\in[j,j+1)$ and any $m\in\mathsf{N}$ we have

$$\frac{d}{dt}\|u\|_m^2 \le -\nu\|u\|_{m+1}^2 + \sum_{|\alpha|=m}\ \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\int_{K^n}|u_1|\cdots|u_4|\,dx,$$

where $u_l=\partial^{\alpha_j}u/\partial x^{\alpha_j}$ for $j\le 3$ and $u_4=\partial^\alpha u/\partial x^\alpha$. Applying the Gagliardo–Nirenberg inequality to estimate the integral on the right-hand side, we get that

$$\frac{d}{dt}\|u\|_m^2 \le -\nu\|u\|_{m+1}^2 + C|u|_\infty^{4-R}\|u\|_{m+1}^R \le -\nu\|u\|_{m+1}^2 + C_1\nu^{R-4}\|u\|_{m+1}^R,$$

where $R=\frac{2m}{m+1}$ (see [**Kuk99**]). Estimating the second term on the right-hand side via the Young inequality we have

$$\frac{d}{dt}\|u\|_m^2 \le -\frac{\nu}{2}\|u\|_{m+1}^2 + C_2\nu^{-3m-4} \le -\frac{\nu}{2}\|u\|_m^2 + C_m\nu^{-3m-4},$$

since $\|u\|_m\le\|u\|_{m+1}$ for any odd periodic function u. Therefore,

$$\|u(j+1-0)\|_m^2 \le e^{-\nu/2}\|u(j)\|_m^2 + C_m'\nu^{-3m-5}.$$

Using (5.4) we arrive at the inequality

$$\|u(j+1)\|_m \le e^{-\nu/4}\|u(j)\|_m + C_m\nu^{-(3m+5)/2} + K_m\,,$$

valid for any $j=0,1,\dots$ and any ω. This relation implies upper bounds for the Sobolev norms of the solution:

$$\|u(t)\|_m \le C_m\nu^{-(3m+7)/2} \quad \forall\, m\ \forall\,\omega, \tag{5.8}$$

for any $t\ge 0$.

On the other hand, the solution for (5.1), (5.2), (5.7) satisfies the following lower bound:

$$L^{-1}\sum_{l=0}^{L}\mathsf{E}\|u(l)\|_m^2 \ge c_m\nu^{-Am}, \tag{5.9}$$

where $L \geq \nu^{-1}$, $m \geq 6$ and $A > 0$ is a constant. This estimate is proven in [**Kuk97**] (see also [**Kuk99**], pp. 171–173), for equation (5.1), (5.2) perturbed by a random force η which is smooth in x, bounded for t in finite intervals, and satisfies some mixing condition. The arguments of these two works apply to equations with random kick-force as above (and even simplify in this case).

Applying (5.5) to the solution u of (5.1), (5.2), (5.7) and using (5.8) we see that in (5.5) the integrating over H^m can be replaced by integrating over the ball $\{\|u\|_m \leq C_m \nu^{-(3m+7)/2}\}$. Accordingly, (5.5) holds for functions f which are bounded and Lipschitz on bounded subsets of H^m, in particular, for $f(u) = \|u\|_m^2$. Using (5.8) and (5.9) with sufficiently large $L = L_\nu$, we arrive at the following result:

THEOREM 5.1. *There exists $A > 0$ such that for any $\nu \in (0, 1]$ and $m \geq 6$ the stationary measure μ_ν satisfies the following estimates:*

$$c_m \nu^{-Am} \leq \int \|u\|_m^2 \, \mu_\nu(du) \leq C_m \nu^{-3m+7}. \tag{5.10}$$

The estimates (5.10) can be reformulated in spectral terms. To do this we write any function $u \in H^m$ as the Fourier series $u = \sum \hat{u}_s e^{i\pi s \cdot x}$. We view the Fourier coefficients $\hat{u}_s$ as functionals on H^m and set

$$E_s = \int |\hat{u}_s|^2 \mu_\nu(du).$$

(In hydrodynamics the quantity E_s is called the *energy of the wave-vector s*.)

Let us take any numbers $a < b$ such that

$$b > \tfrac{3}{2}, \qquad a < \tfrac{1}{2}A,$$

and denote

$$\mathfrak{A} = \{s \in \mathbb{Z}^n \mid \nu^{-a} \leq |s| \leq \nu^{-b}\}.$$

Applying to (5.10) the general results of [**Kuk99**] (see Theorem A2.2 there), we get:

THEOREM 5.2. *For a and b as above and any $M \geq 1$ there exist positive numbers $\nu_0 = \nu_0(a, b, M)$ and $\nu_1 = \nu_1(a, b)$ such that*

$$\sum_{|s| \geq \nu^{-b}} E_s \leq \nu^{Mb}, \quad \forall\, \nu < \nu_0,$$

and

$$\nu^C \leq |\mathfrak{A}|^{-1} \sum_{s \in \mathfrak{A}} E_s \leq \nu^c, \quad \forall\, \nu < \nu_1,$$

where $c \leq C$ are some finite real constants.

This result can be treated as a weak form of the Kolmogorov–Obukhow law from the theory of turbulence [**LL87**]; see more in [**Kuk99**].

The approach of [**Kuk97, Kuk99**] to get results similar to statements of Theorems 5.1, 5.2, does not apply to the NS equations, but applies to many other equations with a small coefficient ν of the Laplacian. E.g., see [**Bir01**], where it is proved that any odd periodic solution of a generalized 1D Burgers equation with zero force, satisfies the estimates

$$c_m \nu^{-2m+1} \leq \frac{1}{T} \int_0^T \|u(t)\|_m^2 \, dt \leq C_m \nu^{-2m+1},$$

where $m \geq 1$ and $T = \text{const}/|u_0|_{L^2}$.

The 1D Burgers equation perturbed by a white in time random force was considered in [**EKMS00**]:

$$\dot{u} - \nu u_{xx} + u u_x = \eta(t, x).$$

It is known that the RDS corresponding to this equation converges as $\nu \to 0$ to the RDS corresponding to viscosity solutions of the Hopf equation

$$\dot{u} + u u_x = \eta(t, x)\,.$$

It is proven in [**EKMS00**] that the latter has a unique stationary measure, and a half-explicit description of this measure is given. In [**IK01**] similar results are obtained for the multidimensional Burgers equation. Unfortunately, techniques of [**EKMS00, IK01**] do not apply to the NS and CGL equations since for them limiting dynamics as $\nu \to 0$ is unknown.

References

[Bir01] A. Biryuk, *Spectral properties of solutions of the Burgers equation with small dissipation*, Funct. Anal. Appl. **35** (2001), 1–12.

[BKL00] J. Bricmont, A. Kupiainen, and R. Lefevere, *Exponential mixing for the 2d stochastic Navier–Stokes dynamics*, Preprint (2000).

[Dud89] R. M. Dudley, *Real analysis and probability*, Wadsworth&Brooks/Cole, Pacific Grove, California, 1989.

[EH01] J.-P. Eckmann and M. Hairer, *Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise*, Preprint (2001).

[EKMS00] W. E, K. Khanin, A. Mazel, and Ya. G. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Annals of Math. **151** (2000), 877–960.

[EMS01] W. E, J. C. Mattingly, and Ya. G. Sinai, *Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation*, Commun. Math. Phys. **224** (2001), 83–106.

[Gal01] G. Gallavotti, *Fluid mechanics*, Springer-Verlag, Berlin, 2001.

[IK01] R. Iturriaga and K. Khanin, *Burgers turbulence and random Lagrangian systems*, Preprint (2001).

[KA77] L. B. Kantorovich and G. P. Akilov, *Functional analysis*, Nauka, Moscow, 1977. (Russian)

[KPS01] S. B. Kuksin, A. Piatnitskii, and A. Shirikyan, *A coupling approach to randomly forced nonlinear PDEs. 2*, Preprint (2001).

[KS00] S. B. Kuksin and A. Shirikyan, *Stochastic dissipative PDEs and Gibbs measure*, Commun. Math. Phys. **213** (2000), 291–330.

[KS01a] ———, *A coupling approach to randomly forced nonlinear PDEs. 1*, Commun. Math. Phys. **221** (2001), 351–366.

[KS01b] ———, *Ergodicity for the randomly forced 2D Navier–Stokes equations*, Mathematical Physics, Analysis and Geometry **4** (2001), 147–195.

[KS02] ———, *On dissipative systems perturbed by bounded random kick-forces*, To appear in "Ergodic Theory and Dynamical Systems" (2002).

[Kuk97] S. B. Kuksin, *On turbulence in nonlinear Shrödinger equation*, Geometric and Functional Analysis **7** (1997), 338–363.

[Kuk99] ———, *Spectral properties of solutions for nonlinear PDEs in the turbulent regime*, Geometric and Functional Analysis **9** (1999), 141–184.

[Kuk01] ———, *On exponential convergence to a stationary measure for a class of random dynamical systems*, Proceedings of Journées Équations aux Dérivées Partielles, Plestin-les-Grèves, Gropement de Recherche 1151 du CNRS, Juin 2001.

[Lan97] E. M. Landis, *Second order equations of elliptic and parabolic type*, Amer. Math. Soc., Providence, 1997.

[Lin92] T. Lindvall, *Lectures on the coupling methods*, Wiley, New York, 1992.

[LL87] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, second ed., Pergamon Press, Oxford, 1987.

[MY01] N. Masmoudi and L.-S. Young, *Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDEs*, Preprint (2001).

[VF88] M. I. Vishik and A. V. Fursikov, *Mathematical problems in statistical hydromechanics*, Kluwer, Dordrecht, 1988.

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland, UK; Steklov Institute of Mathematics, 8 Gubkina St., 117966 Moscow, Russia

URL: www.ma.hw.ac.uk/∼kuksin

E-mail address: kuksin@ma.hw.ac.uk