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Index and secondary index theory for flat bundles with duality

Ulrich Bunke and Xiaonan Ma

ABSTRACT. We discuss some aspects of index and secondary index theory for flat bundles with duality. This theory was first developed by Lott. Our main purpose in the present paper is provide a modification with better functorial properties.

1. Introduction to the paper

This article has its origin in the work of J. Lott [**21**], where he develops two versions of secondary index theory for flat vector bundles.

The basic relation in the first version is the transition from a complex of flat real vector bundles to its cohomology. The primary characteristic classes in this case are the Bott-Chern classes. The secondary analytic information is given by the analytic torsion forms. The main building block of the theory is the construction of a push-forward operation (primary and secondary index map) for fibre bundles with closed fibres. In a sense it is given by considering the fibre-wise de Rham complex twisted by a flat bundle as an infinite-dimensional object of the theory. Then one applies the equivalence relation (take the cohomology) to reduce to finite-dimensional vector bundles again. As shown in [**14**] (based on the analytic results of [**24**]) this first version of secondary index theory has the expected functorial properties with respect to iterated fibre bundles.

Lott's second version of secondary index theory involves flat real vector bundles with parallel non-degenerate quadratic or symplectic forms (flat duality bundles). The basic relation in Lott's approach was that hyperbolic forms were considered to be trivial. The primary characteristic classes in this case are Chern classes and the secondary information is given by eta-forms. Again, the main building block of the theory was the construction of a push-forward for fibre bundles with closed oriented even-dimensional fibres. The infinite-dimensional object in this case is

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the signature operator twisted by the flat vector bundle. By applying the relation one again reduces to cohomology and therefore to the finite-dimensional world. It turns out that Lott's definition is not functorial with respect to iterated fibre bundles already on the primary level. In order to repair this defect one must enlarge the equivalence relation. We replace triviality of hyperbolic bundles by lagrangian reduction. Note that a hyperbolic bundle admits an lagrangian sub-bundle which has a complementary lagrangian sub-bundle. It is the existence of a complementary lagrangian sub-bundle that we must give up.

The set of flat duality bundles together with the secondary information and taken modulo equivalence is now organized in primary L- and secondary $\bar{L}$-groups. These are quotients of the corresponding groups introduced by Lott. For maps out of these groups we are going to use the same formulas involving generators as in Lott's work. One of the achievement of the present paper is the verification that these constructions are still well-defined, i.e., they factor over the enlarged equivalence relation. The other main result is that our version now also enjoys functoriality with respect to iterated fibres bundles.

In Section 2 we define the L-functor on spaces. It only depends on the fundamental group of the space. If the representation theory of the fundamental group is sufficiently well-known then we can explicitely compute the L-group. Then we study certain relations between flat duality bundles which hold in L-theory. This information is needed later in the proof of functoriality with respect to iterated fibre bundles. We also discuss a natural transformation from L- to K-theory.

In Section 3 we introduce the secondary counterpart $\bar{L}$. We relate it with secondary K-theory ($K^{-1}_{\mathbb{R}/\mathbb{Z}}$ in Lott's notation). Then we show how the relations which were already investigated in the primary case now extend to the secondary situation. In fact, the knowledge of many relations in $\bar{L}$ helps in arguments showing well-definedness and functoriality for maps with values in $\bar{L}$.

In Section 4 we study an η-homomorphism from $\bar{L}$ to $\mathbb{R}/\mathbb{Z}$ ("η" since its analytic definition involves η-invariants) and its lift to $\mathbb{R}$. The homomorphism to $\mathbb{R}/\mathbb{Z}$ comes from the natural transformation to $K^{-1}_{\mathbb{R}/\mathbb{Z}}$-theory and the usual pairing with K-homology. In Lott's version it has a lift to $\mathbb{R}$. Unfortunately this lift does not factor over our enlarged equivalence relation. In order to repair this defect we introduce an extended version of L and $\bar{L}$-theory. This extended version now admits a real-valued η-homomorphism. It is possible to define a push-forward for the extended L and $\bar{L}$-groups, but note that these are not contravariant functors of the underlying space. The properties of the extended L-groups deserve further study.

In Section 5 we define the secondary index map. The verification of well-definedness and functoriality is based on the behavior of η-forms under adiabatic limits. So we first state these results without proof and then turn to the details of the secondary index map. The results of this section were the main goal of the present paper.

The last Section 6 is devoted to the proof of the adiabatic limits result needed and stated in Section 5. In contrast to the preceeding sections, where we tried to give complete proofs, in this last section we will only sketch the main steps. It should be clear that keeping the level of depth also across the last section would expand the paper by a factor of three. The arguments in the last sections are in fact very similar to the corresponding proofs for analytic torsion forms. For a specialist it should not be too complicated (but by the experience of the first author it is also not easy) to take the stated theorems, find their counterparts for analytic torsion forms together with the proof in the indicated literature, and then correspondingly modify this proof to show the statements for the eta-forms.

2. The functor L_ϵ

2.1. Introduction and summary. The main object of this section is functor L_ϵ from the category *Top* of topological spaces and continuous maps to the category of $\mathbb{Z}_2$-graded rings and ring homomorphisms. For a space X the elements of the ring $L_\epsilon(X)$ are locally constant sheaves of (anti-)symmetric forms over $\mathbb{R}$ considered up to isotropic reduction. The ring operations are induced by the direct sum and the tensor product.

It turns out that the functor L_ϵ factors over the homotopy category *hTop*. For a path-connected space X the ring $L_\epsilon(X)$ only depends on the fundamental group of X (see Subsection 2.3 for more details).

On nice spaces a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules gives rise to a real vector bundle. The form on the sheaf induces a form on the bundle. This observation leads to a natural transformation from L_ϵ to the complex K-theory functor K^0. In this way we consider L_ϵ as a refinement of K^0.

Given a K-oriented morphism $\pi\colon X \to B$ in *Top*, say a locally trivial fibre bundle with fibre a closed even-dimensional manifold which admits a vertical Spin_c-structure, there is a wrong-way homomorphism of groups $\pi_!^{\mathrm{Spin}_c}\colon K^0(X) \to K^0(B)$. Analytically, it is given by the index of the twisted fibrewise Spin_c-Dirac operator.

If the fibres are merely oriented, then we can use the twisted fibrewise signature operator to define the wrong-way homomorphism $\pi_!^{\mathrm{sign}}\colon K^0(X) \to K^0(B)$. The interesting point about the functor L is now that $\pi_!^{\mathrm{sign}}$ can be lifted to a group homomorphism $\pi_*^L\colon L_\epsilon(X) \to L_{\epsilon'}(B)$. It is essentially given by taking the fibrewise cohomology of the locally constant sheaf on X. This yields a locally constant sheaf on B. Using fibrewise Poincaré duality, we define the (anti-)symmetric form on the cohomology sheaf.

It turns out that this wrong-way map is functorial with respect to iterated fibre bundles and natural with respect to pull-back of fibre bundles. A similar functor $L_\epsilon^{\mathrm{Lott}}$ was previously defined by Lott [**21**] using a smaller equivalence relation so that $L_\epsilon(X)$ is a quotient of $L_\epsilon^{\mathrm{Lott}}(X)$. The corresponding wrong-way maps $\pi_*^{L,\mathrm{Lott}}$ are not functorial with respect to iterated fibre bundles.

In the first three sections we will denote by X a topological space and by M a manifold.

2.2. Definition and first properties.

2.2.1. *Definition of* L_ϵ. We now give details of the definition of the contravariant functor L_ϵ from the category *Top* of topological spaces and continuous maps to $\mathbb{Z}_2$-graded rings.

Let X be a topological space. If R is a ring and E is a R-module, then the constant sheaf of R-modules $\underline{E}_X$ with stalk E is the associated sheaf to the presheaf which is associates to any non-empty open subset $U \subset X$ the space of sections E such that the restriction to subsets is given by the identity. A sheaf $\mathcal{F}$ of R-modules over X is called locally constant, if there is an open covering $\{U_\lambda\}$ of X such that $\mathcal{F}\big|_{U_\lambda}$ is a constant sheaf for all λ.

If R is a field, then we say that $\mathcal{F}$ is a locally constant sheaf of finite-dimensional R-modules if there is a suitable open covering such that $\mathcal{F}\big|_{U_\lambda}$ is the constant sheaf with the stalk being a finite-dimensional vector space over R. If $\mathcal{F}$ is a locally constant sheaf of finite-dimensional R-modules over X, then let $\mathcal{F}^* := \mathrm{Hom}_R(\mathcal{F}, \underline{R}_X)$ be its dual. If $q\colon \mathcal{F} \to \mathcal{E}$ is a homomorphism between two such sheaves, then we have an adjoint $q^*\colon \mathcal{F}^* \to \mathcal{E}^*$.

From now on we consider the case $R := \mathbb{R}$. Let $\epsilon \in \mathbb{Z}_2 = \{-1, 1\}$. An ϵ–*symmetric duality structure* on $\mathcal{F}$ is an isomorphism of sheaves $q\colon \mathcal{F} \xrightarrow{\sim} \mathcal{F}^*$ satisfying $q^* = \epsilon q$.

To define the group $L_\epsilon(X)$ we first consider an abelian semigroup $\hat{L}_\epsilon(X)$ with zero element. Then we construct $L_\epsilon(X)$ by introducing a relation. An element of the semigroup $\hat{L}_\epsilon(X)$ is an isomorphism class of a pair $(\mathcal{F}, q)$ consisting of a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules and an ϵ–symmetric duality structure q. The operation in $\hat{L}_\epsilon(X)$ is given by direct sum of representatives

$$(\mathcal{F}, q) + (\mathcal{F}', q') := (\mathcal{F} \oplus \mathcal{F}', q \oplus q') \ .$$

The relation on $\hat{L}_\epsilon(X)$ is generated by *lagrangian reduction.* If $i\colon \mathcal{L} \hookrightarrow \mathcal{F}$ is an inclusion of a locally constant subsheaf, then we can consider the sheaf $\mathcal{L}^\perp := \ker(i^* \circ q)$. This sheaf is again a locally constant subsheaf of $\mathcal{F}$. The sheaf $\mathcal{L}$ is called *lagrangian* if it is isotropic, i.e., $\mathcal{L} \subset \mathcal{L}^\perp$, and coisotropic, i.e., $\mathcal{L}^\perp \subset \mathcal{L}$.

We say that the element $(\mathcal{F}, q)$ is equivalent to zero by lagrangian reduction, $(\mathcal{F}, q) \sim 0$, if it admits a locally constant lagrangian subsheaf. The equivalence relation on $\hat{L}_\epsilon(X)$ is now the minimal equivalence relation which is compatible with the semigroup structure and contains lagrangian reductions.

DEFINITION 2.1. We define $L_\epsilon(X) := \hat{L}_\epsilon(X)/\sim$.

The class of $(\mathcal{F}, q)$ in $L_\epsilon(X)$ will be denoted by $[\mathcal{F}, q]$.

LEMMA 2.2. *$L_\epsilon(X)$ is a group.*

PROOF. We have $[\mathcal{F},q]+[\mathcal{F},-q]=0$. Indeed, consider the diagonal embedding $\mathcal{F}\to\mathcal{F}\oplus\mathcal{F}$. Its image is a locally constant lagrangian subsheaf of $(\mathcal{F}\oplus\mathcal{F},q\oplus -q)$. □

The $\mathbb{Z}_2$-graded ring structure $L_\epsilon(X)\otimes L_{\epsilon'}(X)\to L_{\epsilon\epsilon'}(X)$ is induced by the tensor product:

$$[\mathcal{F},q][\mathcal{F}',q']:=\frac{\sqrt{\epsilon}\sqrt{\epsilon'}}{\sqrt{\epsilon\epsilon'}}[\mathcal{F}\otimes\mathcal{F}',q\otimes q'].$$

The sign-convention is made such that later we have a natural transformation of rings from L to complex K-theory K^0.

If $f\colon Y\to X$ is a morphism in *Top*, then $f^*\colon L_\epsilon(X)\to L_\epsilon(Y)$ is defined by $f^*[\mathcal{F},q]=[f^*\mathcal{F},f^*q]$. It is easy to check that f^* is well-defined. Furthermore, it follows from the fact that we work with locally constant sheaves that the map f^* only depends on the homotopy class of f. Therefore, L factors over the homotopy category *hTop* in which maps are considered up to homotopy.

REMARK 2.3. A version $L_\epsilon^{\mathrm{Lott}}(X)$ of this ring was first introduced by Lott [**21**]. His definition differs from ours since our relation "lagrangian reduction" is replaced by "hyperbolic reduction" in the definition of Lott. Here a pair $(\mathcal{F},q)$ is called hyperbolic if there is an lagrangian subsheaf $\mathcal{L}\subset\mathcal{F}$ such that this embedding extends to an isomorphism $(\mathcal{L}\oplus\mathcal{L}^*,q_{\mathrm{can}})\cong(\mathcal{F},q)$, where

$$q_{\mathrm{can}}:=\begin{pmatrix}0 & \mathrm{id}_{\mathcal{L}}\\ \epsilon\ \mathrm{id}_{\mathcal{L}} & 0\end{pmatrix}.$$

In particular, $L_\epsilon(X)$ is a quotient of $L_\epsilon^{\mathrm{Lott}}(X)$.

2.2.2. *Some simple properties.* In the definition of the functor L_ϵ we tried to generate the equivalence relation in a certain minimal way. This simplifies the check of the well-definedness of a transformation out of $L_\epsilon(X)$ which is given on representatives. To check the well-definedness of a transformation with values in $L_\epsilon(X)$ it is useful to know some list of further relations which hold in $L_\epsilon(X)$.

For $(\mathcal{F},q)$ as above, we also denote $q(x,y):=q(x)(y)$. If $N\in\mathrm{End}(\mathcal{F})$, then we define its adjoint with respect to q by $N':=q^{-1}\circ N^*\circ q$.

DEFINITION 2.4. A $\mathbb{Z}$-grading of $(\mathcal{F},q)$ of length $n\in\mathbb{Z}$ is a semisimple element $N\in\mathrm{End}(\mathcal{F})$ such that N has integral eigenvalues in $\{0,\dots,n\}$ and $N'=n-N$. We set $\mathcal{F}^k:=\ker(N-k)$. An element $v\in\mathrm{End}(\mathcal{F})$ is called a compatible differential if it is of degree one with respect to the grading, $v^2=0$, and $v'=-v$, i.e., $q(vx,y)+q(x,vy)=0$ for $x,y\in\mathcal{F}$.

If $(\mathcal{F},q)$ has a $\mathbb{Z}$-grading of length n, then it is the sum of the subsheaves $\mathcal{F}^k$. The duality pairs $\mathcal{F}^k$ with $\mathcal{F}^{n-k}$. If it has in addition a compatible differential v, then we can consider the cohomology $\mathcal{H}:=\ker(v)/\mathrm{im}(v)$. It is again a locally constant sheaf of finite-dimensional $\mathbb{R}$-modules with an induced duality structure $q_{\mathcal{H}}$ and $\mathbb{Z}$-grading of length n.

LEMMA 2.5. (1) *If $(\mathcal{F}, q)$ admits a $\mathbb{Z}$-grading of length n, then in $L_\epsilon(X)$ we have*

$$[\mathcal{F}, q] = \begin{cases} 0 & n \text{ odd}, \\ [\mathcal{F}^{n/2}, q|_{\mathcal{F}^{n/2}}] & n \text{ even}. \end{cases}$$

(2) *If $(\mathcal{F}, q)$ admits in addition a compatible differential, then in $L_\epsilon(X)$ we have*

$$[\mathcal{F}, q] = [\mathcal{H}, q_{\mathcal{H}}].$$

PROOF. To show the first assertion note that for $k < n/2$ we can take off summands of the form $(\mathcal{F}^k \oplus \mathcal{F}^{n-k}, q|_{\mathcal{F}^k \oplus \mathcal{F}^{n-k}})$. These summands represent trivial elements of $L_\epsilon(X)$, since they contain the lagrangian subsheaves $\mathcal{F}^k \subset \mathcal{F}^k \oplus \mathcal{F}^{n-k}$.

To show the second assertion first note that $-[\mathcal{H}, q_{\mathcal{H}}] = [\mathcal{H}, -q_{\mathcal{H}}]$. Hence $[\mathcal{F}, q] - [\mathcal{H}, q_{\mathcal{H}}] = [\mathcal{F} \oplus \mathcal{H}, q \oplus (-q_{\mathcal{H}})]$. Let $i\colon \ker(v) \to \mathcal{F} \oplus \mathcal{H}$ be given by $i(x) = x \oplus [x]$. Then $\operatorname{im}(i)$ is a locally constant lagrangian subsheaf of $(\mathcal{F} \oplus \mathcal{H}, q \oplus (-q_{\mathcal{H}}))$. Therefore, $[\mathcal{F} \oplus \mathcal{H}, q \oplus (-q_{\mathcal{H}})] = 0$. □

Let $(F^k\mathcal{F})_{k=0,\dots,n+1}$ be a decreasing filtration of $\mathcal{F}$ with $F^0\mathcal{F} = \mathcal{F}$ and $\mathcal{F}^{n+1}F = 0$. We obtain a dual filtration $(F^l\mathcal{F}^*)_{l=0,\dots n+1}$ by setting $F^l\mathcal{F}^* = \operatorname{Ann}(F^{n+1-l}\mathcal{F}) = \{x \in \mathcal{F}^* \mid x(y) = 0 \text{ for any } y \in F^{n+1-l}\mathcal{F}\}$. Let $\operatorname{Gr}^k(\mathcal{F}) = F^k\mathcal{F}/F^{k+1}\mathcal{F}$. We have a natural isomorphism $\operatorname{Gr}(\mathcal{F}^*) \cong \operatorname{Gr}(\mathcal{F})^*$ which identifies $\operatorname{Gr}^k(\mathcal{F})^*$ with $\operatorname{Gr}^{n-k}(\mathcal{F}^*)$.

DEFINITION 2.6. A compatible decreasing filtration of $(\mathcal{F}, q)$ of length n is a decreasing filtration $(F^k\mathcal{F})_{k=0,\dots,n+1}$ by locally constant subsheaves such that $q\colon \mathcal{F} \to \mathcal{F}^*$ preserves the filtrations.

Given a compatible filtration of $(\mathcal{F}, q)$ we obtain an induced ϵ-symmetric duality structure $\operatorname{Gr}(q)\colon \operatorname{Gr}(\mathcal{F}) \to \operatorname{Gr}(\mathcal{F}^*) \xrightarrow{\cong} \operatorname{Gr}(\mathcal{F})^*$.

LEMMA 2.7. *In $L_\epsilon(X)$ we have $[\mathcal{F}, q] \sim [\operatorname{Gr}(\mathcal{F}), \operatorname{Gr}(q)]$.*

PROOF. Note that $[\operatorname{Gr}(\mathcal{F}), \operatorname{Gr}(q)]$ has a $\mathbb{Z}$-grading of length n. In view of Lemma 2.5 (1), it suffices to show that

$$[\mathcal{F}, q] = \begin{cases} 0 & n \text{ odd}, \\ [\operatorname{Gr}^{n/2}(\mathcal{F}), \operatorname{Gr}(q)|_{\operatorname{Gr}^{n/2}(\mathcal{F})}] & n \text{ even}. \end{cases}$$

By the following procedure we can decrease the length of the filtration by two. Note that

$$[\mathcal{F}, q] = [\mathcal{F} \oplus (\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})), q \oplus q|_{\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})}].$$

We introduce the $\mathbb{Z}$-grading of length 2 on $(\mathcal{F} \oplus (\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})), q \oplus q|_{\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})})$ such that $\operatorname{Gr}^n(\mathcal{F})$ sits in degree zero, $\mathcal{F}$ is in degree one, and $\operatorname{Gr}^0(\mathcal{F})$ is in degree two. There is a compatible differential v given by the inclusion $\operatorname{Gr}^n(\mathcal{F}) \to \mathcal{F}$ and the negative of the projection $\mathcal{F} \to \operatorname{Gr}^0(\mathcal{F})$. Using Lemma 2.5 (2), we have

$$[\mathcal{F} \oplus (\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})), q \oplus q|_{\operatorname{Gr}^0(\mathcal{F}) \oplus \operatorname{Gr}^n(\mathcal{F})}] = [\mathcal{F}', q'],$$

where $\mathcal{F}' = F^1\mathcal{F}/F^n\mathcal{F}$ and q' is the induced ϵ–symmetric duality structure. Note that $(\mathcal{F}', q')$ has an induced decreasing filtration of length $n-2$.

Now we iterate this procedure. If n is odd, then it terminates at $0 \in L_\epsilon(X)$, and if n is even, then we finally obtain $(\mathrm{Gr}^{n/2}(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^{n/2}(\mathcal{F})})$. □

Let $\mathcal{I} \subset \mathcal{F}$ be an isotropic subsheaf. Then we can consider the locally constant sheaf $\mathcal{F}_\mathcal{I} := \mathcal{I}^\perp/\mathcal{I}$. Furthermore, we let $q_\mathcal{I} \colon \mathcal{F}_\mathcal{I} \to \mathcal{F}_\mathcal{I}^*$ be given by $\mathcal{I}^\perp \xrightarrow{q|_{\mathcal{I}^\perp}} \mathcal{F}^* \to (\mathcal{I}^\perp)^*$. Then $q_\mathcal{I}$ is an ϵ–symmetric duality structure on $\mathcal{F}_\mathcal{I}$.

DEFINITION 2.8. We call $(\mathcal{F}_\mathcal{I}, q_\mathcal{I})$ the isotropic reduction of $(\mathcal{F}, q)$ by $\mathcal{I}$.

LEMMA 2.9. *In $L_\epsilon(M)$ we have $[\mathcal{F}, q] = [\mathcal{F}_\mathcal{I}, q_\mathcal{I}]$.*

PROOF. We consider the filtration of length two on $\mathcal{F}$ such that $\mathcal{I} := F^2\mathcal{F}$, $\mathcal{I}^\perp := F^1\mathcal{F}$. Then we can identify $(\mathrm{Gr}^1(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^1(\mathcal{F})}) \cong (\mathcal{F}_\mathcal{I}, q_\mathcal{I})$. We now apply the Lemmas 2.7 and 2.5 (1). □

2.3. Computation of $L_\epsilon(X)$.

2.3.1. *Definition of $L_\epsilon(G)$.* Let G be any group. We define a $\mathbb{Z}_2$-graded ring $L(G)$. First we define the abelian semigroups $\hat{L}_\epsilon(G)$, $\epsilon \in \mathbb{Z}_2$, which consists of isomorphism classes of tuples (F, q, ρ). Here F is a finite-dimensional real vector space, $q \colon F \to F^*$ is an ϵ–symmetric duality structure, and $\rho \colon G \to \mathrm{Aut}(F, q)$ is a representation of G which is compatible with q, i.e., $q \colon F \to F^*$ is G-equivariant, where G acts on F^* by the adjoint representation ρ^* given by $\rho^*(g) = \rho(g^{-1})^*$. The operation in $\hat{L}_\epsilon(G)$ is induced by the direct sum of representatives.

We obtain $L_\epsilon(G)$ as the quotient of $\hat{L}_\epsilon(G)$ with respect to the equivalence relation generated by *lagrangian reduction.* First we declare that $(\mathcal{F}, q, \rho) \sim 0$ if there exists a G-invariant lagrangian subspace $L \subset F$, i.e., a G-invariant isotropic subspace such that $L^\perp = L$. Then we extend $\sim$ to the minimal equivalence relation on $\hat{L}_\epsilon(G)$ which contains lagrangian reduction and which is compatible with the semigroup structure. Let $[F, q, \rho]$ denote the class in $L_\epsilon(G)$ represented by (F, q, ρ).

Let (F, q, ρ) be a generator of $L_\epsilon(G)$ and $i \colon L \to F$ be the inclusion of a G-invariant isotropic subspace. Then $L^\perp$ is G-invariant and the quotient $L^\perp/L =: F_L$ carries an induced ϵ–symmetric form q_L and a representation ρ_L.

DEFINITION 2.10. We say that (F_L, q_L, ρ_L) is the isotropic reduction of (F, q, ρ) with respect to L.

LEMMA 2.11. *In $L_\epsilon(G)$ we have $[F, q, \rho] = [F_L, q_L, \rho_L]$.*

PROOF. We consider $(F \oplus F_L, q \oplus (-q_L), \rho \oplus \rho_L)$. This tuple represents the zero element in $L_\epsilon(G)$, since it contains the invariant lagrangian subspace which is the image of $L^\perp \to F \oplus F_L$, $x \mapsto x \oplus [x]$. Thus $[F, q, \rho] - [F_L, q_L, \rho_L] = 0$. □

The ring structure is given by

$$[F, q, \rho] \bullet [F', q', \rho'] = \frac{\sqrt{\epsilon}\sqrt{\epsilon'}}{\sqrt{\epsilon\epsilon'}}[F \otimes F', q \otimes q', \rho \otimes \rho'].$$

If X is path-connected, then $L_\epsilon(X) \cong L_\epsilon(\pi_1(X,x))$ for any base point $x \in X$. Furthermore, if $f\colon G' \to G$ is a homomorphism of groups, then there is a natural ring homomorphism $f^*\colon L_\epsilon(G) \to L_\epsilon(G')$ given by $f^*[F,q,\rho] = [F,q,\rho\circ f]$.

2.3.2. *Classification of irreducible ϵ–symmetric forms.* Let (F,ρ) be an irreducible representation of G. In the following we classify the invariant ϵ–symmetric duality structures on (F,ρ). We distinguish various cases, and in each case we define a $\mathbb{Z}_2$-graded group $A(F,\rho)$ by the following rule. $A^\epsilon(F,\rho)$ is trivial if (F,ρ) does not admit an ϵ–symmetric form. If it admits one isomorphism class of such forms, then we set $A^\epsilon(F,\rho) := \mathbb{Z}_2$. In the remaining case it admits two isomorphism classes, and we set $A^\epsilon(F,\rho) := \mathbb{Z}$. The group $A^\epsilon(F,\rho)$ can naturally be interpreted as the part of $L_\epsilon(G)$ which is generated by triples with underlying representation of the form $(F\otimes W, \rho\otimes 1)$, where W a finite-dimensional real vector space.

If q,p are two G-equivariant duality structures on F, then $p^{-1}\circ q \in \mathrm{Aut}_G(F)$. Thus there exists $\lambda \in \mathrm{Aut}_G(F)$ such that $p = q^\lambda$, where $q^\lambda(x,y) = q(\lambda x, y)$.

We call (F,ρ)

- *real* if $\mathrm{End}_G(F) \cong \mathbb{R}$,
- *complex* if $\mathrm{End}_G(F) \cong \mathbb{C}$ and
- *quarternionic* if $\mathrm{End}_G(F) \cong \mathbb{H}$

as algebras over $\mathbb{R}$. By Schur's lemma, every non-zero element in $\mathrm{End}_G(F)$ is invertible and hence $\mathrm{End}_G(F)$ is a division algebra over $\mathbb{R}$. By Frobenius' Theorem, $\mathrm{End}_G(F)$ must be one of the above three possibilities.

The real case. In this case $\lambda \in \mathbb{R}^*$. If $\lambda > 0$, then q and p are isomorphic, namely $p(x,y) = q(\sqrt{\lambda}x, \sqrt{\lambda}y)$. If $\lambda < 0$, then p and q are not isomorphic. Thus given a real representation (F,ρ) which admits an ϵ–symmetric duality structure q, then ϵ is determined, and there are two isomorphism classes represented by (F,q,ρ) and $(F,-q,\rho)$.

We define the $\mathbb{Z}_2$-graded groups

$$A(F,\rho) := \begin{cases} \mathbb{Z}\oplus 0 & \text{if } \epsilon = 1, \\ 0\oplus\mathbb{Z} & \text{if } \epsilon = -1. \end{cases}$$

If G is compact, by [**13**, §2.6, Prop.6.5], ϵ must be 1.

The complex case. In this case there is a unique up to sign $I \in \mathrm{Aut}_G(F)$ satisfying $I^2 = -1$. For $X \in \mathrm{End}_G(F)$ we define $X^q := q^{-1}\circ X^* \circ q$. Then $(I^q)^2 = -1$, and therefore we distinguish two subcases:

- Case $\mathbb{C}_+$: $I^q = I$
- Case $\mathbb{C}_-$: $I^q = -I$.

Case $\mathbb{C}_+$. In this case $q(\mu x, y) = q(x, \mu y)$ for all $\mu \in \mathrm{End}_G(F)$. There exists a root $\sqrt{\lambda} \in \mathrm{Aut}_G(F)$ and we can write $p(x,y) = q(\sqrt{\lambda}x, \sqrt{\lambda}y)$. Thus p and q are isomorphic. We conclude that given (F,ρ) in case $\mathbb{C}_+$ admitting an ϵ–symmetric duality structure q, then ϵ is determined and there is one isomorphism class represented by (F,q,ρ).

We define the $\mathbb{Z}_2$-graded groups

$$A(F,\rho) := \begin{cases} \mathbb{Z}_2 \oplus 0 & \text{if } \epsilon = 1, \\ 0 \oplus \mathbb{Z}_2 & \text{if } \epsilon = -1. \end{cases}$$

CASE $\mathbb{C}_-$. In this case $q(Ix,y) = -q(x,Iy) = -\epsilon q(Iy,x)$. Thus if q is ϵ–symmetric, then q^I is $-\epsilon$–symmetric. If p and q are ϵ–symmetric, then we write $\lambda = a + bI$ and $p = aq + bq^I$ and conclude that $b = 0$. Moreover, q is isomorphic to p exactly if $a > 0$. If (F,ρ) admits an ϵ–symmetric duality structure q, then it also admits an $-\epsilon$–symmetric duality structure. The isomorphism classes are represented by (F,q,ρ), $(F,-q,\rho)$ for ϵ and (F,q^I,ρ) and $(F,-q^I,\rho)$ for $-\epsilon$.

We define the $\mathbb{Z}_2$-graded group

$$A(F,\rho) := \mathbb{Z} \oplus \mathbb{Z}.$$

The quarternionic case. Let $S^2 \subset \mathrm{Im}(\mathbb{H})$ be the unit sphere of complex structures. The $\mathbb{R}$-linear involution $X \mapsto X^q$ acts on $\mathrm{Im}(\mathbb{H})$ and restricts to an involution of S^2. We distinguish the following three cases

- Case $\mathbb{H}_0$: The involution is trivial.
- Case $\mathbb{H}_+$: The involution is non-trivial, but has a fixed point on S^2.
- Case $\mathbb{H}_-$: The involution has no fixed points on S^2.

CASE $\mathbb{H}_-$. In this case $X^q = -X$ for any $X \in \mathrm{Im}(H)$. We write $\lambda = a + bI$ for some $I \in S^2$. Then the same discussion as in the case $\mathbb{C}_-$ shows the following: If (F,ρ) admits an ϵ–symmetric duality structure q in case $\mathbb{H}_-$, then it also admits an $-\epsilon$–symmetric duality structure. The isomorphism classes are represented by (F,q,ρ), $(F,-q,\rho)$ for ϵ and (F,q^I,ρ) and $(F,-q^I,\rho)$ for $-\epsilon$.

We define the $\mathbb{Z}_2$-graded group

$$A(F,\rho) = \mathbb{Z} \oplus \mathbb{Z}.$$

CASE $\mathbb{H}_0$. We can write $\lambda = a + bI$ with $I^q = I$. The same discussion as in case $\mathbb{C}_+$ shows the following: If (F,ρ) admits an ϵ–duality structure q in case $\mathbb{H}_0$, then ϵ is determined. There is one isomorphism class represented by (F,q,ρ).

We define the $\mathbb{Z}_2$-graded groups

$$A(F,\rho) := \begin{cases} \mathbb{Z}_2 \oplus 0 & \text{if } \epsilon = 1, \\ 0 \oplus \mathbb{Z}_2 & \text{if } \epsilon = -1. \end{cases}$$

CASE $\mathbb{H}_+$. In this case we can write $\lambda = a + bI + cJ$, where $I^q = I$ and $J^q = -J$. Writing $p = aq + bq^I + cq^J$, as q^J is a $-\epsilon$ symmetric duality structure, we see that $c = 0$. On the one hand we argue as in the case $\mathbb{C}_+$ that p and q are isomorphic. On the other hand (F,ρ) also admits the $-\epsilon$–symmetric duality structure q^J. If (F,ρ) admits an ϵ–symmetric duality structure q in case $\mathbb{H}_+$, then it also admits an $-\epsilon$–symmetric duality structure. The isomorphism classes are represented by (F,q,ρ) for ϵ and (F,q^J,ρ) for $-\epsilon$.

We define the $\mathbb{Z}_2$-graded groups

$$A(F,\rho) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

2.3.3. *Computation of $L_\epsilon(G)$.* Let $\mathrm{Rep}(G)$ be the set isomorphism classes of finite-dimensional irreducible representations (F,ρ) of G on real vector spaces. For each $(F,\rho)\in\mathrm{Rep}(G)$ and $\epsilon=\pm1$ we fix one ϵ–symmetric form q if it exists.

THEOREM 2.12. *We have an isomorphism of $\mathbb{Z}_2$-graded groups*

$$L_1(G)\oplus L_{-1}(G)\cong\bigoplus_{(F,\rho)\in\mathrm{Rep}(G)}A(F,\rho).$$

PROOF. Fix $\epsilon\in\mathbb{Z}_2$ and let $\mathrm{Rep}^\epsilon(G)\subset\mathrm{Rep}(G)$ denote the set of classes admitting an ϵ–symmetric form. We consider a generator (V,p,σ) of $L_\epsilon(G)$.

LEMMA 2.13. *If (V,p,σ) has no invariant isotropic subspace, then it is isomorphic to a direct sum $\bigoplus_{(F,\rho)\in\mathrm{Rep}^\epsilon(G)}n(F,\rho)(F,q,\rho)$, where $n(F,\rho)\in\mathbb{Z}$ is non-zero for at most finitely many (F,ρ). Furthermore, $n(F,\rho)\in\{0,1\}$ if $A^\epsilon(F,\rho)=\mathbb{Z}_2$, while in case $A^\epsilon(F,\rho)=\mathbb{Z}$ we use the convention that $-n(F,q,\rho)$ stands for $n(F,-q,\rho)$.*

PROOF. Let $i\colon W\hookrightarrow V$ be a minimal G-invariant subspace. Then there is a G-invariant decomposition $V=W\oplus W^\perp$. In fact, $W\cap W^\perp=0$, since W was assumed to be minimal and it cannot be isotropic by assumption. Iterating this argument replacing V by $W^\perp$ we obtain the required decomposition into irreducibles.

The multiplicity of (F,ρ) with $A^\epsilon(F,\rho)=\mathbb{Z}_2$ cannot be greater than 1. If there were two summands, then $(F,q,\rho)\oplus(F,q,\rho)$ would admit an invariant isotropic subspace $W=\{x\oplus I(x)\,|\,x\in F\}\subset F\oplus F$, where $I\in\mathrm{Aut}_G(F)$ is an isomorphism such that $q(Ix,Iy)=-q(x,y)$ for any $x,y\in F$.

If $A^\epsilon(F,\rho)=\mathbb{Z}$, then either (F,q,ρ) or $(F,-q,\rho)$ can occur with positive multiplicity. If they occurred both, then the sum $(F,q,\rho)\oplus(F,-q,\rho)$ would admit the invariant isotropic subspace $W:=\big\{x\oplus x\;\big|\;x\in F\big\}\subset F\oplus F$. □

LEMMA 2.14. *If (V,p,σ) does not admit an invariant isotropic subspace, then the multiplicities $n(F,\rho)$ are uniquely determined.*

PROOF. Assume that

$$(V,p,\sigma)\cong\bigoplus_{(F,\rho)\in\mathrm{Rep}^\epsilon(G)}n(F,\rho)(F,q,\rho)\cong\bigoplus_{(F,\rho)\in\mathrm{Rep}^\epsilon(G)}n(F,\rho)'(F,q,\rho)$$

are two decompositions. Consider $(F,\rho)\in\mathrm{Rep}^\epsilon(G)$ with $n(F,\rho)\neq0$ and the inclusion $i\colon\mathrm{sign}(n(F,\rho))(F,q,\rho)\hookrightarrow(V,p,\sigma)$ given by the first decomposition. Then one can check that there is a summand (F',q',ρ') of the second decomposition such that the composition of i with the projection onto this summand is an isomorphism. Therefore we can take off a summand $\mathrm{sign}(n(F,\rho))(F,q,\rho)$ from both decompositions. Repeating this argument finitely many times we obtain the assertion of the lemma. □

LEMMA 2.15. *Isotropic reduction in stages can be combined to a single isotropic reduction.*

PROOF. Fix a generator (F, q, ρ). Given an invariant isotropic subspace $L \subset F$, we form the reduction (F_L, q_L, ρ_L). If $N \subset F_L$ is an invariant isotropic subspace, then we further form $\big((F_L)_N, (q_L)_N, (\rho_L)_N\big)$. The preimage $\tilde{N}$ of N under $L^\perp \to F_L$ is isotropic and G-invariant. There is a natural isomorphism

$$(F_{\tilde{N}}, q_{\tilde{N}}, \rho_{\tilde{N}}) \cong \big((F_L)_N, (q_L)_N, (\rho_L)_N\big).$$

□

So given (F, q, ρ) the possible maximal isotropic reductions are parameterized by maximal invariant isotropic subspaces.

LEMMA 2.16. *Let $L, N \subset F$ be two maximal isotropic invariant subspaces. Then the corresponding reductions (F_L, q_L, ρ_L) and (F_N, q_N, ρ_N) are isomorphic.*

PROOF. First of all $L \cap N$ is an invariant isotropic. After reduction by $L \cap N$ we can assume that $L \cap N = 0$.

We now show that $(L+N)^\perp \cap (L+N) = 0$.

We claim that $N \cap L^\perp = 0$. In fact, $L + (N \cap L^\perp)$ is isotropic and invariant. Since L is maximal, we conclude $L + (N \cap L^\perp) = L$. Thus $N \cap L^\perp \subset N \cap L = 0$.

We have $(L+N)^\perp = L^\perp \cap N^\perp$. Let $l + n \in (L+N) \cap (L+N)^\perp$. From $l \in L^\perp$ we conclude $n \in L^\perp$. By the claim above $n = 0$. Interchanging the roles of L and N, we also conclude $l = 0$.

Thus $(L+N) \oplus (L+N)^\perp = F$ and $L^\perp = L \oplus (L+N)^\perp$. Therefore, we can decompose $(F, q, \rho) = \big(L+N, q_{L+N}, \rho_{L+N}\big) \oplus \big((L+N)^\perp, q_{(L+N)^\perp}, \rho_{(L+N)^\perp}\big)$. The second summand is now naturally isomorphic to both, (F_L, q_L, ρ_L) and (F_N, q_N, ρ_N). □

Given a generator (V, p, σ), we have well-defined multiplicities $n_{(V,p,\sigma)}(F, \rho) \in A(F, \rho)$ given by any maximal isotropic reduction of (V, p, σ). One easily checks that these multiplicities are additive and satisfy $n_{(V,-p,\sigma)}(F, \rho) = -n_{(V,p,\sigma)}(F, \rho)$ and $n_{(V,p,\sigma)}(F, \rho) = n_{(V_L, p_L, \sigma_L)}(F, \rho)$ for any isotropic reduction. They therefore define the isomorphism

$$L_\epsilon(G) \to \bigoplus_{(F,\rho) \in \mathrm{Rep}^\epsilon(G)} A^\epsilon(F, \rho).$$

This finishes the proof of the theorem. □

2.4. The natural transformation to K-theory.

2.4.1. *The* `bundle`*-construction.* By Top_{met} we denote the full subcategory of Top of paracompact metrizable topological spaces. Let $K^0(X)$ be the complex K-theory functor. We construct a natural transformation $b\colon L_\epsilon \to K^0$ of functors from Top_{met} to rings.

A locally constant sheaf of finite-dimensional $\mathbb{R}$-modules on X gives rise to a locally trivial real vector bundle $\mathtt{bundle}(\mathcal{F})$ in a natural way. We will describe $\mathtt{bundle}(\mathcal{F})$ by providing the local trivializations and the transition maps. Let $x \in X$ and $U \subset X$ be a neighborhood of x such that the restriction $\mathcal{F}|_U$ is isomorphic to the constant sheaf $\underline{\mathcal{F}_x}_U$, where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at x. Then we have a local trivialization $\mathtt{bundle}(\mathcal{F})\big|_U \cong U \times \mathcal{F}_x$. Consider another point $x' \in X$ and the corresponding local trivialization $\mathtt{bundle}(\mathcal{F})\big|_{U'} \cong U' \times \mathcal{F}_{x'}$ of this type such that $U \cap U' \neq \emptyset$. The isomorphism

$$(\underline{\mathcal{F}_x}_U)\big|_{U\cap U'} \cong \mathcal{F}\big|_{U'\cap U} \cong (\underline{\mathcal{F}_{x'}}_{U'})\big|_{U\cap U'}$$

induces an isomorphism $\mathcal{F}_x \overset{\phi_{UU'}}{\cong} \mathcal{F}_{x'}$ which we consider as the (constant) transition map $\phi_{UU'}\colon U \cap U' \to \mathrm{Hom}(\mathcal{F}_x, \mathcal{F}_{x'})$ between the two local trivializations.

The correspondence $\mathtt{bundle}$ is functorial with respect to morphisms of sheaves and compatible with the linear operations direct sum, tensor product, and duality on sheaves and vector bundles. Thus applying the bundle construction to $(\mathcal{F}, q)$ we obtain a pair (F, Q) consisting of a finite-dimensional real vector bundle and an isomorphism $Q\colon F \to F^*$. The $\mathtt{bundle}$-construction is also compatible with pull-back, i.e., if $f\colon Y \to X$ is a morphism in Top_{met}, then there is a natural isomorphism $f^*\mathtt{bundle}(\mathcal{F}) \cong \mathtt{bundle}(f^*\mathcal{F})$.

2.4.2. *Metric structures.* Fix $\epsilon \in \mathbb{Z}_2$. Let (F, Q) be a real vector bundle with an isomorphism $Q\colon F \to F^*$ such that $Q^* = \epsilon Q$. Following the language introduced by Lott [**21**] we define the notion of a metric structure.

DEFINITION 2.17. An isomorphism $J\colon F \to F$ is called a metric structure if
(1) $J^* \circ Q$ defines a scalar product on F,
(2) $J^2 = \epsilon\, \mathrm{id}_F$,
(3) $J^* \circ Q = \epsilon\, Q \circ J$, i.e., $Q(x, Jy) = \epsilon\, Q(Jx, y)$.

Since we assume that X is metrizable and paracompact, it admits partitions of unity. This implies that metric structures exist and that the space of all metric structures is contractible.

Given (F, Q) as above, we choose a metric structure J. Let $F_{\mathbb{C}}$ be the complexification of F. Then $z^J := \frac{1}{\sqrt{\epsilon}} J$ is a $\mathbb{Z}_2$-grading of $F_{\mathbb{C}}$, and $F_\pm = \{x \in F_{\mathbb{C}}, z^J x = \pm x\}$ are sub-bundles of $F_{\mathbb{C}}$, thus the pair $(F_{\mathbb{C}}, z^J)$ represents an element $F_+ - F_-$ of $K^0(X)$ which does not depend on the choice of J.

DEFINITION 2.18. We define the natural transformation $b\colon L_\epsilon \to K^0$ by composing the latter construction with $\mathtt{bundle}$.

2.5. Push-forward for L_ϵ.

2.5.1. *Definition of* π_*^L. Let $\pi\colon X \to B$ be a locally trivial fibre bundle where the fibre is a closed topological n-dimensional manifold Z. There is an open covering $\mathcal{U} = \{U_\lambda\}_\lambda$ of B such that $\phi_\lambda\colon X\big|_{U_\lambda} \cong U_\lambda \times Z$. If $U_\lambda \cap U_\mu \neq \emptyset$, then we have an isomorphism

$$U_\lambda \cap U_\mu \times Z \xrightarrow{\phi_\lambda^{-1}} X\big|_{U_\lambda \cap U_\mu} \xrightarrow{\phi_\mu} U_\lambda \cap U_\mu \times Z,$$

which is of the form $(b, z) \mapsto (b, \phi_{\lambda\mu}(b)(z))$, where $\phi_{\lambda\mu}\colon U_\lambda \cap U_\mu \to \mathrm{Aut}(Z)$ is a continuous family of homeomorphisms of Z.

A fibrewise orientation of the bundle $\pi\colon X \to B$ is a choice of an orientation of Z and of an atlas of local trivializations such that the $\phi_{\lambda\mu}$ are orientation preserving.

Set $\epsilon_n := (-1)^{\frac{n(n+1)}{2}} = (-1)^{[\frac{n+1}{2}]}$, $\sqrt{\epsilon_n} := (\sqrt{-1})^{[\frac{n+1}{2}]}$. To define

$$\pi_*^L\colon L_\epsilon(X) \to L_{\epsilon\epsilon_n}(B),$$

we assume that the bundle comes equipped with a fibrewise orientation. Let $[\mathcal{F}, q] \in L_\epsilon(X)$. Then we construct a representative of $\pi_*^L([\mathcal{F}, q])$ as follows. Note that $\pi(\mathcal{F}) := HR\pi_*\mathcal{F} := \bigoplus_{i=0}^n R^i\pi_*(\mathcal{F})$ is a locally constant sheaf of finite dimensional $\mathbb{R}$-modules. In fact, let $b \in U_\lambda$ and $Z_b = \pi^{-1}(\{b\})$. Then we have

$$\pi(\mathcal{F})\big|_{U_\lambda} \cong \underline{H(Z_b, \mathcal{F}\big|_{Z_b})}_{U_\lambda}.$$

By Poincaré duality over Z_b, we have an isomorphism such that $H^k(Z_b, \mathcal{F}\big|_{Z_b}^*) \cong H^{n-k}(Z_b, \mathcal{F}\big|_{Z_b})^*$ for all $k \in \mathbb{N}$. This isomorphism is preserved by the transition maps so that we obtain an isomorphism $\pi(\mathcal{F}^*) \cong \pi(\mathcal{F})^*$. If we compose this isomorphism with the sum of the isomorphisms $R^i\pi_*(q)\colon R^i\pi_*\mathcal{F} \to R^i\pi_*\mathcal{F}^*$, then we obtain an $\epsilon\epsilon_n$-symmetric duality structure $\pi(q)$ on $\pi(\mathcal{F})$ (cf. Subsections 2.5.2, 5.2.2).

DEFINITION 2.19. We define $\pi_*^L([\mathcal{F}, q]) := \big[\pi(\mathcal{F}), \pi(q)\big]$.

LEMMA 2.20. *π_*^L is well-defined.*

PROOF. By construction $\pi(q)$ is an isomorphism with the correct symmetry properties. Thus our prescription $(\mathcal{F}, q) \mapsto [\pi(\mathcal{F}), \pi(q)]$ provides a homomorphism of semigroups $\hat{\pi}_*^L\colon \hat{L}_\epsilon(X) \to L_{\epsilon\epsilon_n}(B)$. We must show that it factors over $\hat{L}_\epsilon(X) \to L_\epsilon(X)$. Let $\mathcal{L} \subset \mathcal{F}$ be a lagrangian subsheaf. It leads to a compatible (see Definition 2.6) filtration $\big(F^i\mathcal{F}\big)_{i=0,1}$ of length 1 by $F^0\mathcal{F} := \mathcal{F}$, $F^1\mathcal{F} := \mathcal{L}$.

We obtain an induced filtration $\big(F^i\pi(\mathcal{F})\big)_{i=0,1}$ such that $F^1\pi(\mathcal{F}) = \mathrm{im}(HR\pi_*\mathcal{L} \to HR\pi_*\mathcal{F})$ of length 1 which is compatible with $\pi(q)$. Here one has to check that $F^1\pi(\mathcal{F})$ is a lagrangian subsheaf. This can be verified either directly by looking at the long exact cohomology sequence associated to $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{F}/\mathcal{L} \to 0$, or by invoking the spectral sequence induced by the filtration and the discussion in Subsection 4.4.1 below.

By a combination of Lemmas 2.5 (1) and 2.7, we have $[\pi(\mathcal{F}), \pi(q)] = 0$. □

Note that $(\pi(\mathcal{F}), \pi(q))$ comes with a natural $\mathbb{Z}$-grading of length n. Thus we can apply Lemma 2.5 to reduce to the middle term. In particular, we have $\pi_*^L = 0$ if n is odd.

REMARK 2.21. In **[21]**, Lott defines $\pi_*^{L,\mathrm{Lott}}\colon L_\epsilon^{\mathrm{Lott}}(X) \to L_{\epsilon\epsilon_n}^{\mathrm{Lott}}(B)$ which induces π_*^L by passing to quotients.

2.5.2. *Functoriality.* Here we will show that π_*^L is functorial with respect to iterated fibre bundles. Let $\pi_1 \colon W \to V$ and $\pi_2 \colon V \to S$ be locally trivial fibre bundles where the fibres are closed topological manifolds. We assume that both bundles come equipped with fibrewise orientations. We set $\pi_3 := \pi_2 \circ \pi_1$. We furthermore assume that $\pi_3 \colon W \to S$ is a locally trivial fibre bundle with fibrewise orientation with fiber Z. The fibre Z_s $(s \in S)$ is itself a fibre bundle. We assume that this bundle structure is preserved by the transition maps between local trivializations of π_3. We assume that the orientation of the fibres Z_s are induced by the orientations of the fibres of π_1 and π_2. We will call this situation an iterated fibre bundle with compatible fibrewise orientations.

In the smooth category, i.e., if W, V, and S are smooth manifolds and if π_1, π_2, and π_3 are smooth maps, to give a locally trivial fibre bundle is the same as to give a proper submersion. Furthermore, to give a fibrewise orientation is the same as to give an orientation of the vertical bundle. The composition of proper submersion is again a proper submersion. Thus if $\pi_1 \colon W \to V$ and $\pi_2 \colon V \to S$ are locally trivial fibre bundles in the smooth category with fibre X, Y, then automatically $\pi_3 \colon W \to S$ is a locally trivial fibre bundle with fiber Z. Using the existence of connections and parallel transport on can produce local trivializations of π_3 preserving the bundle structure of the fibres. The compatible fibrewise orientation is obtained by the orientation of the vertical bundle TZ, which can be identified with the sum of oriented vertical bundles $TX \oplus \pi_1^* TY$.

We now turn back to the general situation.

THEOREM 2.22. *We have equality of homomorphisms $L_\epsilon(W) \to L_{\epsilon\epsilon_n}(S)$,*

$$\pi_{3*}^L = \pi_{2*}^L \circ \pi_{1*}^L.$$

PROOF. Let $(\mathcal{F}, q)$ be a generator of $L_\epsilon(W)$. We show that the equality $[\pi_3(\mathcal{F}), \pi_3(q)] = [\pi_2(\pi_1(\mathcal{F})), \pi_2(\pi_1(q))]$ using the fibrewise Leray-Serre spectral sequence $({}_{LS}E_r, {}_{LS}d_r)$ $(r \geq 2)$ associated to the composition of functors π_{1*} and π_{2*} applied to $\mathcal{F}$ (cf. [**18**, Thm. 3.7.3], [**17**, p. 464]). The term ${}_{LS}E_2$ is given by ${}_{LS}E_2^{p,q} = R^p\pi_{2*}(R^q\pi_{1*}\mathcal{F})$. Furthermore, there are decreasing filtrations $(F^iR^k\pi_{3*}\mathcal{F})_i$ on $R^k\pi_{3*}\mathcal{F}$, $k \in \mathbb{N}$, such that $\mathrm{Gr}(R^k\pi_{3*}\mathcal{F}) \cong \sum_{p+q=k} E_\infty^{p,q}$.

One checks that the filtration $\big(F^i\pi_3(\mathcal{F})\big)_i$ is compatible with $\pi_3(q)$. To do so we can restrict to the fibre over some $s \in S$. Let n be the dimension of the base Y_s of the bundle Z_s. The length of the filtration is n. One has to show that

$$\pi_3(q)(x)(y) = 0 \ \text{ whenever } x \in F^p\pi_3(\mathcal{F})_s,\ y \in F^q\pi_3(\mathcal{F})_s,\ p+q > n. \tag{2.1}$$

If one computes the cohomology $\pi_3(\mathcal{F})_s = H(Z_s, \mathcal{F}|_{Z_s})$ using the chain complexes $\overline{\Omega}(Z_s, \mathcal{F}|_{Z_s})$ which are functorial in sheaves, and on which one can implement Poincaré duality $\int_{Z_s} : \overline{\Omega}(Z_s, \mathcal{F}|_{Z_s}) \wedge \overline{\Omega}(Z_s, \mathcal{F}^*|_{Z_s}) \to \mathbb{R}$ as well as the filtration $\big(F^i\overline{\Omega}(Z_s, \mathcal{F}|_{Z_s})\big)_i$ leading to the Leray-Serre spectral sequence (cf. [**25**]), then one checks on this level that $\int_{Z_s} \omega \wedge \omega' = 0$ if $\omega \in F^p\overline{\Omega}(Z_s, \mathcal{F}|_{Z_s})$, $\omega' \in F^{p'}\overline{\Omega}(Z_s, \mathcal{F}^*|_{Z_s})$, and $p+p' > n$. It follows that the filtration is compatible with the form Q given by

$Q(\omega,\omega') = \frac{\sqrt{\epsilon\epsilon_n}}{\sqrt{\epsilon}\sqrt{\epsilon_n}} \int_{Z_s}((-1)^{\frac{N(N-1)}{2}+N\dim Z} q(\omega)) \wedge \omega'$, $\omega, \omega' \in \overline{\Omega}(Z_s, \mathcal{F}|_{Z_s})$, where N is the $\mathbb{Z}$-grading of $\overline{\Omega}(Z_s, \mathcal{F}|_{Z_s})$. The form Q induces $\pi_3(q)$ in cohomology. This immediately implies (2.1).

To be more explicit in the smooth case we can take for $\overline{\Omega}(Z_s, \mathcal{F}^*|_{Z_s})$ as $\Omega(Z_s, \mathcal{F}|_{Z_s})$ the smooth $\mathcal{F}$-valued forms so that the symbols above aquire their usual meaning. In this case, $F^i\Omega(Z_s, \mathcal{F}|_{Z_s}) = \pi^*_{1|Z_s}\Omega^{\geq i}(Y_s)\Omega(X, \mathcal{F}|_X)$ (also cf. [**24**, §2.1]).

By Lemma 2.7 we have $[\pi_3(\mathcal{F}), \pi_3(q)] = [\mathrm{Gr}(\pi_3(\mathcal{F})), \mathrm{Gr}(\pi_3(q))]$.

Now $\pi_2(\pi_1(\mathcal{F}))$ can be identified with the term ${}_{LS}E_2(\mathcal{F})$ of the spectral sequence and $\pi_2(\pi_1(q))$ is the induced form ${}_{LS}E_2(q)$ (see Subsection 4.4.1 below). The same model as above can be used to check that the n-th term ${}_{LS}E_n(\mathcal{F})$, $n \geq 2$, is a locally constant sheaf on S with an induced form ${}_{LS}E_n(q)$, which carries a compatible $\mathbb{Z}$-grading and a compatible differential. We obtain the $(n+1)$th stage of the spectral sequence by taking cohomology. Thus, by Lemma 2.5 (2). we have $[{}_{LS}E_n(\mathcal{F}), {}_{LS}E_n(q)] = [{}_{LS}E_{n+1}(\mathcal{F}), {}_{LS}E_{n+1}(q)]$. We conclude with

$$\begin{aligned}[\pi_2(\pi_1(\mathcal{F})), \pi_2(\pi_1(q))] &= [{}_{LS}E_2(\mathcal{F}), {}_{LS}E_2(q)] = [{}_{LS}E_\infty(\mathcal{F}), {}_{LS}E_\infty(q)] \\ &= [\mathrm{Gr}(\pi_3(\mathcal{F})), \mathrm{Gr}(\pi_3(q))] = [\pi_3(\mathcal{F}), \pi_3(q)].\end{aligned}$$

□

REMARK 2.23. Note that this functoriality does not hold in general for $\pi_*^{L,\mathrm{Lott}}$.

2.5.3. *Compatibility and naturality.* In the present subsection we work in the smooth category. Let $\pi\colon M \to B$ be a locally trivial fibre bundle over a compact base B such that the fibres are compact even-dimensional smooth manifolds. We further assume that the fibrewise tangent bundle TM/B is oriented. Then the bundle has a fibrewise orientation. We have the following maps:

- $\pi_*^{\mathrm{sign}}\colon H^*(M,\mathbb{R}) \to H^*(B,\mathbb{R})$ defined by $\pi_*(\omega) = \int_{M/B} \omega \cup \mathbf{L}(TM/B)$, where $\int_{M/B}$ is integration over the fibre and $\mathbf{L}(TM/B)$ denotes the Hirzebruch $\mathbf{L}$-class of the fibrewise tangent bundle.
- $\pi_!^{\mathrm{sign}}\colon K^0(M) \to K^0(B)$ defined by $\pi^{\mathrm{sign}}([E]) = \mathrm{ind}(D_E^{\mathrm{sign}})$, where D_E^{sign} is the fibrewise signature operator twisted by the complex vector bundle $E \to M$ and $\mathrm{ind}(D_E^{\mathrm{sign}}) \in K^0(B)$ denotes the class of the index bundle.
- $\pi_*^L\colon L_\epsilon(M) \to L_{\epsilon\epsilon_n}(B)$ given in Definition 2.19.

Let $\mathbf{ch}\colon K^0 \to H^{\mathrm{ev}}(.,\mathbb{R})$ denote the natural transformation of ring-valued functors given by the Chern character.

THEOREM 2.24. *The following diagram commutes*:

$$\begin{array}{ccccc} L_\epsilon(M) & \xrightarrow{b} & K^0(M) & \xrightarrow{\mathbf{ch}} & H^{\mathrm{ev}}(M,\mathbb{R}) \\ \pi_*^L \downarrow & & \pi_!^{\mathrm{sign}} \downarrow & & \pi_*^{\mathrm{sign}} \downarrow \\ L_{\epsilon\epsilon_n}(B) & \xrightarrow{b} & K^0(B) & \xrightarrow{\mathbf{ch}} & H^{\mathrm{ev}}(B,\mathbb{R}) \end{array}$$

PROOF. Recall that the map b is defined in Definition 2.18. Commutativity of the right square is the assertion of the index theorem for families applied to the family of twisted fibrewise signature operators. The commutativity of the left square is a consequence of Hodge theory by which we can identify $\mathtt{bundle}(\pi(\mathcal{F}))$ with the bundle $\ker(D_{\mathtt{bundle}(\mathcal{F})_{\mathbb{C}}})$. □

The following proposition is an immediate consequence of the definition.

PROPOSITION 2.25. *The push-forward π_*^L is natural with respect to pull-back of fibre bundles, i.e., given $f\colon B' \to B$ we consider the pull back*

$$\begin{array}{ccc} f^*M & \xrightarrow{f_\sharp} & M \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array},$$

and we then have $(f^*\pi)_*^L \circ f_\sharp^* = f^* \circ \pi_*^L$.

3. The functor $\bar{L}_\epsilon$

3.1. Introduction and summary. This subsection shall be considered as an introduction to both, the present section and Section 5. Our purpose here is to motivate the introduction of $\bar{L}_\epsilon$.

If we compose the natural transformation $b\colon L_\epsilon \to K^0$ with the Chern character $\mathbf{ch}\colon K^0 \to H^{\mathrm{ev}}(.,\mathbb{R})$, then we obtain a natural transformation $\mathbf{ch} \circ b\colon L_\epsilon \to H^{\mathrm{ev}}(.,\mathbb{R})$. In the present section we study the kernel of this map in detail.

Note that L_ϵ is given by a topological construction. Staying in the topological framework we first define a functor $\bar{L}^{\mathbb{R}/\mathbb{Z}}$ from topological spaces to $\mathbb{Z}_2$-graded groups together with a surjective natural transformation to $\ker(\mathbf{ch} \circ b)$. We can consider $\mathbf{ch}\colon K \to H(.,\mathbb{R})$ as a map between classifying spaces. Its homotopy fibre is again a classifying space of a cohomology theory $K_{\mathbb{R}/\mathbb{Z}}$. The functor $\bar{L}^{\mathbb{R}/\mathbb{Z}}$ is defined by a pull-back of $K_{\mathbb{R}/\mathbb{Z}}^{-1} \to K^0$ via $b\colon L_\epsilon \to K^0$. Using the geometric description of $K_{\mathbb{R}/\mathbb{Z}}$ given in [**20**], one could also obtain a geometric description of $\bar{L}^{\mathbb{R}/\mathbb{Z}}$.

As a cohomology theory $K_{\mathbb{R}/\mathbb{Z}}$ admits wrong-way maps for suitably oriented fibre bundles. The topic of secondary index theory [**20**] is to relate these topological secondary index maps with their analytic counterparts. The main constituent of the construction of the secondary analytic index is the η-form of a family of Dirac operators. The well-definedness and functoriality of the wrong-way maps in the geometric picture encode properties of η-forms. As a formal consequence of its definition we have wrong-way maps for $\bar{L}^{\mathbb{R}/\mathbb{Z}}$ with nice functorial properties.

The functor $\bar{L}^{\mathbb{R}/\mathbb{Z}}$ fits into an exact sequence

$$0 \longrightarrow H^{\mathrm{odd}}(X,\mathbb{R})/\mathrm{im}(\mathbf{ch}\colon K^1(X) \to H^{\mathrm{odd}}(X,\mathbb{R})) \longrightarrow \bar{L}^{\mathbb{R}/\mathbb{Z}}(X) \longrightarrow$$
$$\longrightarrow \ker(\mathbf{ch} \circ b\colon K^0(X) \longrightarrow H^{\mathrm{ev}}(X,\mathbb{R})) \longrightarrow 0.$$

It turns out that, working in the smooth category, we can define a refinement $\bar{L}_\epsilon \to \bar{L}^{\mathbb{R}/\mathbb{Z}}$ which fits into the sequence

$$0 \longrightarrow H^{\mathrm{odd}}(X,\mathbb{R}) \longrightarrow \bar{L}_\epsilon(X) \longrightarrow \ker(\mathbf{ch}\circ b\colon K^0(X) \longrightarrow H^{\mathrm{ev}}(X,\mathbb{R})) \longrightarrow 0.$$

The definition of $\bar{L}_\epsilon$ is geometric and not (obviously) related to a cohomology theory. So it can be considered as a nontrivial fact that $\bar{L}_\epsilon$ still admits wrong-way maps with nice functorial properties. It is in fact *the main purpose of the present paper to construct these maps and verify their functorial properties.* These results thus encode some finer properties of the η-forms of families of signature operators twisted with flat vector bundles.

In the present section we give the definition of the secondary L-functors and discuss their simplest properties and relations to other functors. The wrong-way maps are introduced in Section 5 after a digression to η-forms and η-invariants.

3.2. Secondary K-theory.

3.2.1. *Definition of* $K^{-1}_{\mathbb{R}/\mathbb{Z}}$. We are going to recall the definition of the 2-periodic cohomology theory $K_{\mathbb{R}/\mathbb{Z}}$ introduced in [**19**], [**20**]. Let BU be the classifying space of complex K-theory. The Chern character (with real coefficients) is induced by a map $\mathbf{ch}\colon BU \to \prod_{n=1}^{\infty} K(\mathbb{R},2n)$. The homotopy fibre of this map classifies $K_{\mathbb{R}/\mathbb{Z}}$. In particular, for any space X there is a natural exact sequence of $K^0(X)$-modules

$$\dots \longrightarrow K^{-1}(X) \xrightarrow{\mathbf{ch}} H^{\mathrm{odd}}(X,\mathbb{R}) \longrightarrow K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \xrightarrow{\beta} K^0(X) \xrightarrow{\mathbf{ch}} H^{\mathrm{ev}}(X,\mathbb{R}) \longrightarrow \dots,$$

where $K^0(X)$ acts on cohomology via the Chern character.

3.2.2. *A geometric description.* For a manifold M we recall the definition of $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ in terms of generators and relations as given in [**20**, Defs. 5, 6]. We form the abelian semigroup $\hat{K}^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ consisting of isomorphism classes of tuples (E,h^E,∇^E,ρ), where $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded complex vector bundle, h^E is a hermitian metric and $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a metric connection, both being compatible with the grading, and $\rho \in \Omega^{\mathrm{odd}}(M)/\mathrm{im}(d)$ satisfies $d\rho = \mathbf{ch}(\nabla^E) := \mathbf{ch}(\nabla^{E_+}) - \mathbf{ch}(\nabla^{E_-})$. The semigroup operation is induced by the direct sum of generators. On $\hat{K}^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ we consider the minimal equivalence relation $\sim$ which is compatible with the semigroup structure and such that the following holds:

(1) (Change of connections) We have $(E,h^E,\nabla,\rho) \sim (E,h^{E'},\nabla',\rho')$ if and only if $\rho' = \rho + \tilde{\mathbf{ch}}(\nabla',\nabla)$. (See Section 3.6 for a definition of the transgression Chern form.)
(2) (Trivial elements) If (E,h^E,∇^E) is a $\mathbb{Z}_2$-graded hermitean vector bundle with connection, then $(E\oplus E^{\mathrm{op}}, h^{E\oplus E}, \nabla^E \oplus \nabla^E, 0) \sim 0$, where E^{op} denotes E with the opposite grading.

The group $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ is the quotient of $\hat{K}^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ by $\sim$. By $[E,h^E,\nabla^E,\rho]$ we denote the class of (E,h^E,∇^E,ρ) in $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$.

It was shown by Lott [**20**, §2] that the group given by this geometric definition is naturally isomorphic to the topological defined object of Subsection 3.2.1.

3.3. The functor $\bar{L}^{\mathbb{R}/\mathbb{Z}}$.

3.3.1. *The definition.*

DEFINITION 3.1. We define the functor $X \mapsto \bar{L}^{\mathbb{R}/\mathbb{Z}}(X)$ from Top_{para} to groups by the following pull-back diagram:

$$\begin{array}{ccc} \bar{L}^{\mathbb{R}/\mathbb{Z}}(X) & \longrightarrow & L(X) \\ \downarrow & & \downarrow b \\ K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) & \xrightarrow{\beta} & K^0(X) \end{array}$$

Especially, $\bar{L}^{\mathbb{R}/\mathbb{Z}}(X) = \{(u,v) \in K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \times L(X) \mid \beta u = bv\}$.

On morphisms the functor $\bar{L}^{\mathbb{R}/\mathbb{Z}}$ only depends on homotopy classes. Note that $K^0(X)$ and $K^{-1}_{\mathbb{R}/\mathbb{Z}}(X)$ are $L(X)$-modules via b. This induces an $L(X)$-module structure on $\bar{L}^{\mathbb{R}/\mathbb{Z}}(X)$. We have the following natural exact sequence of $L(X)$-modules

$$K^{-1}(X) \xrightarrow{\mathbf{ch}} H^{\text{odd}}(X,\mathbb{R}) \longrightarrow \bar{L}^{\mathbb{R}/\mathbb{Z}}(X) \longrightarrow L(X) \xrightarrow{\mathbf{ch}\circ b} H^{\text{ev}}(X,\mathbb{R}).$$

3.3.2. *Secondary push-forwards.* Let $\pi\colon X \to B$ be a smooth locally trivial fibre bundle with closed even n-dimensional fibres and equipped with an orientation of the vertical tangent bundle TX/B. In order to define an index map for $K_{\mathbb{R}/\mathbb{Z}}$ we need the further assumption that π is K-oriented. Thus assume that TX/B has a Spin_c-structure. Then there are maps $\pi_!^{\text{Spin}_c}\colon K^0(X) \to K^0(B)$ and $\pi_!^{\text{Spin}_c,\mathbb{R}/\mathbb{Z}}\colon K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$ (the topological secondary index, cf. e.g. [**20**]), such that the following diagram commutes:

$$\begin{array}{ccccccc} H^{\text{odd}}(X,\mathbb{R}) & \longrightarrow & K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) & \longrightarrow & K^0(X) & \longrightarrow & H^{\text{ev}}(X,\mathbb{R}) \\ \pi_*^{\text{Spin}_c}\downarrow & & \pi_!^{\text{Spin}_c,\mathbb{R}/\mathbb{Z}}\downarrow & & \pi_!^{\text{Spin}_c}\downarrow & & \pi_*^{\text{Spin}_c}\downarrow \\ H^{\text{odd}}(B,\mathbb{R}) & \longrightarrow & K^{-1}_{\mathbb{R}/\mathbb{Z}}(B) & \longrightarrow & K^0(B) & \longrightarrow & H^{\text{ev}}(B,\mathbb{R}) \end{array},$$

where $\pi_*^{\text{Spin}_c}(\omega) = \int_{X/B} \hat{\mathbf{A}}(TX/B) \cup e^{c_1/2} \cup \omega$ and c_1 is the first Chern class determined by the Spin_c-structure. There is an unique element $E_{\text{sign}} \in K_0(X)$ such that $\pi_!^{\text{sign}}(x) = \pi_!^{\text{Spin}_c}(E_{\text{sign}} \bullet x)$. Note that $\mathbf{ch}(E_{\text{sign}}) \cup \hat{\mathbf{A}}(TX/B) \cup e^{c_1/2} = \mathbf{L}(TX/B)$.

DEFINITION 3.2. (1) The secondary signature index map

$$\pi_!^{\text{sign},\mathbb{R}/\mathbb{Z}}\colon K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$$

is defined by

$$\pi_!^{\text{sign},\mathbb{R}/\mathbb{Z}}(x) := \pi_!^{\text{Spin}_c,\mathbb{R}/\mathbb{Z}}(E_{\text{sign}} \bullet x).$$

(2) The push-forward $\pi_*^{\bar{L},\mathbb{R}/\mathbb{Z}}\colon \bar{L}^{\mathbb{R}/\mathbb{Z}}(X) \to \bar{L}^{\mathbb{R}/\mathbb{Z}}(B)$ is defined as the map induced by $\pi_!^{\mathrm{sign},\mathbb{R}/\mathbb{Z}}$ and π_*^L.

COROLLARY 3.3. *The following diagrams commute*:

$$\begin{array}{ccccccc} H^{\mathrm{odd}}(X,\mathbb{R}) & \longrightarrow & K^{-1}_{\mathbb{R}/\mathbb{Z}}(X) & \longrightarrow & K^0(X) & \longrightarrow & H^{\mathrm{ev}}(X,\mathbb{R}) \\ \pi_*^{\mathrm{sign}}\downarrow & & \pi_!^{\mathrm{sign},\mathbb{R}/\mathbb{Z}}\downarrow & & \pi_!^{\mathrm{sign}}\downarrow & & \pi_*^{\mathrm{sign}}\downarrow \\ H^{\mathrm{odd}}(B,\mathbb{R}) & \longrightarrow & K^{-1}_{\mathbb{R}/\mathbb{Z}}(B) & \to & K^0(B) & \longrightarrow & H^{\mathrm{ev}}(B,\mathbb{R}), \end{array}$$

$$\begin{array}{ccccccc} H^{\mathrm{odd}}(X,\mathbb{R}) & \longrightarrow & \bar{L}^{\mathbb{R}/\mathbb{Z}}(X) & \longrightarrow & L(X) & \longrightarrow & H^{\mathrm{ev}}(X,\mathbb{R}) \\ \pi_*^{\mathrm{sign}}\downarrow & & \pi_*^{\bar{L},\mathbb{R}/\mathbb{Z}}\downarrow & & \pi_*^{L}\downarrow & & \pi_*^{\mathrm{sign}}\downarrow \\ H^{\mathrm{odd}}(B,\mathbb{R}) & \longrightarrow & \bar{L}^{\mathbb{R}/\mathbb{Z}}(B) & \to & L(B) & \longrightarrow & H^{\mathrm{ev}}(B,\mathbb{R}). \end{array}$$

All push-forward maps are natural with respect to the pull-back of fibre bundles. Moreover, they are functorial with respect to iterated fibre bundles. In greater detail we have the following: Let $\pi_1\colon W \to V$ and $\pi_2\colon V \to S$ be locally trivial smooth fibre bundles with closed even-dimensional fibres X, Y. Further assume that the vertical bundles TX and TY are oriented and carry Spin_c-structures. Then the composition $\pi_3 = \pi_2 \circ \pi_1\colon W \to S$ is a locally trivial fibre bundle with closed even-dimensional fibres Z and the vertical bundle TZ carries an induced orientation and Spin_c-structure. In this situation the index maps on complex K-theory, $K^{-1}_{\mathbb{R}/\mathbb{Z}}$-theory, L-theory (Theorem 2.22), and in cohomology behave functorially with respect to the iterated fibre bundle. It now follows immediately from the definition that

COROLLARY 3.4. $\pi_{3*}^{\bar{L},\mathbb{R}/\mathbb{Z}} = \pi_{2*}^{\bar{L},\mathbb{R}/\mathbb{Z}} \circ \pi_{1*}^{\bar{L},\mathbb{R}/\mathbb{Z}}$.

3.4. The functor $\bar{L}$.

3.4.1. *Definition of* $\bar{L}$. The functor $X \mapsto \bar{L}^{\mathbb{R}/\mathbb{Z}}(X)$ from Top_{met} to $L(X)$-modules was defined by a purely homotopy-theoretic construction as an extension of the functor

$$X \mapsto \ker(\mathbf{ch} \circ b\colon L(X) \to H^{\mathrm{ev}}(X,\mathbb{R}))$$

by $X \mapsto H^{\mathrm{odd}}(X,\mathbb{R})/\mathbf{ch}(K^{-1}(X))$.

Let Top_{smooth} denote the full subcategory of paracompact metrizable spaces Top_{met} which are homotopy equivalent to smooth manifolds. In the present subsection we use a differential geometric construction to define on Top_{smooth} a functor $X \mapsto \bar{L}(X)$ to graded $L(X)$-modules which extends $X \mapsto \ker(\mathbf{ch} \circ b)$ by $X \mapsto H^{\mathrm{odd}}(X,\mathbb{R})$.

It suffices to define $\bar{L}$ as a homotopy invariant functor on the category of smooth manifolds and smooth maps. Again we start with defining a abelian semi-groups $\hat{\bar{L}}_\epsilon(M)$, $\epsilon = \pm 1$, with identity and obtain the group $\bar{L}_\epsilon(M)$ as the quotient of $\hat{\bar{L}}_\epsilon(M)$ by an equivalence relation.

Let $(\mathcal{F}, q)$ be a representative of an element of $\hat{L}_\epsilon(M)$. Then we have a real vector bundle $F := \texttt{bundle}(\mathcal{F})$ which carries a natural flat connection ∇^F such that $\ker(\nabla^F) = \mathcal{F}$. The form $Q \in \mathrm{Hom}(F, F^*)$ which is induced by q is parallel with respect to ∇^F. Let J be a smooth metric structure on (F, Q). It induces a $\mathbb{Z}_2$-grading $z^J := \frac{1}{\sqrt{\epsilon}} J$ of the complexification $F_\mathbb{C}$. In general, since J is not parallel with respect to ∇^F, this grading is not preserved by the connection $\nabla^{F_\mathbb{C}}$ induced by ∇^F. The even part of $\nabla^{F_\mathbb{C}}$ with respect to z^J is a connection on $F_\mathbb{C}$ which preserves the $\mathbb{Z}_2$-grading. It will be denoted by $\nabla^{F_\mathbb{C},J}$. Then $\nabla^{F_\mathbb{C},J} = \nabla^{F_\mathbb{C}} + \frac{1}{2} J^{-1}(\nabla^F J)$. Let $(\Omega^*(M), d)$ be the real de Rham complex of M. We can use this connection to define a characteristic form which represents the Chern class of $F_\mathbb{C}$.

DEFINITION 3.5. We define $p(\nabla^F, J) \in \Omega^{4*-\epsilon+1}(M)$ by

$$
\begin{aligned}
p(\nabla^F, J) := \mathbf{ch}(\nabla^{F_\mathbb{C},J}) &:= \mathrm{Tr}_s \left[\exp(-(\nabla^{F_\mathbb{C},J})^2/2\pi\mathrm{i})\right] \\
&:= \mathrm{Tr}\left[z^J \exp(-(\nabla^{F_\mathbb{C},J})^2/2\pi\mathrm{i})\right].
\end{aligned}
\tag{3.1}
$$

An element of $\hat{\hat{L}}_\epsilon(M)$ is an isomorphism class of tuples $(\mathcal{F}, q, J, \rho)$, where $(\mathcal{F}, q)$ is a representative of an element of $\hat{L}_\epsilon(M)$, J is a metric structure on $F := \texttt{bundle}(\mathcal{F})$, and $\rho \in \Omega^{4*-\epsilon}(M)/\mathrm{im}(d)$ satisfies $d\rho = p(\nabla^F, J)$. The semigroup operation is induced by direct sum of representatives:

$$
(\mathcal{F}, q, J, \rho) + (\mathcal{F}', q', J', \rho') := (\mathcal{F} \oplus \mathcal{F}', q \oplus q', J \oplus J', \rho + \rho').
$$

Before introducing the equivalence relation we recall the definition of the transgression Chern form. Let $E \to M$ be a $\mathbb{Z}_2$-graded complex vector bundle and let ∇, ∇' be two connections on E preserving the grading. Then we consider the bundle $\tilde{E} := \mathrm{pr}^* E \to [0,1] \times M$ with connection $\tilde{\nabla}$ which is given by $\tilde{\nabla}_{\partial t} := \partial_t$ (t is the coordinate in $[0,1]$) and $\tilde{\nabla}_X = (1-t)\nabla_X + t\nabla'_X$ for $X \in TM$. We decompose $\mathbf{ch}(\tilde{\nabla}) = dt \wedge \gamma + r$, where r does not contain dt and $\gamma\colon [0,1] \to \Omega(M)$ is a smooth family of forms.

DEFINITION 3.6. The transgression Chern form is defined by

$$
\tilde{\mathbf{ch}}(E, \nabla', \nabla) := \int_0^1 \gamma(t) dt.
$$

It satisfies

$$
d\tilde{\mathbf{ch}}(E, \nabla', \nabla) = \mathbf{ch}(\nabla') - \mathbf{ch}(\nabla).
$$

We now introduce the equivalence relation which is again generated by *lagrangian reduction.* We consider $(\mathcal{F}, q)$ and a metric structure J on F. Let $\mathcal{L} \subset \mathcal{F}$ be a locally constant lagrangian subsheaf and $L := \texttt{bundle}(\mathcal{L})$. Then we have a decomposition $F = L \oplus J(L)$. Let $\nabla^\oplus$ denote be the part of $\nabla^{F_\mathbb{C},J}$ which preserves this decomposition.

DEFINITION 3.7. We define

$$
\tilde{p}(\mathcal{F}, q, J, \mathcal{L}) := \tilde{\mathbf{ch}}(F_\mathbb{C}, \nabla^\oplus, \nabla^{F_\mathbb{C},J}).
$$

We require that the equivalence relation contains the relation

$$(\mathcal{F}, q, J, \rho) \sim \big(0, 0, 0, \rho + \tilde{p}(\mathcal{F}, q, J, \mathcal{L})\big).$$

Note that $(0, 0, 0, \rho+\tilde{p}(\mathcal{F}, q, J, \mathcal{L}))$ is a generator of $\hat{\hat{L}}_\epsilon(M)$, since $\mathbf{ch}(\nabla^\oplus) = \frac{1}{\sqrt{\epsilon}} \operatorname{Tr}[J \exp(-\nabla^{\oplus,2}/2\pi i)] = 0$ and hence

$$d\rho + d\tilde{p}(\mathcal{F}, q, J, \mathcal{L}) = \mathbf{ch}(\nabla^{F_\mathbb{C},J}) + \mathbf{ch}(\nabla^\oplus) - \mathbf{ch}(\nabla^{F_\mathbb{C},J}) = 0.$$

Then we extend $\sim$ to the minimal equivalence relation on $\hat{\hat{L}}_\epsilon(M)$ which contains lagrangian reduction and which is compatible with the semigroup structure.

DEFINITION 3.8. We define the semigroup $\bar{L}_\epsilon(M) := \hat{\hat{L}}_\epsilon(M)/\sim$.

By $[\mathcal{F}, q, J, \rho]$ we denote the class in $\bar{L}_\epsilon(M)$ represented by $(\mathcal{F}, q, J, \rho)$.

LEMMA 3.9. *$\bar{L}_\epsilon(M)$ is a group.*

PROOF. We have $[\mathcal{F}, q, J, \rho] + [\mathcal{F}, -q, -J, -\rho] = 0$. Indeed, we consider the locally constant lagrangian subsheaf $\mathcal{L} \subset \mathcal{F} \oplus \mathcal{F}$ which is the image of the diagonal embedding $\mathcal{F} \to \mathcal{F} \oplus \mathcal{F} =: \bar{\mathcal{F}}$. Let $\bar{J} := J \oplus (-J)$ be the metric structure on $\bar{F} := \mathtt{bundle}(\bar{\mathcal{F}})$. Then $\nabla^{\bar{F}_\mathbb{C},\bar{J}}$ already preserves the decomposition $\bar{F} = L \oplus \bar{J}L$. Hence $\tilde{p}(\bar{\mathcal{F}}, \bar{q}, \bar{J}, \mathcal{L}) = 0$, where $\bar{q} := q \oplus -q$. Therefore, we have $(\bar{\mathcal{F}}, \bar{q}, \bar{J}, 0) \sim 0$. □

3.4.2. *Change of the metric structure.* We consider $(\mathcal{F}, q)$ and two metric structures J, J' on the associated bundle F. We define the transgression form

$$\tilde{p}(\nabla^F, J', J) := \tilde{\mathbf{ch}}(F_\mathbb{C}, \nabla^{F_\mathbb{C},J'}, \nabla^{F_\mathbb{C},J})$$

such that we have

$$d\tilde{p}(\nabla^F, J', J) = p(\nabla^F, J') - p(\nabla^F, J).$$

LEMMA 3.10. *In $\bar{L}_\epsilon(M)$ we have*

$$[\mathcal{F}, q, J', \rho'] - [\mathcal{F}, q, J, \rho] = [0, 0, 0, \rho' - \rho - \tilde{p}(\nabla^F, J', J)].$$

PROOF. Let $\mathcal{M}$ be the space of all metric structures on M and $\operatorname{pr}\colon M \times \mathcal{M} \to M$ be the projection. Let $\tilde{\mathcal{F}} := \operatorname{pr}^*\mathcal{F}$, $\tilde{q} := \operatorname{pr}^* q$. On the associated bundle $\tilde{F}$ we consider the tautological metric structure $\tilde{J}$ which over $M \times \{J'\}$ restricts to J'.

Furthermore, we consider the sheaf $\hat{\mathcal{F}} := \mathcal{F} \oplus \mathcal{F}$ with the form $\hat{q} = q \oplus (-q)$ and the metric structure $\hat{J} = \tilde{J} \oplus (-J)$ on the associated bundle $\hat{F}$.

Finally, we define $\bar{\mathcal{F}} := \operatorname{pr}^*\hat{\mathcal{F}}$ and $\bar{q} = \operatorname{pr}^*\hat{q}$. As metric structure on $\bar{F}$ we take $\bar{J} = \tilde{J} \oplus (-\operatorname{pr}^* J)$. We set $\bar{J}_0 := \operatorname{pr}^* J \oplus (-\operatorname{pr}^* J)$.

Let $\mathcal{L} \subset \hat{\mathcal{F}}$ be the image of the diagonal $\mathcal{F} \to \mathcal{F} \oplus \mathcal{F}$. Furthermore, we set $\bar{\mathcal{L}} = \operatorname{pr}^*\mathcal{L}$. We define the form $\omega := \tilde{p}(\nabla^{\tilde{F}}, \tilde{J}, \operatorname{pr}^* J) + \tilde{p}(\bar{\mathcal{F}}, \bar{q}, \bar{J}, \bar{\mathcal{L}})$ on $M \times \mathcal{M}$. It is closed, since

$$\begin{aligned} d\omega &= p(\nabla^{\tilde{F}}, \tilde{J}) - p(\nabla^{\tilde{F}}, \operatorname{pr}^* J) - \mathbf{ch}(\nabla^{\bar{F}_\mathbb{C},\bar{J}}) \\ &= p(\nabla^{\tilde{F}}, \tilde{J}) - p(\nabla^{\tilde{F}}, \operatorname{pr}^* J) - p(\nabla^{\tilde{F}}, \tilde{J}) + p(\nabla^{\tilde{F}}, \operatorname{pr}^* J) = 0. \end{aligned}$$

The metric structure J' induces an embedding $i_{J'}\colon M \to M \times \{J'\} \to M \times \mathcal{M}$. We have

$$\tilde{p}(\nabla^F, J', J) = i_{J'}^* \tilde{p}(\nabla^{\tilde{F}}, \tilde{J}, \mathrm{pr}^* J)$$
$$\tilde{p}(\hat{F}, \hat{q}, \hat{J}, \mathcal{L}) = i_{J'}^* \tilde{p}(\bar{\mathcal{F}}, \bar{q}, \bar{J}, \bar{\mathcal{L}}).$$

Therefore,

$$\tilde{p}(\nabla^F, J', J) + \tilde{p}(\hat{F}, \hat{q}, \hat{J}, \mathcal{L}) = i_{J'}^* \omega.$$

Since ω is closed, $i_{J'}$ is homotopic to i_J, and

$$i_J^* \tilde{p}(\nabla^{\tilde{F}}, \mathrm{pr}^* J, \tilde{J}) = 0 = i_J^* \tilde{p}(\bar{\mathcal{F}}, \bar{q}, \bar{J}, \bar{\mathcal{L}}),$$

we conclude that $\tilde{p}(\nabla^F, J', J) + \tilde{p}(\hat{F}, \hat{q}, \hat{J}, \mathcal{L})$ is exact. In $\bar{L}_\epsilon(M)$ we have the identity

$$\begin{aligned}[][\mathcal{F}, q, J', \rho'] - [\mathcal{F}, q, J, \rho] &= [\hat{\mathcal{F}}, \hat{q}, \hat{J}, \rho' - \rho] \\ &= [0,0,0,\rho' - \rho + p(\hat{F}, \hat{q}, \hat{J}, \mathcal{L})] = [0,0,0,\rho' - \rho - \tilde{p}(\nabla^F, J', J)].\end{aligned}$$

□

REMARK 3.11. In [**21**, Def. 23], Lott defined a similar functor $\bar{L}^{\mathrm{Lott}}_\epsilon(M)$. It is also obtained as a quotient of $\hat{\bar{L}}_\epsilon(M)$ by an equivalence relation $\sim^{\mathrm{Lott}}$. This relation is the minimal equivalence relation which is compatible with the semigroup operation and which contains $(\mathcal{F}, q, J', \rho') \sim (\mathcal{F}, q, J, \rho) + (0,0,0,\rho' - \rho - \tilde{p}(\nabla^F, J', J))$, and "lagrangian reduction" in the special case that the lagrangian subsheaf admits a lagrangian complement. As a consequence of Lemma 3.10 the relation $\sim$ is coarser than $\sim^{\mathrm{Lott}}$ so that we have a natural surjective map $\bar{L}^{\mathrm{Lott}}_\epsilon(M) \to \bar{L}_\epsilon(M)$.

3.4.3. *The module structure.*

DEFINITION 3.12. The graded module structure of $\bar{L}(M) = \bigoplus_{\epsilon \in \mathbb{Z}_2} \bar{L}_\epsilon(M)$ over $L(M)$ is defined by

$$[\mathcal{F}, q, J, \rho] \bullet [\mathcal{E}, p] := \frac{\sqrt{\epsilon}\sqrt{\epsilon'}}{\sqrt{\epsilon\epsilon'}} \left[\mathcal{F} \otimes \mathcal{E}, q \otimes p, J \otimes J^E, \rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}}, J^E})\right],$$

where J^E is any metric structure on $\mathtt{bundle}(\mathcal{E})$.

LEMMA 3.13. *The $L(M)$-module structure of $\bar{L}(M)$ is well-defined.*

PROOF. Let J_i^E, $i = 0,1$ be two choices of metric structures on $(\mathcal{E}, q)$. Then we must show that

$$\left[\mathcal{F} \otimes \mathcal{E}, q \otimes p, J \otimes J_0^E, \rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}}, J_0^E})\right] = \left[\mathcal{F} \otimes \mathcal{E}, q \otimes p, J \otimes J_1^E, \rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}}, J_1^E})\right].$$

In view of Lemma 3.10 we must show that

$$\rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}}, J_0^E}) - \rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}}, J_1^E}) - \tilde{p}(\nabla^{F \otimes E}, J \otimes J_0^E, J \otimes J_1^E)$$

is exact. We compute

$$\begin{aligned}\rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}},J_0^E}) - \rho \wedge \mathbf{ch}(\nabla^{E_{\mathbb{C}},J_1^E}) - \tilde{p}(\nabla^{F\otimes E}, J\otimes J_0^E, J\otimes J_1^E)\\ = \rho \wedge d\tilde{\mathbf{ch}}(\nabla^{E_{\mathbb{C}},J_0^E}, \nabla^{E_{\mathbb{C}},J_1^E}) - \tilde{\mathbf{ch}}(\nabla^{F_{\mathbb{C}}\otimes E_{\mathbb{C}},J\otimes J_0^E}, \nabla^{F_{\mathbb{C}}\otimes E_{\mathbb{C}},J\otimes J_1^E})\\ = -d\big(\rho\wedge \tilde{\mathbf{ch}}(\nabla^{E_{\mathbb{C}},J_0^E}, \nabla^{E_{\mathbb{C}},J_1^E})\big) + d\rho \wedge \tilde{\mathbf{ch}}(\nabla^{E_{\mathbb{C}},J_0^E}, \nabla^{E_{\mathbb{C}},J_1^E})\\ - \mathbf{ch}(\nabla^{F_{\mathbb{C}},J}) \wedge \tilde{\mathbf{ch}}(\nabla^{E_{\mathbb{C}},J_0^E}, \nabla^{E_{\mathbb{C}},J_1^E})\\ = -d\big(\rho\wedge \tilde{\mathbf{ch}}(\nabla^{E_{\mathbb{C}},J_0^E}, \nabla^{E_{\mathbb{C}},J_1^E})\big).\end{aligned}$$

□

3.4.4. *Complexes and η-forms.* We consider a pair $(\mathcal{F}, q)$ together with a $\mathbb{Z}$-grading N of length n and a compatible differential v (cf. Definition 2.4).

DEFINITION 3.14. A metric structure J is called compatible with N if $JN + NJ = nJ$.

Equivalently, one could require that the decomposition $F = \bigoplus_{k=0}^n F^k$ of F into eigenspaces of N is orthogonal with respect to the metric induced by J.

The adjoint of v with respect to the metric induced by J is given by $v^{*_J} = -\epsilon J \circ v \circ J$. Set $V = v^{*_J} + v$. By Hodge theory we can canonically identify the cohomology bundle $H = \mathtt{bundle}(\mathcal{H})$ with $\ker(v) \cap \ker(v^{*_J})$. Since the latter is J-invariant we obtain an induced metric structure $J^H := J\big|_H$ for $(\mathcal{H}, q_{\mathcal{H}})$.

The theory of characteristic classes and forms extends to superconnections (cf. [**27**], [**3**, §1.4]). Since we consider several $\mathbb{Z}_2$-gradings at the same time we will speak of z-superconnection in order to indicate that z is the relevant grading. In particular, given a z-superconnection A we set

$$\mathbf{ch}(A) := \varphi \mathrm{Tr}\Big[\, z\, \exp(-A^2)\Big],$$

where φ multiplies a p-form by $(2\pi i)^{-p/2}$. Furthermore, if A' is a $(-1)^N$-superconnection on F, then the odd part A with respect to $z^J = \frac{1}{\sqrt{\epsilon}}J$ is a z^J-superconnection. In this case we define $p(A', J) := \mathbf{ch}(A)$.

Let $\tilde{M} := (0,\infty) \times M$ and $\mathrm{pr}\colon \tilde{M} \to M$ be the projection. We consider $(\tilde{\mathcal{F}}, \nabla^{\tilde{F}}) = \mathrm{pr}^*(\mathcal{F}, \nabla^F)$ and $\tilde{q} := \mathrm{pr}^* q$ on $\tilde{M}$. It has the $\mathbb{Z}$-grading $\tilde{N} := \mathrm{pr}^* N$ of length n and the compatible differential $\tilde{v} := \mathrm{pr}^* v$. We further consider the compatible metric structure $\tilde{J}$ which restricts to $t^{-N/2+n/4} \circ J \circ t^{N/2-n/4}$ on $\{t\}\times M$. We have a $(-1)^{\tilde{N}}$-superconnection $A' := \nabla^{\tilde{F}} + \tilde{v}$. Let us decompose $p(A', \tilde{J}) = dt\wedge\gamma + r$, where r does not contain dt. Here $\gamma\colon (0,\infty) \to \Omega(M)$ is a smooth family of forms on M. More precisely,

$$\gamma(t) = -(2\pi i)^{-1/2} \frac{1}{4\sqrt{t}}\, \varphi\, \mathrm{Tr}\Big[z^J V \exp(-(\nabla^{F_{\mathbb{C}},J} + \frac{\sqrt{t}}{2} V)^2)\Big].$$

Remark that $JV = -VJ$. Thus V is odd with respect to the $\mathbb{Z}_2$-grading z^J. By [**21**, Prop. 28 and Prop. 29], (also cf. [**8**, (2.26)]) the following integral exists.

DEFINITION 3.15. [**21**, Def. 32]. The η-form $\tilde{\eta}(\mathcal{F}, N, J, v) \in \Omega(M)$ is defined by

$$\tilde{\eta}(\mathcal{F}, N, J, v) := -\int_0^\infty \gamma(t)dt.$$

It was shown in [**21**, (216)] (cf. [**7**, Thm. 2.8]) that

$$d\tilde{\eta}(\mathcal{F}, N, J, v) = p(\nabla^F, J) - p(\nabla^H, J^H). \tag{3.2}$$

PROPOSITION 3.16. *In $\bar{L}_\epsilon(M)$ we have*

$$[\mathcal{F}, q, J, \rho] = [\mathcal{H}, q_\mathcal{H}, J^H, \rho - \tilde{\eta}(\mathcal{F}, N, J, v)].$$

PROOF. We consider $(\mathcal{G}, q_\mathcal{G}) := (\mathcal{F} \oplus \mathcal{H}, q \oplus -q_\mathcal{H})$. It admits a locally constant lagrangian subsheaf $\mathcal{L}$ given by the image of $\ker(v) \hookrightarrow \mathcal{F} \oplus \mathcal{H}$, $x \mapsto x \oplus [x]$. We consider the metric structure $J^G := J \oplus -J^H$. Then we have

$$\begin{aligned}
&[\mathcal{F}, q, J, \rho] - [\mathcal{H}, q_\mathcal{H}, J^H, \rho - \tilde{\eta}(\mathcal{F}, N, J, v)] \\
&\qquad = [\mathcal{G}, q_\mathcal{G}, J^G, \tilde{\eta}(\mathcal{F}, N, J, v)] \\
&\qquad = [0, 0, 0, \tilde{\eta}(\mathcal{F}, N, J, v) + \tilde{p}(\mathcal{G}, q_\mathcal{G}, J^G, \mathcal{L})].
\end{aligned}$$

It remains to show that $\tilde{\eta}(\mathcal{F}, N, J, v) + \tilde{p}(\mathcal{G}, q_\mathcal{G}, J^G, \mathcal{L})$ is exact. This will be a consequence of the following general result.

LEMMA 3.17. *Let* **Eta** *be a construction which associates to a tuple $(F, \nabla^F, Q, N, J, v)$ (a real vector bundle with flat connection, parallel ϵ-symmetric form, compatible parallel $\mathbb{Z}$-grading, a metric structure (not necessarily parallel), and compatible parallel differential) over a manifold M a form* $\mathbf{Eta}(F, \nabla^F, Q, N, J, v) \in \Omega(M)$ *such that*
(1) $d\mathbf{Eta}(F, \nabla^F, Q, N, J, v) = p(\nabla^F, J) - p(\nabla^H, J^H)$.
(2) *For a smooth map $f \colon M' \to M$ we have*

$$f^*\mathbf{Eta}(F, \nabla^F, Q, N, J, v) = \mathbf{Eta}(f^*F, f^*\nabla^F, f^*Q, f^*N, f^*J, f^*v).$$

(3) $\mathbf{Eta}(F, \nabla^F, Q, N, J, v)$ *depends smoothly on the data ∇^F, Q, J (note that we fix N and v).*
(4) *If $(F, \nabla^F, Q, N, J, v)$ splits, then* $\mathbf{Eta}(F, \nabla^F, Q, N, J, v) = 0$. *Here $(F, \nabla^F, Q, N, J, v)$ splits if and only if the complex (F, N, v) is of the form*

$$0 \to E^0 \oplus H^0 \to E^0 \oplus H^1 \oplus E^1 \to \dots\dots E^{n-2} \oplus H^{n-1} \oplus E^{n-1} \to \oplus E^{n-1} \oplus H^n \to 0$$

with flat vector bundles E^i, H^i, where the differential v is given by the obvious maps. Furthermore, $E^{n-i-1} = (E^i)^*$, $H^i = (H^{n-i})^*$, and this identification gives (with the suitable signs) the form Q. The metric structure J shall induce a metric such that this splitting is orthogonal and the identifications $E^{n-i-1} = (E^i)^*$, $H^i = (H^{n-i})^*$ are isometries.

Then $\mathbf{Eta}(F, \nabla^F, Q, N, J, v) - \tilde{\eta}(\mathcal{F}, N, J, v)$ *is exact.*

PROOF. The proof is very similar to the axiomatic characterizations of analytic torsion forms [**12**, A 1.2] (in the acyclic case) and its extension to the case with cohomology [**24**, Lemma 3.1].

First one shows that the η-form $\tilde{\eta}(\mathcal{F}, N, J, v)$ has the four properties above. Given another construction $\mathbf{Eta}(F, \nabla^F, Q, N, J, v)$ with these properties one considers the difference $\Delta(F, \nabla^F, Q, N, J, v) := \mathbf{Eta}(F, \nabla^F, Q, N, J, v) - \tilde{\eta}(\mathcal{F}, N, J, v)$. Then one applies the argument of [**12**, A1.2] to show that $\Delta(F, \nabla^F, Q, N, J, v)$ is exact. First observe by property (1) that $\Delta(F, \nabla^F, Q, N, J, v)$ is closed. Then the idea is that we can deform ∇, Q, J to approximate the split case. Using the properties (2), (3), and (4), we see that $\Delta(F, \nabla^F, Q, N, J, v)$ can be smoothly deformed to zero without changing its cohomology class. This implies exactness of $\Delta(F, \nabla^F, Q, N, J, v)$. □

We now finish the proof of Lemma 3.16. Given $(F, \nabla^F, Q, N, J, v)$ we can construct $(\mathcal{G}, q_{\mathcal{G}}, J^G, \mathcal{L})$ as above. Let $\mathbf{Eta}(F, \nabla^F, Q, N, J, v) := -\tilde{p}(\mathcal{G}, q_{\mathcal{G}}, J^G, \mathcal{L})$. One checks that it satisfies the four conditions of the lemma. We conclude that $\tilde{\eta}(\mathcal{F}, N, J, v) + \tilde{p}(\mathcal{G}, q_{\mathcal{G}}, J^G, \mathcal{L})$ is exact. □

3.4.5. *Filtration.* Let $(\mathcal{F}, q, J, \rho)$ be given with a compatible filtration $\left(F^i\mathcal{F}\right)_i$ of length n. We assume that we have chosen a metric structure $J^{\mathrm{Gr}(\mathcal{F})}$ on $\big(\mathrm{Gr}(\mathcal{F}),$ $\mathrm{Gr}(q)\big)$ which is compatible with the $\mathbb{Z}$-grading. For $i < n/2$ we consider the following sheaves

$$\mathcal{E}_i := F^i\mathcal{F}/F^{n-i+1}\mathcal{F} \oplus \mathrm{Gr}^{n-i}(\mathcal{F}) \oplus \mathrm{Gr}^i(\mathcal{F}).$$

We introduce forms q_i and metric structures J_i by induction on i.

The sheaf $\mathcal{E}_i$ admits a $\mathbb{Z}$-grading N_i of length 2 such that $\mathcal{E}_i^0 = \mathrm{Gr}^{n-i}(\mathcal{F})$, $\mathcal{E}_i^1 = F^i\mathcal{F}/F^{n-i+1}$, and $\mathcal{E}_i^2 = \mathrm{Gr}^i(\mathcal{F})$. The differential v_i is given by the embedding $\mathrm{Gr}^{n-i}(\mathcal{F}) \hookrightarrow F^i\mathcal{F}/F^{n-i+1}\mathcal{F}$ and the negative of the projection $F^i\mathcal{F}/F^{n-i+1}\mathcal{F} \to \mathrm{Gr}^i(\mathcal{F})$. For $i = 0$ we have an obvious form $q_0 := q_{\mathcal{F}} \oplus \mathrm{Gr}(q)|_{\mathrm{Gr}^0(\mathcal{F})\oplus\mathrm{Gr}^n(\mathcal{F})}$ and a compatible metric structure $J_0 := J^F \oplus J^{\mathrm{Gr}(F)}|_{\mathrm{Gr}^0(\mathcal{F})\oplus\mathrm{Gr}^n(\mathcal{F})}$ such that v_0 is compatible.

Assume that we already have defined q_i such that the grading and the differential are compatible. We identify $F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}$ with the cohomology of v_i. Therefore, we have an induced form $q_{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}}$. We now define $q_{i+1} := q_{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}}$ $\oplus \mathrm{Gr}(q)|_{\mathrm{Gr}^{n-i-1}(\mathcal{F})\oplus\mathrm{Gr}^{i+1}(\mathcal{F})}$. The grading and the differential of $\mathcal{E}_{i+1}$ are again compatible with q_{i+1}.

Assume that we have already defined J_i such that it is compatible with the $\mathbb{Z}$-grading. Then we have an induced metric structure $J^{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}}$. We now define $J_{i+1} := J^{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}} \oplus J^{\mathrm{Gr}(F)}|_{\mathrm{Gr}^{n-i-1}(\mathcal{F})\oplus\mathrm{Gr}^{i+1}(\mathcal{F})}$. Then J_{i+1} is again compatible with the $\mathbb{Z}$-grading.

DEFINITION 3.18. We define the η-form $\tilde{\eta}(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}) \in \Omega(M)$ of the filtration by

$$\tilde{\eta}(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}) := \sum_{i=0}^{[(n-1)/2]} \tilde{\eta}(\mathcal{E}_i, N_i, J_i, v_i) + \tilde{p}\left(\mathrm{Gr}^{n/2} F, J_{n/2}, J^{\mathrm{Gr}(F)}|_{\mathrm{Gr}^{n/2}(\mathcal{F})}\right).$$

Then

$$d, \tilde{\eta}(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}) = p(\nabla^F, J) - p(\nabla^{\mathrm{Gr}(\mathcal{F})}, J^{\mathrm{Gr}(\mathcal{F})}).$$

LEMMA 3.19. *In $\bar{L}_\epsilon(M)$ we have*

$$[\mathcal{F}, q, J, \rho] = \left[\mathrm{Gr}(\mathcal{F}), \mathrm{Gr}(q), J^{\mathrm{Gr}(\mathcal{F})}, \rho - \tilde{\eta}\big(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}\big)\right]$$
$$= \begin{cases} \left[\mathrm{Gr}^{[\frac{n}{2}]}(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^{[\frac{n}{2}]}(\mathcal{F})}, J^{\mathrm{Gr}(\mathcal{F})}|_{\mathrm{Gr}^{[\frac{n}{2}]}(\mathcal{F})}, \rho - \tilde{\eta}\big(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}\big)\right] & n \equiv 0\,(2), \\ \left[0, 0, 0, \rho - \tilde{\eta}\big(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})}\big)\right] & n \equiv 1\,(2). \end{cases}$$

PROOF. We reduce the length of the filtration by two as in the proof of Lemma 2.7. Note that

$$\left[\mathrm{Gr}^{n-i}(\mathcal{F}) \oplus \mathrm{Gr}^{i}(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F})}, J^{\mathrm{Gr}(\mathcal{F})}|_{\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F})}, 0\right] = 0$$

by lagrangian reduction along the lagrangian subsheaf $\mathrm{Gr}^{n-i}(\mathcal{F})$. In fact, in this case

$$\nabla^{(\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F}))_{\mathbb{C}}, J^{\mathrm{Gr}(\mathcal{F})}|_{\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F})}} = \nabla^{\oplus}$$

so that by Definition 3.7

$$\tilde{p}\Big(\mathrm{Gr}^{n-i}(\mathcal{F}) \oplus \mathrm{Gr}^{i}(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F})}, J^{\mathrm{Gr}(\mathcal{F})}|_{\mathrm{Gr}^{n-i}(\mathcal{F})\oplus \mathrm{Gr}^{i}(\mathcal{F})}, \mathrm{Gr}^{n-i}(\mathcal{F})\Big) = 0.$$

Therefore, we obtain by Lemma 3.16

$$\left[F^i\mathcal{F}/F^{n-i+1}\mathcal{F}, q_{F^i\mathcal{F}/F^{n-i+1}\mathcal{F}}, J^{F^i\mathcal{F}/F^{n-i+1}\mathcal{F}}, \rho - \sum_{j<i} \tilde{\eta}(\mathcal{E}_i, N_i, J_i, v_i)\right]$$
$$= \left[\mathcal{E}_i, q_i, J_i, \rho - \sum_{j<i} \tilde{\eta}(\mathcal{E}_i, N_i, J_i, v_i)\right]$$
$$= \left[F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}, q_{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}}, J^{F^{i+1}\mathcal{F}/F^{n-i}\mathcal{F}}, \rho - \sum_{j<i+1} \tilde{\eta}(\mathcal{E}_i, N_i, J_i, v_i)\right].$$

The assertion of the lemma now follows by induction on i. □

3.4.6. *Isotropic reduction.* Assume that we are given a tuple $(\mathcal{F}, q, J, \rho)$ and an isotropic locally constant subsheaf $\mathcal{I}$, i.e., $\mathcal{I} \subset \mathcal{I}^\perp$. We consider the filtration of length two of $\mathcal{F}$ such that $\mathcal{I} := F^2\mathcal{F}$, $\mathcal{I}^\perp := F^1\mathcal{F}$ so that we can identify $(\mathrm{Gr}^1(\mathcal{F}), \mathrm{Gr}(q)|_{\mathrm{Gr}^1(\mathcal{F})}) \cong (\mathcal{F}_\mathcal{I}, q_\mathcal{I})$. We choose any metric structure $J^{\mathrm{Gr}(\mathcal{F})}$. Let $J^{F_\mathcal{I}} := J^{\mathrm{Gr}(\mathcal{F})}|_{\mathrm{Gr}^1(\mathcal{F})}$. Then we have the form $\tilde{\eta}(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{Gr(\mathcal{F})})$. The following assertion is an immediate consequence of Lemma 3.19.

COROLLARY 3.20. *In $\bar{L}_\epsilon(M)$ we have*

$$[\mathcal{F}, q, J, \rho] = \left[\mathcal{F}_\mathcal{I}, q_\mathcal{I}, J^{F_\mathcal{I}}, \rho - \tilde{\eta}(\mathcal{F}, \mathrm{Gr}(\mathcal{F}), J, J^{\mathrm{Gr}(\mathcal{F})})\right].$$

Sometimes we would like to work with easier transgressed characteristic classes than η-forms. It is an interesting observation that one can do isotropic reduction without appealing to η-forms.

Assume again that we are given a tuple $(\mathcal{F}, q, J, \rho)$ and an isotropic locally constant subsheaf $\mathcal{I}$. Then we obtain an orthogonal decomposition $F = I \oplus (I^\perp \ominus I) \oplus (F \ominus I^\perp)$, here we denote by $F \ominus I^\perp$ the orthogonal complement sub-bundle of $I^\perp$ in F with respect to the metric $J^* \circ q$. Note that J induces an isomorphism $I \cong F \ominus I^\perp$. There is a natural identification of bundles $I^\perp \ominus I = F_\mathcal{I}$. The metric structure J restricts to $I^\perp \ominus I$ and therefore defines a metric structure $J^{F_\mathcal{I}}$ on $F_\mathcal{I}$.

We consider the $\mathbb{Z}$-grading on F which is given with respect to the decomposition above by $N = \mathrm{diag}(0, 1, 2)$. The connection ∇^F is upper-triangular with respect to this decomposition. Therefore,

$$\nabla^F_u := u^{-N} \nabla^F u^N \tag{3.3}$$

is regular at $u = 0$. In fact, $\nabla^F_0 = \nabla^I \oplus \nabla^{F_\mathcal{I}} \oplus \nabla^{F/I^\perp}$. The tuple (F, ∇^F_0, J) is thus isomorphic to the tuple obtained as follows: We consider the sum $\mathcal{F}_\mathcal{I} \oplus (\mathcal{I} \oplus \mathcal{F}/\mathcal{I}^\perp)$ with associated bundle $F_\mathcal{I} \oplus (I \oplus I)$, metric structure

$$J^{\mathcal{F}_\mathcal{I}} \oplus \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix},$$

and connection $\nabla^{\mathcal{F}_\mathcal{I}} \oplus \nabla^I \oplus (\nabla^I)^*$.

We now consider $\tilde{M} = [0, 1] \times M$. Let $\mathrm{pr}\colon \tilde{M} \to M$ be the projection, $\tilde{\mathcal{F}} = \mathrm{pr}^*\mathcal{F}$, $\tilde{q} := \mathrm{pr}^* q$. On the corresponding bundle we consider the metric structure $\tilde{J}$ which on $\{u\} \times M$ restricts to $J_u := u^N J u^{-N}$. We decompose $p(\nabla^{\tilde{F}}, \tilde{J}) = du \wedge \gamma + r$, where r does not contain du and where $\gamma\colon [0, 1] \to \Omega(M)$ is a smooth family of forms. By the discussion above, after shifting the rescaling from the metric structure to the connection we see that γ is regular at $u = 0$ so that we can make the following definition.

DEFINITION 3.21. We define

$$\tilde{p}(\mathcal{F}, q, J, \mathcal{I}) := -\int_0^1 \gamma(u) du.$$

If $\mathcal{I}$ is lagrangian, then it is easy to check that this definition coincides with the former Def. 3.7. Furthermore, note that

$$d\tilde{p}(\mathcal{F},q,J,\mathcal{I}) = p(\nabla^{F_\mathcal{I}}, J^{F_\mathcal{I}}) - p(\nabla^F, J).$$

Next we consider isotropic reduction in stages. Assume that $\mathcal{L}$ is a locally constant lagrangian subsheaf of $\mathcal{F}$ such that $\mathcal{I} \subset \mathcal{L}$. Then we obtain an induced lagrangian subsheaf $\mathcal{L}_\mathcal{I} \subset \mathcal{F}_\mathcal{I}$.

LEMMA 3.22. *Modulo exact forms we have*

$$\tilde{p}(\mathcal{F},q,J,\mathcal{L}) \equiv \tilde{p}(\mathcal{F}_\mathcal{I},q_\mathcal{I},J^{\mathcal{F}_\mathcal{I}},\mathcal{L}_\mathcal{I}) + \tilde{p}(\mathcal{F},q,J,\mathcal{I}).$$

PROOF. We consider the orthogonal decompositions

$$F = I \oplus (L \ominus I) \oplus (I^\perp \ominus L) \oplus (F \ominus I^\perp)$$

and the commuting $\mathbb{Z}$-gradings $M := \mathrm{diag}(0,1,1,2)$ and $K := \mathrm{diag}(0,0,1,1)$. Note that ∇^F is upper triangular with respect to these gradings. Thus, if we define $\nabla_{u,v} := u^{-M}v^{-K}\nabla^F v^K u^M$, then this family of connections extends to $(u,v) \in [0,1] \times [0,1]$. We identify $F \cong I \oplus F_\mathcal{I} \oplus I$ such that $F_\mathcal{I} \cong I^\perp \ominus I$ and the last component is identified with $F \ominus I^\perp$ by J. Furthermore, let $\nabla_v^{F_\mathcal{I}}$ be constructed as in (3.3) using $\mathcal{L}_\mathcal{I} \subset \mathcal{F}_\mathcal{I}$ and $J^{\mathcal{F}_\mathcal{I}}$. In this identification, we can write

$$\nabla_{0,v} = \nabla^I \oplus \nabla_v^{F_\mathcal{I}} \oplus (\nabla^I)^*.$$

We decompose $L := I \oplus (L \ominus I)$ and set $N := \mathrm{diag}(0,1)$. Then we define $\nabla_u^L := u^{-N}\nabla^L u^N$. We now identify $F = L \oplus L$ such that the second component goes to $F \ominus L$ via J. Then we can write

$$\nabla_{u,0} = \nabla_u^L \oplus (\nabla_u^L)^*.$$

Let us now consider the manifold $\bar{M} := [0,1] \times [0,1] \times M$. Let $\bar{F} := \mathrm{pr}^*F$ and $\bar{q} = \mathrm{pr}^*q$. We consider the metric structure $\bar{J}$ on $\bar{F}$ which restricts on $\{(u,v)\} \times M$ to $u^M v^K J v^{-K} u^{-M}$. This works for $uv \neq 0$. Nevertheless, it follows from the discussion above after shifting the rescaling from the metric structure to the connection that $p(\bar{\mathcal{F}},\bar{J})$ is a closed smooth form on $\bar{M}$. We decompose $p(\bar{\mathcal{F}},\bar{J}) = du \wedge dv \wedge \sigma + r$, where r contains at most one of du, dv, and $\sigma\colon [0,1]\times[0,1] \to \Omega(M)$ is a smooth family of forms. It follows from Stokes' theorem and the structure of $\nabla_{0,v}$ and $\nabla_{u,0}$ that

$$d\int_{[0,1]\times[0,1]} \sigma(u,v)dudv = \tilde{p}(\mathcal{F}_\mathcal{I},q_\mathcal{I},J^{\mathcal{F}_\mathcal{I}},\mathcal{L}_\mathcal{I}) + \tilde{p}(\mathcal{F},q,J,\mathcal{I}) - \tilde{p}(\mathcal{F},q,J,\mathcal{L}),$$

where the contributions come from the boundary components $\{u=0\}$, $\{v=1\}$, and $\{u=1\}$, respectively. This proves the assertion. □

LEMMA 3.23. *In $\bar{L}_\epsilon(M)$ we have* $[\mathcal{F},q,J,\rho] = [\mathcal{F}_\mathcal{I},q_\mathcal{I},J^{F_\mathcal{I}},\rho + \tilde{p}(\mathcal{F},q,J,\mathcal{I})]$.

PROOF. We consider the sheaf $\mathcal{G} := \mathcal{F} \oplus \mathcal{F}_\mathcal{I}$ together with the form $q_\mathcal{G} = q \oplus (-q_\mathcal{I})$. We have the lagrangian subsheaf $\mathcal{L}$ of $\mathcal{G}$ which is the image of the diagonal map $\mathcal{I}^\perp \to \mathcal{F} \oplus \mathcal{F}_\mathcal{I}$. It contains the isotropic subsheaf $\mathcal{I} \oplus 0$. We consider the metric structure $J^G := J \oplus -J^{F_\mathcal{I}}$. Then we have by Lemma 3.22

$$\begin{aligned}
[\mathcal{F}, q, J, \rho] &- [\mathcal{F}_\mathcal{I}, q_\mathcal{I}, J^{F_\mathcal{I}}, \rho + \tilde{p}(\mathcal{F}, q, J, \mathcal{I})] \\
&= [\mathcal{G}, q_\mathcal{G}, J^G, -\tilde{p}(\mathcal{F}, q, J, \mathcal{I})] \\
&= [0, 0, 0, \tilde{p}(\mathcal{G}, q_\mathcal{G}, J^G, \mathcal{L}) - \tilde{p}(\mathcal{F}, q, J, \mathcal{I})] \\
&= [0, 0, 0, \tilde{p}(\mathcal{G}, q_\mathcal{G}, J^G, \mathcal{I}) + \tilde{p}(\mathcal{G}_\mathcal{I}, (q_\mathcal{G})_\mathcal{I}, J^{G_\mathcal{I}}, \mathcal{L}_\mathcal{I}) - \tilde{p}(\mathcal{F}, q, J, \mathcal{I})]
\end{aligned}$$

Now note that $\tilde{p}(\mathcal{G}, q_\mathcal{G}, J^G, \mathcal{I}) = \tilde{p}(\mathcal{F}, q, J, \mathcal{I})$ and $\tilde{p}(\mathcal{G}_\mathcal{I}, (q_\mathcal{G})_\mathcal{I}, J^{G_\mathcal{I}}, \mathcal{L}_\mathcal{I}) = 0$. The reason for the first equality is that ∇^G and J^G preserve the decomposition $G = F \oplus F_\mathcal{I}$ and that $I \subset F \oplus 0$. The second identity can be seen by noting that $\nabla^{G_\mathcal{I}}$ preserves the decomposition $G_\mathcal{I} = L_\mathcal{I} \oplus J^{G_\mathcal{I}} L_\mathcal{I}$. □

3.5. Functorial properties.

3.5.1. *Homotopy invariance of pull-back.* If $f\colon M \to N$ is a smooth map of manifolds, then we obtain an induced map $f^*\colon \bar{L}_\epsilon(N) \to \bar{L}_\epsilon(M)$, which is given on generators by pull-back of structures.

LEMMA 3.24. *f^* only depends on the smooth homotopy class of f.*

PROOF. Consider a generator $(\mathcal{F}, q, J, \rho)$. Let $(f_t)_{t\in[0,1]}$ be a smooth homotopy of maps from M to N. Then we have an isomorphism $\phi\colon (f_0^*\mathcal{F}, f_0^* q) \to (f_1^*\mathcal{F}, f_1^* q)$. Let $\tilde{M} := [0,1] \times M$ and $H\colon \tilde{M} \to N$ be induced by (f_t). We compute

$$\begin{aligned}
f_0^*\rho - f_1^*\rho - \tilde{p}(\nabla^{f_0^* F}, f_0^* J, \phi^* f_1^* J) &\equiv f_0^*\rho - f_1^*\rho - \int_{\tilde{M}/M} p(\nabla^{H^*F}, H^*J) \\
&= -d\int_{\tilde{M}/M} H^*\rho + \int_{\tilde{M}/M} dH^*\rho - \int_{\tilde{M}/M} p(\nabla^{H^*F}, H^*J) \\
&\equiv \int_{\tilde{M}/M} (p(\nabla^{H^*F}, H^*J) - p(\nabla^{H^*F}, H^*J)) = 0,
\end{aligned}$$

where $\equiv$ means equality modulo exact forms. It now follows from Lemma 3.10 that

$$[f_0^*\mathcal{F}, f_0^* q, f_0^* J, f_0^*\rho] = [f_1^*\mathcal{F}, f_1^* q, f_1^* J, f_1^*\rho].$$

□

One can also check that the pull-back f^* is compatible with the $L(M)$ (resp. $L(N)$)-module structures.

3.5.2. *Exact sequences.* We define natural maps $H^{\mathrm{odd}}(M, \mathbb{R}) \to \bar{L}(M)$ and $\bar{L}(M) \to L(M)$ by $[\rho] \mapsto [0, 0, 0, \rho]$ and $[\mathcal{F}, q, J, \rho] \mapsto [\mathcal{F}, q]$, respectively. We consider $H^{\mathrm{odd}}(M, \mathbb{R})$ as an $L(M)$-module such that every element of $L(M)$ acts trivially. $H^{\mathrm{ev}}(M, \mathbb{R})$ becomes an $L(M)$-module via the ring structure of $H^{\mathrm{ev}}(M, \mathbb{R})$ and the homomorphism $\mathbf{ch} \circ b\colon L(M) \to H^{\mathrm{ev}}(M, \mathbb{R})$.

LEMMA 3.25. *We have the exact sequence of $L(M)$-modules*

$$H^{\mathrm{odd}}(M,\mathbb{R}) \longrightarrow \bar{L}(M) \longrightarrow L(M) \longrightarrow H^{\mathrm{ev}}(M,\mathbb{R}).$$

PROOF. It is obvious from the definition of the maps that the composition of two of them vanishes. Therefore, it remains to check exactness. We have the following commutative diagram

$$\begin{array}{ccccccc} H^{\mathrm{odd}}(M,\mathbb{R}) & \longrightarrow & \bar{L}^{\mathrm{Lott}}(M) & \longrightarrow & L^{\mathrm{Lott}}(M) & \longrightarrow & H^{\mathrm{ev}}(M,\mathbb{R}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^{\mathrm{odd}}(M,\mathbb{R}) & \longrightarrow & \bar{L}(M) & \longrightarrow & L(M) & \longrightarrow & H^{\mathrm{ev}}(M,\mathbb{R}). \end{array}$$

The upper sequence is exact by [**21**, Prop. 21]. The vertical maps are all surjective. It follows by diagram chasing that the lower sequence is exact at $L(M)$.

We now show exactness at $\bar{L}(M)$. Let $(\mathcal{F},q,J,\rho)$ be given such that $[\mathcal{F},q]=0$. Then there exists $(\mathcal{F}_1,q_1)$ which admits a lagrangian subsheaf $\mathcal{L}_1$ such that $(\mathcal{F}\oplus\mathcal{F}_1,q\oplus q_1)$ also admits a lagrangian subsheaf $\mathcal{L}$. We choose a metric structure J_1 and define ρ_1 such that $[\mathcal{F}_1,q_1,J_1,\rho_1]=0$. In fact we must take $\rho_1 := -\tilde{p}(\mathcal{F}_1,q_1,J_1,\mathcal{L}_1)$. Lagrangian reduction by $\mathcal{L}$ shows that $[\mathcal{F}\oplus\mathcal{F}_1,q\oplus q_1,J\oplus J_1,\rho+\rho_1]=[0,0,0,\omega]$ for the form $\omega:=\rho+\rho_1+\tilde{p}(\mathcal{F}\oplus\mathcal{F}_1,q\oplus q_1,J\oplus J_1,\mathcal{L})$. It follows that $[\mathcal{F},q,J,\rho]=[0,0,0,\omega]$ comes from $H^{\mathrm{odd}}(M,\mathbb{R})$. □

3.5.3. *Injectivity of $H^{\mathrm{odd}}(M,\mathbb{R})\to\bar{L}(M)$.*

PROPOSITION 3.26. *The map $H^{\mathrm{odd}}(M,\mathbb{R})\to\bar{L}(M)$ is injective.*

PROOF. Let $\omega\in\Omega^{\mathrm{odd}}(M)$ be a closed form. If $[0,0,0,\omega]=0$ in $\bar{L}(M)$, then, by Definition 3.8, there exists $(\mathcal{F},q,J,\rho)$ together with two lagrangian subsheaves $\mathcal{L}_0$, $\mathcal{L}_1$ such that

$$[\omega]=\left[\tilde{p}(\mathcal{F},q,J,\mathcal{L}_0)-\tilde{p}(\mathcal{F},q,J,\mathcal{L}_1)\right].$$

We claim that $\left[\tilde{p}(\mathcal{F},q,J,\mathcal{L}_0)-\tilde{p}(\mathcal{F},q,J,\mathcal{L}_1)\right]$ is independent of J. To show this we consider the space $\mathcal{M}$ of all metric structures and $\tilde{M}:=M\times\mathcal{M}$. Furthermore, let $(\tilde{\mathcal{F}},\tilde{q},\tilde{\rho}):=\mathrm{pr}^*(\mathcal{F},q,\rho)$. We define the metric structure $\tilde{J}$ on $\tilde{F}$ such that it restricts to J' on $M\times\{J'\}$. Using the two lagrangian subspaces $\tilde{\mathcal{L}}_i:=\mathrm{pr}^*\mathcal{L}_i$ we define the closed form

$$\alpha:=\tilde{p}(\tilde{\mathcal{F}},\tilde{q},\tilde{J},\tilde{\mathcal{L}}_0)-\tilde{p}(\tilde{\mathcal{F}},\tilde{q},\tilde{J},\tilde{\mathcal{L}}_1).$$

The metric structure J provides an embedding $i_J\colon M\to\tilde{M}$, and we have $i_J^*\alpha=\tilde{p}(\mathcal{F},q,J,\mathcal{L}_0)-\tilde{p}(\mathcal{F},q,J,\mathcal{L}_1)$. Since the embeddings I_J and $I_{J'}$ for two metric structures J,J' are homotopic, the classes $[i_J^*\alpha]$ and $[i_{J'}^*\alpha]$ coincide. This proves the claim.

We next claim that we can assume that $\mathcal{L}_0\cap\mathcal{L}_1=\{0\}$. Note that $\mathcal{I}:=\mathcal{L}_0\cap\mathcal{L}_1$ is an isotropic subsheaf. The reduction $(\mathcal{F}_{\mathcal{I}},q_{\mathcal{I}})$ admits two induced lagrangian subsheaves $(\mathcal{L}_i)_{\mathcal{I}}$. We have by Lemma 3.22 that

$$\left[\tilde{p}(\mathcal{F},q,J,\mathcal{L}_0)-\tilde{p}(\mathcal{F},q,J,\mathcal{L}_1)\right]=\left[\tilde{p}(\mathcal{F}_{\mathcal{I}},q_{\mathcal{I}},J^{F_{\mathcal{I}}},(\mathcal{L}_0)_{\mathcal{I}})-\tilde{p}(\mathcal{F}_{\mathcal{I}},q_{\mathcal{I}},J^{F_{\mathcal{I}}},(\mathcal{L}_1)_{\mathcal{I}})\right].$$

Thus we can replace $(\mathcal{F}, q, J, \rho)$ by $(\mathcal{F}_{\mathcal{I}}, q_{\mathcal{I}}, J^{F_{\mathcal{I}}}, \rho + p(\mathcal{F}, q, J, \mathcal{I}))$ and $\mathcal{L}_i$ by $(\mathcal{L}_i)_{\mathcal{I}}$.

Let us now assume that $L_0 \cap L_1 = \{0\}$. Then (cf. e.g. [**26**, Prop. 2.50]) we can choose J such that $JL_i = L_{1-i}$, $i = 0, 1$. With this choice $\tilde{p}(\mathcal{F}, q, J, \mathcal{L}_0) - \tilde{p}(\mathcal{F}, q, J, \mathcal{L}_1) = 0$ so that $[\omega] = 0$. □

3.5.4. *The transformation to $K^{-1}_{\mathbb{R}/\mathbb{Z}}$-theory.* We construct a natural transformation $\bar{\gamma}\colon \bar{L} \to K^{-1}_{\mathbb{R}/\mathbb{Z}}$. Let M be a manifold. Note that $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ is a module over the ring $K^0(M)$ and, therefore, by $b\colon L(M) \to K^0(M)$, a $L(M)$-module. We in fact construct morphism of $L(M)$-modules $\bar{\gamma}_M : \bar{L}(M) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ which depends naturally on M.

To define this morphism on generators $(\mathcal{F}, q, J, \rho)$ we use the geometric definition of $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ in terms of generators and relations which was recalled in Subsection 3.2.2.

We define $\bar{\gamma}_M\colon \bar{L}(M) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ by $\bar{\gamma}_M[\mathcal{F}, q, J, \rho] = [F_{\mathbb{C}}, h^{F_{\mathbb{C}}}, \nabla^{F_{\mathbb{C}}, J}, \rho]$, where $h^{F_{\mathbb{C}}}$ is the hermitian extension of the metric $J^* \circ Q$ on F.

Lemma 3.27. *$\bar{\gamma}_M$ is well-defined and has the properties as stated above.*

Proof. Since $\bar{\gamma}_M$ is induced by an obvious homomorphism $\hat{\bar{\gamma}}_M\colon \hat{\bar{L}}(M) \to K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ of semigroups (which is natural in M) it suffices to show that $\hat{\bar{\gamma}}_M$ is compatible with lagrangian reduction and that it induces a $L(M)$-module isomorphism. The latter property we leave as an exercise.

Let $\mathcal{L} \subset \mathcal{F}$ be a locally constant lagrangian subsheaf. Then $[\mathcal{F}, q, J, \rho] = [0, 0, 0, \rho + \tilde{p}(\mathcal{F}, q, J, \mathcal{L})]$. In $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ we have (using $h^{F_{\mathbb{C}}} = h^L \oplus h^L$, $\nabla^{\oplus} = \nabla^L \oplus \nabla^L$) that

$$\begin{aligned}[F_{\mathbb{C}}, h^{F_{\mathbb{C}}}, \nabla^{F_{\mathbb{C}}, J}, \rho] &= [F_{\mathbb{C}}, h^{F_{\mathbb{C}}}, \nabla^{\oplus}, \rho + \tilde{\mathbf{ch}}(\nabla^{\oplus}, \nabla^{F_{\mathbb{C}}, J})] \\ &= [L \oplus L, h^L \oplus h^L, \nabla^L \oplus \nabla^L, \rho + \tilde{p}(\mathcal{F}, q, J, \mathcal{L})] \\ &= [0, 0, 0, \rho + \tilde{p}(\mathcal{F}, q, J, \mathcal{L})].\end{aligned}$$

This shows that γ_M is well-defined. □

As an immediate consequence of the definition we get:

Corollary 3.28. *The following diagram commutes:*

$$\begin{array}{ccc} \bar{L}(M) & \longrightarrow & L(M) \\ \downarrow & & \downarrow \\ K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) & \longrightarrow & K^0(M) \end{array}$$

It follows from Definition 3.1 of $\bar{L}^{\mathbb{R}/\mathbb{Z}}(M)$ as a pull-back that:

Corollary 3.29. *There is a natural surjective map*

$$\bar{L}(M) \to \bar{L}^{\mathbb{R}/\mathbb{Z}}(M).$$

4. Eta homomorphisms

4.1. Introduction and summary. It is an interesting problem to detect non-trivial elements of $\bar{L}_\epsilon(M)$ for a manifold M.

From the homotopy theoretic definition of $K^{-1}_{\mathbb{R}/\mathbb{Z}}$ there is a natural pairing

$$K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) \otimes K_1(M) \to \mathbb{R}/\mathbb{Z},$$

where $K_1(M)$ denotes K-homology of M. In **[20]**, Lott gave an analytic description of this pairing. If M is odd-dimensional and the K-homology class is represented by a Dirac operator, then its pairing with a class in $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ given in the geometric picture can be expressed through spectral invariants of the corresponding twisted Dirac operator, in particular through the η-invariant. In the present section we consider the K-homology class given by the signature operator, which leads to a homomorphism

$$\eta^{-1}_{\mathbb{R}/\mathbb{Z}} \colon K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) \to \mathbb{R}/\mathbb{Z}.$$

The latter pulls back to

$$\eta^{\mathbb{R}/\mathbb{Z}} \colon \bar{L}^{\mathbb{R}/\mathbb{Z}}(M) \to \mathbb{R}/\mathbb{Z}.$$

It further induces a homomorphism from $\bar{L}(M)$ to $\mathbb{R}/\mathbb{Z}$.

The main objective of the present section is to refine the homomorphism to an $\mathbb{R}$-valued one. From the analytic definition of $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}$ it is quite obvious that it cannot be lifted to $\mathbb{R}$ since the relevant η-invariants jump by integers if the kernels of the corresponding operators change. But the dimension of the kernel is not an invariant of the data given by the classes $K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$ and $K_1(M)$.

However, in our special case we work with the signature operator and with flat vector bundles. The kernels are tied to cohomology and thus have stronger invariance properties. Lott has defined a lift $\eta^{\mathrm{Lott}} \colon \bar{L}^{\mathrm{Lott}}(M) \to \mathbb{R}$. Unfortunately, this homomorphism does not factor over $\bar{L}(M)$. But we can analyze this failure in detail. These considerations lead to the definition of the extended groups $\bar{L}^{\mathrm{ex}}_\epsilon(M)$, $\epsilon = \pm 1$, fitting into exact sequences

$$0 \to \mathbb{Z} \to \bar{L}^{\mathrm{ex}}_\epsilon(M) \to \bar{L}_\epsilon(M) \to 0.$$

The homomorphism from $\bar{L}(M) \to \mathbb{R}/\mathbb{Z}$ lifts to a homomorphism $\eta \colon \bar{L}^{\mathrm{ex}}(M) \to \mathbb{R}$.

The extended groups are only defined for closed odd-dimensional oriented manifolds M. Note that they are not functorial with respect to smooth maps (only with respect to homotopy equivalences). But they admit extended secondary index maps.

4.2. $\eta^{-1}_{\mathbb{R}/\mathbb{Z}} \colon K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) \to \mathbb{R}/\mathbb{Z}$.

4.2.1. *The definition.* Let M be an oriented n-dimensional closed manifold. If we choose a Riemannian metric, then we can define the signature operator D^{sign}. Assume now that n is odd. Then D^{sign} is a self-adjoint operator which induces a class $[M^{\mathrm{sign}}] \in K_1(M)$, and which is independent of the choice of the metric.

DEFINITION 4.1. We define $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}\colon K^{-1}_{\mathbb{R}/\mathbb{Z}}(M) \to \mathbb{R}/\mathbb{Z}$ by $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}(x) := \langle x, [M^{\mathrm{sign}}]\rangle$. Furthermore, we define $\eta^{\mathbb{R}/\mathbb{Z}}\colon \bar{L}^{\mathbb{R}/\mathbb{Z}}(M) \to \mathbb{R}/\mathbb{Z}$ as the composition of $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}$ with the natural transformation $\bar{L}^{\mathbb{R}/\mathbb{Z}} \to K^{-1}_{\mathbb{R}/\mathbb{Z}}$.

4.2.2. *Geometric description of* $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}$. We fix a Riemannian metric g^{TM} on M. Let $[E, h^E, \nabla^E, \rho] \in K^{-1}_{\mathbb{R}/\mathbb{Z}}(M)$. Then one can express $\eta^{-1}_{\mathbb{R}/\mathbb{Z}}(x)$ in analytic terms. Let $D^{\mathrm{sign}}_E : \Omega(M, E) \to \Omega(M, E)$ be the odd signature operator of M twisted by the bundle E. Then its η-invariant is given by

$$\eta(D^{\mathrm{sign}}_E) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}\left[D^{\mathrm{sign}}_E \mathrm{e}^{-t(D^{\mathrm{sign}}_E)^2}\right] \frac{dt}{t^{1/2}}.$$

Let $L(\nabla^{TM}) \in \Omega(M)$ be the L-form of the tangent bundle TM equipped with the Levi-Civita connection ∇^{TM}. Then we have

$$\eta^{\mathbb{R}/\mathbb{Z}}([E, h^E, \nabla^E, \rho]) := \left[\eta(D^{\mathrm{sign}}_E) - 2\int_M L(\nabla^{TM}) \wedge \rho\right],$$

where $[.]$ on the right-hand side takes the class in $\mathbb{R}/\mathbb{Z}$. Using the local variation formula for the η-invariant we see that this combination indeed is independent of the Riemannian metric and only depends on the class of $[E, h^E, \nabla^E, \rho]$. For details we refer to Lott [**20**, Prop. 3].

Note that this homomorphisms cannot be refined to have values in $\mathbb{R}$, since the dimension of the kernel of D^{sign}_E depends on the choice of the connection for the representative $[E, h^E, \nabla^E, \rho]$.

4.3. $\mathbb{R}$-valued η-homomorphisms. Since the $\bar{L}_\epsilon$-groups involve flat bundles $F = \texttt{bundle}(\mathcal{F})$, the kernel of the twisted signature operator D^{sign}_F defined in (4.2) is isomorphic to the sheaf cohomology $H^*(M, \mathcal{F})$. In particular, it is independent of the choice of a metric structure. This eventually allows us to define $\mathbb{R}$-valued η-homomorphisms.

4.3.1. η^{Lott}. We start with describing the homomorphism $\eta^{\mathrm{Lott}}\colon \bar{L}^{\mathrm{Lott}}_\epsilon(M) \to \mathbb{R}$ which has been defined by Lott in [**21**, §3.3]. Let M be a closed smooth m-dimensional oriented manifold. The $\mathbb{Z}$-graded vector space of real differential forms $\Omega(M)$ carries the $\epsilon_m = (-1)^{[\frac{m+1}{2}]}$—symmetric duality structure (cf. (4.4))

$$q(\omega, \omega') := \int_M \left((-1)^{\frac{N_M(N_M-1)}{2} + mN_M}\omega\right) \wedge \omega',$$

where N_M denotes the $\mathbb{Z}$-grading on $\Omega(M)$. We fix the convention $\sqrt{\epsilon_m} := i^{[\frac{m+1}{2}]}$.

A choice of a Riemannian metric g^{TM} and of the orientation on M induces the Hodge-$*$ operator which is characterized by

$$\omega \wedge *\omega' = (\omega, \omega')_{g^{TM}} \mathrm{vol}.$$

Here vol is the Riemannian volume element of (M, g^{TM}). We define the N_M-compatible metric structure

$$J^M := * \, (-1)^{\frac{N_M(N_M-1)}{2}+mN_M}.$$

Let now $(\mathcal{F}, q)$ be ϵ–symmetric. By $\Omega(M,F)$ we denote the space of F-valued smooth forms on M, where $F := \mathtt{bundle}(\mathcal{F})$. If $\omega \in \Omega(M)$ and $\phi \in C^\infty(M,F)$, then we have a product $\omega \otimes \phi \in \Omega(M,F)$. The space $\Omega(M,F)$ carries the $\epsilon_m\epsilon$–duality structure

$$(4.1)\quad q_{M,F}(\omega\otimes\phi,\omega'\otimes\phi') := \frac{\sqrt{\epsilon\epsilon_m}}{\sqrt{\epsilon}\sqrt{\epsilon_m}}\int_M \left((-1)^{\frac{N_M(N_M-1)}{2}+mN_M}\omega\right)\wedge\omega'\, q(\phi,\phi')_F.$$

It further has a natural $\mathbb{Z}$-grading N of length m. If J^F is a metric structure on F, then $J^{M,F}(\omega\otimes\phi) := \frac{\sqrt{\epsilon\epsilon_m}}{\sqrt{\epsilon}\sqrt{\epsilon_m}} J^M\omega\otimes J^F\phi$ is a N-compatible metric structure on $\Omega(M,F)$ (cf. Definition 3.14). Let d^F be the twisted de Rham differential on $\Omega(M,F)$ induced by the flat connection ∇^F on F. Let $(d^F)^*$ be the adjoint of d^F with respect to this metric on $\Omega(M,F)$. Then we have $(d^F)^* = (-1)^{m+1}\epsilon\epsilon_m J^{M,F} d^F J^{M,F}$.

We now assume that $m \equiv -\epsilon \bmod 4$, then $\epsilon\epsilon_m = 1$. Then we define the self-adjoint operator (cf. [**2**, (4.6)], [**7**, (1.38)])

$$(4.2)\qquad D_F^{\mathrm{sign}} := J^{M,F}d^F + d^F J^{M,F}.$$

DEFINITION 4.2. The homomorphism $\eta^{\mathrm{Lott}}\colon \bar{L}_\epsilon^{\mathrm{Lott}}(M) \to \mathbb{R}$ is defined by

$$\eta^{\mathrm{Lott}}(\mathcal{F},q,J,\rho) := \eta(D_F^{\mathrm{sign}}) - 2\int_M L(\nabla^{TM})\wedge\rho.$$

By the argument in [**21**, Prop. 24], η^{Lott} is well-defined and independent of the choice of the Riemannian metric g^{TM} of M. It also follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \bar{L}_\epsilon^{\mathrm{Lott}}(M) & \xrightarrow{\eta^{\mathrm{Lott}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ K_{\mathbb{R}/\mathbb{Z}}^{-1}(M) & \xrightarrow{\eta^{\mathbb{R}/\mathbb{Z}}} & \mathbb{R}/\mathbb{Z} \end{array}$$

4.3.2. *Motivation of extended L-groups.* A natural problem is now to lift the composition

$$\bar{L}_\epsilon(M) \to K_{\mathbb{R}/\mathbb{Z}}^{-1}(M) \xrightarrow{\eta^{\mathbb{R}/\mathbb{Z}}} \mathbb{R}/\mathbb{Z}$$

to a $\mathbb{R}$-valued homomorphism "η" $: \bar{L}_\epsilon(M) \to \mathbb{R}$. This problem could be solved by factoring η^{Lott} over the quotient $\bar{L}_\epsilon^{\mathrm{Lott}}(M) \to \bar{L}_\epsilon(M)$. Unfortunately, this factorization does not exist in general. For this reason we will define modified L- and

$\bar{L}$-groups which will be denoted by L_ϵ^{ex} and $\bar{L}_\epsilon^{\text{ex}}$. They fit into the diagram

(4.3)

$$\begin{array}{ccccccc} H^{4*-\epsilon}(M,\mathbb{R}) & \longrightarrow & \bar{L}_\epsilon^{\text{Lott}}(M) & \longrightarrow & L_\epsilon^{\text{Lott}}(M) & \longrightarrow & H^{4*+1-\epsilon}(M,\mathbb{R}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^{4*-\epsilon}(M,\mathbb{R}) & \longrightarrow & \bar{L}_\epsilon^{\text{ex}}(M) & \longrightarrow & L_\epsilon^{\text{ex}}(M) & \longrightarrow & H^{4*+1-\epsilon}(M,\mathbb{R}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^{4*-\epsilon}(M,\mathbb{R}) & \longrightarrow & \bar{L}_\epsilon(M) & \longrightarrow & L_\epsilon(M) & \longrightarrow & H^{4*+1-\epsilon}(M,\mathbb{R}). \end{array}$$

The advantage of $\bar{L}_\epsilon^{\text{ex}}$ is that we can extend the homomorphism η^{Lott} to an homomorphism $\eta\colon \bar{L}_\epsilon^{\text{ex}}(M) \to \mathbb{R}$. The draw-back of the extended L_ϵ-groups is that they are *not functors* on the category of manifolds.

4.4. Definition of the extended L-groups.

4.4.1. *Some linear algebra.* Let V be a real vector space with a non-degenerate form $r\colon V \to V^*$. We will call (V,r) a duality space.

DEFINITION 4.3. A $\mathbb{Z}$-graded duality space of length $n \in \mathbb{Z}$ is a triple (V,r,N), where (V,r) is a duality space and N is a $\mathbb{Z}$-grading operator such that $r^{-1} \circ N^* \circ r = n - N$.

If (V_i, r_i, N_i), $i = 1,2$, are $\mathbb{Z}$-graded duality spaces of length n_i, then their tensor product is a $\mathbb{Z}$-graded duality space of length $n_1 + n_2$.

Let (V,r,N) be a $\mathbb{Z}$-graded duality space of length n. Assume further that r is *graded symmetric*, i.e., $r(v,w) = (-1)^{|v||w|} r(w,v)$. Then we can define an $\epsilon_n := (-1)^{[\frac{n+1}{2}]}$–duality structure on V by

$$q(v,w) := r\big((-1)^{\frac{N(N-1)}{2}+nN} v, w\big). \tag{4.4}$$

Let $\mathbb{R}[-n]$ be the graded vector space which has $\mathbb{R}$ in degree n and 0 in all remaining degrees. The duality structure r induces a map of graded vector spaces $\hat{r}\colon V \otimes V \to \mathbb{R}[-n]$ by $v \otimes w \to r(v,w)$.

We now consider a differential $d\colon V \to V$ of degree one. It induces a differential on $V \otimes V$ by $d(v \otimes w) = dv \otimes w + (-1)^{|v|} v \otimes dw$, where v is homogeneous of degree $|v|$. On $\mathbb{R}[-n]$ we consider the trivial differential.

DEFINITION 4.4. The differential d is called compatible with r if $\hat{r}\colon V \otimes V \to \mathbb{R}[-n]$ is a map of complexes. Equivalently, $r(dv,w) + (-1)^{|v|} r(v,dw) = 0$ for all v, $w \in V$ with v homogeneous of degree $|v|$.

We now assume that d is compatible with r. Let $H(V)$ denote the cohomology of (V,d). It is a $\mathbb{Z}$-graded vector space. The map $H(\hat{r})\colon H(V \otimes V) \to H(\mathbb{R}[-n]) = \mathbb{R}[-n]$ together with the Künneth formula $H(V) \otimes H(V) \xrightarrow{\cong} H(V \otimes V)$ induces a graded–symmetric duality structure r_H on $H(V)$. In order to see that this pairing is non-degenerate, note that $\operatorname{im}(d)^\perp = \ker(d)$ and hence $\ker(d)^\perp = \operatorname{im}(d)$.

Let (V, r, N) be a $\mathbb{Z}$-graded duality space of length n with compatible differential d, where r is graded symmetric. Then we have a $(-1)^{n+1}\epsilon_n$—symmetric form

$$Q(v, w) := r((-1)^{\frac{N(N-1)}{2}+nN} v, dw) = q(v, dw).$$

In particular, if V is finite-dimensional and $(-1)^{n+1}\epsilon_n = 1$, then we can consider the signature of the quadratic form Q.

DEFINITION 4.5. $\tau(V, r, N, d) := 2\,\mathrm{sign}(Q)$.

Let J be a metric structure on (V, q). Let d^{*_J} denote the adjoint of d with respect to the scalar product $q(., J.)$ induced by J. We have $d^{*_J} = (-1)^{n+1}\epsilon_n JdJ$. Assume that $(-1)^{n+1}\epsilon_n = 1$. Then we can form the self-adjoint operator $D := Jd + (-1)^{n+1}dJ = J(d + d^{*_J})$.

LEMMA 4.6. *We have* $\mathrm{sign}(D) = \frac{1}{2}\left(1 + (-1)^{n+1}\right)\tau(V, r, N, d)$.

PROOF. By Hodge theory we can identify $H(V)$ with $\mathcal{H} := \ker(d) \cap \ker(d^{*_J})$. We decompose V into D-invariant subspaces $V = \mathrm{im}(d) \oplus \mathrm{im}(d^{*_J}) \oplus \mathcal{H}$. The metric structure J maps $\mathrm{im}(d)$ isomorphically to $\mathrm{im}(d^{*_J})$. Furthermore, for $x, y \in \mathrm{im}(d)$ we have $q(DJx, JJy) = (-1)^{n+1}q(JdJx, y) = q(dJx, Jy) = (-1)^{n+1}q(Dx, Jy)$. Let P be the projection to $\mathrm{im}(d^{*_J})$. Then $\mathrm{sign}(D) = \left(1 + (-1)^{n+1}\right)\mathrm{sign}(DP)$. For all x, y we have $q(x, JDPy) = (-1)^{n+1}q(x, dy)$. Thus we have $\mathrm{sign}(DP) = \frac{1}{2}(-1)^{n+1}\,\tau(V, r, N, d)$. □

Let (V, r) be a duality space with grading N of length n and compatible differential d. Furthermore, let $\ldots \subset F^{p+1}V \subset F^pV \subset \ldots$ be a decreasing filtration of the complex (V, d, N) by subcomplexes. Further we assume that there exists n_B such that $r(F^pV, F^qV) = 0$ if $p + q > n_B$. On $V \otimes V$ we consider the induced filtration $F^r(V \otimes V) = \sum_{p+q=r} F^pV \otimes F^qV$. Let $E_u^{p,q}(V)$ be the corresponding spectral sequence (cf. [**17**, §3.5]).

We filter $\mathbb{R}[-n]$ such that $F^s\mathbb{R}[-n] = \mathbb{R}[-n]$ if $s \le n_B$ and $F^s\mathbb{R}[-n] = 0$ for $s > n_B$. The map $\hat{r}$ is then a map of filtered complexes. We therefore obtain a morphism of associated spectral sequences, which we also denote by $\hat{r}$. The target $E_s^{p,q}(\mathbb{R}[-n])$ is easy to describe. We have $E_s^{p,q}(\mathbb{R}[-n]) = 0$ for all $s \ge 0$, p, q except for $p + q = n$ and $p = n_B$, where $E_s^{p,q} \cong \mathbb{R}$. All differentials vanish.

The Künneth formula for spectral sequences yields a map

$$E_u^{p,q}(V) \otimes E_u^{s,t}(V) \to E_u^{p+s,q+t}(V \otimes V).$$

If we compose this with the morphism of spectral sequences induced by $\hat{r}$ we obtain pairings $r_u\colon E_u^{p,q}(V) \otimes E_u^{s,t}(V) \to \mathbb{R}$ if $p + q + s + t = n$ and $p + s = n_B$. In particular, we obtain a pairing on $E_u(V)$ which is compatible with the $\mathbb{Z}$-grading N_{E_u} (induced by N) of length n and has the same symmetry properties as r.

The differential d_u is compatible with the duality structure r_u on E_u and the duality structure r_{u+1} on E_{u+1} is obtained from r_u by considering E_{u+1} as the cohomology of E_u as described above. If r_0 is non-degenerate, then so are the r_u for all $u \ge 0$.

Assume now that r is graded symmetric and $(-1)^{n+1}\epsilon_n = 1$. Then the form r_u induced on $E_u(V)$ is graded symmetric as well. If $E_u(V)$ is finite-dimensional for $u \geq i$, then we define

DEFINITION 4.7. $\tau_i(E_*) := \sum_{u\geq i} \tau(E_u(V), r_u, N_{E_u}, d_u)$.

4.4.2. *Hypercohomology and spectral sequences.* Let M be a smooth manifold. Let $\mathcal{F}$ be a locally constant sheaf of finite-dimensional real vector spaces and (F, ∇^F) be the associated flat vector bundle. We further assume that $\mathcal{F}$ is $\mathbb{Z}$-graded by N_F and has a differential v (cf. Definition 2.4). We use the same symbols in order to denote the corresponding parallel endomorphisms of F. The cohomology $\mathcal{H}$ of $(\mathcal{F}, v)$ is again a locally constant sheaf of real vector spaces with $\mathbb{Z}$-grading N_H.

There are two spectral sequences which converge to the hypercohomology $\mathbf{H}(M, (\mathcal{F}, N_F, v))$ of this complex, the local-global spectral sequence and the hypercohomology spectral sequence. Let us take the soft resolution of $(\mathcal{F}, N, v)$ given by the twisted de Rham complex $(\Omega(M, F), d^F + v)$ with the total $\mathbb{Z}$-grading $N_{M,F} = N_M + N_F$. Its cohomology is naturally isomorphic to $\mathbf{H}(M, (\mathcal{F}, N_F, v))$. The two spectral sequences are associated to the two natural filtrations of this complex.

We first describe the local-global spectral sequence which is associated to the filtration

$$F^p\Omega(M, F) = \sum_{q\geq p} \Omega^q(M, F).$$

We have ${}_{lg}E_0^{p,q} = \Omega^p(M, F^q)$ and ${}_{lg}d_0 = v\colon {}_{lg}E_0^{p,q} \to {}_{lg}E_0^{p,q+1}$. The first term is given by ${}_{lg}E_1^{p,q} = \Omega^p(M, H^q)$ and ${}_{lg}d_1 = d^H\colon {}_{lg}E_1^{p,q} \to {}_{lg}E_1^{p+1,q}$. We obtain ${}_{lg}E_2^{p,q} = H^p(M, \mathcal{H}^q)$. The local-global spectral sequences is natural and finite-dimensional starting with its second term.

We now describe the hypercohomology spectral sequence which is associated to the filtration

$$F^p\Omega(M, F) = \sum_{q\geq p} \Omega(M, F^q).$$

We have ${}_{hc}E_0^{p,q} = \Omega^q(M, F^p)$ and ${}_{hc}d_0 = d^F\colon {}_{hc}E_0^{p,q} \to {}_{hc}E_0^{p,q+1}$. The first term is given by ${}_{hc}E_1^{p,q} = H^q(M, \mathcal{F}^p)$ and ${}_{hc}d_1 := H(v)\colon {}_{hc}E_1^{p,q} \to {}_{hc}E_1^{p+1,q}$, where $H(v)$ is induced by v. The hypercohomology spectral sequence is natural and finite-dimensional starting with its first term.

Now assume that q is an ϵ–duality structure on $\mathcal{F}$ such that N_F and v are compatible with q (cf. Definition 2.4). Let M be closed and oriented, and set $m := \dim M$. On $\Omega(M, F)$ we define the $\epsilon\epsilon_m$–duality structure $q_{M,F}$ by (4.1). The differential $d_F = d^F + v$ and the total grading $N_{M,F} = N_M + N_F$ are compatible with $q_{M,F}$. As explained in Subsection 4.4.1. we obtain induced pairings on the spectral sequences.

We first consider the local-global spectral sequence. We put $n_B := m$ and obtain pairings

$$ {}_{lg}q_r \colon {}_{lg}E_r^{p,q} \otimes {}_{lg}E_r^{s,t} \to \mathbb{R} $$

for $p+q+s+t = m+n$ and $p+s = m$. On ${}_{lg}E_0^{p,q} \otimes {}_{lg}E_0^{s,t} = \Omega^p(M, F^q) \otimes \Omega^s(M, F^t)$ this pairing is just $q_{M,F}$. In particular, it is non-degenerate. As in Subsection 4.4.1, the $(-1)^{m+1}\epsilon\epsilon_m$–symmetric form $q_{M,F}(x, (d^F + v)y)$ induces the corresponding $(-1)^{m+1}\epsilon\epsilon_m$–symmetric forms on ${}_{lg}E_r$. If $\epsilon\epsilon_m = 1$, and m is odd, then we can define

Definition 4.8. ${}_{lg}\tau_2(\mathcal{F}, q, N_F, v) := \tau_2({}_{lg}E_*)$.

In the case of the hypercohomology spectral sequence we put $n_B = n$. We have parings

$$ {}_{hc}q_r \colon {}_{hc}E_r^{p,q} \otimes {}_{lg}E_r^{s,t} \to \mathbb{R} $$

for $p+q+s+t = m+n$ and $p+s = n$. On ${}_{hc}E_0^{p,q} \otimes {}_{hc}E_0^{s,t} = \Omega^q(M, F^p) \otimes \Omega^t(M, F^s)$ this pairing is again just $q_{M,F}$. In particular, it is non-degenerate. If $\epsilon\epsilon_m = 1$, and m is odd, then we can define

Definition 4.9. ${}_{hc}\tau_1(\mathcal{F}, q, N_F, v) := \tau_1({}_{hc}E_*)$.

Finally, we define

Definition 4.10.

$$ \tau(\mathcal{F}, q, N_F, v) := {}_{hc}\tau_1(\mathcal{F}, q, N_F, v) - {}_{lg}\tau_2(\mathcal{F}, q, N_F, v). $$

4.4.3. *Definition of $L_\epsilon^{\mathrm{ex}}(M)$ and $\bar{L}_\epsilon^{\mathrm{ex}}(M)$.* We fix closed odd-dimensional oriented manifold M. We assume that $\epsilon \in \mathbb{Z}_2$ is such that $\epsilon\epsilon_m = 1$, where $\epsilon_m := (-1)^{[\frac{m+1}{2}]} \in \mathbb{Z}_2$ and $m = \dim M$.

Let $(\mathcal{F}, q)$ be ϵ–symmetric and $\mathcal{L}$ be a locally constant lagrangian subsheaf of $\mathcal{F}$. Then we define an integer $\tau(\mathcal{F}, q, \mathcal{L})$ by the following construction.

We consider the sheaf $\mathcal{G} := \mathcal{L} \oplus \mathcal{F} \oplus \mathcal{F}/\mathcal{L}$. It carries a natural ϵ–symmetric duality structure $q_{\mathcal{G}}$. Note that q induces maps $q_{\mathcal{L}} \colon \mathcal{L} \to (\mathcal{F}/\mathcal{L})^*$ and $q_{\mathcal{F}/\mathcal{L}} \colon \mathcal{F}/\mathcal{L} \to \mathcal{L}^*$. Then $q_{\mathcal{G}} \colon \mathcal{G} \to \mathcal{G}^*$ is given by

$$ \begin{pmatrix} 0 & 0 & q_{\mathcal{F}/\mathcal{L}} \\ 0 & q & 0 \\ q_{\mathcal{L}} & 0 & 0 \end{pmatrix}. $$

A compatible $\mathbb{Z}$-grading is given by $N_{\mathcal{G}} = \mathrm{diag}(0, 1, 2)$. We define a differential on $\mathcal{G}$ by

$$ v_{\mathcal{G}} := \begin{pmatrix} 0 & 0 & 0 \\ i_{\mathcal{L}} & 0 & 0 \\ 0 & -\mathrm{pr} & 0 \end{pmatrix}, $$

where $i_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{F}$ is the inclusion and $\mathrm{pr} \colon \mathcal{F} \to \mathcal{F}/\mathcal{L}$ is the projection. We introduce the sign to make the differential compatible with $q_{\mathcal{G}}$. We now define the following integer.

Definition 4.11. $\tau(\mathcal{F}, q, \mathcal{L}) := \tau(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}})$.

Lemma 4.12. *If $\mathcal{L}$ has a lagrangian complement, then we have $\tau(\mathcal{F}, q, \mathcal{L}) = 0$.*

Proof. In this case the spectral sequences degenerate. □

We now define the group L_ϵ^{ex} (note that $\epsilon\epsilon_m = 1$). First we define the semigroup $\hat{L}_\epsilon^{\mathrm{ex}}(M)$ of isomorphism classes of tuples $(\mathcal{F}, q, z)$, where $(\mathcal{F}, q)$ is locally constant sheaf of real vector spaces with ϵ–symmetric duality structure q and $z \in \mathbb{Z}$. The semigroup operation is given by the sum of the entries. Next we define an equivalence relation $\sim$ generated by lagrangian reduction. Let $(\mathcal{F}, q, z)$ as above and $\mathcal{L} \subset \mathcal{F}$ a locally constant lagrangian subsheaf. Then we require that

$$(\mathcal{F}, q, z) \sim (0, 0, z + \tau(\mathcal{F}, q, \mathcal{L})).$$

We extend $\sim$ to the minimal equivalence relation which contains lagrangian reduction and which is compatible with the sum.

Definition 4.13. We define $L_\epsilon^{\mathrm{ex}}(M) := \hat{L}_\epsilon^{\mathrm{ex}}(M)/\sim$.

The semigroup $L_\epsilon^{\mathrm{ex}}(M)$ is in fact a group. Let $[\mathcal{F}, q, z]$ denote the class represented by $(\mathcal{F}, q, z)$. Its inverse is given by $[\mathcal{F}, -q, -z]$, since by Lemma 4.12 we have

$$\tau(\mathcal{F} \oplus \mathcal{F}, q \oplus -q, \mathcal{F}) = 0.$$

There is a surjective homomorphism $L_\epsilon^{\mathrm{ex}}(M) \to L_\epsilon(M)$ induced by $(\mathcal{F}, q, z) \mapsto (\mathcal{F}, q)$. Furthermore, there is homomorphism $\mathbb{Z} \to L_\epsilon^{\mathrm{ex}}(M)$ induced by $z \mapsto (0, 0, z)$.

Lemma 4.14. *The following sequence is exact:*

$$0 \longrightarrow \mathbb{Z} \longrightarrow L_\epsilon^{\mathrm{ex}}(M) \longrightarrow L_\epsilon(M) \longrightarrow 0.$$

Proof. We must show that $\mathbb{Z} \to L_\epsilon^{\mathrm{ex}}(M)$ is injective. This will be a consequence of the fact that $\eta\colon L_\epsilon^{\mathrm{ex}}(M) \to \mathbb{R}$ is well-defined which will be proved below (cf. Lemma 4.19). □

Of course, there should be a purely algebraic proof of Lemma 4.14. Another open problem is to turn $M \to L_\epsilon^{\mathrm{ex}}(M)$ into a functor.

Using Lemma 4.12 we see that $(\mathcal{F}, q) \mapsto (\mathcal{F}, q, 0)$ defines a homomorphism

$$L_\epsilon^{\mathrm{Lott}}(M) \to L_\epsilon^{\mathrm{ex}}(M).$$

Definition 4.15. We define $\bar{L}_\epsilon^{\mathrm{ex}}(M)$ by the following pull-back diagram:

$$\begin{array}{ccc} \bar{L}_\epsilon^{\mathrm{ex}}(M) & \longrightarrow & L_\epsilon^{\mathrm{ex}}(M) \\ \downarrow & & \downarrow \\ \bar{L}_\epsilon(M) & \longrightarrow & L_\epsilon(M) \end{array}$$

An element of $\bar{L}_\epsilon^{\mathrm{ex}}(M)$ can be written as $[\mathcal{F}, q, J, \rho, z]$. The following assertions follow immediately from the definition, Lemma 3.25, and Proposition 3.26.

COROLLARY 4.16. *We have the exact sequences*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \bar{L}^{\mathrm{ex}}_{\epsilon}(M) \longrightarrow \bar{L}_{\epsilon}(M) \longrightarrow 0$$

and

$$0 \longrightarrow H^{4*-\epsilon}(M,\mathbb{R}) \longrightarrow \bar{L}^{\mathrm{ex}}_{\epsilon}(M) \longrightarrow L^{\mathrm{ex}}_{\epsilon}(M) \longrightarrow H^{4*+1-\epsilon}(M,\mathbb{R}).$$

By $(\mathcal{F},q,J,\rho) \mapsto (\mathcal{F},q,J,\rho,0)$ we define a morphism

$$\bar{L}^{\mathrm{Lott}}_{\epsilon}(M) \to \bar{L}^{\mathrm{ex}}_{\epsilon}(M).$$

It is now easy to check that the diagram (4.3) commutes.

4.5. Construction of $\eta\colon \bar{L}^{\mathrm{ex}}_{\epsilon}(M) \to \mathbb{R}$.

4.5.1. *The η-invariant of a complex.* Let $\epsilon \in \mathbb{Z}_2$. We assume that M is a smooth closed oriented manifold of dimension m such that m is odd and $\epsilon\epsilon_m = 1$, equivalently $m \equiv -\epsilon \bmod 4$. Let $(\mathcal{F},q)$ be a locally constant sheaf of finite dimensional real vector spaces with ϵ–symmetric form. Let furthermore N_F be a compatible $\mathbb{Z}$-grading and v be a compatible differential. By $(\mathcal{H},q_{\mathcal{H}})$ we denote the associated cohomology. We choose a Riemannian metric g^{TM}, a compatible metric structure J^F, and let J^H be the induced metric structure. Then we can define the operators D^{sign}_F and D^{sign}_H as in (4.2). The following theorem is the main ingredient in the construction of η. Recall Definitions 3.15 and 4.10 of the η-form $\tilde{\eta}(\mathcal{F},N_F,J^F,v)$ and of the integer $\tau(\mathcal{F},q,N_F,v)$.

THEOREM 4.17. *We have*

$$\eta(D^{\mathrm{sign}}_F) - \eta(D^{\mathrm{sign}}_H) = 2\int_M L(\nabla^{TM}) \wedge \tilde{\eta}(\mathcal{F},N_F,J^F,v) - \tau(\mathcal{F},q,N_F,v).$$

PROOF. A sketch of the proof is given in Subsection 6.2. □

4.5.2. *Definition and well-definedness of η.* Let $[\mathcal{F},q,J,\rho,z] \in \bar{L}^{\mathrm{ex}}_{\epsilon}(M)$. We choose a Riemannian metric g^{TM}.

DEFINITION 4.18. We define

$$\begin{aligned} \eta(\mathcal{F},q,J,\rho,z) &:= \eta^{\mathrm{Lott}}(\mathcal{F},q,J^F,\rho) - z \\ &= \eta(D^{\mathrm{sign}}_F) - 2\int_M L(\nabla^{TM}) \wedge \rho - z. \end{aligned}$$

LEMMA 4.19. *The map $(\mathcal{F},q,J,\rho,z) \mapsto \eta(\mathcal{F},q,J,\rho,z)$ induces a well-defined homomorphism $\eta\colon \bar{L}^{\mathrm{ex}}_{\epsilon}(M) \to \mathbb{R}$.*

PROOF. It follows from the well-definedness of η^{Lott} [**21**, Prop.24] that η is independent of the choice of the Riemannian metric on M. It therefore suffices to show that $(\mathcal{F},q,J,\rho,z) \sim 0$ implies that $\eta(\mathcal{F},q,J,\rho,z) = 0$. Thus let $\mathcal{L} \subset \mathcal{F}$ be a locally constant lagrangian subsheaf of $\mathcal{F}$ such that $\rho + \tilde{p}(\mathcal{F},q,J,\mathcal{L}) \equiv 0$ and $z + \tau(\mathcal{F},q,\mathcal{L}) = 0$.

We construct the complex $(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}})$ as in Subsection 4.4.3. Using the compositions $J_L\colon L \longrightarrow F \xrightarrow{J} F \xrightarrow{\mathrm{pr}} F/L$ and $J_{F/L}\colon F/L \longrightarrow F \ominus L \xrightarrow{J} L$, we define the metric structure

$$J^{\mathcal{G}} := \begin{pmatrix} 0 & 0 & J_L \\ 0 & J & 0 \\ J_{F/L} & 0 & 0 \end{pmatrix}.$$

As in the proof of Proposition 3.16, we see that modulo exact forms

$$\tilde{\eta}(\mathcal{G}, N_{\mathcal{G}}, J^{\mathcal{G}}, v_{\mathcal{G}}) \equiv -\tilde{p}(\mathcal{F}, q, J, \mathcal{L}).$$

By definition we have

$$\tau(\mathcal{F}, q, \mathcal{L}) = \tau(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}}).$$

The complex $\mathcal{G}$ is exact. By Theorem 4.17, we have

$$\eta(D^{\mathrm{sign}}_{\mathcal{G}}) = 2\int_M L(\nabla^{TM}) \wedge \tilde{\eta}(\mathcal{G}, N_{\mathcal{G}}, J^{\mathcal{G}}, v_{\mathcal{G}}) - \tau(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}}).$$

Thus

$$\begin{aligned} \eta(\mathcal{F}, q, J, \rho, z) &= \eta(D^{\mathrm{sign}}_{F}) - 2\int_M L(\nabla^{TM}) \wedge \rho - z \\ &= 2\int_M L(\nabla^{TM}) \wedge \tilde{\eta}(\mathcal{G}, N_{\mathcal{G}}, J^{\mathcal{G}}, v_{\mathcal{G}}) - \tau(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}}) \\ &\qquad\qquad - 2\int_M L(\nabla^{TM}) \wedge \rho - z \\ &= -2\int_M L(\nabla^{TM}) \wedge (\tilde{p}(\mathcal{F}, q, J, \mathcal{L}) + \rho) - (z + \tau(\mathcal{G}, q_{\mathcal{G}}, N_{\mathcal{G}}, v_{\mathcal{G}})) \\ &= 0. \end{aligned}$$

□

5. The secondary index map

5.1. Introduction and summary. In this section we consider the secondary index map (i.e., the wrong-way or push-forward map) for $\bar{L}_\epsilon$ associated to fibre bundles. It is constructed by refining the geometric construction of $\pi_!^{\mathrm{sign},\mathbb{R}/\mathbb{Z}}$. This construction naturally involves η-forms for fibre bundles. The proofs of the facts that $\pi_*^{\bar{L}}$ is well-defined, and that it has nice functorial properties, are all based on the study of various adiabatic limits of these η-forms. We start this section with the introduction of the η-form associated to a fibre bundle and the statements of the adiabatic limit results. Then we introduce the secondary index maps and discuss their functorial properties. We give the algebraic parts of the proofs in detail using the corresponding adiabatic limit results.

5.2. Adiabatic limits of eta invariants, the eta form.

5.2.1. *Generalized connections.* Let M be a smooth manifold and $F \to M$ be a smooth real vector bundle. Let ∇ be a connection on F. Using the Leibniz rule we extend ∇ to $\Omega(M, F)$ (cf. [**3**, Def. 1.14]).

DEFINITION 5.1. A generalized connection on F is an operator $A\colon \Omega(M,F) \to \Omega(M,F)$ of the form $A = \nabla + S$ with $S \in \Omega(M, \mathrm{End}(F))$.

Let $q\colon F \to F^*$ be a duality structure on F. We extend q to a form $q\colon \Omega(M,F) \otimes \Omega(M,F) \to \Omega(M)$ by $q(u \otimes x, w \otimes y) = u \wedge wq(x,y)$.

DEFINITION 5.2. We say that a generalized connection A on $\Omega(M,F)$ is compatible with q if $dq(\phi,\psi) = q(A\phi,\psi) + q(\phi, A\psi)$.

5.2.2. *The η-form of a fibre bundle.* Let $\pi\colon M \to B$ be a smooth locally trivial fibre bundle with closed fibres Z_b, $b \in B$, of dimension n. Set $\epsilon_n = (-1)^{[\frac{n+1}{2}]}$, $\sqrt{\epsilon_n} = (\sqrt{-1})^{[\frac{n+1}{2}]}$. We assume that the vertical bundle $TZ := \ker(d\pi) \subset TM$ is oriented. Let $(\mathcal{F}, q_{\mathcal{F}})$ be a locally constant sheaf of finite dimensional real vector spaces over M with a ϵ–symmetric duality structure $q_{\mathcal{F}}$ and let F be the corresponding flat vector bundle (cf. Subsection 2.4.1).

We consider the infinite-dimensional $\mathbb{Z}$-graded vector bundle $\Omega(Z,F) \to B$ (with grading N_Z by form degree) with fibre $\Omega(Z,F)_b = \Omega(Z_b, F|_{Z_b})$ such that its space of smooth sections on B is $C^\infty(M, \Lambda^*(T^*Z) \otimes F)$. The space $\Omega(Z,F)$ carries an $\epsilon\epsilon_n$–duality structure $q_{Z,F}$ induced by $q_{\mathcal{F}}$ as in (4.1). Then the $\mathbb{Z}$-grading N_Z on $\Omega(Z,F)$ is a compatible $\mathbb{Z}$-grading of length n.

Our next goal is the interpretation of the twisted de Rham differential d^F on $\Omega(M,F)$ induced by the flat connection ∇^F on F as a superconnection (cf. [**12**, §III (a)]). We choose a horizontal distribution $T^H M \subset TM$, i.e., a complement to TZ. This choice induces an identification $\Omega(M,F) \cong \Omega(B, \Omega(Z,F))$. Then d^F can be viewed as a generalized connection (see Def. 5.2) on $\Omega(Z,F)$. For a vector field $X \in C^\infty(B, TB)$ we denote by $X^H \in C^\infty(M, T^H M)$ its horizontal lift. If $\omega \otimes \phi \in C^\infty(B, \Omega(Z,F)) \subset \Omega(M,F)$, then we set

$$\nabla_X^{Z,F} \omega \otimes \phi := L_{X^H}\omega \otimes \phi + \omega \otimes \nabla^F_{X^H}\phi,$$

where L_{X^H} denotes the Lie derivative. In this way we define a connection on $\Omega(Z,F)$. We extend $\nabla^{Z,F}$ to $\Omega(B, \Omega(Z,F))$ by using the Leibniz rule. Now we decompose (cf. [**12**, Prop. 3.4])

$$d^F = d^{Z,F} + \nabla^{Z,F} + i_T,$$

where $d^{Z,F}$ is the fibrewise twisted de Rham differential along the fiber Z and i_T is the insertion of a tensor field $T \in C^\infty(M, \Lambda^2(T^H M)^* \otimes TZ)$. To be precise, $\Lambda^2(T^H M)^*$ is considered here as a subspace of $\Lambda^2(T^*M)$ and i_T is interpreted as an element of $C^\infty(B, \Lambda^2(T^*B) \otimes \mathrm{End}(\Omega(Z,F)))$. It turns out that d^F is a $(-1)^{N_Z}$-superconnection of N_Z-degree one (cf. Subsection 3.4.4) which is flat because of $d^F \circ d^F = 0$.

We can form the (finite-dimensional) cohomology $H(Z,F)$ of $d^{Z,F}$ which comes equipped with a flat connection $\nabla^{H(Z,F)}$ (cf. [**12**, §III (f)]) and a parallel $\epsilon\epsilon_n$-duality structure $q_{H(Z,F)}$. The sheaf of parallel sections of $H(Z,F)$ is naturally isomorphic to $\mathcal{H}(Z,F) := \pi(\mathcal{F})$ so that $q_{H(Z,F)}$ corresponds to the form $q_{\mathcal{H}(Z,F)} := \pi(q_{Z,F})$ on $\mathcal{H}(Z,F)$ (cf. Subsection 2.5.1 for notation).

We now choose a vertical Riemannian metric g^{TZ} on TZ and a metric structure J^F on F. The vertical metric and orientation on TZ together induce a Hodge $*$-operator $*\colon \Omega(Z) \to \Omega(Z)$ as in Subsection 4.3.1. We define a metric structure on $\Omega(Z,F)$ by

$$J^{Z,F} := \frac{\sqrt{\epsilon\epsilon_n}}{\sqrt{\epsilon}\sqrt{\epsilon_n}}(*(-1)^{\frac{N_Z(N_Z-1)}{2}+nN_Z}) \otimes J^F. \tag{5.1}$$

This metric structure is compatible with the $\mathbb{Z}$-grading N_Z. On the cohomology $H(Z,F)$ we obtain a metric structure $J^{H(Z,F)}$ induced by $J^{Z,F}$. Let $(d^F)^*$, $(d^{Z,F})^*$, $(i_T)^*$, $(\nabla^{Z,F})^*$ be the adjoints of d^F, $d^{Z,F}$, i_T, $\nabla^{Z,F}$, respectively, with respect to the scalar product $q_{Z,F}(\cdot, J^{Z,F}\cdot)$ on $\Omega(Z,F)$ defined by $J^{Z,F}$.

• If n is even, by an easy computation we see that d^F is compatible with $q_{Z,F}$ in the sense of Def. 5.2. This in particular implies that the differential $d^{Z,F}$ is compatible with $q_{Z,F}$. Though $\Omega(Z,F)$ is infinite-dimensional, the theory of characteristic classes and forms extends to certain nice superconnections. In particular, $p(d^F, J^{Z,F}) \in \Omega(B)$ is well-defined. We consider the grading $z^{J^{Z,F}} := \frac{1}{\sqrt{\epsilon\epsilon_n}} J^{Z,F}$. Then the odd part of d^F with respect to this grading is given by $A = \frac{1}{2}\left(d^F + (d^F)^*\right)$. It is a $z^{J^{Z,F}}$-superconnection and we have

$$p(d^F, J^{Z,F}) := \varphi \operatorname{Tr}\left[z^{J^{Z,F}} \exp(-A^2)\right], \tag{5.2}$$

here φ multiplies a p-form by $(2\pi i)^{-p/2}$.

We now introduce the rescaling. Let $\tilde{B} := (0,\infty) \times B$ and $\mathrm{pr}\colon \tilde{B} \to B$ be the projection. We consider the bundle $\tilde{M} := \mathrm{pr}^* M$ over $\tilde{B}$ together with the canonical projection $\mathrm{Pr}\colon \tilde{M} \to M$. We define $(\tilde{\mathcal{F}}, \tilde{q}_{\mathcal{F}}, T^H\tilde{M}, \tilde{J}^F) := \mathrm{Pr}^*(\mathcal{F}, q_{\mathcal{F}}, T^H M, J^F)$. We obtain the $\mathbb{Z}$-graded $\epsilon\epsilon_n$-duality bundle $\Omega(Z,\tilde{F})$ over $\tilde{B}$ with $(-1)^{\tilde{N}_Z}$-superconnection $A' := \tilde{d}^F$, which is the twisted de Rham differential on $\Omega(\tilde{M},\tilde{F})$ induced by ∇^F. We fix the vertical metric $\tilde{g}^{TZ}$ which restricts to $t^{-1}g^{TZ}$ over $\{t\} \times M$. It induces the metric structure $\tilde{J}$ on $\Omega(Z,\tilde{F})$. The form $p(A',\tilde{J}) \in \Omega(\tilde{B})$ is now well-defined as in (5.2). Let us decompose $p(A',\tilde{J}) = dt \wedge \gamma + r$, where r does not contain dt. Here $\gamma\colon (0,\infty) \to \Omega(B)$ is a smooth family of forms on B. More precisely, for $t > 0$ we set

$$C_t = \frac{1}{2}\left(t^{N_Z/2} d^F t^{-N_Z/2} + t^{-N_Z/2}(d^F)^* t^{N_Z/2}\right). \tag{5.3}$$

Then

$$\gamma(t) = -(2\pi i)^{-1/2} \varphi \operatorname{Tr}\left[z^{J^{Z,F}} (\frac{\partial}{\partial t} C_t) \exp(-C_t^2)\right]. \tag{5.4}$$

By [**21**, Prop. 31], we have $\gamma(t) = O(t^{-3/2})$ as $t \to \infty$ and $\gamma(t) = O(1)$ as $t \to 0$ so that we can define ([**21**, Def. 33], [**8**, Def. 4.33])

DEFINITION 5.3. $\tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F) := -\int_0^\infty \gamma(t)dt$.

Note that g^{TZ} and $T^H M$ induce a canonical connection ∇^{TZ} on TZ (cf. [**5**, Thm. 1.9], [**3**, Prop. 10.2]). Let $L(\nabla^{TZ}) \in \Omega(M)$ be the L-form of TZ as in Subsection 4.2.2. We have

$$(5.5)\quad d\tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F) = \int_Z L(\nabla^{TZ}) \wedge p(\nabla^F, J^F) - p(\nabla^{H(Z,F)}, J^{H(Z,F)}).$$

• If n is odd, then A, $A^{(1)} = \frac{1}{2}(\nabla^{Z,F} + (\nabla^{Z,F})^*)$ commute with $J^{Z,F}$. To stay in the superconnection formalism we introduce an extra odd variable σ such that $\sigma^2 = 1$. We multiply all components of A with even form degree of $\Lambda(T^*B)$ by this variable and denote the result still A. This modified A is then again a superconnection (cf. [**9**, §II (f)]).

If B_0, B_1 are trace class in $\mathrm{End}(\Omega(Z,F))$, $\omega_0, \omega_1 \in \Lambda(T^*B)$, we put

$$(5.6)\quad \mathrm{Tr}'_\sigma\left[\omega_0 B_0 + \omega_1 B_1 \sigma\right] := \omega_0 \,\mathrm{Tr}\left[B_0\right], \quad \mathrm{Tr}_\sigma\left[\omega_0 B_0 + \omega_1 B_1 \sigma\right] := \omega_1 \,\mathrm{Tr}\left[B_1\right].$$

By [**9**, Thm. 2.10], the form

$$(5.7)\qquad p(A', J) = (2i)^{1/2} \varphi \,\mathrm{Tr}_\sigma [z^{J^{Z,F}} \exp(-A^2)]$$

is a closed odd form. Let us decompose $p(A', \tilde{J}) = dt \wedge \gamma + r$, where r does not contain dt. Then

$$(5.8)\qquad \gamma(t) = -\frac{1}{\sqrt{\pi}} \varphi \,\mathrm{Tr}_\sigma \left[z^{J^{Z,F}} (\frac{\partial}{\partial t} C_t) \exp(-C_t^2) \right].$$

By the same argument as in [**10**, Thm. 2.11], we have $\gamma(t) = O(1)$ as $t \to 0$ as well as $\gamma(t) = O(t^{-3/2})$ as $t \to \infty$ as in [**3**, Thm. 9.23]. So we can define (cf. [**8**, Def. 4.93]):

DEFINITION 5.4. $\tilde{\eta}\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big) := -\int_0^\infty \gamma(t)dt$.

Then the degree zero part $\tilde{\eta}^{(0)}$ of $\tilde{\eta}$ is $\frac{1}{2}\eta(D^{\mathrm{sign}}_{Z,F})$ for the fibrewise operator in (4.2). By the argument in [**9**, Thm. 2.10], as in [**8**, Thm. 4.95] we have

$$(5.9)\qquad d\tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F) = \int_Z L(\nabla^{TZ}) \wedge p(\nabla^F, J^F).$$

Since the fibres Z are odd-dimensional, we must specify our sign conventions when integrating differential form along the fibres Z. If $\alpha \in \Omega(B)$, β a section of $\Lambda(T^*Z)$ on M with compact support, then $\int_Z \pi^*(\alpha) \wedge \beta = \alpha \int_Z \beta$. This sign convention is compatible with the sign convention of Tr in Subsection 3.4.4 and (5.7).

REMARK 5.5. Assume that n is odd. In [**21**, (233)], Lott defined also an even form $p^{\mathrm{Lott}}(A', J^W(t))$. Actually, $p^{\mathrm{Lott}}(A', J^W(t)) = 0$. In fact, we only need to show $p^{\mathrm{Lott}}(A', J^W(1)) = 0$. Set $\bar{A} = A^{(1)} + z^{J^{Z,F}}(A - A^{(1)})$. If we use our notation,

$$\begin{aligned} p^{\mathrm{Lott}}(A', J^W(1)) &= \sqrt{\epsilon_n}\,\varphi\,\mathrm{Tr}'_\sigma\big[z^{J^{Z,F}}(-1)^{N_Z}\exp(-A^2)\big] \\ &= \sqrt{\epsilon_n}\,\varphi\,\mathrm{Tr}'_\sigma\big[z^{J^{Z,F}}(-1)^{N_Z}\exp(-\bar{A}^2)\big]. \end{aligned}$$

Now $\bar{A}$ preserves the parity of the N_Z-grading, while $z^{J^{Z,F}}$ changes the parity of the N_Z-grading. Thus the above trace must be zero. It also follows from this observation that the eta form in [**21**, (238)] is zero if n is odd.

We now describe how $\tilde{\eta}\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big)$ depends on its arguments. Let (T^HM, g^{TZ}, J^F), (T'^HM, g'^{TZ}, J'^F) be two triples of geometric data. We will mark the objects associated to the second triple by $'$.

Let $\tilde{L}(TZ, \nabla^{TZ}, \nabla'^{TZ})$ be the transgression of the L-form such that

$$d\tilde{L}(TZ, \nabla^{TZ}, \nabla'^{TZ}) = L(\nabla^{TZ}) - L(\nabla'^{TZ}).$$

THEOREM 5.6. *Modulo exact forms on B, we have*

$$\begin{aligned} &\tilde{\eta}\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big) - \tilde{\eta}\big(\Omega(Z,F), N_Z, J'^{Z,F}, d^F\big) \\ &= \int_Z \tilde{L}(TZ, \nabla^{TZ}, \nabla'^{TZ}) \wedge p(\nabla^F, J^F) + \int_Z L(TZ, \nabla'^{TZ}) \wedge \tilde{p}(\nabla^F, J^F, J'^F) \\ &\qquad - \tilde{p}(\nabla^{H(Z,F)}, J^{H(Z,F)}, J'^{H(Z,F)}). \end{aligned}$$

PROOF. Note that if n is odd, $J^{H(Z,F)}$ changes the parity of the N_Z-grading on $H(Z,F)$, thus $p(\nabla^{H(Z,F)}, J^{H(Z,F)})$, $\tilde{p}(\nabla^{H(Z,F)}, \ldots)$ are zero. Now this equation is a formal consequence of (5.5) and (5.9). □

5.2.3. *Adiabatic limits and the formula of Dai.* We keep the notation which was introduced in Subsection 5.2.2. Furthermore, we assume that $\dim Z := n$ is even and that B is closed, oriented, and of odd dimension n_B such that $\epsilon\epsilon_{n+n_B} = 1$. We choose a Riemannian metric g^{TB} on the base B. Using the horizontal distribution T^HM, we define the family of Riemannian metrics $g_T^{TM} := \pi^* g^{TB} \oplus \frac{1}{T^2} g^{TZ}$, $T > 0$. The orientations of B and TZ induce an orientation of M. Let $D_T^F := D_F^{\mathrm{sign}}$ be defined as in (4.2) using the metric g_T^{TM} and let $\eta(D_T^F)$ be its η-invariant. On the base B we consider the twisted signature operator $D^{H(Z,F)} := D^{\mathrm{sign}}_{H(Z,F)}$ and its η-invariant $\eta(D^{H(Z,F)})$ with respect to $g^{TB}, J^{H(Z,F)}$.

There is a decreasing filtration of the cohomological group $H(M, \mathcal{F})$ and a Leray spectral sequence $({}_{LS}E_r, {}_{LS}d_r)$ $(r \geq 2)$ with ${}_{LS}E_2^{p,q} = H^p(B, \mathcal{H}^q(Z,F))$ converging to $\mathrm{Gr}H^*(M, \mathcal{F})$. In the present smooth model this spectral sequence is associated to the filtration of $\Omega(M,F)$ given by $F^p\Omega(M,F) = \oplus_{q \geq p} \pi^*\Omega^q(B, \Omega(Z,F))$ (cf. [**17**, §3.5]). Then ${}_{LS}E_0^{p,q} = \Omega^p(B, \Omega^q(Z,F))$. The induced 1-symmetric pairing ${}_{LS}E_0^{p,q} \otimes {}_{LS}E_0^{s,t} \to \mathbb{R}$, $p+q+s+t = \dim M$, $p+s = N_B = \dim B$, is given by (4.1). In particular, it is non-degenerate. This spectral sequence with duality structure gives rise to the integer ${}_{LS}\tau_2(Z,F) := \tau_2({}_{LS}E_*)$ as in Definition 4.8.

Let ∇^{TB} be the Levi-Civita connection on TB. Let $L(\nabla^{TB})$ be the corresponding L-form on TB. We can now state the main result of [**16**].

THEOREM 5.7 (Dai [**16**, Thm. 0.3]).

$$\lim_{T\to\infty}\eta(D^F_T) = \eta(D^{H(Z,F)}) + 2\int_B L(\nabla^{TB})\wedge\tilde\eta(\Omega(Z,F),N_Z,J^{Z,F},d^F) + {}_{LS,F}\tau_2(Z,F).$$

5.2.4. *The η-form and complexes.* We make the same assumptions as in Subsection 5.2.2. In addition we assume that $\mathcal{F}$ admits a $\mathbb{Z}$-grading N_F of length n_F and a compatible differential v. Let $\mathcal{H}$ be the cohomology of v. Let H be the flat vector bundle corresponding to $\mathcal{H}$.

We choose a metric structure J^F on F such that it is compatible with N_F. Then we obtain an induced metric structure J^H on H.

We modify the construction of the eta form in Subsection 5.2.2 by replacing d^F by $\mathbf{d} := d^F + v$. This differential is now a flat $(-1)^{\mathbf{N}}$-superconnection, where $\mathbf{N} := N_Z + N_F$ is the total grading. Here $\mathbf{N}$ is a grading of length $n + n_F$ which is compatible with the form $q_{Z,F}$.

The cohomology of the zero part $d^{Z,F}+v$ of $\mathbf{d}$ is the fibrewise hypercohomology $\mathcal{H}H$ of the complex $(\mathcal{F},v)$ along the fibre Z. By HH we denote the corresponding bundle which acquires an metric structure J^{HH} induced by $J^{Z,F}$.

We now introduce the rescaling. Let again $\tilde B := (0,\infty)\times B$ and $\mathrm{pr}\colon \tilde B\to B$ be the projection. We consider the bundle $\tilde M := \mathrm{pr}^*M$ over $\tilde B$ together with the canonical projection $\mathrm{Pr}\colon \tilde M\to M$. We define $(\tilde{\mathcal{F}},\tilde q_{\mathcal{F}},T^H\tilde M,\tilde v,\tilde N_F) := \mathrm{Pr}^*(\mathcal{F},q_{\mathcal{F}}, T^HM,v,N_F)$. We obtain the (by $\tilde N := \tilde N_Z + \tilde N_F$) $\mathbb{Z}$-graded $\epsilon\epsilon_n$-duality bundle $\Omega(Z,\tilde F)$ over $\tilde B$ with the $(-1)^{\tilde N}$-superconnection $A' := \tilde{\mathbf{d}} = d^{\tilde F} + \tilde v$. We fix the vertical metric $\tilde g^{TZ}$ which restricts to $t^{-1}g^{TZ}$ over $\{t\}\times M$. Furthermore, we consider the metric structure $\tilde J^F$ which restricts to $t^{-\tilde N_F/2+n_F/4}J^Ft^{\tilde N_F/2-n_F/4} = J^Ft^{\tilde N_F-n_F/2}$ (cf. Definition 3.14) on $\{t\}\times M$. These choices induce the metric structure $\tilde J$ on $\Omega(Z,\tilde F)$. The form $p(A',\tilde J)\in\Omega(\tilde B)$ is now well-defined as in Subsection 5.2.2. Let us decompose $p(A',\tilde J) = dt\wedge\gamma + r$, where r does not contain dt. Here $\gamma\colon (0,\infty)\to\Omega(B)$ is a smooth family of forms on B.

LEMMA 5.8. *We have $\gamma(t) = O(t^{-3/2})$ as $t\to\infty$ and $\gamma = O(1)$ as $t\to 0$. The form*

$$\tilde\eta(\Omega(Z,F),\mathbf{N},J^{Z,F},\mathbf{d}) := -\int_0^\infty \gamma(t)dt$$

satisfies

$$d\tilde{\eta}(\Omega(Z,F),\mathbf{N},J^{Z,F},\mathbf{d}) = \begin{cases} \int_Z L(\nabla^{TZ})\wedge p(\nabla^F,J^F) - p(\nabla^{HH},J^{HH}) & \text{if } n \text{ is even,} \\ \int_Z L(\nabla^{TZ})\wedge p(\nabla^F,J^F) & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. This lemma can be shown using the techniques of the proof of [**12**, Thm. 3.21], [**9**, Thm. 2.10]. □

There are two spectral sequences of locally constant sheaves of finite-dimensional real vector spaces which converge to the hypercohomology $\mathcal{H}H$ of $(\mathcal{F},v)$ along the fibre Z, namely the fibrewise hypercohomology spectral sequence $({}_{hc}\mathcal{E}_r, {}_{hc}d_r)$ and the fibrewise local-global spectral sequence $({}_{lg}\mathcal{E}_r, {}_{lg}d_r)$. Both are obtained from filtrations of $\Omega(Z,F)$ which are compatible with the duality. Therefore, we obtain $\epsilon\epsilon_n$ duality structures $q_{{}_{hc}E_r}$ and $q_{{}_{lg}E_r}$ induced by $q_{Z,F}$. Let $N_{{}_{hc}E_r}$, $N_{{}_{gl}E_r}$ be the $\mathbb{Z}$-gradings on ${}_{hc}E_r, {}_{gl}E_r$ induced by $\mathbf{N}$. We further obtain corresponding metric structures $J^{{}_{hc}E_r}$ and $J^{{}_{lg}E_r}$. Let ${}_{hc}d_r^*$ be the adjoint of ${}_{hc}d_r$ with respect to $J^{{}_{hc}E_r}$. We identify ${}_{hc}\mathrm{Gr}(\mathcal{H}H) \cong {}_{hc}\mathcal{E}_\infty$ and ${}_{lg}\mathrm{Gr}(\mathcal{H}H) \cong {}_{lg}\mathcal{E}_\infty$ (cf. Subsection 4.4.2).

To consider the case of even and odd n in parallel we adopt the following convention. If n is even, then the eta form of a complex or a filtration was defined in Subsections 3.4.4 and 3.4.5, respectively.

If n is odd, then we proceed as in (5.7). Note that ${}_{hc}d_r + {}_{hc}d_r^*$ commutes with $J^{{}_{hc}E_r}$. We form ${}_{hc}A := \nabla^{{}_{hc}E_r, J^{{}_{hc}E_r}} + ({}_{hc}d_r + {}_{hc}d_r^*)\sigma$. By [**7**, Prop. 2.12], [**27**, §5], the form

$$p(\nabla^{{}_{hc}E_r} + {}_{hc}d_r, J^{{}_{hc}E_r}) := (2i)^{1/2}\varphi\,\mathrm{Tr}_\sigma\left[z^{J^{{}_{hc}E_r}}\exp(-{}_{hc}A^2)\right] \tag{5.10}$$

is an exact odd form. To define the η-form $\tilde{\eta}({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r)$ we now employ the usual rescaling induced by the $\mathbb{Z}$-grading $N_{{}_{hc}E_r}$. As in [**8**, Thm. 2.43], [**7**, Thm. 2.17], it is a closed form on B, and its cohomology class is $\frac{1}{2}[\mathbf{ch}({}_{hc}E_{r,>0}) - \mathbf{ch}({}_{hc}E_{r,<0})]$, where ${}_{hc}E_{r,>0}$ and ${}_{hc}E_{r,<0}$ are the sub-bundles of ${}_{hc}E_r$ associated to positive or negative eigenvalues of $z^{J^{{}_{hc}E_r}}({}_{hc}d_r + {}_{hc}d_r^*)$. In a similar manner we define $\tilde{\eta}({}_{lg}\mathcal{E}_r, N_{{}_{lg}\mathcal{E}_r}, J^{{}_{lg}\mathcal{E}_r}, {}_{lg}d_r)$ associated to the local-global spectral sequence. We do not change the definition of the η-forms of the filtrations.

The following theorems are the main ingredients of the proof of well-definedness of the secondary index map.

THEOREM 5.9. *Modulo exact forms on B we have*

$$\tilde{\eta}(\Omega(Z,F),\mathbf{N},J^{Z,F},\mathbf{d}) \equiv \tilde{\eta}(\Omega(Z,F),N_Z,J^{Z,F},d^F) + \sum_{r\geq 1}\tilde{\eta}({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r) - \tilde{\eta}(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{hc}\mathrm{Gr}(HH)}).$$

THEOREM 5.10. *Modulo exact forms on B we have*

$$\tilde{\eta}(\Omega(Z,F),\mathbf{N},J^{Z,F},\mathbf{d}) \equiv \tilde{\eta}(\Omega(Z,H),N_Z,J^{Z,H},d^H) + \sum_{r\geq 2}\tilde{\eta}({}_{lg}\mathcal{E}_r,N_{{}_{lg}\mathcal{E}_r},J^{{}_{lg}\mathcal{E}_r},{}_{lg}d_r) - \tilde{\eta}(\mathcal{H}H,{}_{lg}\mathrm{Gr}(\mathcal{H}H),J^{HH},J^{{}_{lg}\mathrm{Gr}(HH)}) + \int_Z L(\nabla^{TZ})\wedge\tilde{\eta}(F,N_F,J^F,v).$$

The proof of both results will be sketched in Section 6. Note that if n_F is odd, then $\tilde{\eta}(\Omega(Z,F),N_Z,J^{Z,F},d^F) = 0$ and $\tilde{\eta}(\Omega(Z,H),N_Z,J^{Z,H},d^H) = 0$, as $J^{Z,F}$ changes the parity of N_F-grading.

5.2.5. *The η-form of an iterated fibre bundle.* Let W,V,S be smooth manifolds. Let $\pi_1\colon W\to V$, $\pi_2\colon V\to S$ be smooth fibrations with compact fibres X and Y, respectively. Then $\pi_3=\pi_2\circ\pi_1\colon W\to S$ is a smooth fibration with compact fiber Z of dimension n. Let $n_0=\dim X, m_0=\dim Y$. Let TX,TY,TZ be the corresponding vertical tangent bundles. We assume that TX,TY,TZ are compatibly oriented such that we have an iterated fibre bundle with compatible orientation as in Subsection 2.5.2.

Let $\mathcal{F}$ be a locally constant sheaf of finite dimensional real vector spaces over W with a ϵ–symmetric duality structure $q_{\mathcal{F}}$. We choose a metric structure J^F on $F:=\mathtt{bundle}(\mathcal{F})$.

By taking fibrewise cohomology we obtain locally constant sheaves of finite-dimensional real vector spaces $\mathcal{H}(X,\mathcal{F})$ over V, and $\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F}))$ and $\mathcal{H}(Z,\mathcal{F})$ over S. These come with induced duality structures $q_{\mathcal{H}(X,\mathcal{F})}, q_{\mathcal{H}(Z,\mathcal{F})}, q_{\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F}))}$.

We choose horizontal sub-bundles T_1^HW, T_2^HV, T_3^HW, i.e., complements in TW, TV, TW to TX, TY, TZ. Furthermore, we choose vertical metrics g^{TZ}, g^{TX}, g^{TY} on TZ, TX, TY. Then we obtain induced metric structures $J^{H(X,\mathcal{F})}$, $J^{H(Z,\mathcal{F})}$, $J^{H(Y,\mathcal{H}(X,\mathcal{F}))}$.

Let ∇^{TX}, ∇^{TY}, ∇^{TZ} be the connections on TX, TY, TZ which are induced by the choice of horizontal subspaces and vertical metrics (cf. [**5**, Def. 1.6], [**3**, Prop. 10.2]). We get a further connection ${}^0\nabla^{TZ} := \pi_1^*\nabla^{TY}\oplus\nabla^{TX}$ on $TZ = TX\oplus(T_1^HW\cap TZ)$. Let $L(\nabla^{TX})$, $L(\nabla^{TY})$, $L(\nabla^{TZ})$, $L({}^0\nabla^{TZ})$ be the associated L-forms as in Subsection 5.2.2. By $\tilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})$ we denote the transgression L-form such that

$$d\widetilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ}) = L(\nabla^{TZ}) - L({}^0\nabla^{TZ}). \tag{5.11}$$

Since the fibre Z has the structure of a fibre bundle, we have a fibrewise Leray spectral sequence $({}_{LS}\mathcal{E}_r,{}_{LS}d_r)$ $(r\geq 2)$ of locally constant sheaves on S (cf. also [**24**, Prop. 2.1]). All terms have $\epsilon\epsilon_n$–duality structures $q_{{}_{LS}\mathcal{E}_r}$ induced by $q_{Z,F}$ on $\Omega(Z,F)$. The $\mathbb{Z}$-gradings $N_{{}_{LS}\mathcal{E}_r}$ (induced by N_Z) and the differentials ${}_{LS}d_r$ are compatible with the duality. In particular, we have ${}_{LS}\mathcal{E}_2 = H(Y,\mathcal{H}(X,\mathcal{F}))$. Hence, we have an induced compatible metric structure $J^{{}_{LS}\mathcal{E}_2} := J^{H(Y,\mathcal{H}(X,\mathcal{F}))}$. Since the other terms are obtained by taking cohomology successively, we get induced metric structures $J^{{}_{LS}\mathcal{E}_r}$ on ${}_{LS}\mathcal{E}_r$ for all $r \geq 2$. Thus we can define $\tilde{\eta}({}_{LS}\mathcal{E}_r, N_{{}_{LS}\mathcal{E}_r}, J^{{}_{LS}E_r}, {}_{LS}d_r)$ (with the usual modifications if one or two of the dimensions n_0, m_0 are odd, see the remark below).

There is a filtration on $\mathcal{H}(Z,\mathcal{F})$ and a natural isomorphism $\mathrm{Gr}\mathcal{H}(Z,\mathcal{F}) \cong {}_{LS}\mathcal{E}_\infty$. Therefore, we have the form $\tilde{\eta}(\mathcal{H}(Z,\mathcal{F}), {}_{LS}\mathcal{E}_\infty, J^{H(Z,\mathcal{F})}, J^{{}_{LS}E_\infty})$ as in Definition 3.18.

The following theorem provides the relation of the η-forms of the total fibration π_3 with the η-forms of the partial fibrations π_1 and π_2. It is the main ingredient in the proof of the fact that the secondary index map is functorial with respect to iterated fibre bundles.

THEOREM 5.11. *The following identity holds modulo exact forms on S:*

$$\begin{aligned}\tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F) \equiv \int_Y & L(\nabla^{TY}) \wedge \tilde{\eta}(\Omega(X,F), N_X, J^{X,F}, d^F) \\ & + \tilde{\eta}(\Omega(Y, H(X,\mathcal{F})), N_Y, J^{Y,H(X,\mathcal{F}))}, d^{H(X,\mathcal{F})}) \\ & - \tilde{\eta}(\mathcal{H}(Z,\mathcal{F}), {}_{LS}\mathcal{E}_\infty, J^{H(Z,\mathcal{F})}, J^{{}_{LS}E_\infty}) \\ & + \sum_{r=2}^{\infty} \tilde{\eta}({}_{LS}\mathcal{E}_r, N_{{}_{LS}\mathcal{E}_r}, J^{{}_{LS}E_r}, {}_{LS}d_r) \\ & + \int_Z \tilde{L}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \wedge p(\nabla^F, J^F).\end{aligned}$$

PROOF. The proof of this theorem will be sketched in Subsection 6.5. The formula above arrises from a detailed investigation of the adiabatic limit of $\tilde{\eta}(\Omega(Z,F)$, N_Z, $J^{Z,F}, d^F)$. □

REMARK 5.12. Theorem 5.11 has four cases. Assume first that n is even. Then $\tilde{\eta}(\Omega(Z,F),\ldots)$ and $\tilde{\eta}({}_{LS}\mathcal{E}_r,\ldots)$ are defined as in Definitions 5.3, 3.15. If n_0 is even, then $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\ldots)$, $\tilde{\eta}(\Omega(X,F),\ldots)$ are defined as in Definition 5.3. If n_0 is odd, then $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\ldots)$, $\tilde{\eta}(\Omega(X,F),\ldots)$ are defined as in Definition 5.4 and $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\ldots)$ is zero by the remark following Theorem 5.10.

Assume now that n is odd. Then $\tilde{\eta}(\Omega(Z,F),\ldots)$ and $\tilde{\eta}({}_{LS}\mathcal{E}_r,\ldots)$ are defined as in Definition 5.4 and Subsection 5.2.4. If n_0 is odd, then $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\ldots)$ and $\tilde{\eta}(\Omega(X,F),\ldots)$ are defined as in Definitions 5.3 and 5.4 and $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\cdot)$ is zero by the remark following Theorem 5.10. If n_0 is even, then $\tilde{\eta}(\Omega(X,F),\ldots)$ is defined as in Definition 5.3 and $\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),\ldots)$ is defined as in Definition 5.4.

5.3. Definition of the secondary index maps.

5.3.1. *The map* $\pi_*^{\bar{L},\mathrm{Lott}}$. Let $M \to B$ be a smooth locally trivial fibre bundle with even-dimensional closed fibres Z over a compact base B such that the vertical bundle TZ is oriented. We set $n := \dim Z$, $\epsilon_n = (-1)^{[\frac{n+1}{2}]}$.

In [**21**, §3.6], Lott constructed a secondary index map

$$\pi_*^{\bar{L},\mathrm{Lott}} \colon \bar{L}_\epsilon^{\mathrm{Lott}}(M) \to \bar{L}_{\epsilon\epsilon_n}^{\mathrm{Lott}}(B)$$

which fits into the commutative diagram

$$\begin{array}{ccccccc} H^{4*-\epsilon}(M,\mathbb{R}) & \longrightarrow & \bar{L}_\epsilon^{\mathrm{Lott}}(M) & \longrightarrow & \bar{L}_\epsilon^{\mathbb{R}/\mathbb{Z}}(M) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^{-1}(M) \\ \pi_*^{\mathrm{sign}}\downarrow & & \pi_*^{\bar{L},\mathrm{Lott}}\downarrow & & \pi_*^{\bar{L},\mathbb{R}/\mathbb{Z}}\downarrow & & \pi_*^{\mathrm{sign},\mathbb{R}/\mathbb{Z}}\downarrow \\ H^{4*-\epsilon\epsilon_n}(B,\mathbb{R}) & \longrightarrow & \bar{L}_{\epsilon\epsilon_n}^{\mathrm{Lott}}(B) & \longrightarrow & \bar{L}_{\epsilon\epsilon_n}^{\mathbb{R}/\mathbb{Z}}(B) & \longrightarrow & K_{\mathbb{R}/\mathbb{Z}}^{-1}(B). \end{array}$$

We recall the definition of $\pi_*^{\bar{L},\mathrm{Lott}}$. We choose a metric g^{TZ} and a horizontal distribution $T^H M$. This fixes a canonical connection ∇^{TZ} on the vertical bundle TZ.

DEFINITION 5.13. Given a class $\left[\mathcal{F}, q_{\mathcal{F}}, J^F, \rho\right]_{\mathrm{Lott}} \in \bar{L}_\epsilon^{\mathrm{Lott}}(M)$, its secondary index is defined by

$$\pi_*^{\bar{L},\mathrm{Lott}}[\mathcal{F}, q_F, J^F, \rho]_{\mathrm{Lott}} :=$$
$$\left[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\right]_{\mathrm{Lott}}.$$

By [**21**, Prop. 32], $\pi_*^{\bar{L},\mathrm{Lott}}$ is well-defined. In particular, it is independent of the additional choices.

5.3.2. *Construction of* $\pi_*^{\bar{L}}$. We can now define the secondary analytic index map

$$\pi_*^{\bar{L}} \colon \bar{L}_\epsilon(M) \to \bar{L}_{\epsilon\epsilon_n}(B).$$

Recall that we have surjective homomorphisms

$$\bar{L}_\epsilon^{\mathrm{Lott}}(M) \to \bar{L}_\epsilon(M), \quad \bar{L}_{\epsilon\epsilon_n}^{\mathrm{Lott}}(B) \to \bar{L}_{\epsilon\epsilon_n}(B).$$

DEFINITION 5.14. We define $\pi_*^{\bar{L}}$ by the condition that the following diagram commutes:

$$\begin{array}{ccc} \bar{L}_\epsilon^{\mathrm{Lott}}(M) & \xrightarrow{\pi_*^{\bar{L},\mathrm{Lott}}} & \bar{L}_{\epsilon\epsilon_n}^{\mathrm{Lott}}(B) \\ \downarrow & & \downarrow \\ \bar{L}_\epsilon(M) & \xrightarrow{\pi_*^{\bar{L}}} & \bar{L}_{\epsilon\epsilon_n}(B). \end{array}$$

THEOREM 5.15. *The index map* $\pi_*^{\bar{L},\mathrm{Lott}}$ *induces a well-defined secondary index map*

$$\pi_*^{\bar{L}} \colon \bar{L}_\epsilon(M) \to \bar{L}_{\epsilon\epsilon_n}(B).$$

PROOF. Let $[\mathcal{F}, q_{\mathcal{F}}, J^F, \rho] \in \bar{L}_\epsilon(M)$ and $\mathcal{L} \subset \mathcal{F}$ be a locally constant lagrangian subsheaf. We must show that

$$
\begin{aligned}
(5.12)\quad \Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big] \\
= \Big[0,0,0, \int_Z L(\nabla^{TZ}) \wedge (\rho + \tilde{p}(\mathcal{F}, q_{\mathcal{F}}, J^F, \mathcal{L}))\Big].
\end{aligned}
$$

We repeat the construction of $\mathcal{G}$, q_{cG}, J^{cG}, $v_{\mathcal{G}}$, etc. from Subsection 4.5.2 to write

$$
(5.13)\qquad -\tilde{p}(\mathcal{F}, q_{\mathcal{F}}, J^F, \mathcal{L}) = \tilde{\eta}(\mathcal{G}, N_{\mathcal{G}}, J^{\mathcal{G}}, v_{\mathcal{G}}).
$$

We consider the sum $\mathcal{G}^0 \oplus \mathcal{G}^2$ together with the corresponding form and metric structure. Note that

$$
[\mathcal{H}(Z,\mathcal{G}^0) \oplus \mathcal{H}(Z,\mathcal{G}^2), q_{\mathcal{H}(Z,\mathcal{G}^0)\oplus\mathcal{H}(Z,\mathcal{G}^2)}, J^{H(Z,\mathcal{G}^0)\oplus H(Z,\mathcal{G}^2)}, 0] = 0.
$$

Moreover, we have $\tilde{\eta}(\Omega(Z, G^0 \oplus G^2), N_Z, J^{G^0\oplus G^2}, d^{Z,G^0\oplus G^2}) = 0$. Thus

$$
\begin{aligned}
&\Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big] \\
&\quad = \Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big] \\
&\qquad + [\mathcal{H}(Z,\mathcal{G}^0) \oplus \mathcal{H}(Z,\mathcal{G}^2), q_{\mathcal{H}(Z,\mathcal{G}^0)\oplus\mathcal{H}(Z,\mathcal{G}^2)}, J^{H(Z,\mathcal{G}^0)\oplus H(Z,\mathcal{G}^2)}, 0] \\
&\quad = \Big[\mathcal{H}(Z,\mathcal{G}), q_{\mathcal{H}(Z,\mathcal{G})}, J^{H(Z,\mathcal{G})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,G), N_Z, J^{Z,G}, d^G)\Big].
\end{aligned}
$$

Next observe that we have $\mathcal{H}(Z,\mathcal{G}) = {}_{hc}\mathcal{E}_1$. This relation respects the other structures. Let $\mathcal{H}H$ be the fibrewise hypercohomology of the complex $\mathcal{G}$ along the fibre Z. Using Lemma 3.16 repeatedly and finally Lemma 3.19, we get

$$
\begin{aligned}
&\Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big] \\
&\quad = \Big[{}_{hc}\mathcal{E}_2, q_{{}_{hc}\mathcal{E}_2}, J^{{}_{hc}E_2}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,G), N_Z, J^{Z,G}, d^G) \\
&\qquad - \tilde{\eta}({}_{hc}\mathcal{E}_1, N_{{}_{hc}E_1}, J^{{}_{hc}E_1}, {}_{hc}d_1)\Big] \\
&\quad = \Big[{}_{hc}\mathcal{E}_\infty, q_{{}_{hc}\mathcal{E}_\infty}, J^{{}_{hc}E_\infty}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,G), N_Z, J^{Z,G}, d^G) \\
&\qquad - \sum_{r\geq 1} \tilde{\eta}({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r)\Big] \\
&\quad = \Big[\mathcal{H}H, q_{\mathcal{H}H}, J^{HH}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,G), N_Z, J^{Z,G}, d^G) \\
&\qquad - \sum_{r\geq 1} \tilde{\eta}({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r) + \tilde{\eta}(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{\mathrm{Gr}(HH)})\Big].
\end{aligned}
$$

(Note that in fact $\mathcal{H}H = 0$.) We now apply first Theorem 5.9 and then Theorem 5.10 to conclude that

$$\begin{aligned}
&\Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big] \\
&= \Big[\mathcal{H}H, q_{\mathcal{H}H}, J^{\mathcal{H}H}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,G), \mathbf{N}, J^{Z,G}, \mathbf{d})\Big] \\
&= \Big[\mathcal{H}H, q_{\mathcal{H}H}, J^{HH}, \int_Z L(\nabla^{TZ}) \wedge (\rho - \tilde{\eta}(\mathcal{G}, N_G, J^G, v_{\mathcal{G}})) \\
&\qquad - \sum_{r\geq 2} \tilde{\eta}({}_{lg}\mathcal{E}_r, N_{{}_{lg}E_r}, J^{{}_{lg}E_r}, {}_{lg}d_r) + \tilde{\eta}(\mathcal{H}H, {}_{lg}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{lg}\mathrm{Gr}(HH)})\Big].
\end{aligned}$$

Remark that in this case, ${}_{lg}E_r = 0$ for $r \geq 2$ and $\mathcal{H}H = 0$, as $\mathcal{G}$ is a short exact sequence. By (5.13) and the above equation, we get (5.12). The proof of Theorem 5.15 is complete. □

5.4. Functorial properties.

5.4.1. *Compatibility and naturality.* The following two propositions are immediate consequences of the definition of the secondary index map.

PROPOSITION 5.16. *The following diagram commutes:*

$$\begin{array}{ccccc}
H^{4*-\epsilon}(M,\mathbb{R}) & \longrightarrow & \bar{L}_\epsilon(M) & \longrightarrow & L_\epsilon(M) \\
\Big\downarrow{\scriptstyle \pi_*^{\mathrm{sign}}} & & \Big\downarrow{\scriptstyle \pi_*^{\bar{L}}} & & \Big\downarrow{\scriptstyle \pi_*^{L}} \\
H^{4*-\epsilon\epsilon_n}(B,\mathbb{R}) & \longrightarrow & \bar{L}_{\epsilon\epsilon_n}(B) & \longrightarrow & L_{\epsilon\epsilon_n}(B).
\end{array}$$

PROPOSITION 5.17. *The secondary index map is natural with respect to pull-back of fibre bundles, i.e., given $f\colon B' \to B$ we consider the pull back*

$$\begin{array}{ccc}
f^*M & \xrightarrow{f_\sharp} & M \\
\Big\downarrow{\scriptstyle f^*\pi} & & \Big\downarrow{\scriptstyle \pi} \\
B' & \xrightarrow{f} & B,
\end{array}$$

where $f^*M \to B'$ has the induced fibrewise orientation, and we have $(f^*\pi)_*^{\bar{L}} \circ f_\sharp^* = f^* \circ \pi_*^{\bar{L}}$.

5.4.2. *Functoriality.* We adopt the notation of Subsection 5.2.5. In particular, we have smooth fibre bundles $\pi_1\colon W \to V$, $\pi_2\colon V \to S$ with closed fibres X, Y. The composition $\pi_3 = \pi_2 \circ \pi_1\colon W \to S$ is a fibre bundle with closed fibre Z. Let $n = \dim Z, n_0 = \dim X, m_0 = \dim Y$ such that $n = n_0 + m_0$. We assume that n, n_0 are even. Furthermore, we assume that the relative tangent bundles TX, TY, TZ are compatibly oriented.

Note that $\epsilon_n = \epsilon_{n_0}\epsilon_{m_0}$. We have well-defined secondary index maps

$$\pi^{\bar{L}}_{3,*}\colon \bar{L}_\epsilon(W) \to \bar{L}_{\epsilon\epsilon_n}(S),$$
$$\pi^{\bar{L}}_{1,*}\colon \bar{L}_\epsilon(W) \to \bar{L}_{\epsilon\epsilon_{n_0}}(V),$$
$$\pi^{\bar{L}}_{2,*}\colon \bar{L}_{\epsilon\epsilon_{n_0}}(V) \to \bar{L}_{\epsilon\epsilon_n}(S).$$

THEOREM 5.18. *We have the equality of homomorphisms* $\bar{L}_\epsilon(W) \to \bar{L}_{\epsilon\epsilon_n}(S)$:

$$\pi^{\bar{L}}_{3,*} = \pi^{\bar{L}}_{2,*} \circ \pi^{\bar{L}}_{1,*}.$$

PROOF. Let $(\mathcal{F}, q_\mathcal{F}, J^F, \rho)$ represent some element of $\bar{L}_\epsilon(W)$. Then the element $\pi^{\bar{L}}_{3,*}[\mathcal{F}, q_\mathcal{F}, J^F, \rho]$ is represented by

$$\Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big].$$

There is a filtration of $\mathcal{H}(Z,\mathcal{F})$ such that the corresponding graded sheaf is the limit ${}_{LS}\mathcal{E}_\infty$ of the fibrewise Leray-Serre spectral sequence. Using Lemma 3.19, we get

$$\begin{aligned}
&\pi^{\bar{L}}_{3,*}[\mathcal{F}, q_\mathcal{F}, J^F, \rho]\\
&= \Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\Big]\\
&= \Big[{}_{LS}\mathcal{E}_\infty, q_{{}_{LS}\mathcal{E}_\infty}, J^{{}_{LS}E_\infty}, \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\\
&\qquad - \tilde{\eta}(\mathcal{H}(Z,\mathcal{F}), {}_{LS}\mathcal{E}_\infty, J^{H(Z,\mathcal{F})}, J^{{}_{LS}E_\infty})\Big].
\end{aligned}$$

Note that $\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F})) = {}_{LS}\mathcal{E}_2$ with all induced structures. Next we use Lemma 3.16 several times to get

$$\begin{aligned}
\pi^{\bar{L}}_{3,*}[\mathcal{F}, q_\mathcal{F}, J^F, \rho] &= \Big[\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F})), q_{\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F}))}, J^{\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F}))},\\
&\qquad \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)\\
&\quad - \tilde{\eta}(\mathcal{H}(Z,F), {}_{LS}\mathcal{E}_\infty, J^{H(Z,\mathcal{F})}, J^{{}_{LS}E_\infty}) + \sum_{r=2}^{\infty} \tilde{\eta}({}_{LS}\mathcal{E}_r, N_{{}_{LS}E_r}, J^{{}_{LS}E_r}, {}_{LS}d_r)\Big].
\end{aligned}$$

Now we start with the other side. We have

$$\begin{aligned}
&\pi^{\bar{L}}_{1,*}[\mathcal{F}, q_\mathcal{F}, J^F, \rho]\\
&= \Big[\mathcal{H}(X,\mathcal{F}), q_{\mathcal{H}(X,\mathcal{F})}, J^{H(X,\mathcal{F})}, \int_X L(\nabla^{TX}) \wedge \rho - \tilde{\eta}(\Omega(X,F), N_X, J^{X,F}, d^F)\Big].
\end{aligned}$$

Furthermore, we get

$$(\pi^{\bar{L}}_{2,*}\circ\pi^{\bar{L}}_{1,*})[\mathcal{F},q_{\mathcal{F}},J^F,\rho]$$
$$=\Big[\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F})),q_{\mathcal{H}(Y,\mathcal{H}(X,\mathcal{F}))},J^{H(Y,\mathcal{H}(X,\mathcal{F}))},$$
$$+\int_Y L(\nabla^{TY})\wedge\Big(\int_X L(\nabla^{TX})\wedge\rho-\tilde{\eta}(\Omega(X,F),N_X,J^{X,F},d^F)\Big)$$
$$-\tilde{\eta}(\Omega(Y,H(X,\mathcal{F})),N_Y,J^{Y,H(X,\mathcal{F})},d^{H(X,\mathcal{F})})\Big].$$

Thus the assertion of the theorem is proved if we show that

$$\int_Z L(\nabla^{TZ})\wedge\rho-\tilde{\eta}(\Omega(Z,F),N_Z,J^{Z,F},d^F)\tag{5.14}$$
$$-\tilde{\eta}\big(\mathcal{H}(Z,\mathcal{F}),{}_{LS}\mathcal{E}_\infty,J^{H(Z,\mathcal{F})},J^{{}_{LS}E_\infty}\big)+\sum_{r=2}^{\infty}\tilde{\eta}\big({}_{LS}\mathcal{E}_r,N_{{}_{LS}E_r},J^{{}_{LS}E_r},{}_{LS}d_r\big)$$
$$\equiv\int_Y L(\nabla^{TY})\wedge\Big(\int_X L(\nabla^{TX})\wedge\rho-\tilde{\eta}(\Omega(X,F),N_X,J^{X,F},d^F)\Big)$$
$$-\tilde{\eta}\big(\Omega(Y,H(X,\mathcal{F})),N_Y,J^{Y,H(X,\mathcal{F})},d^{H(X,\mathcal{F})}\big)$$

modulo exact forms. Note that we have

$$\int_Y L(\nabla^{TY})\wedge\int_X L(\nabla^{TX})\wedge\rho$$
$$=\int_Z \pi_2^*L(\nabla^{TY})\wedge L(\nabla^{TX})\wedge\rho$$
$$=\int_Z\Big(L(\nabla^{TZ})\wedge\rho-d\tilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge\rho\Big)$$
$$\equiv\int_Z\Big(L(\nabla^{TZ})\wedge\rho-\tilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge d\rho\Big)$$
$$\equiv\int_Z\Big(L(\nabla^{TZ})\wedge\rho-\tilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge p(\nabla^F,J^F)\Big).$$

Using this identity, we see that (5.14) is equivalent to

$$-\tilde{\eta}(\Omega(Z,F),N_Z,J^{Z,F},d^F)-\tilde{\eta}\big(\mathcal{H}(Z,\mathcal{F}),{}_{LS}\mathcal{E}_\infty,J^{H(Z,\mathcal{F})},J^{{}_{LS}E_\infty}\big)$$
$$+\sum_{r=2}^{\infty}\tilde{\eta}\big({}_{LS}\mathcal{E}_r,N_{{}_{LS}E_r},J^{{}_{LS}E_r},{}_{LS}d_r\big)$$
$$\equiv-\int_Z\tilde{L}(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge p(\nabla^F,J^F)$$
$$-\int_Y L(\nabla^{TY})\wedge\tilde{\eta}\big(\Omega(X,F),N_X,J^{X,F},d^F\big)$$
$$-\tilde{\eta}\big(\Omega(Y,H(X,\mathcal{F})),N_Y,J^{Y,H(X,\mathcal{F})},d^{Y,H(X,\mathcal{F})}\big).$$

The latter relation, however, is exactly the assertion of Theorem 5.11. □

5.5. The index map for L^{ex}_ϵ and $\bar{L}^{\mathrm{ex}}_\epsilon$.

5.5.1. *Definition.* Let $\pi\colon M \to B$ be a fibre bundle with even-dimensional closed fibres Z of dimension n. We assume that the relative tangent bundle TZ is oriented. Furthermore, we assume that B is closed, oriented, and of dimension m such that $\epsilon\epsilon_n\epsilon_m = 1$ and m is odd. Let $(\mathcal{F}, q_{\mathcal{F}})$ be a locally constant sheaf of finite-dimensional real vector spaces on M with an ϵ–symmetric duality structure $q_{\mathcal{F}}$. Recall from Subsection 5.2.3 that in this situation we have the Leray spectral sequence $({}_{LS}E_r, {}_{LS}d_r)$ $(r \geq 2)$ of finite dimensional vector spaces which carries induced duality structures $q_{{}_{LS}E_r}$. It gives rise to the integer

$$ {}_{LS}\tau_2(Z,F) := \tau_2({}_{LS}E_*) := \sum_{r\geq 2} \tau\big({}_{LS}E_r, q_{{}_{LS}E_r}, N_{{}_{LS}E_r}, {}_{LS}d_r\big). $$

DEFINITION 5.19. (1) We introduce the extended primary index map $\pi^{L^{\mathrm{ex}}}_* : L^{\mathrm{ex}}_\epsilon(M) \to L^{\mathrm{ex}}_{\epsilon\epsilon_n}(B)$ by

$$ \pi^{L^{\mathrm{ex}}}_*[\mathcal{F}, q_{\mathcal{F}}, z] := \Big[\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, z - {}_{LS}\tau_2(Z,F)\Big]. $$

(2) We introduce the extended secondary index map $\pi^{\bar{L}^{\mathrm{ex}}}_* : \bar{L}^{\mathrm{ex}}_\epsilon(M) \to \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_n}(B)$ by

$$ \begin{aligned} \pi^{\bar{L}^{\mathrm{ex}}}[\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z] := \Big[&\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \\ &\int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big), z - {}_{LS}\tau_2(Z,F)\Big]. \end{aligned} $$

THEOREM 5.20. *$\pi^{L^{\mathrm{ex}}}_*$ and $\pi^{\bar{L}^{\mathrm{ex}}}_*$ are well-defined.*

PROOF. Let $\hat{\pi}^{L^{\mathrm{ex}}}_* : \hat{L}^{\mathrm{ex}}_\epsilon(M) \to \hat{L}^{\mathrm{ex}}_{\epsilon\epsilon_n}(B)$ and $\hat{\pi}^{\bar{L}^{\mathrm{ex}}}_* : \hat{\bar{L}}^{\mathrm{ex}}_\epsilon(M) \to \hat{\bar{L}}^{\mathrm{ex}}_{\epsilon\epsilon_n}(B)$ be given by the above formulas.

LEMMA 5.21. *We have*

$$ \eta\big(\hat{\pi}^{\bar{L}^{\mathrm{ex}}}_*(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z)\big) = \eta\big(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z\big). $$

PROOF. This is just a reformulation of Theorem 5.7. In fact, since the η-homomorphism of Definition 4.18 is independent of the metric, we can perform the adiabatic limit and obtain (using that in this limit $L(\nabla^{TM}_T) \to \pi^* L(\nabla^{TB}) \wedge$

$L(\nabla^{TZ})$ (cf. [**24**, Thm. 5.1]))

$$\begin{aligned}\eta(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z) &= \eta(D_F^{\mathrm{sign}}) - 2\int_Z L(\nabla^{TZ}) \wedge \rho - z \\ &= \eta(D^{\mathrm{sign}}_{H(Z,\mathcal{F})}) + 2\int_Z L(\nabla^{TB}) \wedge \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F) \\ &\qquad + {}_{LS}\tau_2(Z,F) - 2\int_B L(\nabla^{TB}) \wedge \int_Z L(\nabla^{TZ}) \wedge \rho - z \\ &= \eta(\mathcal{H}(Z,\mathcal{F}), q_{\mathcal{H}(Z,\mathcal{F})}, J^{H(Z,\mathcal{F})}, \\ &\qquad \int_Z L(\nabla^{TZ}) \wedge \rho - \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F), z - {}_{LS}\tau_2(Z,F)) \\ &= \eta\big(\hat{\pi}_*^{\bar{L}^{\mathrm{ex}}}(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z)\big).\end{aligned}$$

□

LEMMA 5.22. *$\pi_*^{\bar{L}^{\mathrm{ex}}}$ is well-defined.*

PROOF. Assume that $(\mathcal{F}, q_F, J, \rho, z) \in \hat{\bar{L}}_\epsilon^{\mathrm{ex}}(M)$ satisfies $(\mathcal{F}, q_F, J, \rho, z) \sim 0$. Then we have $[\mathcal{F}, q_F, J, \rho] = 0$ in $\bar{L}_\epsilon(M)$. Since $\pi_*^{\bar{L}}$ is well-defined, we have $\pi_*^{\bar{L}}([\mathcal{F}, q_F, J, \rho]) = 0$. By an inspection of the definitions we further observe that $\hat{\pi}_*^{\bar{L}^{\mathrm{ex}}}(\mathcal{F}, q_F, J, \rho, z) \sim (0,0,0,0,u)$ for some $u \in \mathbb{Z}$. We must show that $u = 0$. In fact, since the η-homomorphism is well-defined, we can compute (using also Lemma 5.21)

$$0 = \eta(\mathcal{F}, q_F, J, \rho, z) = \hat{\pi}_*^{\bar{L}^{\mathrm{ex}}}(\mathcal{F}, q_F, J, \rho, z) = \eta(0,0,0,0,u) = -u.$$

□

LEMMA 5.23. *$\pi_*^{L^{\mathrm{ex}}}$ is well-defined.*

PROOF. Let $(\mathcal{F}, q_{\mathcal{F}}, z) \in \hat{L}_\epsilon^{\mathrm{ex}}(M)$ satisfy $(\mathcal{F}, q, z) \sim 0$. Then we can find a metric structure J^F and a form ρ such that $(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z) \sim 0$ in $\hat{\bar{L}}_\epsilon^{\mathrm{ex}}(M)$. It follows from Lemma 5.22 that

$$\hat{\pi}_*^{\bar{L}^{\mathrm{ex}}}(\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z) \sim 0.$$

This implies $\hat{\pi}_*^{L^{\mathrm{ex}}}(\mathcal{F}, q, z) \sim 0$. □

REMARK 5.24. The assertion of Lemma 5.23 should have a purely algebraic proof. In particular, such a proof should be independent of analytic results about η-invariants and η-forms. We were not able to find such an argument.

The proof of Theorem 5.20 is now finished. □

Let us state as a corollary the following consequence of Lemma 5.21.

COROLLARY 5.25. *The following diagram commutes:*

$$\begin{array}{ccc} \bar{L}^{\mathrm{ex}}_{\epsilon}(M) & \xrightarrow{\eta} & \mathbb{R} \\ {\scriptstyle \pi^{\bar{L}^{\mathrm{ex}}}_{*}}\downarrow & & \| \\ \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_n}(B) & \xrightarrow{\eta} & \mathbb{R}. \end{array}$$

5.5.2. *Functoriality.* We adopt the notation and assumptions of Subsection 5.4.2. In addition we assume that S is compact, the dimension n_S is odd, and $\epsilon\epsilon_{n+n_S} = 1$. We have well-defined extended secondary index maps

$$\begin{aligned} \pi^{\bar{L}^{\mathrm{ex}}}_{3,*} &: \bar{L}^{\mathrm{ex}}_{\epsilon}(W) \to \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_n}(S) \\ \pi^{\bar{L}^{\mathrm{ex}}}_{1,*} &: \bar{L}^{\mathrm{ex}}_{\epsilon}(W) \to \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_{n_0}}(V) \\ \pi^{\bar{L}^{\mathrm{ex}}}_{2,*} &: \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_{n_0}}(V) \to \bar{L}^{\mathrm{ex}}_{\epsilon\epsilon_n}(S). \end{aligned}$$

THEOREM 5.26. *We have the equality of homomorphisms $\bar{L}_{\epsilon}(W) \to \bar{L}_{\epsilon\epsilon_n}(S)$:*

$$\pi^{\bar{L}^{\mathrm{ex}}}_{3,*} = \pi^{\bar{L}^{\mathrm{ex}}}_{2,*} \circ \pi^{\bar{L}^{\mathrm{ex}}}_{1,*}.$$

PROOF. Let $[\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z] \in \bar{L}_{\epsilon}(W)$ be given. Then we have in view of Theorems 2.22 and 5.18

$$\pi^{\bar{L}}_{3,*}([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho]) - (\pi^{\bar{L}}_{2,*} \circ \pi^{\bar{L}}_{1,*})([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho]) = 0.$$

Thus

$$\pi^{\bar{L}^{\mathrm{ex}}}_{3,*}([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z]) - (\pi^{\bar{L}^{\mathrm{ex}}}_{2,*} \circ \pi^{\bar{L}^{\mathrm{ex}}}_{1,*})([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z]) = [0,0,0,0,u]$$

for some $u \in \mathbb{Z}$. We must show that $u = 0$. We again use the η-homomorphism and Lemma 5.25:

$$\begin{aligned} -u &= \eta(0,0,0,0,u) \\ &= \eta\big(\pi^{\bar{L}^{\mathrm{ex}}}_{3,*}([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z])\big) - \eta\big((\pi^{\bar{L}^{\mathrm{ex}}}_{2,*} \circ \pi^{\bar{L}^{\mathrm{ex}}}_{1,*})([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z])\big) \\ &= \eta\big([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z]\big) - \eta\big(\pi^{\bar{L}^{\mathrm{ex}}}_{1,*}([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z])\big) \\ &= \eta\big([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z]\big) - \eta\big([\mathcal{F}, q_{\mathcal{F}}, J^F, \rho, z]\big) \\ &= 0. \end{aligned}$$

□

REMARK 5.27. We conjecture that we also have

$$\pi^{L^{\mathrm{ex}}}_{3,*} = \pi^{L^{\mathrm{ex}}}_{2,*} \circ \pi^{L^{\mathrm{ex}}}_{1,*}.$$

There should be a purely algebraic-topological proof of this identity. This together with Theorem 2.22 would again imply Theorem 5.26. Unfortunately, we were not able to find such a proof. The difficulties are very similar to the problems in an algebraic approach to Lemma 5.23.

In fact, we could conclude the well-definedness and the functoriality of $\pi^{\bar{L}}$ from a combination of these hypothetical algebraic-topological results with Dai's formula for adiabatic limits of η-invariants and the well-definedness of the η-homomorphisms. In this case we would obtain an independent verification of the adiabatic limit results for η-forms.

6. Adiabatic limits - sketches of proofs

6.1. Remarks. This section contains the proofs of the assertions about adiabatic limits of η–forms and invariants which were formulated earlier. The general techniques were developed mainly in the work of Bismut and coworkers, but also by Dai and others. Unfortunately, the details were worked out in specific cases which are similar to the situations of the present paper, but not exactly the same. In order to show the results needed in the present paper one can use the methods after adaptation.

We decided to choose for the present section a coarser level of detailedness of our arguments. While Subsection 6.2 is still rather detailed, in the remaining subsections we just stated the main intermediate results with references to the literature, where proofs of similar results in slightly different situations can be found, which can be adapted to the present cases. It is not by coincidence that the formulations of these intermediate results in the last three subsections almost agree.

6.2. The proof of Theorem 4.17. Let $\epsilon \in \mathbb{Z}_2$. We assume that M is a smooth closed oriented Riemannian manifold of odd dimension m such that $\epsilon\epsilon_m = 1$, i.e., $m \equiv -\epsilon \bmod 4$. Let $(\mathcal{F}, q_{\mathcal{F}})$ be a locally constant sheaf of finite-dimensional real vector spaces with ϵ–symmetric form. Furthermore, let N_F be a compatible $\mathbb{Z}$-grading of length n_F and v be a compatible differential on $\mathcal{F}$. By $(\mathcal{H}, q_{\mathcal{H}})$ we denote the associated cohomology. We choose a compatible metric structure J^F on F and let J^H be the induced metric structure on the cohomology H. Then we can define the operators D_F^{sign} and D_H^{sign} as in (4.2). In the present section we sketch the proof of the following formula.

THEOREM 6.1 (Theorem 4.17). *We have*

$$\eta(D_F^{\text{sign}}) - \eta(D_H^{\text{sign}}) = 2 \int_M L(\nabla^{TM}) \wedge \tilde{\eta}(\mathcal{F}, N_F, J^F, v) - \tau(\mathcal{F}, q_{\mathcal{F}}, N_F, v).$$

PROOF. We are going give a detailed sketch of the proof. The methods have been developed in connection with similar questions about analytic torsion (forms) and for the study of adiabatic limits of η-invariants. The proof of Theorem 4.17 is achieved by adapting these methods correspondingly to the present situation.

We abbreviate $D := D_F^{\text{sign}}$. Let $J^{M,F}$ be the metric structure on $\Omega(M, F)$ induced by J^F and the Riemannian metric g^{TM} as in Subsection 4.3.1. We proceed with v in the same manner as we did for d^F to define D. We introduce $W := J^{M,F}v + vJ^{M,F}$ and set $\mathcal{D} := D + W$. Note that $\mathcal{D}$ is not a compatible Dirac operator. Therefore, we must define its η-invariant by zeta function regularization. Thus, we define the η-invariant $\eta(\mathcal{D})$ as the value of the function

$$s \mapsto \frac{1}{\Gamma(s+\frac{1}{2})}\int_0^\infty \operatorname{Tr} \mathcal{D}e^{-t\mathcal{D}^2} t^{s-\frac{1}{2}}\,dt$$

at $s = 0$, where the integral converges for $\operatorname{Re}(s) \gg 0$ and has a meromorphic continuation which is regular at $s = 0$. In fact, we will see below (Proposition 6.4) that the integral converges locally uniformly for $\operatorname{Re}(s) > -\frac{1}{2}$.

For $T > 0$, we define the rescaled metric structure

$$J^F(T) := T^{-N_F/2+n_F/4} J^F T^{N_F/2-n_F/4}.$$

Let $J^{M,F}(T)$ be the metric structure on $\Omega(M,F)$ induced by $J^F(T)$ and g^{TM} and define $\mathcal{D}(T)$ using $J^{M,F}(T)$. Then we have

$$\mathcal{D}_T := T^{N_F/2}\,\mathcal{D}(T)\,T^{-N_F/2} = D + T^{1/2}W.$$

Note that $\mathcal{D}_0 = D$ is well-defined. Furthermore, we have $\eta(\mathcal{D}_T) = \eta(\mathcal{D}(T))$. Below we show that $\eta(\mathcal{D}_T)$ is independent of $T \in (0,\infty)$. We obtain the proof of Theorem 4.17 by considering the limits $T \to 0$ and $T \to \infty$.

PROPOSITION 6.2. *$\eta(\mathcal{D}_T)$ is constant for $T \in (0,\infty)$.*

PROOF. By Hodge theory the kernel of $\mathcal{D}(T)$ for $T \in (0,\infty)$ can be identified with the cohomology of the total complex $(\Omega(M,F), d^F + v)$, i.e., with the hypercohomology $HH(M,F)$. The kernel of $\mathcal{D}_T$ is isomorphic to the kernel of $\mathcal{D}(T)$ and thus has constant dimension. It follows that $\eta(\mathcal{D}_T)$ is a smooth function of T, and its derivative is given by the coefficient $-\frac{2}{\sqrt{\pi}}\,b_{-1/2}$, where $b_{-1/2}$ is a coefficient of the asymptotic expansion

$$\operatorname{Tr} \frac{\partial \mathcal{D}_T}{\partial T}\, e^{-t\mathcal{D}_T^2} \underset{t\to 0}{\sim} \sum_{i\in\frac{1}{2}\mathbb{Z}} b_i t^i.$$

We now show that we can apply [**8**, Lemma 2.11] which states that $b_{-1/2} = 0$. Let $\mathbf{D}$ be the restriction of $D^M := J^M d + dJ^M$ to $\Omega(M)^{\mathrm{ev}}$, where J^M denotes the metric structure on $\Omega(M)$ induced by the Riemannian metric g^{TM}. By $\mathbf{D}^F$ we denote the twist with F, or, what is the same, the restriction of D to $\Omega^{\mathrm{ev}}(M,F)$ (the superscript refers to the form degree). We have an isomorphism $z^{J^{M,F}} : \Omega^{\mathrm{odd}}(M,F) \cong \Omega^{\mathrm{ev}}(M,F)$ such that we can identify $\Omega(M,F) \cong \Omega^{\mathrm{ev}}(M,F) \oplus \Omega^{\mathrm{ev}}(M,F)$. In this identification D and W correspond to

$$\begin{pmatrix} \mathbf{D}^F & 0 \\ 0 & \mathbf{D}^F \end{pmatrix}, \quad \begin{pmatrix} 0 & W_- \\ W_+ & 0 \end{pmatrix}.$$

Up to the sign, $\mathcal{D}$ has the form [**8**, (2.5)]. The role of that sign in the argument of [**8**, Lemma 2.11] is to assure that the anti-commutator of the Dirac operator and the potential is a zeroth-order operator. Since in our situation $DW + WD$ is still of zero order, the argument of [**8**] applies. □

Let ${}_{hc}\tau_1 := {}_{hc}\tau_1(\mathcal{F}, q_{\mathcal{F}}, N_F, v)$ be as in Paragraph 4.4.2 associated with the hypercohomology spectral sequence.

PROPOSITION 6.3. *For $T \in (0, \infty)$ we have $\eta(\mathcal{D}_T) - \eta(\mathcal{D}_0) = {}_{hc}\tau_1$.*

PROOF. $\eta(\mathcal{D}_T) - \eta(\mathcal{D}_0)$ is the difference of the numbers of eigenvalues (counted with multiplicity) of $\mathcal{D}_T$ which become positive and negative when T moves from 0 to positive values.

The eigenvalues of $\mathcal{D}_T$ which tend to zero as $T \to 0$ can be described in terms of the hypercohomology spectral sequence $({}_{hc}E_r, {}_{hc}d_r)$. We employ the method developed in [**4**, §VI]. In particular, we realize the spaces ${}_{hc}E_r$ using Hodge theory to obtain natural metric structures $J^{{}_{hc}E_r}$ induced by $J^{Z,F}$.

Fix $r \geq 1$ and $\varepsilon > 0$ sufficiently small. We can find $a > 0$ such that $\pm a T^{r/2}$ is not in the spectrum of $\mathcal{D}_T$ for $T \in (0, \varepsilon)$. Let $P^a_{r,T} := E_{T^{-r/2}\mathcal{D}_T}(-a, a)$ be the spectral projection, i.e., the orthogonal projection from $\Omega(M, F)$ on the direct sum of the eigenspaces of $T^{-r/2}\mathcal{D}_T$ associated to eigenvalues lying in $(-a, a)$. Then as $T \to 0$ the spectrum of $T^{-r/2}\mathcal{D}_T P^a_{r,T}$ converges to the spectrum of ${}_{hc}D_r := {}_{hc}d_r J^{{}_{hc}E_r} + J^{{}_{hc}E_r}{}_{hc}d_r$ (cf. [**4**, (6.55)]). This implies (using Definition 4.7 and Lemma 4.6 for the second equality) that

$$\eta(\mathcal{D}_T) - \eta(\mathcal{D}_0) = \sum_{r \geq 1} \operatorname{sign}({}_{hc}D_r) = {}_{hc}\tau_1.$$

□

Recall the definition of the families of forms $\gamma(t)$ in $\Omega(M)$ which enter the Definition 3.15 of $\tilde{\eta}(\mathcal{F}, N_F, J^F, v)$. Recall also that $L(\nabla^{TM})$ is defined in Subsection 4.2.2.

PROPOSITION 6.4. *There is some $N \in \mathbb{N}$ such that*

$$\operatorname{Tr} \mathcal{D}_T e^{-t\mathcal{D}_T^2} = 2\sqrt{\pi} \int_M L(\nabla^{TM}) \wedge \gamma(Tt) + t^{1/2} O(1 + T^N t^N).$$

PROOF. This follows from [**8**, (3.1)] and some local index theory calculation. In fact, our operator is (locally) the spin Dirac operator twisted with the tensor product of the spinor bundle and F. This explains the appearance of the L-form above and the matrix structure of D, W in the proof of Proposition 6.2. □

It follows from Proposition 6.4 and the fact that $\gamma(t) = O(1)$ as $t \to 0$ (see [**21**, Prop. 29]) that

$$\eta(\mathcal{D}_T) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr} \mathcal{D}_T e^{-t\mathcal{D}_T^2} \frac{dt}{t^{1/2}}.$$

If we choose $\alpha > 0$ so small that $1 - \alpha > \frac{1/2+N}{1+N}$, then we have

$$\lim_{T \to \infty} \int_0^{T^{-1+\alpha}} T^{1/2} t^{1/2} O(1 + T^N t^N) \frac{dt}{t^{1/2}} = 0$$

and thus after some transformation

COROLLARY 6.5.

$$\lim_{T\to\infty} \frac{1}{\sqrt{\pi}} \int_0^{T^{-1+\alpha}} \operatorname{Tr} \mathcal{D}_T e^{-t\mathcal{D}_T^2} \frac{dt}{t^{1/2}} = 2 \int_M L(\nabla^{TM}) \wedge \tilde{\eta}(\mathcal{F}, N_F, J^F, v).$$

LEMMA 6.6. *Given $N \in \mathbb{N}$, there are functions $a_i(z)$ bounded for $z \in [1,\infty)$ and $r_N(t,T)$ bounded for $T \in [1,\infty)$, $t \geq T^{-1}$ such that*

$$T^{-1/2} \operatorname{Tr} \mathcal{D}_T e^{-t\mathcal{D}_T^2} = \sum_{i=-n}^{N-1} a_{i/2}(Tt) t^{i/2} + t^{N/2} r_N(t,T).$$

PROOF. This is essentially [**16**, Thm. 1.7] (cf. [**8**, (4.81)]). In [**16**], the situation is more complicated, since it corresponds to an infinite-dimensional bundle F and unbounded differential v, namely a fibrewise de Rham complex. But the proof given in [**16**, §3] can be applied in our situation with many simplifications. First we write $\mathcal{D}_T = T^{1/2}(T^{-1/2}D + W)$ so that the parameters x and t in [**16**] correspond to $T^{-1/2}$ and Tt in the present paper.

The localization part [**16**, §3.1] is just the usual finite propagation speed argument. Now we construct the rough parametrix as in [**16**, §3.2]. Then we verify that the proof of [**16**, Lemma 3.4] goes through in our situation. Note that formula [**16**, (3.5)] simplifies a lot in the present situation. The remaining argument using Duhamel's principle can be taken without change. □

We employ the suggestive notation $\mathcal{D}_\infty := D_H^{\text{sign}} = J^{M,H} d^H + d^H J^{M,H}$ and establish the estimate which corresponds to [**16**, (1.14)].

PROPOSITION 6.7. *Given $\alpha \in (0,1)$, there exists constants $C > 0$, $N \in \mathbb{N}$ such that, for $T \geq 1$, $t \geq T^{-1+\alpha}$, we have*

$$\left|\operatorname{Tr} \mathcal{D}_T e^{-t\mathcal{D}_T^2} - \operatorname{Tr} \mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2}\right| \leq \frac{C}{T^{1/2} \min(1, t^N)}.$$

PROOF. We decompose $\Lambda^* T^* M \otimes F := E_0 \oplus E_1$, where $E_0 := \ker W$ and $E_1 := E_0^\perp$. Let Q be the projection onto E_0 and $Q^\perp = 1 - Q$. With respect to this decomposition we write

$$\mathcal{D}_T := \begin{pmatrix} \mathcal{D}_{T,1} & \mathcal{D}_{T,2} \\ \mathcal{D}_{T,3} & \mathcal{D}_{T,4} \end{pmatrix}.$$

$\mathcal{D}_{T,1}$ is independent of T and can be identified with $\mathcal{D}_\infty$. We extend $\mathcal{D}_\infty$ by zero to $C^\infty(M, E_1)$.

Let $\sigma(A) \subset \mathbb{C}$ denote the spectrum of the operator A. Let

$$U_T := \left\{\lambda \in \mathbb{C} \;\middle|\; |\lambda| \leq c_1\sqrt{T},\ \inf_{\mu\in\sigma(\mathcal{D}_\infty)} |\lambda - \mu| \geq c_2\right\},$$

where $c_1, c_2 > 0$ are sufficiently small and will be fixed below. Following [**11**, §9], we first show the following lemma:

LEMMA 6.8. *For $p \geq \dim(M)+1$, there exist constants $C>0$, $T_1 \geq 1$ such that, for $T \geq T_1$ and $\lambda \in U_T$,*

$$\left\|(\lambda - \mathcal{D}_T)^{-p} - (\lambda - \mathcal{D}_\infty)^{-p} Q\right\|_1 \leq \frac{C}{\sqrt{T}}(1+|\lambda|)^p, \tag{6.1}$$

where $\|.\|_1$ denotes the norm on trace class operators.

PROOF. In some sense, this is a special case of [**11**, §9]. Instead of writing all details we explain how the the arguments given in [**11**] can be employed in the present case.

The operator $\mathcal{D}_{T,1} = QDQ$ is as in [**11**, §9 (b)], but much simpler. Furthermore, $\mathcal{D}_{T,2} = -Q[D,Q] \in C^\infty(M, \mathrm{Hom}(E_0^\perp, E_0))$, $\mathcal{D}_{T,3} = Q^\perp[D,Q] \in C^\infty(M, \mathrm{Hom}(E_0, E_0^\perp))$ are independent of T and bounded. Therefore, we have a stronger version of the estimate [**11**, (9.38)].

If $\phi \in C^\infty(M, E_1)$, then using elliptic regularity, the fact that $DW + WD$ is bounded, and that $W\big|_{E_1}$ is injective, we obtain the estimate [**11**, (9.48)]:

$$\begin{aligned} \|\mathcal{D}_T\phi\|^2 &= \langle \mathcal{D}_T\phi, \mathcal{D}_T\phi\rangle \\ &= \langle \phi, \left(D^2 + T^{1/2}(DW+WD) + TW^2\right)\phi\rangle \\ &= \|D\phi\|^2 + T^{1/2}\langle\phi, (DW+WD)\phi\rangle + T\|W\phi\|^2 \\ &\geq d_1(\|\phi\|^2_{W^1(M,E)} + (T-d_2)\|\phi\|^2), \end{aligned} \tag{6.2}$$

where $d_1, d_2 > 0$ are independent of $T \geq 1$ and ϕ, $\|.\|$ denotes the L^2-norm, and $W^1(M, E_1)$ is the L^2-based Sobolev space of order one. We now conclude [**11**, (9.104)]: There are constants $T_0 \geq 1$ and $d_3 > 0$ such that, for $T \geq T_0$, $\phi \in C^\infty(M, E_1)$,

$$\|\mathcal{D}_{T,4}\phi\| \geq d_3(\|\phi\|_{W^1(M,E_1)} + \sqrt{T}\|\phi\|). \tag{6.3}$$

Thus there exists a constant C_1 such that, for $T \geq T_0$, $|\lambda| \leq \frac{d_3}{2}\sqrt{T}$, we have [**11**, (9.106)],

$$\begin{aligned} \|(\lambda - \mathcal{D}_{T,4})^{-1}\phi\| &\leq \frac{C_1}{\sqrt{T}}\|\phi\|, \\ \|(\lambda - \mathcal{D}_{T,4})^{-1}\phi\|_{W^1(M,E_1)} &\leq C_1\|\phi\|, \end{aligned}$$

for all $\phi \in C^\infty(M, E_1)$. The following estimates are proved exactly as in [**11**, Prop. 9.18]. For $p \geq \dim(M)+1$, there is a constant $C_2 > 0$ such that, for $T \geq T_0$, $|\lambda| \leq \frac{d_3}{2}\sqrt{T}$,

$$\begin{aligned} \|(\lambda - \mathcal{D}_{T,4})^{-1}\|_\infty &\leq \frac{C_2}{\sqrt{T}}, \\ \|(\lambda - \mathcal{D}_{T,4})^{-1}\|_p &\leq C_2, \\ \|\mathcal{D}_{T,2}(\lambda - \mathcal{D}_{T,4})^{-1}\|_\infty &\leq \frac{C_2}{\sqrt{T}}, \end{aligned}$$

where $\|.\|_p$ denotes the norm of the pth Schatten class.

We now fix c_2 such that $\sigma(\mathcal{D}_\infty) \cap (-2c_2, 2c_2) \subset \{0\}$. Furthermore, we assume that at least $2c_1 \leq d_3$. Following [**11**, (9.120)], for $\lambda \in U_T$ and $T \geq T_0$, we define

$$M_T(\lambda) := \lambda - \mathcal{D}_{T,1} - \mathcal{D}_{T,2}(\lambda - \mathcal{D}_{T,4})^{-1}\mathcal{D}_{T,3}.$$

We have the estimates which correspond to [**11**, (9.124)]: If c_1 is sufficiently small, then there are constants $C_3 > 0$, $T_1 \geq T_0$ such that, for $T \geq T_1$, $\lambda \in U_T$, we have

$$\begin{aligned} \|M_T(\lambda)^{-1}\|_\infty &\leq C_3, \\ \|\mathcal{D}_{T,3} M_T(\lambda)^{-1}\|_\infty &\leq C_3, \\ \|M_T(\lambda)^{-1}\|_p &\leq C_3(1 + |\lambda|), \\ \|M_T(\lambda)^{-p} - (\lambda - \mathcal{D}_{1,T})^{-p}\|_1 &\leq \frac{C_3}{\sqrt{T}}(1 + |\lambda|)^p. \end{aligned}$$

Note that in our case the operator defined in [**11**, (9.125)] is trivial, and this gives the better power p instead of $p + 1$ in the last estimate above. Now we follow the proof of [**11**, Thm. 9.23] to finish the proof of (6.1). □

Let $\delta \subset \mathbb{C}$ be the circle of radius c_2 centered at the origin and oriented counterclockwise. For $T \geq T_1$, we have $\sigma(\mathcal{D}_T) \cap \delta = \emptyset$. Let P_T and P_∞ be the spectral projections of $\mathcal{D}_T$ and $\mathcal{D}_\infty$ corresponding to the interval $(-c_2/2, c_2/2)$. Recall that Q is the projection onto E_0. Then, for $T \geq T_1$, we have

$$P_T - P_\infty Q = \frac{1}{2\pi \mathrm{i}} \int_\delta \lambda^{p-1} \left((\lambda - \mathcal{D}_T)^{-p} - (\lambda - \mathcal{D}_\infty)^{-p} Q \right) d\lambda.$$

We conclude from (6.1) that $\|P_T - P_\infty Q\|_1 \leq C_4/\sqrt{T}$ for some C_4 independent of T. Since $\mathcal{D}_T P_\infty Q = 0$ and $P_\infty \mathcal{D}_\infty = 0$, there exists a constant C_5 such that, for $T \geq T_1$,

$$\|\mathcal{D}_T P_T - \mathcal{D}_\infty P_\infty\|_1 \leq \|P_T \mathcal{D}_T (P_T - P_\infty Q)\|_1 \leq \frac{C_5}{\sqrt{T}}. \tag{6.4}$$

We conclude that there is a constant C_6 such that, for $T \geq T_1$,

$$\left| \operatorname{Tr} P_T \mathcal{D}_T \mathrm{e}^{-t\mathcal{D}_T^2} - \operatorname{Tr} P_\infty \mathcal{D}_\infty \mathrm{e}^{-t\mathcal{D}_\infty^2} \right| \leq \frac{C_6}{T^{1/2}}. \tag{6.5}$$

Let Δ be the oriented path in $\mathbb{C}$ which goes parallel to the real axis from $-\infty - \mathrm{i}c_2$ to $c_2 - \mathrm{i}c_2$, then parallel to the imaginary axis to $c_2 + \mathrm{i}c_2$, and then parallel to the real axis to $-\infty + \mathrm{i}c_2$, and which goes parallel to the real axis from $\infty + \mathrm{i}c_2$ to $c_2 + \mathrm{i}c_2$, then parallel to the imaginary axis to $c_2 - \mathrm{i}c_2$, and then parallel to the real axis to $\infty - \mathrm{i}c_2$.

Let $h_p(\lambda)$ be holomorphic on $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$ such that $h_p^{(p-1)}(\lambda) = (p-1)!\, \lambda \mathrm{e}^{-\lambda^2}$. So up to a constant it is the function f_{p-1} as defined in [**11**, (9.165)]. In particular, we have the estimate [**11**, (9.169)],

$$|h_p(\lambda)| \leq C_7\, \mathrm{e}^{-c_4|\lambda|^2}, \quad \lambda \in \Delta,$$

where $c_4 > 0$ and C_7 are independent of λ. We have

$$\begin{aligned}(1-P_T)\,\mathcal{D}_T e^{-t\mathcal{D}_T^2} - (1-P_\infty)\,\mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2} &= \frac{1}{2\pi i}\int_\Delta \lambda e^{-t\lambda^2}\left((\lambda-\mathcal{D}_T)^{-1} - (\lambda-\mathcal{D}_\infty)^{-1}Q\right)d\lambda \\ &= \frac{1}{2\pi i}\int_\Delta \frac{h_p(t^{1/2}\lambda)}{t^{(p+1)/2}}\left((\lambda-\mathcal{D}_T)^{-p} - (\lambda-\mathcal{D}_\infty)^{-p}Q\right)d\lambda.\end{aligned}$$

We split the integral into the parts $\Delta_1(T) := \Delta\cap\{\lambda\in U_T\} = \Delta\cap\{|\lambda|\le c_1\sqrt{T}\}$ and $\Delta_2(T) := \Delta\cap\{|\lambda|\ge c_1\sqrt{T}\}$. Let $p := \dim(M)+1$. By Lemma 6.8, there are constants C_8, C_9 such that, for $T\ge T_1$, $t>0$,

$$\begin{aligned}(6.6)\quad &\left\|\frac{1}{2\pi i}\int_{\Delta_1(T)} \frac{h_p(t^{1/2}\lambda)}{t^{(p+1)/2}}\left((\lambda-\mathcal{D}_T)^{-p} - (\lambda-\mathcal{D}_\infty)^{-p}Q\right)d\lambda\right\|_1 \\ &\qquad\le \frac{C_8}{T^{1/2}t^{(p+1)/2}}\int_{\Delta_1(T)} e^{-c_4 t|\lambda|^2}(1+|\lambda|)^p\,d\lambda \le \frac{C_9}{T^{1/2}t^{p+1}}.\end{aligned}$$

It follows from (6.2) that there is a constant C_{10} such that, for all $T\ge T_1$, $\lambda\in\Delta_2(T)$,

$$\|(\lambda-\mathcal{D}_T)^{-p}\|_1\le C_{10},\quad \|(\lambda-\mathcal{D}_\infty)^{-p}Q\|_1\le C_{10}.$$

We obtain constants C_{11}, C_{12}, C_{13} such that, for all $T\ge T_1$ and $t\ge T^{-1+\alpha}$,

$$\begin{aligned}(6.7)\quad &\left\|\frac{1}{2\pi i}\int_{\Delta_2(T)} \frac{h_p(t^{1/2}\lambda)}{t^{(p+1)/2}}\left((\lambda-\mathcal{D}_T)^{-p} - (\lambda-\mathcal{D}_\infty)^{-p}Q\right)d\lambda\right\|_1 \\ &\qquad\le \frac{C_{11}}{t^{(p+1)/2}}\int_{\Delta_2(T)} e^{-c_4 t|\lambda^2|}\,d\lambda \le \frac{C_{12}}{t^{(p+2)/2}}e^{-c_4Tt/2} \le \frac{C_{13}}{\sqrt{T}t^{(p+2)/2}}.\end{aligned}$$

We combine (6.6) and (6.7) to obtain

$$(6.8)\qquad \left\|(1-P_T)\,\mathcal{D}_T e^{-t\mathcal{D}_T^2} - (1-P_\infty)\,\mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2}\right\|_1 \le \frac{C_{14}}{\sqrt{T}\min(t^{p+1},1)}$$

for $T\ge T_1$ and $t\ge T^{-1+\alpha}$, where C_{14} is independent of T, t. Together with (6.5) this implies the assertion of the proposition. □

The estimates in Proposition 6.4, Lemma 6.6, and Proposition 6.7 are all the ingredients which are necessary to perform the proof of [**16**, Prop. 1.8]. We conclude

COROLLARY 6.9. *For $\beta > 0$ sufficiently small, we have*

$$\lim_{T\to\infty}\int_{T^{-1+\alpha}}^{T^{-\beta}} \operatorname{Tr}\mathcal{D}_T e^{-t\mathcal{D}_T^2}\frac{dt}{t^{1/2}} = 0.$$

Fix $1-\alpha > \beta > 0$ such that Corollary 6.9 holds. Note that $\operatorname{Tr}\mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2} = O(1)$ as $t\to 0$ (cf. [**9**, Thm. 2.4]). The following is an immediate consequence of Proposition 6.7:

COROLLARY 6.10.

$$\lim_{T\to\infty}\int_{T^{-\beta}}^{1} \mathrm{Tr}\, \mathcal{D}_T e^{-t\mathcal{D}_T^2} \frac{dt}{t^{1/2}} = \int_0^1 \mathrm{Tr}\, \mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2} \frac{dt}{t^{1/2}}.$$

It follows from (6.8) that $\|(1-P_T)\mathcal{D}_T e^{-\mathcal{D}_T^2}\|_1$ is uniformly bounded for $T \geq 1$. Since $P_T\mathcal{D}_T \geq c_2$ for all $T \geq T_1$, we have a constant C such that $\|(1-P_T)\, e^{-(t-1)\mathcal{D}_T^2}\|_1 \leq Ce^{-tc_2}$ for all $T \geq T_1$, $t \geq 1$. We conclude that there is a constant C_1 such that, for all $T \geq T_1$,

$$\|(1-P_T)\,\mathcal{D}_T e^{-t\mathcal{D}_T^2}\|_1 \leq C_1 e^{-tc_2}.$$

Using (6.8) and Lebesgue's theorem about majorized convergence, we obtain

COROLLARY 6.11.

$$\lim_{T\to\infty}\int_1^\infty \mathrm{Tr}\,(1-P_T)\mathcal{D}_T e^{-t\mathcal{D}_T^2} \frac{dt}{t^{1/2}} = \int_1^\infty \mathrm{Tr}\,(1-P_\infty)\mathcal{D}_\infty e^{-t\mathcal{D}_\infty^2} \frac{dt}{t^{1/2}}.$$

Note that P_T and P_∞ are finite-dimensional projections. Since $\mathcal{D}_\infty P_\infty = 0$ it follows from (6.4) that the spectrum of $P_T\mathcal{D}_T$ converges to zero as $T \to \infty$. We have

$$\lim_{T\to\infty}\frac{1}{\sqrt{\pi}}\int_1^\infty \mathrm{Tr}\, P_T\mathcal{D}_T e^{-t\mathcal{D}_T^2} \frac{dt}{t^{1/2}} = \lim_{T\to\infty}\sum_{\mu\in\sigma(\mathcal{D}_T P_T)} \frac{1}{\sqrt{\pi}}\int_1^\infty \mu e^{-t\mu^2} \frac{dt}{t^{1/2}}$$

$$= \lim_{T\to\infty}\sum_{\mu\in\sigma(\mathcal{D}_T P_T)} \frac{1}{\sqrt{\pi}}\int_{\mu^2}^\infty \mathrm{sign}(\mu)\, e^{-t} \frac{dt}{t^{1/2}} = \lim_{T\to\infty}\sum_{\mu\in\sigma(\mathcal{D}_T P_T)} \mathrm{sign}(\mu).$$

Let ${}_{lg}\tau_2 := {}_{lg}\tau_2(\mathcal{F}, q, N, v)$ be associated to the local-global spectral sequence as in Definition 4.8.

PROPOSITION 6.12. *We have*

$$\lim_{T\to\infty}\sum_{\mu\in\sigma(\mathcal{D}_T P_T)} \mathrm{sign}(\mu) = {}_{lg}\tau_2.$$

PROOF. The eigenvalues of $\mathcal{D}_T$ which tend to zero as $T \to \infty$ can be described in terms of the local-global spectral sequence $({}_{lg}E_r, {}_{lg}d_r)$. We apply the method of [**4**, §VI] to $\mathcal{D}_T = T^{1/2}W + D$. In particular, we realize the spaces ${}_{lg}E_r$ using Hodge theory to obtain natural metric structures $J^{{}_{lg}E_r}$ (induced by $J^{M,F}$).

Fix $r \geq 2$. We can find $a > 0$, $T_1 \geq 1$ such that $\pm aT^{-(r-1)/2}$ is not in the spectrum of $\mathcal{D}_T$ for $T \geq T_1$. Let $P^a_{r,T} := E_{T^{(r-1)/2}\mathcal{D}_T}(-a,a)$ be the spectral projection. Then as $T \to 0$ the spectrum of $T^{(r-1)/2}\mathcal{D}_T P^a_{r,T}$ converges to the spectrum of ${}_{lg}D_r := {}_{lg}d_r J^{{}_{lg}E_r} + J^{{}_{lg}E_r}{}_{lg}d_r$. This implies

$$\lim_{T\to\infty}\sum_{\mu\in\sigma(\mathcal{D}_T P_T)} \mathrm{sign}(\mu) = \sum_{r\geq 2}\mathrm{sign}({}_{lg}D_r) = {}_{lg}\tau_2$$

and hence the proposition. □

Combining Propositions 6.2, 6.3, 6.5, 6.9, 6.10, 6.11, and 6.12 we finish the proof of Theorem 4.17. □

6.3. Proof of Theorem 5.9. We consider a family of superconnections depending on two parameters $(T,u) \in (0,\infty) \times (0,\infty)$. The parameter u is the usual rescaling parameter associated to the total grading of $\Omega(Z,F)$. The parameter T is associated to the form degree. If T becomes large, then the de Rham differential is scaled large compared to the differential v of the complex. To stay in our formalism we are going to work over the base $\hat{B} := (0,\infty) \times (0,\infty) \times B$. Let $\mathrm{pr}\colon \hat{B} \to B$ denote the projection and define $\hat{\pi}\colon \hat{M} := \mathrm{pr}^*M \to \hat{B}$ with fiber $\hat{Z}$. Let $\mathrm{pr}_M\colon \hat{M} \to M$ be the projection. Let $T\hat{Z}$ be the relative tangent bundle of $\hat{\pi}$, i.e., $T\hat{Z} = \mathrm{pr}_M^* TZ$. This bundle is equipped with the vertical metric $\hat{g}^{TZ}$ on $T\hat{Z}$ which restricts to $(Tu)^{-2} g^{TZ}$ on the fibre over $(T,u) \times \{b\} \in \hat{B}$. We define the metric structure $\hat{J}^F$ on $\mathrm{pr}_M^* F$ such that it restricts to $u^{-N_F+n_F/2} J^F u^{N_F-n_F/2}$ over $(T,u) \times M$. The metric $\hat{g}^{TZ}$ and $\hat{J}^F$ together induce the metric structure $\hat{J}^{Z,F}$ on $\Omega(\hat{Z},\mathrm{pr}_M^* F)$ (cf. (5.1)). Let $d^{\mathrm{pr}_M^* F}$ be the twisted de Rham differential on $\Omega(\hat{M},\mathrm{pr}_M^* F)$. We define $\hat{\mathbf{d}} = d^{\mathrm{pr}_M^* F} + \mathrm{pr}_M^* v$ and consider the form $p(\hat{\mathbf{d}},\hat{J}^{Z,F})$ on $\hat{B}$ defined in (5.2) if n is even (resp. in (5.7) if n is odd).

DEFINITION 6.13. We define $\beta := du \wedge \beta^u + dT \wedge \beta^T$ to be the part of $p(\hat{\mathbf{d}},\hat{J}^{Z,F}) \in \Omega(\hat{B})$ of degree one with respect to the coordinates (T,u), with functions $\beta^u, \beta^T\colon (0,\infty) \times (0,\infty) \to \Omega(B)$.

The following corollary is an immediate consequence of the fact that $p(\hat{\mathbf{d}},\hat{J}^{Z,F})$ is closed. Let $d = d_{T,u} + d^B$ be the decomposition of the de Rham differential on $(0,\infty) \times (0,\infty) \times B$.

COROLLARY 6.14. *There exists a smooth family* $\alpha\colon (0,\infty) \times (0,\infty) \to \Omega(B)$ *such that*

$$d_{T,u}\beta = dT \wedge du\, d^B \alpha. \tag{6.9}$$

The following theorems can be shown by adapting the methods of the corresponding references to our present situation. Actually, the following results are analogues of some related properties of Bismut–Lott's real analytic torsion forms, see especially [**24**, Thms. 4.3-4.9], where we work in a much more complicated situation (instead of the finite-dimensional flat vector bundle here we have an infinite-dimensional vector bundle there). See also Subsection 6.5 for more details.

Let $\check{B} := (0,\infty) \times B$. Let $N_{H(Z,\mathcal{F})}$ be the $\mathbb{Z}$-grading on $H(Z,\mathcal{F})$ induced by N_F. We consider the metric structure $\check{J}^{H(Z,\mathcal{F})}$ on $\check{H}(Z,\mathcal{F}) := \mathrm{pr}^* H(Z,\mathcal{F})$ which restricts to $u^{-N_{H(Z,\mathcal{F})}+n_{H(Z,\mathcal{F})}/2} J^{H(Z,\mathcal{F})} u^{N_{H(Z,\mathcal{F})}-n_{H(Z,\mathcal{F})}/2}$ over $\{u\} \times B$. We consider the flat $(-1)^{N_{H(Z,\mathcal{F})}}$-superconnection $B' := \nabla^{H(Z,\mathcal{F})} + v_{H(Z,\mathcal{F})}$, where the differential $v_{H(Z,\mathcal{F})}$ on $H(Z,\mathcal{F})$ is induced by v.

Let $p(d^{H(Z,\mathcal{F})},\check{J}^{H(Z,\mathcal{F})})$ be the form on $\check{B}$ defined in Subsection 3.4.4 if n is even (resp. in (5.10) if n is odd). Let $\gamma\colon (0,\infty) \to \Omega(B)$ be such that

$$p\big(d^{H(Z,\mathcal{F})},\check{J}^{H(Z,\mathcal{F})}\big) = du \wedge \gamma + r,$$

where the remainder r does not contain du.

THEOREM 6.15. (1) *For any* $u > 0$,

$$\lim_{T\to\infty} \beta^u(T,u) = \gamma(u). \tag{6.10}$$

(2) *For* $0 < u_1 < u_2$ *fixed, there exists* $C > 0$ *such that, for* $u \in [u_1, u_2]$, $T \geq 1$, *we have*

$$|\beta^u(T,u)| \leq C. \tag{6.11}$$

(3) *We have the following identity*:

$$\lim_{T\to\infty} \int_1^\infty \beta^u(T,u)du = \int_1^\infty \gamma(u)du - \sum_{r\geq 2} \tilde{\eta}\big({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r\big). \tag{6.12}$$

THEOREM 6.16. (1) *There exists a smooth family* $\sigma\colon (0,\infty) \to \Omega(B)$ *such that, for* $T \geq 1$, *we have*

$$\lim_{u\to\infty} \beta^T(T,u) = \sigma(T).$$

(2) *There exist constants* $C > 0$, $\delta > 0$ *such that, for* $T \geq 1$, *we have the following estimate*:

$$|\sigma(T)| \leq \frac{C}{T^{1+\delta}}. \tag{6.13}$$

(3) *Modulo exact forms on* B *we have the following identity*:

$$\int_1^\infty \sigma(T)dT = -\tilde{\eta}\big(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{hc}\mathrm{Gr}(HH)}\big). \tag{6.14}$$

Note that (6.14) follows from Lemma 3.17 as in the proof of [**24**, Thm. 4.5].

Let $\check{M} = (0,\infty)\times M$, $\mathrm{Pr}\colon \check{M} \to M$ be the canonical projection. We consider the vertical metric $\check{g}^{TZ}$ on $\check{M}$ for the fibration $\check{M} \to \check{B}$ which restricts to $u^{-2}\mathrm{Pr}^* g^{TZ}$ on the fibre over $(u,b) \in (0,\infty) \times B$. $\mathrm{Pr}^* J^F$,$\check{g}^{TZ}$ induce a metric structure $\check{J}^{Z,F}$ on $\mathrm{Pr}^*\,\Omega(Z,F)$ on $\check{B}$. We define the smooth family $\theta\colon (0,\infty) \to \Omega(B)$ such that (cf. Subsection 5.2.2)

$$p\big(d^{\mathrm{Pr}^* F}, \check{J}^{Z,F}\big) = du \wedge \theta + r$$

and r does not contain du. Note that by Definitions 5.3 and 5.4

$$\int_0^\infty \theta(u)\, du = -\tilde{\eta}\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big).$$

THEOREM 6.17. (1) *For any* $u > 0$, *there exist* $C > 0$,$\delta > 0$ *such that, for* $T \geq 1$,

$$\big|\beta^T(T,u)\big| \leq \frac{C}{T^{\delta+1}}. \tag{6.15}$$

(2) *For any* $T > 0$, *we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta^T(T\varepsilon^{-1}, \varepsilon) = \theta(T). \tag{6.16}$$

(3) *There exists $C > 0$ such that, for $\varepsilon \in (0,1]$, $\varepsilon \le T \le 1$,*

$$\varepsilon^{-1}\left|\beta^T(T\varepsilon^{-1},\varepsilon)\right| \le C. \tag{6.17}$$

(4) *There exist $\delta \in (0,1]$, $C > 0$ such that, for $\varepsilon \in (0,1]$, $T \ge 1$,*

$$\varepsilon^{-1}\left|\beta^T(T\varepsilon^{-1},\varepsilon)\right| \le \frac{C}{T^{1+\delta}}. \tag{6.18}$$

PROOF OF THEOREM 5.9. We now finish the proof of Theorem 5.9. For $0 < \varepsilon < A$ and $1 < T_0$, we consider the rectangle $(T,u) \in R := [1,T_0] \times [\varepsilon, A]$. By Corollary 6.14 we have $\int_{\partial R}\beta = d^B \int_R \alpha$. Thus

$$\int_\varepsilon^A \beta^u(T_0,u)du - \int_1^{T_0} \beta^T(T,A)dT - \int_\varepsilon^A \beta^u(1,u)du + \int_1^{T_0} \beta^T(T,\varepsilon)dT$$
$$= I_1 + I_2 + I_3 + I_4$$

is an exact form on B. We take the limits $A \to \infty$, $T_0 \to \infty$, and then $\varepsilon \to 0$ in the indicated order ([**4**, §4 (c)], [**22**, §4 (c)]). Let I_j^k, $j = 1,\dots 4$, $k = 1,2,3$ denote the value of the part I_j after the kth limit. Note that by [**28**, §22, Thm. 17], if α_k is a family of smooth exact forms on B which converges uniformly on any compact set $K \subset B$ to a smooth form α, then α is exact. Thus modulo exact forms on B, $\sum_{j=1}^4 I_j^3 \equiv 0$. We obtain by the definition of $\tilde\eta(\Omega(Z,F), q, J^{Z,F}, \mathbf{d})$ that

$$I_3^3 = \tilde\eta\big(\Omega(Z,F), N_{Z,F}, J^{Z,F}, \mathbf{d}\big).$$

Furthermore, by Theorem 6.16 and in particular (6.14) we get

$$I_2^2 = I_2^3 = \tilde\eta\big(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{hc}\mathrm{Gr}(HH)}\big).$$

Now $\gamma(u) = O(1)$ as $u \to 0$, and by Definition 3.15, (5.10),

$$\int_0^\infty \gamma(u)\,du = -\tilde\eta\big({}_{hc}\mathcal{E}_1, N_{{}_{hc}E_1}, J^{{}_{hc}E_1}, {}_{hc}d_1\big).$$

From (6.10), (6.11), and (6.12) we conclude that

$$I_1^3 = -\sum_{r\ge 1} \tilde\eta\big({}_{hc}\mathcal{E}_r, N_{{}_{hc}E_r}, J^{{}_{hc}E_r}, {}_{hc}d_r\big).$$

Finally, using Theorem 6.17, we get

$$I_4^3 = -\tilde\eta\big(\Omega(Z,F), N_Z, J^{Z,F}, d^F\big).$$

These four equations imply Theorem 5.9. □

6.4. Proof of Theorem 5.10. We consider again two parameters (T, u). Here u is the rescaling parameter corresponding to the total grading of $\Omega(Z, F)$ and T is associated to the grading of F. For large T, the rescaled differential v becomes large in comparison with the de Rham differential.

Let $\hat{B} := (0,\infty)\times(0,\infty)\times B$. Let $\mathrm{pr}\colon \hat{B} \to B$ denote the projection and define $\hat{M} := \mathrm{pr}^*M \to \hat{B}$ with fiber $\hat{Z}$. This bundle is equipped with the vertical metric $\hat{g}^{TZ}$ on $T\hat{Z}$ which restricts to $u^{-2}g^{TZ}$ on the fibre over $(T,u)\times\{b\} \in \hat{B}$. We define the metric structure $\hat{J}^F$ such that it restricts to $(Tu)^{-N_F+n_F/2}J^F(Tu)^{N_F-n_F/2}$ over $(T,u)\times M$. The metric $\hat{g}^{TZ}$ and $\hat{J}^F$ together induce the metric structure $\hat{J}^{Z,F}$ on $\Omega\big(\hat{Z}, \mathrm{pr}_M^*F\big)$ (cf. (5.1)).

As in Section 6.3, let $\hat{\mathbf{d}} = d^{\mathrm{pr}_M^*F} + \mathrm{pr}_M^* v$ be the pull-back of $\mathbf{d}$. Let $p\big(\hat{\mathbf{d}}, \hat{J}^{Z,F}\big)$ be the form on $\hat{B}$ defined in (5.2) if n is even (resp. in (5.7) if n is odd).

Definition 6.18. We define $\beta = du\wedge\beta^u + dT\wedge\beta^T$ to be the part of $p\big(\hat{\mathbf{d}}, \hat{J}^{Z,F}\big)$ of degree one with respect to the coordinates (T,u), with functions $\beta^u, \beta^T : (0,\infty)\times(0,\infty) \to \Omega(B)$.

The following corollary is an immediate consequence of the fact that $p\big(\hat{\mathbf{d}}, \hat{J}^{Z,F}\big)$ is closed.

Corollary 6.19. *There exists a smooth family $\alpha\colon (0,\infty)\times(0,\infty) \to \Omega(B)$ such that*

$$d_{T,u}\beta = dT \wedge du\, d^B\alpha. \tag{6.19}$$

Let $\check{B} := (0,\infty)\times B$ and $\mathrm{pr}_1 : \check{B} \to B$ be the projection. We consider the bundle $\check{M} := \mathrm{pr}_1^* M \to \check{B}$ with fiber $\check{Z}$. Let $\mathrm{Pr}\colon \check{M} \to M$ be the induced map and $\check{H} = \mathrm{Pr}^* H$ be the pull-back of the flat cohomology bundle H of (F, v) on M. We consider the metric structure $\check{J}^H$ on $\check{H}$ which restricts to $u^{-N_H+n_H/2}(\mathrm{Pr}^* J^H)u^{N_H-n_H/2}$ over $\{u\}\times M$. Furthermore, we consider the vertical metric $\check{g}^{TZ}$ on $T\check{Z}$ which restricts to $u^{-2}\mathrm{Pr}^* g^{TZ}$ on the fibre over $(u,b) \in \check{B}$. They induce the metric structure $\check{J}^{Z,H}$ on $\mathrm{pr}_1^*\,\Omega(Z,H)$ as in (5.1). Let N_H be the $\mathbb{Z}$-grading on H induced by N_F. The total $\mathbb{Z}$-grading on $\Omega(Z,H)$ is $N_{Z,H} = N_Z + N_H$. Let $\check{d}^H$ be the twisted de Rham differential on $\Omega(\check{M}, \check{H})$. Then $\check{d}^H$ is a flat $(-1)^{\check{N}_{Z,H}}$-superconnection on $\mathrm{pr}_1^*\,(\Omega(Z,H))$; here $\check{N}_{Z,H} = \mathrm{pr}_1^* N_{Z,H}$. We define the family $\gamma\colon (0,\infty) \to \Omega(B)$ such that

$$p(\check{d}^H, \check{J}^{Z,H}) = du\wedge\gamma + r,$$

where r does not contain du. By Definitions 5.3 and 5.4,

$$\begin{aligned}\int_0^\infty \gamma(u)\,du &= -\tilde{\eta}({}_{hc}\mathcal{E}_1, N_{{}_{hc}E_1}, J^{{}_{hc}E_1}, {}_{hc}d_1) \\ &= -\tilde{\eta}\big(\Omega(Z,H), N_Z, J^{Z,H}, d^H\big).\end{aligned} \tag{6.20}$$

Theorem 6.20. (1) *For any $u > 0$, we have*

$$\lim_{T\to\infty} \beta^u(T,u) = \gamma(u). \tag{6.21}$$

(2) *For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 1$,*

$$|\beta^u(T,u)| \leq C. \tag{6.22}$$

(3) *We have the following identity:*

$$\lim_{T\to\infty} \int_1^\infty \beta^u(T,u)\,du = \int_1^\infty \gamma(u)\,du - \sum_{r\geq 2} \tilde{\eta}\big({}_{lg}\mathcal{E}_r, N_{{}_{lg}E_r}, J^{{}_{lg}E_r}, {}_{lg}d_r\big). \tag{6.23}$$

THEOREM 6.21. (1) *There exists a smooth family $\sigma\colon (0,\infty) \to \Omega(B)$ such that, for $T \geq 1$, we have*

$$\lim_{u\to\infty} \beta^T(T,u) = \sigma(T).$$

(2) *There exist constants $C > 0$, $\delta > 0$ such that, for $T \geq 1$, we have the following estimate:*

$$|\sigma(T)| \leq \frac{C}{T^{1+\delta}}. \tag{6.24}$$

(3) *We have the following equality modulo exact forms on B:*

$$\int_1^\infty \sigma(T)\,dT = -\tilde{\eta}\big(\mathcal{H}H, {}_{lg}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{lg}\mathrm{Gr}(HH)}\big). \tag{6.25}$$

We consider the metric structure $\check{J}^F$ on Pr^*F which restricts to $u^{-N_F+n_F/2}J^F u^{N_F-n_F/2}$ over $\{u\} \times M$. We consider $\nabla^F + v$ as $(-1)^{N_F}$-superconnection on F. We define the smooth family $\theta\colon (0,\infty) \to \Omega(M)$ such that

$$p\big(\mathrm{Pr}^*(\nabla^F + v), \check{J}^F\big) = du \wedge \theta + r,$$

where r does not contain du. Note that by Definition 3.15

$$\int_0^\infty \theta(u)\,du = -\tilde{\eta}\big(\mathcal{F}, N_F, J^F, v\big).$$

.

THEOREM 6.22. (1) *For any $u > 0$, there exist $C > 0$, $\delta > 0$ such that, for $T \geq 1$, we have*

$$\big|\beta^T(T,u)\big| \leq \frac{C}{T^{\delta+1}}. \tag{6.26}$$

(2) *For any $T > 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta^T(T\varepsilon^{-1},\varepsilon) = \int_Z L(\nabla^{TZ}) \wedge \theta(T). \tag{6.27}$$

(3) *There exists $C > 0$ such that, for $\varepsilon \in (0,1]$, $\varepsilon \leq T \leq 1$, we have*

$$\varepsilon^{-1}\big|\beta^T(T\varepsilon^{-1},\varepsilon)\big| \leq C. \tag{6.28}$$

(4) *There exist $\delta \in (0,1]$, $C > 0$ such that, for $\varepsilon \in (0,1]$, $T \geq 1$, we have*

$$\varepsilon^{-1}\big|\beta^T(T\varepsilon^{-1},\varepsilon)\big| \leq \frac{C}{T^{1+\delta}}. \tag{6.29}$$

PROOF OF THEOREM 5.10. We now finish the proof of Theorem 5.10. For $0 < \varepsilon < A$ and $1 < T_0$, we consider the rectangle $(T,u) \in R := [1,T_0] \times [\varepsilon, A]$. By Corollary 6.19 we have $\int_{\partial R} \beta = d^B \int_R \alpha$. Thus

$$\int_\varepsilon^A \beta^u(T_0,u)\,du - \int_1^{T_0} \beta^T(T,A)\,dT - \int_\varepsilon^A \beta^u(1,u)du + \int_1^{T_0} \beta^T(T,\varepsilon)\,dT$$
$$= I_1 + I_2 + I_3 + I_4$$

is an exact form. We take the limits $A \to \infty$, $T_0 \to \infty$, and then $\varepsilon \to 0$ in the indicated order. Let I_j^k, $j = 1,\dots 4$, $k = 1,2,3$ denote the value of the part I_j after the kth limit. Then modulo exact forms on B, $\sum_{j=1}^4 I_j^3 \equiv 0$. We obtain by the definition of $\tilde\eta\big(\Omega(Z,F), N_{Z,F}, J^{Z,F}, \mathbf{d}\big)$ that

$$I_3^3 = \tilde\eta\big(\Omega(Z,F), N_{Z,F}, J^{Z,F}, \mathbf{d}\big).$$

Furthermore, by Theorem 6.21 and in particular (6.25) we get

$$I_2^2 = I_2^3 = \tilde\eta\big(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{hc}\mathrm{Gr}(HH)}\big).$$

From (6.20), (6.21), (6.22), and (6.23), we conclude that

$$I_1^3 = -\tilde\eta\big(\Omega(Z,H), N_Z, J^{Z,H}, d^Z\big) - \sum_{r\geq 2} \tilde\eta\big({}_{lg}\mathcal{E}_r, N_{{}_{lg}E_r}, J^{{}_{lg}E_r}, {}_{lg}d_r\big).$$

Finally, using Theorem 6.22, we get

$$I_4^3 = -\int_Z L(\nabla^{TZ}) \wedge \tilde\eta\big(\mathcal{F}, N_F, J^F, v\big).$$

These four equations imply the theorem. □

6.5. Proof of Theorem 5.11. By the variation formula for eta forms (Theorem 5.6) it suffices to prove Theorem 5.11 for a particular choice of $T_1^H W$, $T_2^H V$, $T_3^H W$, and g^{TZ}, g^{TX}, g^{TY}. We will suppose that

$$T_3^H W \subset T_1^H W, \qquad g^{TZ} = g^{TX} \oplus \pi_1^* g^{TY}. \tag{6.30}$$

We consider a family of superconnections depending on two parameters $(T,u) \in (0,\infty) \times (0,\infty)$. The parameter u is the usual rescaling parameter associated to the total grading of $\Omega(Z,F)$. In the present case the fibre Z is the total space of a fibre bundle $Z \to Y$ with fibre X. The parameter T is introduced to perform an adiabatic limit in this fibration. For large T the vertical part corresponding to $d^{X,F}$ of the differential d^F is scaled to become large with respect to the horizontal part.

Let us now fit this idea into the formalism. We consider the space $\hat S := (0,\infty) \times (0,\infty) \times S$. Let $\mathrm{pr}\colon \hat S \to S$ denote the projection and define $\hat W := \mathrm{pr}^* W \to \hat S$ with fiber $\hat Z$. Let $\mathrm{pr}_W\colon \hat W \to W$ be the canonical projection. We consider the decomposition of the vertical bundle $TZ = TX \oplus T^H Z$, where $T^H Z := T_1^H W \cap TZ$.

Then $T^H Z \cong \pi_1^* TY$. We define the metric $\hat{g}^{TZ}$on $\hat{Z} = \mathrm{pr}_W^*(TX \oplus \pi_1^* TY)$ such that it restricts to $u^{-2}(T^{-2} g^{TX} \oplus \pi_1^* g^{TY})$ over $(T, u) \times S$. The metric $\hat{g}^{TZ}$ and $\mathrm{pr}_W^* J^F$ together induce the metric structure $\hat{J}^{Z,F}$ on $\mathrm{pr}^* \Omega(Z, F)$ as in (5.1).

Let $d^{\mathrm{pr}_W^* F}$ be the twisted de Rham differential on $\Omega(\hat{W}, \mathrm{pr}_W^* F)$. Further let $p\big(d^{\mathrm{pr}_W^* F}, \hat{J}^{Z,F}\big)$ be the form on $\hat{S}$ defined by (5.2) if n is even (resp. by (5.7) if n is odd).

DEFINITION 6.23. We define $\beta = du \wedge \beta^u + dT \wedge \beta^T$ to be the part of $p\big(d^{\mathrm{pr}_W^* F}, \hat{J}^{Z,F}\big)$ of degree one with respect to the coordinates (T, u), with functions $\beta^u, \beta^T \colon (0, \infty) \times (0, \infty) \to \Omega(S)$.

The following fact is an immediate consequence of the fact that $p\big(\hat{\mathbf{d}}, \hat{J}^{Z,F}\big)$ is closed. Let $d = d_{T,u} + d^S$ be the decomposition of the de Rham differential on $(0, \infty) \times (0, \infty) \times S$.

COROLLARY 6.24. *There exists a smooth family* $\alpha \colon (0, \infty) \times (0, \infty) \to \Omega(S)$ *such that*

$$d_{T,u} \beta = dT \wedge du\, d^S \alpha. \tag{6.31}$$

To compare easily to [**24**, §4–§9], in the following we will write down explicitly β^u, β^T. For α_1, α_2 two differential forms on S, we denote $\{\alpha_1 + du\, \alpha_2\}^{du} = \alpha_2$, and $\{\alpha_1 + dT\, \alpha_2\}^{dT} = \alpha_2$.

Let $J_T^{Z,F}$, $*_T$ be the metric structure and the Hodge star operator on $\Omega(Z, F)$ with respect to the metrics $T^{-2} g^{TX} \oplus \pi_1^* g^{TY}$ and J^F (cf. Subsection 4.3.1). Then we have $J_T^{Z,F} = T^{-N_X + \dim X/2} J^{Z,F} T^{N_X - \dim X/2}$. Denote $z^{J_T^{Z,F}} = \sqrt{\epsilon \epsilon_n}^{\,-1} J_T^{Z,F}$. Let $d^{Z,F}$, $\nabla^{Z,F}$, i_T be the operators defined in Subsection 5.2.2. Let $(d^F)_T^*$, $(d^{Z,F})_T^*$, $(i_T)_T^*$, $(\nabla^{Z,F})_T^*$ be the formal adjoints of d^F, $d^{Z,F}$, i_T, $\nabla^{Z,F}$, respectively, with respect to the metric structure $J_T^{Z,F}$ on $\Omega(Z, F)$. Then

$$\begin{aligned} *_T^{-1} \frac{\partial *_T}{\partial T} &= \frac{1}{T} \left(2N_X - \dim X\right), \\ (d^F)_T^* &= T^{-2N_X} (d^F)_1^* \, T^{2N_X}. \end{aligned} \tag{6.32}$$

For $u > 0$, we set

$$C'_{3,u^2,T} = u^{N_Z} d^F u^{-N_Z}, \qquad C''_{3,u^2,T} = u^{-N_Z} (d^F)_T^* u^{N_Z},$$
$$C_{3,u^2,T} = \frac{1}{2}(C'_{3,u^2,T} + C''_{3,u^2,T}), \qquad D_{3,u^2,T} = \frac{1}{2}(C''_{3,u^2,T} - C'_{3,u^2,T}).$$

We denote by $[A, B] = AB - BA$ the commutator. For $T \geq 1$, set $A_{u,T} = T^{N_X} C_{3,u^2,T} T^{-N_X}$. Then from the above equations we get

$$\big[T^{N_X} D_{3,u^2,T} T^{-N_X}, N_X\big] = T \frac{\partial}{\partial T} A_{u,T}.$$

• If n is even, then

$$\beta^u = (2\pi i)^{-1/2}\varphi\,\mathrm{Tr}\Big[z^{J_T^{Z,F}}\big(\frac{\partial}{\partial u}C_{3,u^2,T}\big)\exp(-C^2_{3,u^2,T})\Big], \tag{6.33}$$

$$\beta^T = (2\pi i)^{-1/2}\varphi\,\mathrm{Tr}\Big[z^{J_T^{Z,F}}\big[\frac{1}{2}D_{3,u^2,T}, *_T^{-1}\frac{\partial *_T}{\partial T}\big]\exp(-C^2_{3,u^2,T})\Big].$$

Thus we get

$$\beta^u = (2\pi i)^{-1/2}\varphi\Big\{\mathrm{Tr}\Big[z^{J^{Z,F}}\exp(-A^2_{u,T} + du\,(\frac{\partial}{\partial u}A_{u,T}))\Big]\Big\}^{du}, \tag{6.34}$$

$$\beta^T = (2\pi i)^{-1/2}\varphi\Big\{\mathrm{Tr}\Big[z^{J^{Z,F}}\exp(-A^2_{u,T} + dT\,(\frac{\partial}{\partial T}A_{u,T}))\Big]\Big\}^{dT}.$$

• If n is odd, we introduce an extra odd variable σ as in Subsection 5.2.2. Then

$$\beta^u = \frac{1}{\sqrt{\pi}}\varphi\mathrm{Tr}_\sigma\Big[z^{J_T^{Z,F}}\big(\frac{\partial}{\partial u}C_{3,u^2,T}\big)\exp(-C^2_{3,u^2,T})\Big], \tag{6.35}$$

$$\beta^T = \frac{1}{\sqrt{\pi}}\varphi\mathrm{Tr}_\sigma\Big[z^{J_T^{Z,F}}\big[\frac{1}{2}D_{3,u^2,T}, *_T^{-1}\frac{\partial *_T}{\partial T}\big]\exp(-C^2_{3,u^2,T})\Big].$$

Remark that $J_T^{Z,F}$ commute with $(\nabla^{Z,F}+(\nabla^{Z,F})^*_T$, $d^{Z,F}+(d^{Z,F})^*_T$, $i_T+(i_T)^*_T$. Let $A^{(i)}_{u,T}$ be the part of $A_{u,T}$ of degree i in $\Lambda(T^*S)$. Then $A^{(1)}_{u,T} = \frac{1}{2}(\nabla^{Z,F}+(\nabla^{Z,F})^*_T)$. Set $\bar{A}_{u,T} = A^{(1)}_{u,T} + z^{J^{Z,F}}(A^{(0)}_{u,T} + A^{(2)}_{u,T})$. Then

$$\beta^u = \frac{1}{\sqrt{\pi}}\varphi\Big\{\mathrm{Tr}_\sigma\Big[\exp(-\bar{A}^2_{u,T} + du(\frac{\partial}{\partial u}\bar{A}_{u,T}))\Big]\Big\}^{du}, \tag{6.36}$$

$$\beta^T = \frac{1}{\sqrt{\pi}}\varphi\Big\{\mathrm{Tr}_\sigma\Big[\exp(-\bar{A}^2_{u,T} + dT(\frac{\partial}{\partial T}\bar{A}_{u,T}))\Big]\Big\}^{dT}.$$

The following theorems can be shown by adapting the the method [**24**] to our present situation.

Let $\check{S} := (0,\infty)\times S$ and $\mathrm{pr}_1\colon \check{S}\to S$ be the projection. We consider the bundle $\check{V} := \mathrm{pr}^*V \to \check{S}$. Let $\mathrm{pr}_V\colon \check{V}\to V$ be the induced map and $H(\check{X},\mathcal{F}) = \mathrm{pr}_V^* H(X,\mathcal{F})$ be the pull-back of the flat cohomology bundle $H(X,\mathcal{F})$ of $(\Omega(X,F)$, $d^{X,F})$. We consider the metric structure $\check{J}^{H(X,\mathcal{F})}$ which restricts to

$$u^{-N_{H(X,\mathcal{F})}+n_{H(X,\mathcal{F})}/2}\mathrm{pr}_V^* J^{H(X,\mathcal{F})}u^{N_{H(X,\mathcal{F})}-n_{H(X,\mathcal{F})}/2}$$

over $\{u\}\times V$. Furthermore, we consider the vertical metric $\check{g}^{TY}$, which restricts to $\frac{1}{u^2}\mathrm{pr}_V^* g^{TY}$ on $\{u\}\times V$. They induce the metric structure $\check{J}^{Y,H(X,\mathcal{F})}$ on $\mathrm{pr}_1^*\,\Omega(Y, H(X,\mathcal{F}))$ as in (5.1). Let $\check{d}^{H(X,\mathcal{F})}$ be the twisted de Rham differential on $\Omega(\check{V}, \mathrm{pr}_V^* H(X,\mathcal{F}))$. Note that the $\mathbb{Z}$-grading operators N_Z, N_X, N_Y act naturally on $\Omega(Y,H(X,\mathcal{F}))$. Then $\check{d}^{H(X,\mathcal{F})}$ is a flat $(-1)^{\check{N}_Z}$-superconnection on $\mathrm{pr}_1^*\,\Omega(Y, H(X,\mathcal{F}))$. We define the family $\gamma\colon (0,\infty)\to\Omega(S)$ such that

$$p\big(\check{d}^{H(X,\mathcal{F})}, \check{J}^{Y,H(X,\mathcal{F})}\big) = du\wedge\gamma + r,$$

where r does not contain du. Note that $p\big(\check{d}^{H(X,\mathcal{F})}, \check{J}^{Y,H(X,\mathcal{F})}\big) = 0$ if $\dim X$ is odd. Furthermore, note that by Definitions 5.3 and 5.4

$$(6.37)\quad \int_0^\infty \gamma(u)\,du = -\tilde{\eta}\big({}_{hc}\mathcal{E}_1, N_{{}_{hc}E_1}, J^{{}_{hc}E_1}, {}_{hc}d_1\big) = -\tilde{\eta}\big(\Omega(Y, H(X,\mathcal{F})), N_Y, J^{Y,H(X,\mathcal{F})}, d^{H(X,\mathcal{F})}\big).$$

THEOREM 6.25. (1) *For any $u > 0$, we have*

$$(6.38)\quad \lim_{T\to\infty} \beta^u(T,u) = \gamma(u).$$

(2) *For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 1$, we have*

$$(6.39)\quad |\beta^u(T,u)| \leq C.$$

(3) *We have the following identity*:

$$(6.40)\quad \lim_{T\to\infty} \int_1^\infty \beta^u(T,u)\,du = \int_1^\infty \gamma(u)\,du - \sum_{r\geq 2} \tilde{\eta}\big({}_{LS}\mathcal{E}_r, N_{{}_{LS}E_r}, J^{{}_{LS}E_r}, {}_{LS}d_r\big).$$

THEOREM 6.26. (1) *There exists a smooth family $\sigma\colon (0,\infty) \to \Omega(S)$ such that, for $T \geq 1$, we have*

$$\lim_{u\to\infty} \beta^T(T,u) = \sigma(T).$$

(2) *There exist constants $C > 0$, $\delta > 0$ such that, for $T \geq 1$, we have the following estimate*:

$$(6.41)\quad |\sigma(T)| \leq \frac{C}{T^{1+\delta}}$$

(3) *We have the following identity*:

$$(6.42)\quad \int_1^\infty \sigma(T)\,dT = -\tilde{\eta}\big(\mathcal{H}(Z,\mathcal{F}), {}_{LS}\mathrm{Gr}(\mathcal{H}(Z,\mathcal{F})), J^{H(Z,\mathcal{F})}, J^{{}_{LS}\mathrm{Gr}(H(Z,\mathcal{F}))}\big).$$

We consider the fibration $\check{W} = (0,\infty) \times W \to \check{V}$ equipped with the vertical metric $\check{g}^{TX}$ which restricts to $u^{-2}g^{TX}$ on $\{u\} \times W$. Let $\mathrm{Pr}_W\colon \check{W} \to W$ be the projection. $\mathrm{Pr}_W^* J^F$ and $\check{g}^{TX}$ induce a metric structure $\check{J}^{X,F}$ on $\mathrm{pr}_V^*\Omega(X,F)$ as in (5.1). Let $d^{\mathrm{Pr}_W^* F}$ be the twisted de Rham differential on $\Omega(\check{W}, \mathrm{Pr}_W^* F)$. We define the smooth family $\theta\colon (0,\infty) \to \Omega(V)$ such that

$$p(d^{\mathrm{Pr}_W^* F}, \check{J}^{X,F}) = du \wedge \theta + r,$$

where r does not contain du. Note that by Definitions 5.3 and 5.4

$$\int_0^\infty \theta(u)\,du = -\tilde{\eta}\big(\Omega(X,F), N_X, J^{X,F}, d^F\big).$$

The adiabatic family of metrics $g^{TX} \oplus \frac{1}{T^2}\pi_1^* g^{TY}$ on $\mathrm{Pr}_W^* TZ$ on $\check{W}$ induces a family of connections $\widetilde{\nabla}^{TZ}$ of $\mathrm{Pr}_W^* TZ$ which after restriction to $\{T\} \times W$ is the connection ∇_T^{TZ} on TZ with respect to g_T^{TZ}, $T_3^H W$, π_3 defined by [**5**, Def. 1.6].

By [**24**, Thm. 5.1], for large T, ∇_T^{TZ} converges to ${}^0\nabla^{TZ} + A_{3,\infty}$, where $A_{3,\infty} \in T^*W \otimes \mathrm{End}(T^H Z, TX)$. We define the smooth family $\lambda\colon (0,\infty) \to \Omega(W)$ such that

$$L(\widetilde{\nabla}^{TZ}) = dT \wedge \lambda + r$$

and r does not contain dT. Now, by the same argument as in [**24**, § 5.7], we have the following proposition for the transgression L class.

PROPOSITION 6.27. *When $T \to \infty$, then we have $\lambda(T) = O(T^{-2})$. Modulo exact forms on W we have*

$$\tilde{L}(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \equiv -\int_1^\infty \lambda(T)\, dT. \tag{6.43}$$

THEOREM 6.28. (1) *For any $u > 0$, there exist $C > 0$, $\delta > 0$ such that, for $T \geq 1$, we have*

$$\left|\beta^T(T,u)\right| \leq \frac{C}{T^{\delta+1}}. \tag{6.44}$$

(2) *For any $T > 0$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta^T(T\varepsilon^{-1},\varepsilon) = \int_Y L(\nabla^{TY}) \wedge \theta(T). \tag{6.45}$$

(3) *There exists $C > 0$ such that, for $\varepsilon \in (0,1]$, $\varepsilon \leq T \leq 1$,*

$$\varepsilon^{-1}\left|\beta^T(T\varepsilon^{-1},\varepsilon) - \int_Z p(\nabla^F, J^F) \wedge \lambda(T\varepsilon^{-1})\right| \leq C. \tag{6.46}$$

(4) *There exist $\delta \in (0,1]$, $C > 0$ such that, for $\varepsilon \in (0,1]$, $T \geq 1$,*

$$\varepsilon^{-1}\left|\beta^T(T\varepsilon^{-1},\varepsilon)\right| \leq \frac{C}{T^{1+\delta}}. \tag{6.47}$$

PROOF OF THEOREM 5.11. We now finish the proof of Theorem 5.11. For $0 < \varepsilon < A$ and $1 < T_0$, we consider the rectangle $(T,u) \in R := [1,T_0] \times [\varepsilon, A]$. By Corollary 6.24 we have $\int_{\partial R} \beta = d^S \int_R \alpha$. Thus

$$\int_\varepsilon^A \beta^u(T_0,u)\, du - \int_1^{T_0} \beta^T(T,A) dT - \int_\varepsilon^A \beta^u(1,u) du + \int_1^{T_0} \beta^T(T,\varepsilon) dT$$
$$= I_1 + I_2 + I_3 + I_4$$

is an exact form. We take the limits $A \to \infty$, $T_0 \to \infty$, and then $\varepsilon \to 0$ in the indicated order. Let I_j^k, $j = 1,\dots 4$, $k = 1,2,3$ again denote the value of the part I_j after the kth limit. By [**28**, §22, Thm. 17], $d\Omega(S)$ is closed under uniformly convergence on compact sets of S. Thus modulo exact forms on S, $\sum_{j=1}^4 I_j^3 \equiv 0$. We obtain from the definition of $\tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F)$ that

$$I_3^3 = \tilde{\eta}(\Omega(Z,F), N_Z, J^{Z,F}, d^F).$$

Furthermore, by Theorem 6.26 and in particular (6.42) we get

$$I_2^2 = I_2^3 = \tilde{\eta}(\mathcal{H}H, {}_{hc}\mathrm{Gr}(\mathcal{H}H), J^{HH}, J^{{}_{hc}\mathrm{Gr}(HH)}).$$

From (6.37), (6.38), (6.39), and (6.40), we conclude that

$$I_1^3 = -\tilde{\eta}\big(\Omega(Y, H(X,\mathcal{F})), N_Y, J^{Y,H(X,\mathcal{F})}, d^{H(X,\mathcal{F})}\big) - \sum_{r\geq 2} \tilde{\eta}\big({}_{lg}\mathcal{E}_r, N_{{}_{lg}E_r}, J^{{}_{lg}E_r}, {}_{lg}d_r\big).$$

Finally, using Theorem 6.28, we get

$$I_4^3 = -\int_Y L(\nabla^{TY}) \wedge \tilde{\eta}(\Omega(X,F), N_X, J^{X,F}, d^F) - \int_Z \tilde{L}\big(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}\big) \wedge p(\nabla^F, J^F)$$

as follows: Convergence of the integrals below is granted by (6.44). We write

$$\int_1^\infty \beta^T(T,\epsilon)\, dT = \int_\epsilon^\infty \epsilon^{-1}\beta^T(T\epsilon^{-1},\epsilon)\, dT.$$

Using Proposition 6.27, (6.45), and (6.47), we get

$$\lim_{\epsilon\to 0}\int_1^\infty \epsilon^{-1}\beta^T(T\epsilon^{-1},\epsilon)\, dT = \int_Y L(\nabla^{TY}) \wedge \int_1^\infty \theta(T)\, dT,$$

$$\lim_{\epsilon\to 0}\int_\epsilon^1 \epsilon^{-1}\left[\beta^T(T\epsilon^{-1},\epsilon) - \int_Z p(\nabla^F, J^F) \wedge \lambda(T\epsilon^{-1})\right] dT = \int_Y L(\nabla^{TY}) \wedge \int_0^1 \theta(T)\, dT.$$

The remaining part of the integral yields by (6.43)

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\epsilon^1 \epsilon^{-1}\int_Z p(\nabla^F, J^F) \wedge \lambda(T\epsilon^{-1}) dT &= \int_Z p(\nabla^F, J^F) \wedge \int_1^\infty \lambda(T) dT \\ &= -\int_Z \tilde{L}\big(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}\big) \wedge p(\nabla^F, J^F).\end{aligned}$$

These four equations for I_k^3, $k = 1, \ldots, 4$, imply Theorem 5.11. □

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UNIVERSITÄT GÖTTINGEN, BUNSENSTR. 3-5, 37073 GÖTTINGEN, GERMANY
E-mail address: `bunke@uni-math.gwdg.de`

CNRS UMR 7640, CENTRE DE MATHÉMATIQUES, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE
E-mail address: `ma@math.polytechnique.fr`