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ON THE ASYMPTOTIC EXPANSION OF BERGMAN KERNEL

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Abstract

We study the asymptotic of the Bergman kernel of the spin^c Dirac operator on high tensor powers of a line bundle.

1. Introduction

The Bergman kernel in the context of several complex variables (i.e., for pseudoconvex domains) has long been an important subject (cf, for example, [**2**]). Its analogue for complex projective manifolds is studied in [**32**], [**29**], [**34**], [**14**], [**26**], establishing the diagonal asymptotic expansion for high powers of an ample line bundle. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the recent work of [**22**] where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow–Mumford stability.

Borthwick and Uribe [**10**], Shiffman and Zelditch [**30**] were the first ones to study the corresponding symplectic versions. Note that they use the almost holomorphic sections based on a construction of Boutet de Monvel–Guillemin [**12**] of a first order pseudodifferential operator D_b associated to the line bundle L on a compact symplectic manifold, which mimic the $\overline{\partial}_b$ operator on the circle bundle in the holomorphic case. The Szegö kernels are well defined modulo smooth operators on the associated circle bundle, even though D_b is neither canonically defined nor unique. (Indeed, Boutet de Monvel–Guillemin define the Szegö kernels first, and construct the operator D_b from the Szegö kernels.) Moreover, in the holomorphic case, the Szegö kernels are exactly (modulo smooth operators) the Szegö kernel associated to the holomorphic sections by Boutet de Monvel–Sjöstrand [**13**]. In the very important paper [**30**], Shiffman and Zelditch also gave a simple way to construct first the Szegö kernels, then the operator D_b from the construction of Boutet de Monvel–Guillemin [**12**], and in [**30**, Theorem 1], they studied the near diagonal asymptotic expansion and small ball Gaussian estimate (for $d(x,y) \leq C/\sqrt{p}$ where p is the power of the line bundle L). On the

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other hand, in the holomorphic setting, in [**19**], Christ (and Lindholm in [**25**]) proved an Agmon type estimate for the Szegö kernel on $\mathbb{C}^1$, but they did not treat the asymptotic expansions.

In this paper, we establish the full off-diagonal asymptotic expansion and Agmon estimate for the Bergman kernel of the spin^c Dirac operator associated to high powers of an ample line bundle in the general context of symplectic manifolds and orbifolds (Cf. Theorem 4.18; note the important factors on the right-hand side of the estimate (4.119) which make our estimate uniform for Z and Z'). Our motivations are to extend Donaldson's work [**22**] to orbifolds and to understand the relationship between heat kernel, index formula and stability. Moreover, the spin^c Dirac operator is a natural geometric operator associated to the symplectic structure. As a result, the coefficients in the asymptotic expansion are naturally polynomials of the curvatures and their derivatives.

Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L). Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u, v) = g^{TX}(\mathbf{J}u, v) \tag{1.1}$$

for $u, v \in TX$. Let J be an almost complex structure which is separately compatible with g^{TX} and ω, and $\omega(\cdot, J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$ and $-J\mathbf{J} \in \mathrm{End}(TX)$ is positive, thus $-J\mathbf{J} = (-\mathbf{J}^2)^{1/2}$. Let ∇^{TX} be the Levi–Civita connection on (TX, g^{TX}) with curvature R^{TX}, and ∇^{TX} induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$ (cf. Section 3). The spin^c Dirac operator D_p acts on $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^{n} \Omega^{0,q}(X, L^p \otimes E)$, the direct sum of spaces of $(0, q)$–forms with values in $L^p \otimes E$.

Let $\{S_i^p\}_{i=1}^{d_p}$ ($d_p = \dim \mathrm{Ker}\, D_p$) be any orthonormal basis of $\mathrm{Ker}\, D_p$ with respect to the inner product (3.2). We define the diagonal of the Bergman kernel of D_p (the distortion function) by

$$B_p(x) = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x))^* \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x. \tag{1.2}$$

Clearly, $B_p(x)$ does not depend on the choice of $\{S_i^p\}$. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) = \mathbb{C} \oplus \Lambda^{>0}(T^{*(0,1)}X)$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \mathrm{End}(T_xX)$, and $|\mathbf{J}| = (-\mathbf{J}^2)^{1/2} \in \mathrm{End}(T_xX)$. A simple corollary of Theorem 4.18′ is:

Theorem 1.1. *There exist smooth coefficients*

$$b_r(x) \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$$

which are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r-1$ (resp. $2r$) and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, and $b_0 = (\det \mathbf{J})^{1/2} I_{\mathbb{C}\otimes E}$, such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\left| B_p(x) - \sum_{r=0}^{k} b_r(x) p^{n-r} \right|_{\mathscr{C}^l} \leq C_{k,l} p^{n-k-1}. \tag{1.3}$$

Moreover, the expansion is uniform in that for any $k, l \in \mathbb{N}$, there is an integer s such that if all data (g^{TX}, h^L, ∇^L, h^E, ∇^E) run over a set which are bounded in $\mathscr{C}^s$ and with g^{TX} bounded below, there exists the constant $C_{k,\,l}$ independent of g^{TX}, and the $\mathscr{C}^l$-norm in (1.3) includes also the derivatives on the parameters.

We also study the asymptotic expansion of the corresponding heat kernel and relates it to that of the Bergman kernel. Let $\exp(-\frac{u}{p}D_p^2)(x,x')$ be the smooth kernel of $\exp(-\frac{u}{p}D_p^2)$ with respect to the Riemannian volume form $dv_X(x')$. We introduce in (3.4), $\omega_d(x) \in \mathrm{End}(\Lambda(T_x^{*(0,1)}X))$.

Theorem 1.2. *There exist smooth sections $b_{r,u}$ of $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)$ on X which are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r-1$ (resp. $2r$) and functions on the eigenvalues of $\mathbf{J}$ at x, and $b_{0,u} = \left(\det(\frac{|\mathbf{J}|}{1-e^{-4\pi u|\mathbf{J}|}})\right)^{1/2} e^{2u\omega_d}$, such that for each $u > 0$ fixed, we have the asymptotic expansion in the sense of (1.3) as $p \to \infty$,*

$$\exp\left(-\frac{u}{p}D_p^2\right)(x,x) = \sum_{r=0}^{k} b_{r,u}(x) p^{n-r} + \mathscr{O}(p^{n-k-1}). \tag{1.4}$$

Moreover, there exists $c > 0$ such that as $u \to +\infty$,

$$b_{r,u}(x) = b_r(x) + \mathscr{O}(e^{-cu}). \tag{1.5}$$

Note that the coefficient $b_{0,u}$ in Theorem 1.2 was first obtained in [**4**, (f)]. Theorems 1.1, 1.2 give us a way to compute the coefficient $b_r(x)$, as it is relatively easy to compute $b_{r,u}(x)$ (cf. (4.107), (4.125)). As an example, we compute b_1 which plays an important role in Donaldson's recent work [**22**]. Note if (X, ω) is Kähler and $\mathbf{J} = J$, then $B_p(x) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ for p large enough, thus $b_r(x) \in \mathrm{End}(E)_x$.

Theorem 1.3. *If (X,ω) is Kähler and $\mathbf{J}=J$, then there exist smooth functions $b_r(x)\in \mathrm{End}(E)_x$ such that we have (1.3), and b_r are polynomials in R^{TX}, R^E and their derivatives with order $\leq 2r-1$ at x. Moreover,*

$$b_0 = \mathrm{Id}_E, \quad b_1 = \frac{1}{4\pi}\left[\sqrt{-1}\sum_i R^E(e_i, Je_i) + \frac{1}{2} r^X \,\mathrm{Id}_E\right]. \tag{1.6}$$

Here, r^X is the scalar curvature of (X, g^{TX}), and $\{e_i\}$ is an orthonormal basis of (X, g^{TX}).

Theorem 1.3 was essentially obtained in [**26**], [**33**] by applying the peak section trick, and in [**14**], [**34**] and [**16**] by applying the Boutet de Monvel–Sjöstrand parametrix for the Szegö kernel [**13**]. We refer the reader to [**22**], [**33**] for its interesting applications.

Our proof of Theorems 1.1, and 1.2 is inspired by local Index Theory, especially by [**7**, Section 11], and we derive Theorem 1.1 from Theorem 1.2. In particular, with the help of the heat kernel, we get the full off-diagonal asymptotic expansion for the Bergman kernel and the Agmon estimate for the remainder term of the asymptotic expansion (Cf. Theorem 4.18). And when (X,ω) is a Kähler manifold, $\mathbf{J}=J$ on X and $E=\mathbb{C}$, we recover [**30**, Theorem 1] if we restrict Theorem 4.18′ to $|Z|, |Z'| < C/\sqrt{p}$.

One of the advantages of our method is that it can be easily generalized to the orbifold situation, and indeed, in (5.25), we deduce the explicit asymptotic expansion near the singular set of the orbifold.

Theorem 1.4. *If (X,ω) is a symplectic orbifold with the singular set X', and L, E are corresponding proper orbifold vector bundles on X as in Theorem 1.1, then there exist smooth coefficients $b_r(x)\in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_x$ with $b_0 = (\det \mathbf{J})^{1/2} I_{\mathbb{C}\otimes E}$, and $b_r(x)$ are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r-1$ (resp. $2r$) and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, such that for any $k,l\in\mathbb{N}$, there exist $C_{k,l}>0$, $N\in\mathbb{N}$ such that for any $x\in X$, $p\in\mathbb{N}$,*

$$\begin{aligned} &\left|\frac{1}{p^n} B_p(x) - \sum_{r=0}^{k} b_r(x) p^{-r}\right|_{\mathscr{C}^l} \\ &\qquad \leq C_{k,l}\Big(p^{-k-1} + p^{l/2}(1+\sqrt{p}\,d(x,X'))^N e^{-C\sqrt{p}\,d(x,X')}\Big). \end{aligned} \tag{1.7}$$

Moreover, if the orbifold (X,ω) is Kähler, $\mathbf{J}=J$ and the proper orbifold vector bundles E, L are holomorphic on X, then $b_r(x)\in\mathrm{End}(E)_x$ with b_0, b_1 still given by (1.6) and $b_r(x)$ are polynomials in R^{TX}, R^E and their derivatives with order $\leq 2r-1$ at x.

This paper is organized as follows. In Section 3, we recall a result on the spectral gap of the spin^c Dirac operator [**28**]. In Section 4, we localize the problem by finite propagation speed and use the rescaling in local index theorem to prove Theorems 1.1, and 1.2. In Section 5, we compute the coefficients of the asymptotic expansion and explain how to generalize our method to the orbifold situation.

The results of this paper have been announced in [**20**].

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3. The spectral gap of the spin^c Dirac operator

The almost complex structure J induces a splitting $T_{\mathbb{R}}X\otimes_{\mathbb{R}}\mathbb{C} = T^{(1,0)}X\oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. For any $v\in TX$ with decomposition $v=v_{1,0}+v_{0,1}\in T^{(1,0)}X\oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0}\in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v)=\sqrt{2}(\overline{v}^*_{1,0}\wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product respectively. Set

$$\mu_0=\inf_{u\in T_x^{(1,0)}X,\, x\in X} R^L_x(u,\overline{u})/|u|^2_{g^{TX}}>0. \tag{3.1}$$

Let ∇^{TX} be the Levi–Civita connection of the metric g^{TX}. By [**24**, pp. 397–398], ∇^{TX} induces canonically a Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ (cf. also [**28**, Section 2]). Let ∇^{E_p} be the connection on $E_p=\Lambda(T^{*(0,1)}X)\otimes L^p\otimes E$ induced by ∇^{Cliff}, ∇^L and ∇^E.

Let $\langle\ \ \rangle_{E_p}$ be the metric on E_p induced by g^{TX}, h^L and h^E. Let dv_X be the Riemannian volume form of (TX, g^{TX}). The L^2–scalar product on $\Omega^{0,\bullet}(X,L^p\otimes E)$, the space of smooth sections of E_p, is given by

$$\langle s_1,s_2\rangle=\int_X\langle s_1(x),s_2(x)\rangle_{E_p}\,dv_X(x)\,. \tag{3.2}$$

We denote the corresponding norm with $\|\cdot\|_{L^2}$. Let $\{e_i\}_i$ be an orthonormal basis of TX.

Definition 3.1. The spin^c Dirac operator D_p is defined by

$$D_p=\sum_{j=1}^{2n}c(e_j)\nabla^{E_p}_{e_j}:\Omega^{0,\bullet}(X,L^p\otimes E)\longrightarrow\Omega^{0,\bullet}(X,L^p\otimes E). \tag{3.3}$$

D_p is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^p\otimes E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, L^p\otimes E)$ and $\Omega^{0,\mathrm{odd}}(X, L^p\otimes E)$.

We denote by $P^{T^{(1,0)}X}$ the projection from $T_{\mathbb{R}}X\otimes_{\mathbb{R}}\mathbb{C}$ to $T^{(1,0)}X$. Let $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ be the Hermitian connection on $T^{(1,0)}X$ induced by ∇^{TX} with curvature $R^{T^{(1,0)}X}$. Let $\nabla^{\det(T^{(1,0)}X)}$ be the connection on $\det(T^{(1,0)}X)$ induced by $\nabla^{T^{(1,0)}X}$ with curvature $R^{\det} = \mathrm{Tr}[R^{T^{(1,0)}X}]$. Let $\{w_i\}$ be an orthonormal frame of $(T^{(1,0)}X, g^{TX})$. Set

$$\omega_d = -\sum_{l,m} R^L(w_l,\overline{w}_m)\,\overline{w}^m\wedge i_{\overline{w}_l}\,, \qquad \tau(x) = \sum_j R^L(w_j,\overline{w}_j). \tag{3.4}$$

Let r^X be the scalar curvature of (TX, g^{TX}), and

$$\mathbf{c}(R) = \sum_{l<m}\Big(R^E + \tfrac{1}{2}\,\mathrm{Tr}\left[R^{T^{(1,0)}X}\right]\Big)(e_l,e_m)\,c(e_l)\,c(e_m).$$

Then, the Lichnerowicz formula [**3**, Theorem 3.52] (cf. [**28**, Theorem 2.2]) for D_p^2 is

$$D_p^2 = \left(\nabla^{E_p}\right)^*\nabla^{E_p} - 2p\omega_d - p\tau + \tfrac{1}{4}r^X + \mathbf{c}(R), \tag{3.5}$$

If A is any operator, we denote by Spec(A) the spectrum of A.

The following simple result was obtained in [**28**, Theorems 1.1, 2.5] by applying the Lichnerowicz formula (cf. also [**8**, Theorem 1] in the holomorphic case).

Theorem 3.2. *There exists $C_L>0$ such that for any $p\in\mathbb{N}$ and any $s\in\Omega^{>0}(X, L^p\otimes E) = \bigoplus_{q\geqslant 1}\Omega^{0,q}(X, L^p\otimes E)$,*

$$\|D_p s\|^2_{L^2} \geqslant (2p\mu_0 - C_L)\|s\|^2_{L^2}\,. \tag{3.6}$$

Moreover, $\mathrm{Spec}D_p^2 \subset \{0\}\cup [2p\mu_0 - C_L, +\infty[$.

4. Bergman kernel

In this Section, we will study the uniform estimate with its derivatives on $t=\frac{1}{\sqrt{p}}$ of the heat kernel and the Bergman kernel of D_p^2 as $p\to\infty$. The first difficulty is that the space $\Omega^{0,\bullet}(X, L^p\otimes E)$ depends on p. To overcome this, we will localize the problem to a problem on $\mathbb{R}^{2n}$. Now, after rescaling, another substantial difficulty appears, which is the lack of the usual elliptic estimate on $\mathbb{R}^{2n}$ for the rescaled Dirac operator. Thus, we introduce a family of Sobolev norms defined by the rescaled connection on L^p, then we can extend the functional analysis technique developed in [**7**, Section 11], and in this way, we can even get the estimate on its derivatives on $t=\frac{1}{\sqrt{p}}$.

This section is organized as follows. In Section 4.1, we establish the fact that the asymptotic expansion of $B_p(x)$ is local on X. In Section

4.2, we derive an asymptotic expansion of D_p in normal coordinate. In Section 4.3, we study the uniform estimate with its derivatives on t of the heat kernel and the Bergman kernel associated to the rescaled operator L_2^t from D_p^2. In Theorem 4.16, we estimate uniformly the remainder term of the Taylor expansion of $e^{-uL_2^t}$ for $u \geq u_0 > 0, t \in [0,1]$. In Section 4.4, we identify $J_{r,u}$ the coefficient of the Taylor expansion of $e^{-uL_2^t}$ with the Volterra expansion of the heat kernel, thus giving us a way to compute the coefficient b_j in Theorem 1.1. In Section 4.4, we prove Theorems 1.1, and 1.2.

4.1. Localization of the problem. Let a^X be the injectivity radius of (X, g^{TX}), and $\varepsilon \in (0, a^X/4)$. We denote by $B^X(x,\epsilon)$ and $B^{T_xX}(0,\epsilon)$ the open balls in X and T_xX with center x and radius ϵ, respectively. Then, the map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\epsilon)$ on $B^X(x,\epsilon)$ for $\epsilon \leq a^X$. From now on, we identify $B^{T_xX}(0,\epsilon)$ with $B^X(x,\epsilon)$ for $\epsilon \leq a^X$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that:

$$f(v) = \begin{cases} 1 & \text{for} \quad |v| \leq \varepsilon/2, \\ 0 & \text{for} \quad |v| \geq \varepsilon. \end{cases} \tag{4.1}$$

Set

$$F(a) = \left(\int_{-\infty}^{+\infty} f(v)dv \right)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{4.2}$$

Then $F(a)$ lies in Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$.

Let P_p be the orthogonal projection from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $\operatorname{Ker} D_p$, and let $P_p(x,x')$, $F(D_p)(x,x')$ $(x,x' \in X)$, be the smooth kernels of P_p, $F(D_p)$ with respect to the volume form $dv_X(x')$. The kernel $P_p(x,x')$ is called the Bergman kernel of D_p. By (1.2),

$$B_p(x) = P_p(x,x). \tag{4.3}$$

Proposition 4.1. *For any $l, m \in \mathbb{N}$, $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for $p \geq 1$, $x, x' \in X$,*

$$\begin{aligned} &|F(D_p)(x,x') - P_p(x,x')|_{\mathscr{C}^m(X\times X)} \leq C_{l,m,\varepsilon}\, p^{-l}, \\ &|P_p(x,x')|_{\mathscr{C}^m(X\times X)} \leq C_{l,m,\varepsilon}\, p^{-l} \quad \text{if } d(x,x') \geq \varepsilon. \end{aligned} \tag{4.4}$$

Here, the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E and ∇^{Cliff}.

Proof. For $a \in \mathbb{R}$, set

$$\phi_p(a) = 1_{[\sqrt{p\mu_0}, +\infty[}(|a|) F(a). \tag{4.5}$$

Then by Theorem 3.2, for $p > C_L/\mu_0$,

$$F(D_p) - P_p = \phi_p(D_p). \tag{4.6}$$

By (4.2), for any $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\sup_{a \in \mathbb{R}} |a|^m |F(a)| \leq C_m. \tag{4.7}$$

As X is compact, there exist $\{x_i\}_{i=1}^r$ such that $\{U_i = B^X(x_i,\varepsilon)\}_{i=1}^r$ is a covering of X. We identify $B^{T_{x_i}X}(0,\varepsilon)$ with $B^X(x_i,\varepsilon)$ by geodesic as above. We identify $(TX)_Z, (E_p)_Z$ for $Z \in B^{T_{x_i}X}(0,\varepsilon)$ to $T_{x_i}X, (E_p)_{x_i}$ by parallel transport with respect to the connections ∇^{TX}, ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_i}(uZ)$. Let $\{e_i\}_i$ be an orthonormal basis of $T_{x_i}X$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve. Let $\Gamma^E, \Gamma^L, \Gamma^{\mathrm{Cliff}}$ be the corresponding connection forms of ∇^E, ∇^L and ∇^{Cliff} with respect to any fixed frame for E, L, $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve γ_Z under the trivialization on U_i.

Denote by ∇_U the ordinary differentiation operator on $T_{x_i}X$ in the direction U. Then,

$$D_p = \sum_j c(\widetilde{e}_j)\Big(\nabla_{\widetilde{e}_j} + p\Gamma^L(\widetilde{e}_j) + \Gamma^{\mathrm{Cliff}}(\widetilde{e}_j) + \Gamma^E(\widetilde{e}_j)\Big). \tag{4.8}$$

Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$. For $l \in \mathbb{N}$, we define a Sobolev norm on the l-th Sobolev space $H^l(X,E_p)$ by

$$\|s\|^2_{H^l_p} = \sum_i \sum_{k=0}^{l} \sum_{i_1\cdots i_k=1}^{2n} \|\nabla_{e_{i_1}}\cdots\nabla_{e_{i_k}}(\varphi_i s)\|^2_{L^2}. \tag{4.9}$$

Then by (4.8), there exists $C>0$ such that for $p\geq 1$, $s\in H^1(X,E_p)$,

$$\|s\|_{H^1_p} \leq C(\|D_p s\|_{L^2} + p\|s\|_{L^2}). \tag{4.10}$$

Let Q be a differential operator of order $m\in\mathbb{N}$ with scalar principal symbol and with compact support in U_i, then

$$\begin{aligned}[D_p, Q] &= \sum_j p[c(\widetilde{e}_j)\Gamma^L(\widetilde{e}_j), Q] \\ &\quad + \sum_j \Big[c(\widetilde{e}_j)\Big(\nabla_{\widetilde{e}_j} + \Gamma^{\mathrm{Cliff}}(\widetilde{e}_j) + \Gamma^E(\widetilde{e}_j)\Big), Q\Big]\end{aligned} \tag{4.11}$$

which are differential operators of order $m-1$, m respectively. By (4.10), (4.11),

$$\begin{aligned}\|Qs\|_{H^1_p} &\leq C(\|D_pQs\|_{L^2} + p\|Qs\|_{L^2}) \\ &\leq C(\|QD_ps\|_{L^2} + p\|s\|_{H^m_p}).\end{aligned} \tag{4.12}$$

From (4.12), for $m\in\mathbb{N}$, there exists $C'_m>0$ such that for $p\geq 1$,

$$\|s\|_{H^{m+1}_p} \leq C'_m(\|D_ps\|_{H^m_p} + p\|s\|_{H^m_p}). \tag{4.13}$$

This means

$$\|s\|_{H^{m+1}_p} \leq C'_m \sum_{j=0}^{m+1} p^{m+1-j}\|D^j_p s\|_{L^2}. \tag{4.14}$$

Moreover, from $\langle D_p^{m'}\,\phi_p\,(D_p)\,Qs,\ s'\,\rangle = \langle s, Q^*\phi_p(D_p)D_p^{m'}s'\rangle$, (4.5), (4.7) and (4.14), we know that for $l, m' \in \mathbb{N}$, there exists $C_{l,m'} > 0$ such that for $p \geq 1$,

$$\|D_p^{m'}\phi_p(D_p)Qs\|_{L^2} \leq C_{l,m'}p^{-l+m}\|s\|_{L^2}. \tag{4.15}$$

We deduce from (4.14) and (4.15) that if P, Q are differential operators of order m, m' with compact support in U_i, U_j respectively, then for any $l > 0$, there exists $C_l > 0$ such that for $p \geq 1$,

$$\|P\phi_p(D_p)Qs\|_{L^2} \leq C_l p^{-l}\|s\|_{L^2}. \tag{4.16}$$

On $U_i \times U_j$, by using Sobolev inequality and (4.6), we get the first inequality of (4.4).

By the finite propagation speed of solutions of hyperbolic equations [**17**], [**18**], [**15**, Section 7.8], [**31**, Section 4.4], $F(D_p)(x,x')$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$, and is zero if $d(x,x') \geq \varepsilon$. Thus, we get the second inequality of (4.4). The proof of Proposition 4.1 is complete. q.e.d.

From Proposition 4.1 and the finite propagation speed as above, we know that the asymptotic of $P_p(x,x')$ as $p \to \infty$ is localized on a neighborhood of x.

To compare the coefficients of the expansion of $P_p(x,x')$ with the heat kernel expansion of $\exp(-\frac{u}{p}D_p^2)$ in Theorem 1.2, we will use again the finite propagation speed to localize the problem.

Definition 4.2. For $u > 0, a \in \mathbb{C}$, set

$$\begin{aligned} G_u(a) &= \int_{-\infty}^{+\infty} e^{iva}\exp\left(-\frac{v^2}{2}\right) f(\sqrt{u}v)\frac{dv}{\sqrt{2\pi}}, \\ H_u(a) &= \int_{-\infty}^{+\infty} e^{iva}\exp\left(-\frac{v^2}{2u}\right)(1-f(v))\frac{dv}{\sqrt{2\pi u}}. \end{aligned} \tag{4.17}$$

The functions $G_u(a), H_u(a)$ are even holomorphic functions. The restrictions of G_u, H_u to $\mathbb{R}$ lie in the Schwartz space $\mathcal{S}(\mathbb{R})$. Clearly,

$$G_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}}D_p\right) + H_{\frac{u}{p}}(D_p) = \exp\left(-\frac{u}{2p}D_p^2\right). \tag{4.18}$$

Let $G_{\frac{u}{p}}(\sqrt{\frac{u}{p}}D_p)(x,x')$, $H_{\frac{u}{p}}(D_p)(x,x')$ $(x,x' \in X)$ be the smooth kernels associated to $G_{\frac{u}{p}}(\sqrt{\frac{u}{p}}D_p)$, $H_{\frac{u}{p}}(D_p)$ calculated with respect to the volume form $dv_X(x')$.

Proposition 4.3. *For any $m \in \mathbb{N}$, $u_0 > 0, \varepsilon > 0$, there exists $C > 0$ such that for any $x, x' \in X$, $p \in \mathbb{N}$, $u \geq u_0$,*

$$\left|H_{\frac{u}{p}}(D_p)(x,x')\right|_{\mathscr{C}^m} \leq Cp^{2m+2n+2}\exp\left(-\frac{\varepsilon^2 p}{10u}\right). \tag{4.19}$$

Proof. By (4.17), for any $m \in \mathbb{N}$, there exists $C_m > 0$ (which depends on ε) such that

$$\sup_{a\in\mathbb{R}} |a|^m |H_u(a)| \le C_m \exp\left(-\frac{\varepsilon^2}{10u}\right). \tag{4.20}$$

As (4.16), we deduce from (4.14) and (4.20) that if P, Q are differential operators of order m, m' with compact support in U_i, U_j respectively, then there exists $C > 0$ such that for $p \ge 1, u \ge u_0$,

$$\|PH_{\frac{u}{p}}(D_p)Qs\|_{L^2} \le Cp^{m+m'} \exp\left(-\frac{\varepsilon^2 p}{10u}\right) \|s\|_{L^2}. \tag{4.21}$$

On $U_i \times U_j$, by using Sobolev inequality, we get our Proposition 4.3.

q.e.d.

Using (4.17) and finite propagation speed [**15**, Section 7.8], [**31**, Section 4.4], it is clear that for $x, x' \in X$, $G_{\frac{u}{p}}(\sqrt{\frac{u}{p}}D_p)(x,x')$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$, and is zero if $d(x,x') \ge \varepsilon$.

4.2. Rescaling and a Taylor expansion of the operator D_p. Now we fix $x_0 \in X$. We identify L_Z, E_Z and $(E_p)_Z$ for $Z \in B^{T_{x_0}X}(0,\varepsilon)$ to L_{x_0}, E_{x_0} and $(E_p)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$. We also denote by $\{e^i\}_i$ the dual basis of $\{e_i\}$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve.

Now, for $\varepsilon > 0$ small enough, we will extend the geometric objects on $B^{T_{x_0}X}(0,\varepsilon)$ to $\mathbb{R}^{2n} \simeq T_{x_0}X$ (here we identify $(Z_1,\ldots,Z_{2n}) \in \mathbb{R}^{2n}$ to $\sum_i Z_i e_i \in T_{x_0}X$) such that D_p is the restriction of a spin^c Dirac operator on $\mathbb{R}^{2n}$ associated to a Hermitian line bundle with positive curvature. In this way, we can replace X by $\mathbb{R}^{2n}$.

First of all, we denote L_0, E_0 the trivial bundles L_{x_0}, E_{x_0} on $X_0 = \mathbb{R}^{2n}$. And we still denote by ∇^L, ∇^E, h^L etc. the connections and metrics on L_0, E_0 on $B^{T_{x_0}X}(0,4\varepsilon)$ induced by the above identification. Then h^L, h^E is identified with the constant metrics $h^{L_0} = h^{L_{x_0}}$, $h^{E_0} = h^{E_{x_0}}$. Let $\mathcal{R} = \sum_i Z_i e_i = Z$ be the radial vector field on $\mathbb{R}^{2n}$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = 1 \ \text{ if } \ |v| < 2; \quad \rho(v) = 0 \ \text{ if } |v| > 4. \tag{4.22}$$

Let $\varphi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined by $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. Let $g^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(Z))$, $J_0(Z) = J(\varphi_\varepsilon(Z))$ be the metric and almost complex structure on X_0. Let $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$, then ∇^{E_0} is the extension of ∇^E on $B^{T_{x_0}X}(0,\varepsilon)$. Let ∇^{L_0} be the Hermitian connection on (L_0, h^{L_0}) defined by

$$\nabla^{L_0}|_Z = \varphi_\varepsilon^* \nabla^L + \frac{1}{2}(1 - \rho^2(|Z|/\varepsilon)) R^L_{x_0}(\mathcal{R}, \cdot). \tag{4.23}$$

Then, we calculate easily that its curvature $R^{L_0} = (\nabla^{L_0})^2$ is

$$\begin{aligned}(4.24)\quad R^{L_0}(Z) &= \varphi_\varepsilon^* R^L + \frac{1}{2} d\Big((1-\rho^2(|Z|/\varepsilon)) R^L_{x_0}(\mathcal{R},\cdot)\Big)\\ &= \Big(1-\rho^2(|Z|/\varepsilon)\Big) R^L_{x_0} + \rho^2(|Z|/\varepsilon) R^L_{\varphi_\varepsilon(Z)}\\ &\quad - (\rho\rho')(|Z|/\varepsilon)\sum_i \frac{Z_i e^i}{\varepsilon|Z|}\wedge\Big[R^L_{x_0}(\mathcal{R},\cdot) - R^L_{\varphi_\varepsilon(Z)}(\mathcal{R},\cdot)\Big].\end{aligned}$$

Thus, R^{L_0} is positive in the sense of (3.1) for ε small enough, and the corresponding constant μ_0 for R^{L_0} is bigger than $\frac{4}{5}\mu_0$. From now on, we fix ε as above.

Let $T^{*(0,1)}X_0$ be the anti-holomorphic cotangent bundle of (X_0, J_0). Since $J_0\,(Z) = J\,(\varphi_\varepsilon\,(Z))$, $T^{*(0,\,1)}_{Z,\,J_0}\,X_0$ is naturally identified with $T^{*(0,1)}_{\varphi_\varepsilon(Z),J}X_0$ (obviously, here the second subscript indicates the almost complex structure with respect to which the splitting is done). Let $\nabla^{\mathrm{Cliff}_0}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_0)$ induced by the Levi–Civita connection ∇^{TX_0} on (X_0, g^{TX_0}). Let R^{E_0}, R^{TX_0}, R^{Cliff_0} be the corresponding curvatures on E_0, TX_0 and $\Lambda(T^{*(0,1)}X_0)$.

We identify $\Lambda\,(T^{*(0,\,1)}\,X_0)_Z$ with $\Lambda\,(T^{*(0,\,1)}_{x_0}\,X)$ by identifying first $\Lambda(T^{*(0,1)}X_0)_Z$ with $\Lambda(T^{*(0,1)}_{\varphi_\varepsilon(Z),J}X_0)$, which in turn is identified with $\Lambda(T^{*(0,1)}_{x_0}X)$ by using parallel transport along $u \to u\varphi_\varepsilon(Z)$ with respect to $\nabla^{\mathrm{Cliff}_0}$. We also trivialize $\Lambda(T^{*(0,1)}X_0)$ in this way. Let S_L be an unit vector of L_{x_0}. Using S_L and the above discussion, we get an isometry $E_{0,p} := \Lambda(T^{*(0,1)}X_0)\otimes E_0 \otimes L_0^p \simeq (\Lambda(T^{*(0,1)}X)\otimes E)_{x_0} =: \mathbf{E}_{x_0}$.

Let $D_p^{X_0}$ (resp. $\nabla^{E_{0,p}}$) be the Dirac operator on X_0 (resp. the connection on $E_{0,p}$) associated to the above data by the construction in Section 3. By the argument in [**28**, pp. 656–657], we know that Theorem 3.2 still holds for $D_p^{X_0}$. In particular, there exists $C>0$ such that

$$\mathrm{Spec}(\mathrm{D}_\mathrm{p}^{\mathrm{X}_0})^2 \subset \{0\} \cup \Big[\frac{8}{5}\mathrm{p}\mu_0 - \mathrm{C}, +\infty\Big[. \tag{4.25}$$

Let P_p^0 be the orthogonal projection from $\Omega^{0,\bullet}\,(X_0,\, L_0^p \otimes E_0) \simeq \mathscr{C}^\infty(X_0, \mathbf{E}_{x_0})$ on $\mathrm{Ker}\, D_p^{X_0}$, and let $P_p^0(x,x')$ be the smooth kernel of P_p^0 with respect to the volume form $dv_{X_0}(x')$.

Proposition 4.4. *For any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$,*

$$\Big|(P_p^0 - P_p)(x,x')\Big|_{\mathscr{C}^m} \leq C_{l,m}\, p^{-l}. \tag{4.26}$$

Proof. Using (4.2) and (4.25), we know that $P_p^0 - F(D_p)$ verifies also (4.4) for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$, thus we get (4.26). q.e.d.

To be complete, we prove the following result in [**3**, Proposition 1.28].

Lemma 4.5. *The Taylor expansion of $\widetilde{e}_i(Z)$ with respect to the basis $\{e_i\}$ to order r is a polynomial of the Taylor expansion of the coefficients of R^{TX} to order $r-2$. Moreover, we have*

$$\widetilde{e}_i(Z) = e_i - \frac{1}{6}\sum_j \left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle e_j + \sum_{|\alpha|\geq 3}\Big(\frac{\partial^\alpha}{\partial Z^\alpha}\widetilde{e}_i\Big)(0)\frac{Z^\alpha}{\alpha!}. \tag{4.27}$$

Proof. Let Γ^{TX} be the connection form of ∇^{TX} with respect to the frame $\{\widetilde{e}_i\}$ of TX. Let $\partial_i = \nabla_{e_i}$ be the partial derivatives along e_i. By the definition of our fixed frame, we have $i_{\mathcal{R}}\Gamma^{TX}=0$. As in [**3**, (1.12)],

$$\mathcal{L}_{\mathcal{R}}\Gamma^{TX} = [i_{\mathcal{R}}, d]\Gamma^{TX} = i_{\mathcal{R}}(d\Gamma^{TX} + \Gamma^{TX}\wedge\Gamma^{TX}) = i_{\mathcal{R}}R^{TX}. \tag{4.28}$$

Let $\Theta(Z) = (\theta^i_j(Z))_{i,j=1}^{2n}$ be the $2n\times 2n$-matrix such that

$$e_i = \sum_j \theta^j_i(Z)\widetilde{e}_j(Z), \quad \widetilde{e}_j(Z) = (\Theta(Z)^{-1})^k_j e_k. \tag{4.29}$$

Set $\theta^j(Z) = \sum_i \theta^j_i(Z)e^i$ and

$$\theta = \sum_j e^j\otimes e_j = \sum_j \theta^j\widetilde{e}_j \in T^*X\otimes TX. \tag{4.30}$$

As ∇^{TX} is torsion free, $\nabla^{TX}\theta = 0$, thus the $\mathbb{R}^{2n}$-valued one-form $\theta = (\theta^j(Z))$ satisfies the structure equation,

$$d\theta + \Gamma^{TX}\wedge\theta = 0. \tag{4.31}$$

Observe first that (cf. [**3**, Proposition 1.27])

$$\mathcal{R} = \sum_j Z_j\widetilde{e}_j(Z), \quad i_{\mathcal{R}}\theta = \sum_j Z_je_j = \mathcal{R}. \tag{4.32}$$

Substituting (4.32) and $(\mathcal{L}_{\mathcal{R}}-1)\mathcal{R}=0$, into the identity $i_{\mathcal{R}}(d\theta + \Gamma^{TX}\wedge\theta)=0$, we obtain via (4.28)

$$(\mathcal{L}_{\mathcal{R}}-1)\mathcal{L}_{\mathcal{R}}\theta = (\mathcal{L}_{\mathcal{R}}-1)(d\mathcal{R} + \Gamma^{TX}\mathcal{R}) = (\mathcal{L}_{\mathcal{R}}\Gamma^{TX})\mathcal{R} = (i_{\mathcal{R}}R^{TX})\mathcal{R}. \tag{4.33}$$

Where we consider the curvature R^{TX} as a matrix of two forms and θ is a $\mathbb{R}^{2n}$-valued one form. The i-th component of $R^{TX}\mathcal{R}$, θ is $\left\langle R^{TX}\mathcal{R}, \widetilde{e}_i\right\rangle$, θ^i, from (4.33), we get

$$i_{e_j}(\mathcal{L}_{\mathcal{R}}-1)\mathcal{L}_{\mathcal{R}}\theta^i(Z) = \left\langle R^{TX}(\mathcal{R}, e_j)\mathcal{R}, \widetilde{e}_i\right\rangle(Z). \tag{4.34}$$

By (4.32), $\mathcal{L}_{\mathcal{R}}e^j = e^j$. Thus from the Taylor expansion of $\theta^i_j(Z)$, we get

$$\sum_{|\alpha|\geq 1}(|\alpha|^2+|\alpha|)(\partial^\alpha\theta^i_j)(0)\frac{Z^\alpha}{\alpha!} = \left\langle R^{TX}(\mathcal{R}, e_j)\mathcal{R}, \widetilde{e}_i\right\rangle(Z). \tag{4.35}$$

Now, by (4.29) and $\theta^i_j(x_0) = \delta_{ij}$, (4.35) determines the Taylor expansion of $\theta^i_j(Z)$ to order m in terms of the Taylor expansion of R^{TX} to order $m-2$. And

$$(\Theta^{-1})^i_j = \delta_{ij} - \frac{1}{6}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle + \mathscr{O}(|Z|^3). \tag{4.36}$$

By (4.29), (4.36), we get (4.27). q.e.d.

For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$ and $Z \in \mathbb{R}^{2n}$, for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}(S_t s)(Z) = s(Z/t), \quad \nabla_t = S_t^{-1} t \nabla^{E_{0,p}} S_t,\\ \mathbf{D}_t = S_t^{-1} t D_p^{X_0} S_t, \quad L_2^t = S_t^{-1} t^2 D_p^{X_0,2} S_t.\end{aligned} \tag{4.37}$$

Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U. Set

$$\mathcal{O}_0 = \sum_j c(e_j)\left(\nabla_{e_j} + \frac{1}{2} R^L_{x_0}(Z, e_j)\right). \tag{4.38}$$

Theorem 4.6. *There exist $\mathcal{B}_{i,r}$ (resp. $\mathcal{A}_{i,r}$, resp. $\mathcal{C}_{i,r}$) ($r \in \mathbb{N}, i \in \{1, \dots, 2n\}$) homogeneous polynomials in Z of degree r with coefficients polynomials in R^{TX}, $R^{\det}$, R^E (resp. R^{TX}, resp. R^L, R^{TX}) and their derivatives at x_0 to order $r-1$ (resp. $r-2$, resp. $r-1$, $r-2$) such that if we denote by*

$$\mathcal{O}_r = \sum_{i=1}^{2n} c(e_i)\Big(\mathcal{A}_{i,r}\nabla_{e_i} + \mathcal{B}_{i,r-1} + \mathcal{C}_{i,r+1}\Big), \tag{4.39}$$

then

$$\mathbf{D}_t = \mathcal{O}_0 + \sum_{r=1}^m t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}). \tag{4.40}$$

Moreover, there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $t \le 1$, $|tZ| \le \varepsilon$, the derivatives of order $\le k$ of the coefficients of the operator $\mathscr{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$.

Proof. By the definition of ∇^{Cliff}, $\widetilde{e}_j$, for $Z \in \mathbb{R}^{2n}$, $|Z| \le \varepsilon$,

$$[\nabla^{\mathrm{Cliff}}_Z, c(\widetilde{e}_j)(Z)] = c(\nabla^{TX}_Z \widetilde{e}_j)(Z) = 0. \tag{4.41}$$

Thus, we know that under our trivialization, for $Z \in \mathbb{R}^{2n}$, $|Z| \le \varepsilon$,

$$c(\widetilde{e}_j)(Z) = c(e_j). \tag{4.42}$$

We identify $(\det(T^{(1,0)}X))_Z$ for $Z \in B^{T_{x_0}X}(0,\varepsilon)$ to $(\det(T^{(1,0)}X))_{x_0}$ by parallel transport with respect to the connection $\nabla^{\det(T^{(1,0)}X)}$ along the curve γ_Z. Let Γ^E, $\Gamma^{\det}$ and Γ^L be the connection forms of ∇^E, $\nabla^{\det(T^{(1,0)}X)}$ and ∇^L with respect to any fixed frames for E, $\det(T^{(1,0)}X)$ and L which are parallel along the curve γ_Z under our trivialization on $B^{T_{x_0}X}(0,\varepsilon)$. Then Γ^E is $\mathrm{End}(\mathbb{C}^{\dim E})$-valued one form on $\mathbb{R}^{2n}$ and

$\Gamma^L, \Gamma^{\det}$ are one forms on $\mathbb{R}^{2n}$. The corresponding connection form of $\Lambda(T^{*(0,1)}X)$ is

$$\Gamma^{\mathrm{Cliff}} = \frac{1}{4}\left\langle \Gamma^{TX}\widetilde{e}_k, \widetilde{e}_l\right\rangle c(\widetilde{e}_k)c(\widetilde{e}_l) + \frac{1}{2}\Gamma^{\det}. \tag{4.43}$$

Now, for $\Gamma^\bullet = \Gamma^E, \Gamma^L$ or $\Gamma^{\det}$ and $R^\bullet = R^E, R^L$ or $R^{\det}$ respectively, by the definition of our fixed frame, we have as in (4.28)

$$i_{\mathcal{R}}\Gamma^\bullet = 0, \quad \mathcal{L}_{\mathcal{R}}\Gamma^\bullet = [i_{\mathcal{R}}, d]\Gamma^\bullet = i_{\mathcal{R}}(d\Gamma^\bullet + \Gamma^\bullet\wedge\Gamma^\bullet) = i_{\mathcal{R}}R^\bullet. \tag{4.44}$$

Using $\mathcal{L}_{\mathcal{R}}e^j = e^j$ and expanding the Taylor's series of both sides of (4.44) at $Z=0$, we obtain

$$\sum_\alpha (|\alpha|+1)(\partial^\alpha\Gamma^\bullet)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \sum_\alpha (\partial^\alpha R^\bullet)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}. \tag{4.45}$$

By equating coefficients of Z^α on both sides, we see from this formula

$$\sum_{|\alpha|=r}(\partial^\alpha\Gamma^\bullet)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^\bullet)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}. \tag{4.46}$$

Especially,

$$\partial_i\Gamma^\bullet_{x_0}(e_j) = \frac{1}{2}R^\bullet_{x_0}(e_i, e_j). \tag{4.47}$$

Furthermore, it follows that the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at x_0 to order r are determined by those of $R^\bullet$ to order $r-1$.

By (4.38), (4.42), for $t = 1/\sqrt{p}$, for $|Z| \le \sqrt{p}\varepsilon$, then

$$\begin{aligned} \nabla_t|_Z &= \nabla + \left(t\Gamma^{\mathrm{Cliff}} + t\Gamma^E + \frac{1}{t}\Gamma^L\right)(tZ), \\ \mathbf{D}_t &= \sum_{j=1}^{2n} c(e_j)\nabla_{t,\widetilde{e}_j(tZ)}|_Z. \end{aligned} \tag{4.48}$$

By Lemma 4.5, (4.46) and (4.48), we get our Theorem. q.e.d.

4.3. Uniform estimate on the heat kernel and the Bergman kernel. Recall that the operators L_2^t, ∇_t were defined in (4.37). We also denote by $\langle\ ,\ \rangle_{0,L^2}$ and $\|\ \|_{0,L^2}$ the scalar product and the L^2 norm on $\mathscr{C}^\infty(X_0, \mathbf{E}_{x_0})$ induced by g^{TX_0}, h^{E_0} as in (3.2).

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z)dv_{TX}(Z), \tag{4.49}$$

with $k(0)=1$. For $s\in \mathscr{C}^\infty(T_{x_0}X,\mathbf{E}_{x_0})$, set

$$\|s\|^2_{t,0}=\int_{\mathbb{R}^{2n}}|s(Z)|^2_{h^{\Lambda(T^{*(0,1)}X_0)\otimes E_0}(tZ)}dv_{X_0}(tZ)=t^{-2n}\|S_t s\|^2_{0,L^2}, \tag{4.50}$$

$$\|s\|^2_{t,m}=\sum_{l=0}^{m}\sum_{i_1,\dots,i_l=1}^{2n}\|\nabla_{t,e_{i_1}}\cdots\nabla_{t,e_{i_l}}s\|^2_{t,0}.$$

We denote by $\langle s',s\rangle_{t,0}$ the inner product on $\mathscr{C}^\infty(X_0,\mathbf{E}_{x_0})$ corresponding to $\|\ \ \|^2_{t,0}$. Let H^m_t be the Sobolev space of order m with norm $\|\ \ \|_{t,m}$. Let H^{-1}_t be the Sobolev space of order -1 and let $\|\ \ \|_{t,-1}$ be the norm on H^{-1}_t defined by $\|s\|_{t,-1}=\sup_{0\neq s'\in H^1_t}|\langle s,s'\rangle_{t,0}|/\|s'\|_{t,1}$. If $A\in\mathscr{L}(H^m,H^{m'})$ $(m,m'\in\mathbb{Z})$, we denote by $\|A\|^{m,m'}_t$ the norm of A with respect to the norms $\|\ \ \|_{t,m}$ and $\|\ \ \|_{t,m'}$.

Then L^t_2 is a formally self adjoint elliptic operator with respect to $\|\ \ \|^2_{t,0}$, and is a smooth family of operators with parameter $x_0\in X$.

Theorem 4.7. *There exist constants $C_1,C_2,C_3>0$ such that for $t\in]0,1]$ and any $s,s'\in\mathscr{C}^\infty_0(\mathbb{R}^{2n},\mathbf{E}_{x_0})$,*

$$\begin{aligned}\left\langle L^t_2 s,s\right\rangle_{t,0}&\geq C_1\|s\|^2_{t,1}-C_2\|s\|^2_{t,0},\\ |\left\langle L^t_2 s,s'\right\rangle_{t,0}|&\leq C_3\|s\|_{t,1}\|s'\|_{t,1}.\end{aligned}\tag{4.51}$$

Proof. Now, from (3.5),

$$\begin{aligned}&\left\langle D^{X_0,2}_p s,s\right\rangle_{0,L^2}\\ &\quad=\|\nabla^{E_{0,p}}s\|^2_{0,L^2}+\left\langle\left(-2p\omega_d-p\tau+\tfrac14 r^X+\mathbf{c}(R)\right)s,s\right\rangle_{0,L^2}.\end{aligned}\tag{4.52}$$

Thus, from (4.37), (4.50), and (4.52),

$$\begin{aligned}&\left\langle L^t_2 s,s\right\rangle_{t,0}\\ &\quad=\|\nabla_t s\|^2_{t,0}+\left\langle\left(-2S^{-1}_t\omega_d-S^{-1}_t\tau+\tfrac{t^2}{4}S^{-1}_t r^X+t^2S^{-1}_t\mathbf{c}(R)\right)s,s\right\rangle_{t,0}.\end{aligned}\tag{4.53}$$

From (4.53), we get (4.51). q.e.d.

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$, and let Δ be the oriented path in $\mathbb{C}$ which goes parallel to the real axis from $+\infty+i$ to $\frac{\mu_0}{2}+i$ then parallel to the imaginary axis to $\frac{\mu_0}{2}-i$ and the parallel to the real axis to $+\infty-i$. By (4.25), (4.37), for t small enough,

$$\operatorname{Spec}\mathrm{L}^{\mathrm{t}}_2\subset\{0\}\cup[\mu_0,+\infty[. \tag{4.54}$$

Thus, $(\lambda-L^t_2)^{-1}$ exists for $\lambda\in\delta\cup\Delta$.

Theorem 4.8. *There exists $C > 0$ such that for $t \in]0,1]$, $\lambda \in \delta \cup \Delta$, and $x_0 \in X$,*

$$\begin{aligned} &\|(\lambda - L_2^t)^{-1}\|_t^{0,0} \leq C, \\ &\|(\lambda - L_2^t)^{-1}\|_t^{-1,1} \leq C(1 + |\lambda|^2). \end{aligned} \tag{4.55}$$

Proof. The first inequality of (4.55) is from (4.54). Now, by (4.51), for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leq -2C_2$, $(\lambda_0 - L_2^t)^{-1}$ exists, and we have $\|(\lambda_0 - L_2^t)^{-1}\|_t^{-1,1} \leq \frac{1}{C_1}$. Now,

$$(\lambda - L_2^t)^{-1} = (\lambda_0 - L_2^t)^{-1} - (\lambda - \lambda_0)(\lambda - L_2^t)^{-1}(\lambda_0 - L_2^t)^{-1}. \tag{4.56}$$

Thus, for $\lambda \in \delta \cup \Delta$, from (4.56), we get

$$\|(\lambda - L_2^t)^{-1}\|_t^{-1,0} \leq \frac{1}{C_1}\left(1 + \frac{4}{\mu_0}|\lambda - \lambda_0|\right). \tag{4.57}$$

Now, we change the last two factors in (4.56), and apply (4.57), we get

$$\begin{aligned} \|(\lambda - L_2^t)^{-1}\|_t^{-1,1} &\leq \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{{C_1}^2}\left(1 + \frac{4}{\mu_0}|\lambda - \lambda_0|\right) \\ &\leq C(1 + |\lambda|^2). \end{aligned} \tag{4.58}$$

The proof of our Theorem is complete. q.e.d.

Proposition 4.9. *Take $m \in \mathbb{N}^*$. There exists $C_m > 0$ such that for $t \in]0,1]$, $Q_1, \ldots, Q_m \in \{\nabla_{t,e_i}, Z_i\}_{i=1}^{2n}$ and $s, s' \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$,*

$$\left|\left\langle [Q_1, [Q_2, \ldots, [Q_m, L_2^t]] \ldots] s, s' \right\rangle_{t,0}\right| \leq C_m \|s\|_{t,1} \|s'\|_{t,1}. \tag{4.59}$$

Proof. Set $g_{ij}(Z) = g^{TX_0}(e_i, e_j)(Z)$. Let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. Let $\nabla^{TX_0}_{e_i} e_j = \Gamma^k_{ij}(Z) e_k$, then by (3.5),

$$\begin{aligned} L_2^t(Z) = &-g^{ij}(tZ)(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_i}) \\ &- 2\omega_d(tZ) - \tau(tZ) + t^2(\tfrac{1}{4}r^X + \mathbf{c}(R))(tZ). \end{aligned} \tag{4.60}$$

Note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$. Thus by (4.60), we know that $[Z_j, L_2^t]$ verifies (4.59).

Note that by (4.37),

$$[\nabla_{t,e_i}, \nabla_{t,e_j}] = \left(R^{L_0} + t^2 R^{\mathrm{Cliff}_0} + t^2 R^{E_0}\right)(tZ)(e_i, e_j). \tag{4.61}$$

Thus, from (4.60) and (4.61), we know that $[\nabla_{t,e_k}, L_2^t]$ has the same structure as L_2^t for $t \in]0,1]$, i.e., $[\nabla_{t,e_k}, L_2^t]$ has the type as

$$\sum_{ij} a_{ij}(t, tZ)\nabla_{t,e_i}\nabla_{t,e_j} + \sum_i b_i(t, tZ)\nabla_{t,e_i} + c(t, tZ), \tag{4.62}$$

and $a_{ij}(t,Z), b_i(t,Z), c(t,Z)$ and their derivatives on Z are uniformly bounded for $Z \in \mathbb{R}^{2n}, t \in [0,1]$; moreover, they are polynomial in t.

Let $(\nabla_{t,e_i})^*$ be the adjoint of ∇_{t,e_i} with respect to $\langle\ ,\ \rangle_{t,0}$, then by (4.50),

$$(\nabla_{t,e_i})^* = -\nabla_{t,e_i} - t(k^{-1}\nabla_{e_i}k)(tZ), \tag{4.63}$$

the last term of (4.63) and its derivatives in Z are uniformly bounded in $Z \in \mathbb{R}^{2n}, t \in [0,1]$.

By (4.62) and (4.63), (4.59) is verified for $m = 1$.

By iteration, we know that $[Q_1, [Q_2, \ldots, [Q_m, L_2^t]] \ldots]$ has the same structure (4.62) as L_2^t. By (4.63), we get Proposition 4.9. q.e.d.

Theorem 4.10. *For any $t \in]0,1]$, $\lambda \in \delta \cup \Delta$, $m \in \mathbb{N}$, the resolvent $(\lambda - L_2^t)^{-1}$ maps H_t^m into H_t^{m+1}. Moreover, for any $\alpha \in \mathbb{Z}^{2n}$, there exist $N \in \mathbb{N}$, $C_{\alpha,m} > 0$ such that for $t \in]0,1]$, $\lambda \in \delta \cup \Delta$, $s \in \mathscr{C}_0^\infty(X_0, \mathbf{E}_{x_0})$,*

$$\|Z^\alpha(\lambda - L_2^t)^{-1}s\|_{t,m+1} \leq C_{\alpha,m}(1+|\lambda|^2)^N \sum_{\alpha' \leq \alpha} \|Z^{\alpha'}s\|_{t,m}. \tag{4.64}$$

Proof. For $Q_1, \ldots, Q_m \in \{\nabla_{t,e_i}\}_{i=1}^{2n}$, $Q_{m+1}, \ldots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n}$, We can express $Q_1 \cdots\ Q_{m+|\alpha|}(\lambda - L_2^t)^{-1}$ as a linear combination of operators of the type

(4.65)

$$[Q_1, [Q_2, \ldots [Q_{m'}, (\lambda - L_2^t)^{-1}]] \ldots] Q_{m'+1} \cdots Q_{m+|\alpha|} \quad m' \leq m + |\alpha|.$$

Let $\mathscr{R}_t$ be the family operators $\mathscr{R}_t = \{[Q_{j_1}, [Q_{j_2}, \ldots, [Q_{j_l}, L_2^t]] \ldots]\}$. Clearly, any commutator $[Q_1, [Q_2, \ldots [Q_{m'}, (\lambda - L_2^t)^{-1}]] \ldots]$ is a linear combination of operators of the form

$$(\lambda - L_2^t)^{-1} R_1 (\lambda - L_2^t)^{-1} R_2 \cdots R_{m'} (\lambda - L_2^t)^{-1} \tag{4.66}$$

with $R_1, \ldots, R_{m'} \in \mathscr{R}_t$.

By Proposition 4.9, the norm $\|\ \ \|_t^{1,-1}$ of the operators $R_j \in \mathscr{R}_t$ is uniformly bound by C. By Theorem 4.8, we find that there exist $C > 0$, $N \in \mathbb{N}$ such that the norm $\|\ \ \|_t^{0,1}$ of operators (4.66) is dominated by $C(1+|\lambda|^2)^N$. q.e.d.

Let $e^{-uL_2^t}(Z, Z')$, $(L_2^t e^{-uL_2^t})(Z, Z')$ be the smooth kernels of the operators $e^{-uL_2^t}$, $L_2^t e^{-uL_2^t}$ with respect to $dv_{TX}(Z')$. Note that L_2^t are families of differential operators with coefficients in $\mathrm{End}(\mathbf{E}_{x_0}) = \mathrm{End}\,(\Lambda\,(T^{*(0,1)}X) \otimes E)_{x_0}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then, we can view $e^{-uL_2^t}(Z, Z')$, $(L_2^t e^{-uL_2^t})(Z, Z')$ as smooth sections of $\pi^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ on $TX \times_X TX$. Let $\nabla^{\mathrm{End}(\mathbf{E})}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)$ induced by ∇^{Cliff} and ∇^E. And $\nabla^{\mathrm{End}(\mathbf{E})}$ induces naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in X$.

Theorem 4.11. *There exists $C''>0$ such that for any $m,m',r\in\mathbb{N}$, $u_0>0$, there exist $C>0$, $N\in\mathbb{N}$ such that for $t\in]0,1]$, $u\geq u_0$, $Z,Z'\in T_{x_0}X$,*

$$\begin{aligned}(4.67)\qquad &\sup_{|\alpha|,|\alpha'|\leq m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}e^{-uL_2^t}\left(Z,Z'\right)\Big|_{\mathscr{C}^{m'}(X)}\\ &\qquad\leq C(1+|Z|+|Z'|)^N\exp\left(\frac{1}{2}\mu_0u-\frac{2C''}{u}|Z-Z'|^2\right),\\ &\sup_{|\alpha|,|\alpha'|\leq m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}(L_2^te^{-uL_2^t})\left(Z,Z'\right)\Big|_{\mathscr{C}^{m'}(X)}\\ &\qquad\leq C(1+|Z|+|Z'|)^N\exp\left(-\frac{1}{4}\mu_0u-\frac{2C''}{u}|Z-Z'|^2\right).\end{aligned}$$

Here, $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0\in X$.

Proof. By (4.54), for any $k\in\mathbb{N}^*$,

(4.68)
$$\begin{aligned}e^{-uL_2^t}&=\frac{(-1)^{k-1}(k-1)!}{2\pi iu^{k-1}}\int_{\delta\cup\Delta}e^{-u\lambda}(\lambda-L_2^t)^{-k}d\lambda,\\ L_2^te^{-uL_2^t}&=\frac{(-1)^{k-1}(k-1)!}{2\pi iu^{k-1}}\int_{\Delta}e^{-u\lambda}\Big[\lambda(\lambda-L_2^t)^{-k}-(\lambda-L_2^t)^{-k+1}\Big]d\lambda.\end{aligned}$$

For $m\in\mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\{\nabla_{t,e_{i_1}}\cdots\nabla_{t,e_{i_j}}\}_{j\leq m}$. From Theorem 4.10, we deduce that if $Q\in\mathcal{Q}^m$, there are $M\in\mathbb{N}$, $C_m>0$ such that for any $\lambda\in\delta\cup\Delta$,

$$(4.69)\qquad \|Q(\lambda-L_2^t)^{-m}\|_t^{0,0}\leq C_m(1+|\lambda|^2)^M.$$

Next we study L_2^{t*}, the formal adjoint of L_2^t with respect to (4.50). Then, L_2^{t*} has the same structure (4.62) as the operator L_2^t, especially,

$$(4.70)\qquad \|Q(\lambda-L_2^{t*})^{-m}\|_t^{0,0}\leq C_m(1+|\lambda|^2)^M.$$

After taking the adjoint of (4.70), we get

$$(4.71)\qquad \|(\lambda-L_2^{t})^{-m}Q\|_t^{0,0}\leq C_m(1+|\lambda|^2)^M.$$

From (4.68), (4.69) and (4.71), we have, for $Q,Q'\in\mathcal{Q}^m$,

$$\begin{aligned}(4.72)\qquad \|Qe^{-uL_2^t}Q'\|_t^{0,0}&\leq C_me^{\frac{1}{4}\mu_0u},\\ \|Q(L_2^te^{-uL_2^t})Q'\|_t^{0,0}&\leq C_me^{-\frac{1}{2}\mu_0u}.\end{aligned}$$

Let $|\ \ |_m$ be the usual Sobolev norm on $\mathscr{C}^\infty(\mathbb{R}^{2n},\mathbf{E}_{x_0})$ induced by $h^{\mathbf{E}_{x_0}}=h^{\Lambda(T^{*(0,1)}_{x_0}X)\otimes E_{x_0}}$ and the volume form $dv_{TX}(Z)$ as in (4.50).

Observe that by (4.48), (4.50), there exists $C > 0$ such that for $s \in \mathscr{C}^\infty(X_0, \mathbf{E}_{x_0})$, $\operatorname{supp} s \subset B^{T_{x_0}X}(0,q)$, $m \geq 0$,

$$\frac{1}{C}(1+q)^{-m}\|s\|_{t,m} \leq |s|_m \leq C(1+q)^m\|s\|_{t,m}. \tag{4.73}$$

Now (4.72), (4.73) together with Sobolev's inequalities implies that if $Q, Q' \in \mathcal{Q}^m$,

$$\begin{aligned} &\sup_{|Z|,|Z'|\leq q} |Q_Z Q'_{Z'} e^{-uL_2^t}(Z,Z')| \leq C(1+q)^{2n+2}\ e^{\frac{1}{4}\mu_0 u}, \\ &\sup_{|Z|,|Z'|\leq q} |Q_Z Q'_{Z'} (L_2^t e^{-uL_2^t})(Z,Z')| \leq C(1+q)^{2n+2}\ e^{-\frac{1}{2}\mu_0 u}. \end{aligned} \tag{4.74}$$

Thus by (4.48), (4.74), we derive (4.67) with the exponential $e^{\frac{1}{4}\mu_0 u}$, $e^{-\frac{1}{2}\mu_0 u}$ for the case when $r = m' = 0$ and $C'' = 0$.

To obtain (4.67) in general, we proceed as in the proof of [**6**, Theorem 11.14]. Note that the function f is defined in (4.1). For $h > 1$, put

$$K_{u,h}(a) = \int_{-\infty}^{+\infty} \exp(iv\sqrt{2u}a)\exp\left(-\frac{v^2}{2}\right)\left(1 - f\left(\frac{1}{h}\sqrt{2u}v\right)\right)\frac{dv}{\sqrt{2\pi}}. \tag{4.75}$$

Then, there exist $C', C_1 > 0$ such that for any $c > 0$, $m, m' \in \mathbb{N}$, there is $C > 0$ such that for $u \geq u_0$, $h > 1$, $a \in \mathbb{C}$, $|\mathrm{Im}(a)| \leq c$, we have

$$|a|^m |K_{u,h}^{(m')}(a)| \leq C\exp\Big(C'c^2u - \frac{C_1}{u}h^2\Big). \tag{4.76}$$

For any $c > 0$, let V_c be the images of $\{\lambda \in \mathbb{C}, |\mathrm{Im}\lambda| \leq c\}$ by the map $\lambda \to \lambda^2$. Then $V_c = \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \geq \frac{1}{4c^2}\mathrm{Im}(\lambda)^2 - c^2\}$, and $\delta \cup \Delta \subset V_c$ for c big enough. Let $\widetilde{K}_{u,h}$ be the holomorphic function such that $\widetilde{K}_{u,h}(a^2) = K_{u,h}(a)$. Then by (4.76), for $\lambda \in V_c$,

$$|\lambda|^m |\widetilde{K}_{u,h}^{(m')}(\lambda)| \leq C\exp\Big(C'c^2u - \frac{C_1}{u}h^2\Big). \tag{4.77}$$

Using finite propagation speed of solutions of hyperbolic equations and (4.75), we find that there exists a fixed constant (which depends on ε) $c' > 0$ such that

$$\widetilde{K}_{u,h}(L_2^t)(Z,Z') = e^{-uL_2^t}(Z,Z') \quad \text{if } |Z - Z'| \geq c'h. \tag{4.78}$$

By (4.77), we see that given $k \in \mathbb{N}$, there is a unique holomorphic function $\widetilde{K}_{u,h,k}(\lambda)$ defined on a neighborhood of V_c such that it verifies the same estimates as $\widetilde{K}_{u,h}$ in (4.77) and $\widetilde{K}_{u,h,k}(\lambda) \to 0$ as $\lambda \to +\infty$; moreover

$$\widetilde{K}_{u,h,k}^{(k-1)}(\lambda)/(k-1)! = \widetilde{K}_{u,h}(\lambda). \tag{4.79}$$

Thus, as in (4.68),

$$(4.80)\quad \widetilde{K}_{u,h}(L_2^t) = \frac{1}{2\pi i}\int_{\delta\cup\Delta} \widetilde{K}_{u,h,k}(\lambda)(\lambda - L_2^t)^{-k} d\lambda,$$
$$L_2^t\widetilde{K}_{u,h}(L_2^t) = \frac{1}{2\pi i}\int_{\Delta} \widetilde{K}_{u,h,k}(\lambda)\Big[\lambda(\lambda - L_2^t)^{-k} - (\lambda - L_2^t)^{-k+1}\Big] d\lambda.$$

By (4.69), (4.71) and by proceeding as in (4.72)–(4.74), we find that for $\mathbf{K}(a) = \widetilde{K}_{u,h}(a)$ or $a\widetilde{K}_{u,h}(a)$, for $|Z|, |Z'| \leq q$,

(4.81)

$$\sup_{|\alpha|,|\alpha'|\leq m}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\mathbf{K}(L_2^t)(Z,Z')\right| \leq C(1+q)^N \exp\left(C'c^2u - \frac{C_1}{u}h^2\right).$$

Setting $h = \frac{1}{c'}|Z - Z'|$ in (4.81), we get for α, α' verified $|\alpha|, |\alpha'| \leq m$,

$$(4.82)\qquad \left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\mathbf{K}(L_2^t)(Z,Z')\right|$$
$$\leq C(1+|Z|+|Z'|)^N \exp\left(C'c^2u - \frac{C_1}{2c'^2u}|Z-Z'|^2\right).$$

By (4.67) with the exponential $e^{\frac{1}{4}\mu_0 u}$, $e^{-\frac{1}{2}\mu_0 u}$ for $r = m' = C'' = 0$, (4.78), (4.82), we get (4.67) for $r = m' = 0$.

To get (4.67) for $r \geq 1$, note that from (4.68), for $k \geq 1$

$$(4.83)\qquad \frac{\partial^r}{\partial t^r}e^{-uL_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}\frac{\partial^r}{\partial t^r}(\lambda - L_2^t)^{-k} d\lambda.$$

We have the similar equation for $\frac{\partial^r}{\partial t^r}(L_2^t e^{-uL_2^t})$. Set

(4.84)

$$I_{k,r} = \left\{(\mathbf{k},\mathbf{r}) = (k_i, r_i)\Big|\sum_{i=0}^j k_i = k + j, \sum_{i=1}^j r_i = r,\ k_i, r_i \in \mathbb{N}^*\right\}.$$

Then, there exist $a_{\mathbf{r}}^{\mathbf{k}} \in \mathbb{R}$ such that

$$(4.85)\quad A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t) = (\lambda - L_2^t)^{-k_0}\frac{\partial^{r_1}L_2^t}{\partial t^{r_1}}(\lambda - L_2^t)^{-k_1}\cdots\frac{\partial^{r_j}L_2^t}{\partial t^{r_j}}(\lambda - L_2^t)^{-k_j},$$
$$\frac{\partial^r}{\partial t^r}(\lambda - L_2^t)^{-k} = \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}}A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t).$$

We claim that $A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)$ is well defined and for any $m \in \mathbb{N}$, $k > 2(m + r + 1)$, $Q, Q' \in \mathcal{Q}^m$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $\lambda \in \delta\cup\Delta$,

$$(4.86)\qquad \|QA_{\mathbf{r}}^{\mathbf{k}}(\lambda,t)Q's\|_{t,0} \leq C(1+|\lambda|)^N \sum_{|\beta|\leq 2r}\|Z^\beta s\|_{t,0}.$$

In fact, by (4.60), $\frac{\partial^r}{\partial t^r}L_2^t$ is combination of $\frac{\partial^{r_1}}{\partial t^{r_1}}(g^{ij}(tZ))$ $(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i})$ $(\frac{\partial^{r_3}}{\partial t^{r_3}}\nabla_{t,e_j})$, $\frac{\partial^{r_1}}{\partial t^{r_1}}(d(tZ))$, $\frac{\partial^{r_1}}{\partial t^{r_1}}(d_i(tZ))(\frac{\partial^{r_2}}{\partial t^{r_2}}\nabla_{t,e_i})$. Now, $\frac{\partial^{r_1}}{\partial t^{r_1}}(d(tZ))$ (resp.

$\frac{\partial^{r_1}}{\partial t^{r_1}}\nabla_{t,e_i})$ ($r_1 \geq 1$), are functions of the type as $d'(tZ)Z^\beta$, $|\beta| \leq r_1$ (resp. r_1+1) and $d'(Z)$ and its derivatives on Z are bounded smooth functions on Z.

Let $\mathscr{R}'_t$ be the family of operators of the type

$$\mathscr{R}'_t = \{[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \ldots [f_{j_l}Q_{j_l}, L^t_2]] \ldots]\}$$

with f_{j_i} smooth bounded (with its derivatives) functions and $Q_{j_i} \in \{\nabla_{t,e_l}\}_{l=1}^{2n}$.

Now, for the operator $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$, we will move first all the term Z^β in $d'(tZ)Z^\beta$ as above to the right-hand side of this operator. To do so, we always use the commutator trick, i.e., each time, we consider only the commutation for Z_i, not for Z^β with $|\beta| > 1$. Then, $A^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is as the form $\sum_{|\beta|\leq 2r} L^t_\beta Q''_\beta Z^\beta$, and Q''_β is obtained from Q' and its commutation with Z^β. Now, we move all the terms ∇_{t,e_i} in $\frac{\partial^{r_j} L^t_2}{\partial t^{r_j}}$ to the right-hand side of the operator L^t_β. Then as in the proof of Theorem 4.10, we get finally that $QA^{\mathbf{k}}_{\mathbf{r}}(\lambda,t)Q'$ is as the form $\sum_{|\beta|\leq 2r}\mathscr{L}^t_\beta Z^\beta$ where $\mathscr{L}^t_\beta$ is a linear combination of operators of the form

$$Q(\lambda - L^t_2)^{-k'_0}R_1(\lambda - L^t_2)^{-k'_1}R_2\cdots R_{l'}(\lambda - L^t_2)^{-k'_{l'}}Q'''Q'',$$

with $R_1,\ldots,R_{l'} \in \mathscr{R}'_t$, $Q''' \in \mathscr{Q}^l$, $Q'' \in \mathscr{Q}^m$, $|\beta| \leq 2r$, and Q'' is obtained from Q' and its commutation with Z^β. By the argument as in (4.69) and (4.71), as $k > 2(m+r+1)$, we can split the above operator to two parts

$$\begin{aligned}&Q(\lambda - L^t_2)^{-k'_0}R_1(\lambda - L^t_2)^{-k'_1}R_2\cdots R_i(\lambda - L^t_2)^{-k''_i};\\ &(\lambda - L^t_2)^{-(k'_i-k''_i)}\cdots R_{l'}(\lambda - L^t_2)^{-k'_{l'}}Q'''Q'',\end{aligned}$$

and the $\|\quad\|^{0,0}_t$-norm of each part is bounded by $C(1+|\lambda|^2)^N$. Thus, the proof of (4.86) is complete.

By (4.83), (4.85) and (4.86), we get the similar estimates (4.67) with $m' = C'' = 0$, (4.82) for $\frac{\partial^r}{\partial t^r}e^{-uL^t_2}$, $\frac{\partial^r}{\partial t^r}(L^t_2 e^{-uL^t_2})$. Thus, we get (4.67) for $m' = 0$.

Finally, for U a vector on X,

(4.87)
$$\nabla_U^{\pi^*\operatorname{End}(\mathbf{E})}e^{-uL^t_2} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda}\nabla_U^{\pi^*\operatorname{End}(\mathbf{E})}(\lambda - L^t_2)^{-k}d\lambda.$$

Now, by using the similar formula (4.85) for $\nabla_U^{\pi^*\operatorname{End}(\mathbf{E})}(\lambda - L^t_2)^{-k}$ by replacing $\frac{\partial^{r_1} L^t_2}{\partial t^{r_1}}$ by $\nabla_U^{\pi^*\operatorname{End}(\mathbf{E})}L^t_2$, and remark that $\nabla_U^{\pi^*\operatorname{End}(\mathbf{E})}L^t_2$ is a differential operator on $T_{x_0}X$ with the same structure as L^t_2. Then, by the above argument, we get (4.67) for $m' \geq 1$. q.e.d.

Let $P_{0,t}$ be the orthogonal projection from $\mathscr{C}^\infty(X_0, \mathbf{E}_{x_0})$ to the kernel of L_2^t with respect to $\langle\ ,\ \rangle_{t,0}$. Set

$$F_u(L_2^t) = \frac{1}{2\pi i}\int_\Delta e^{-u\lambda}(\lambda - L_2^t)^{-1} d\lambda. \tag{4.88}$$

Let $P_{0,t}(Z,Z')$, $F_u(L_2^t)(Z,Z')$ be the smooth kernels of $P_{0,t}, F_u(L_2^t)$ with respect to $dv_{TX}(Z')$. Then by (4.54),

$$F_u(L_2^t) = e^{-uL_2^t} - P_{0,t} = \int_u^{+\infty} L_2^t e^{-u_1 L_2^t} du_1. \tag{4.89}$$

Corollary 4.12. *With the notation in Theorem* 4.11*,*

$$\begin{aligned} \sup_{|\alpha|,|\alpha'|\le m} \Big| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} F_u(L_2^t)\left(Z,Z'\right) \Big|_{\mathscr{C}^{m'}(X)} \\ \le C(1+|Z|+|Z'|)^N \exp\left(-\frac{1}{8}\mu_0 u - \sqrt{C''\mu_0}|Z-Z'|\right). \end{aligned} \tag{4.90}$$

Proof. Note that $\frac{1}{8}\mu_0 u + \frac{2C''}{u}|Z-Z'|^2 \ge \sqrt{C''\mu_0}|Z-Z'|$, thus

$$\begin{aligned} \int_u^{+\infty} e^{-\frac{1}{4}\mu_0 u_1 - \frac{2C''}{u_1}|Z-Z'|^2} du_1 \le e^{-\sqrt{C''\mu_0}|Z-Z'|}\int_u^{+\infty} e^{-\frac{1}{8}\mu_0 u_1} du_1 \\ = \frac{8}{\mu_0} e^{-\frac{1}{8}\mu_0 u - \sqrt{C''\mu_0}|Z-Z'|}. \end{aligned} \tag{4.91}$$

By (4.67), (4.89), and (4.91), we get (4.90). q.e.d.

Remark 4.13. Under the condition of Lindholm [**25**], the metric on the trivial holomorphic line bundle on $\mathbb{C}^n$ is $\|1\| = e^{-\varphi/2}$. Now, we use the unit section $S_L = e^{\varphi/2}1$ to trivialize this line bundle. Then, if φ is $\mathscr{C}^\infty$ and $\frac{\partial^\alpha}{\partial Z^\alpha}\varphi$ is bounded for $|\alpha| \ge 3$, from (4.67), (4.89), (4.90) with $r = 0$, we can derive the off-diagonal estimate of the Bergman kernel on $\mathbb{C}^n$. Actually, the $\mathscr{C}^0$-estimate was obtained by Lindholm [**25**, Prop. 9].

For k large enough, set

$$\begin{aligned} F_{r,u} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i\, r!\, u^{k-1}}\int_\Delta e^{-u\lambda} \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_\mathbf{r}^\mathbf{k} A_\mathbf{r}^\mathbf{k}(\lambda,0)d\lambda, \\ J_{r,u} &= \frac{(-1)^{k-1}(k-1)!}{2\pi i\, r!\, u^{k-1}}\int_{\delta\cup\Delta} e^{-u\lambda} \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_\mathbf{r}^\mathbf{k} A_\mathbf{r}^\mathbf{k}(\lambda,0)d\lambda, \\ F_{r,u,t} &= \frac{1}{r!}\frac{\partial^r}{\partial t^r} F_u(L_2^t) - F_{r,u}, \quad J_{r,u,t} = \frac{1}{r!}\frac{\partial^r}{\partial t^r} e^{-uL_2^t} - J_{r,u}. \end{aligned} \tag{4.92}$$

Certainly, as $t \to 0$, the limit of $\|\ \ \|_{t,m}$ exists, and we denote it by $\|\ \ \|_{0,m}$.

Theorem 4.14. *$r \geq 0, k > 0$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $t \in [0,1], \lambda \in \delta \cup \Delta$,*

$$\left\| \Big(\frac{\partial^r L_2^t}{\partial t^r} - \frac{\partial^r L_2^t}{\partial t^r}|_{t=0} \Big) s \right\|_{t,-1} \leq Ct \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1}, \tag{4.93}$$

$$\left\| \Big(\frac{\partial^r}{\partial t^r} (\lambda - L_2^t)^{-k} - \sum_{(\mathbf{k},\mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) \Big) s \right\|_{0,0}$$
$$\leq Ct(1 + |\lambda|^2)^N \sum_{|\alpha| \leq 4r+3} \|Z^\alpha s\|_{0,0}.$$

Proof. Note that by (4.48), (4.50), for $t \in [0,1]$, $k \geq 1$,

$$\|s\|_{t,0} \leq C\|s\|_{0,0}, \quad \|s\|_{t,k} \leq C \sum_{|\alpha| \leq k} \|Z^\alpha s\|_{0,k}. \tag{4.94}$$

An application of Taylor expansion for (4.60) leads to the following equation, if s, s' have compact support,

$$\Big| \Big\langle \Big(\frac{\partial^r L_2^t}{\partial t^r} - \frac{\partial^r L_2^t}{\partial t^r}|_{t=0} \Big) s, s' \Big\rangle_{0,0} \Big| \leq Ct\|s'\|_{t,1} \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1}. \tag{4.95}$$

Thus, we get the first inequality of (4.93). Note that

$$(\lambda - L_2^t)^{-1} - (\lambda - L_2^0)^{-1} = (\lambda - L_2^t)^{-1}(L_2^t - L_2^0)(\lambda - L_2^0)^{-1}. \tag{4.96}$$

After taking the limit, we know that Theorems 4.8–4.10 still hold for $t = 0$. From (4.55), (4.95) and (4.96),

$$\big\| \big((\lambda - L_2^t)^{-1} - (\lambda - L_2^0)^{-1} \big) s \big\|_{0,0} \leq Ct(1 + |\lambda|^4) \sum_{|\alpha| \leq 3} \|Z^\alpha s\|_{0,0}. \tag{4.97}$$

Now, from the first inequality of (4.93) for $r = 0$, (4.85) and (4.97), we get (4.93). q.e.d.

Theorem 4.15. *There exist $C > 0$, $N \in \mathbb{N}$ such that for $t \in]0,1]$, $u \geq u_0$, $q \in \mathbb{N}$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq q$,*

$$\Big| F_{r,u,t}(Z, Z') \Big| \leq Ct^{1/2(2n+1)}(1+q)^N e^{-\frac{1}{8}\mu_0 u}, \tag{4.98}$$
$$\Big| J_{r,u,t}(Z, Z') \Big| \leq Ct^{1/2(2n+1)}(1+q)^N e^{\frac{1}{2}\mu_0 u}.$$

Proof. Let $J_{x_0,q}^0$ be the vector space of square integrable sections of $\mathbf{E}_{x_0}$ over $\{Z \in T_{x_0}X, |Z| \leq q+1\}$. If $s \in J_{x_0,q}^0$, put $\|s\|_{(q)}^2 = \int_{|Z| \leq q+1} |s|_{\mathbf{E}_{x_0}}^2 dv_{TX}(Z)$. Let $\|A\|_{(q)}$ be the operator norm of $A \in \mathscr{L}(J_{x_0,q}^0)$

with respect to $\| \ \|_{(q)}$. By (4.83), (4.92) and (4.93), we get: There exist $C>0, N\in\mathbb{N}$ such that for $t\in]0,1], u\geq u_0$,

$$\begin{aligned}\|F_{r,u,t}\|_{(q)} &\leq Ct(1+q)^N e^{-\frac{1}{2}\mu_0 u},\\ \|J_{r,u,t}\|_{(q)} &\leq Ct(1+q)^N \ e^{\frac{1}{4}\mu_0 u}.\end{aligned} \tag{4.99}$$

Let $\phi:\mathbb{R}\to[0,1]$ be a smooth function with compact support, equal 1 near 0, such that $\int_{T_{x_0}X}\phi(Z)dv_{TX}(Z)=1$. Take $\nu\in]0,1]$. By the proof of Theorem 4.11, $F_{r,u}$ verifies the similar inequality as in (4.90). Thus by (4.90), there exists $C>0$ such that if $|Z|,|Z'|\leq q$, $U,U'\in\mathbf{E}_{x_0}$,

$$\begin{aligned}&\Big|\big\langle F_{r,u,t}(Z,Z')U,U'\big\rangle-\int_{T_{x_0}X\times T_{x_0}X}\big\langle F_{r,u,t}(Z-W,Z'-W')U,U'\big\rangle\\ &\frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big|\leq C\nu(1+q)^N e^{-\frac{1}{8}\mu_0 u}|U||U'|.\end{aligned} \tag{4.100}$$

On the other hand, by (4.99),

$$\begin{aligned}\Big|\int_{T_{x_0}X\times T_{x_0}X}&\big\langle F_{r,u,t}(Z-W,Z'-W')U,U'\big\rangle\\ &\cdot\frac{1}{\nu^{4n}}\phi(W/\nu)\phi(W'/\nu)dv_{TX}(W)dv_{TX}(W')\Big|\\ &\leq Ct\frac{1}{\nu^{2n}}(1+q)^N e^{-\frac{1}{2}\mu_0 u}|U||U'|.\end{aligned} \tag{4.101}$$

By taking $\nu=t^{1/2(2n+1)}$, we get (4.98). In the same way, we get (4.98) for $J_{r,u,t}$. q.e.d.

Theorem 4.16. *There exists $C''>0$ such that for any $k,m,m'\in\mathbb{N}$, there exist $N\in\mathbb{N}$, $C>0$ such that if $t\in]0,1], u\geq u_0$, $Z,Z'\in T_{x_0}X$,*

$$\begin{aligned}&\sup_{|\alpha|,|\alpha'|\leq m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\Big(F_u(L_2^t)-\sum_{r=0}^k F_{r,u}t^r\Big)(Z,Z')\Big|_{\mathscr{C}^{m'}(X)}\\ &\quad\leq Ct^{k+1}(1+|Z|+|Z'|)^N\exp\left(-\frac{1}{8}\mu_0 u-\sqrt{C''\mu_0}|Z-Z'|\right),\\ &\sup_{|\alpha|,|\alpha'|\leq m}\Big|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\Big(e^{-uL_2^t}-\sum_{r=0}^k J_{r,u}t^r\Big)(Z,Z')\Big|_{\mathscr{C}^{m'}(X)}\\ &\quad\leq Ct^{k+1}(1+|Z|+|Z'|)^N\exp\left(\frac{1}{2}\mu_0 u-\frac{2C''}{u}|Z-Z'|^2\right).\end{aligned} \tag{4.102}$$

Proof. By (4.92), and (4.98),

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r}F_u(L_2^t)|_{t=0} = F_{r,u}, \quad \frac{1}{r!}\frac{\partial^r}{\partial t^r}e^{-uL_2^t}|_{t=0} = J_{r,u}. \tag{4.103}$$

Now, by Theorem 4.11 and (4.92), $J_{r,u}, F_{r,u}$ have the same estimates as $\frac{\partial^r}{\partial t^r}e^{-uL_2^t}$, $\frac{\partial^r}{\partial t^r}F_u(L_2^t)$, in (4.67), (4.90). Again from (4.67), (4.90), (4.92), and the Taylor expansion $G(t) - \sum_{r=0}^k \frac{1}{r!}\frac{\partial^r G}{\partial t^r}(0)t^r = \frac{1}{k!}\int_0^t (t-t_0)^k \frac{\partial^{k+1}G}{\partial t^{k+1}}(t_0)dt_0$, we get (4.102). q.e.d.

4.4. Evaluation of $J_{r,u}$.

For $u>0$, we will write $u\Delta_j$ for the rescaled simplex $\{(u_1,\ldots,u_j)|\ 0\le u_1\le u_2\le\cdots\le u_j\le u\}$. By (4.40),

$$\mathbf{D}_t^2 = \mathscr{O}_0^2 + \sum_{r=1}^{\infty}\sum_{r_1+r_2=r}\mathscr{O}_{r_1}\mathscr{O}_{r_2}t^r = L_2^0 + \sum_{r=1}^{\infty}\mathscr{Q}_r t^r. \tag{4.104}$$

Set $\mathcal{J} = -2\pi\sqrt{-1}\mathbf{J}$. By (1.1), $\mathcal{J}\in \mathrm{End}(T^{(1,0)}X)$ is positive, and the $\mathcal{J}$ action on TX is skew-symmetric. We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,0)}X$. We denote by $|\mathcal{J}_{x_0}| = (\mathcal{J}_{x_0}^2)^{1/2}$, and by $L_{2,\mathbb{C}}^0$ the restriction of L_2^0 on $\mathscr{C}^\infty(\mathbb{R}^{2n},\mathbb{C})$, then by (3.4), (4.60),

$$\begin{aligned} L_{2,\mathbb{C}}^0 &= -\sum_j \Big(\nabla_{e_j} + \frac{1}{2}R_{x_0}^L(Z,e_j)\Big)^2 - \tau_{x_0}, \\ L_2^0 &= L_{2,\mathbb{C}}^0 - 2\omega_{d,x_0}. \end{aligned} \tag{4.105}$$

Let $e^{-uL_{2,\mathbb{C}}^0}(Z,Z')$, $e^{-uL_2^0}(Z,Z')$ be the smooth kernels of $e^{-uL_{2,\mathbb{C}}^0}$, $e^{-uL_2^0}$ with respect to $dv_{TX}(Z')$. Now, from (4.105) (cf. [**5**, (6.37), (6.38)]),

$$\begin{aligned} & e^{-uL_{2,\mathbb{C}}^0}(Z,Z') \\ &= \frac{1}{(2\pi)^n}\det{}_{\mathbb{C}}\Big(\frac{\mathcal{J}_{x_0}}{1-e^{-2u\mathcal{J}_{x_0}}}\Big)\exp\Big(-\frac{1}{2}\Big\langle\frac{\mathcal{J}_{x_0}/2}{\tanh(u\mathcal{J}_{x_0})}Z,Z\Big\rangle \\ &\quad -\frac{1}{2}\Big\langle\frac{\mathcal{J}_{x_0}/2}{\tanh(u\mathcal{J}_{x_0})}Z',Z'\Big\rangle + \Big\langle\frac{\mathcal{J}_{x_0}/2}{\sinh(u\mathcal{J}_{x_0})}e^{u\mathcal{J}_{x_0}}Z,Z'\Big\rangle\Big), \end{aligned} \tag{4.106}$$

$$e^{-uL_2^0}(Z,Z') = e^{-uL_{2,\mathbb{C}}^0}(Z,Z')e^{2u\omega_{d,x_0}}.$$

Theorem 4.17. *For $r\ge 0$, we have*

$$\begin{aligned} J_{r,u} = \sum_{\sum_{i=1}^j r_i=r,\, r_i\ge 1}(-1)^j\int_{u\Delta_j} & e^{-(u-u_j)L_2^0}\mathscr{Q}_{r_j}e^{-(u_j-u_{j-1})L_2^0} \\ & \cdots\mathscr{Q}_{r_1}e^{-u_1L_2^0}du_1\cdots du_j, \end{aligned} \tag{4.107}$$

where the product in the integrand is the convolution product. Moreover, there exist $J_{r,\beta,\beta'}(u) \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ *smooth on* $u \in]0, +\infty[$ *such that*

$$J_{r,u}(Z, Z') = \sum_{|\beta|+|\beta'|\leq 3r} J_{r,\beta,\beta'}(u) Z^{\beta} Z'^{\beta'} e^{-uL^0_{2,\mathbb{C}}}(Z, Z'), \tag{4.108}$$

and $\sum_{|\beta|+|\beta'|\leq 3r} J_{r,\beta,\beta'}(u) Z^{\beta} Z'^{\beta'}$ *as polynomial of* Z, Z' *is even or odd according to whether* r *is even or odd.*

Proof. We introduce an even extra-variable σ such that $\sigma^{r+1} = 0$. Set $[\ \ ]^{[r]}$ the coefficient of σ^r, $L_\sigma = L^0_2 + \sum_{j=1}^r \mathcal{Q}_j \sigma^j$. From (4.92), (4.103), we know

$$J_{r,u}(Z, Z') = \frac{1}{r!} \frac{\partial^r}{\partial t^r} e^{-uL^t_2}(Z, Z')|_{t=0} = [e^{-uL_\sigma}]^{[r]}(Z, Z'). \tag{4.109}$$

Now, from (4.109) and the Volterra expansion of e^{-uL_σ} (cf. [**3**, Section 2.4]), we get (4.107).

We prove (4.108) by iteration. By (4.106), (4.107) and Theorem 4.6, we immediately derive (4.108). By the iteration, (4.106) and Theorem 4.6, the polynomial of Z, Z' has the same parity with r. q.e.d.

4.5. Proof of Theorems 1.1, 1.2. By (4.89), (4.102), for any $u > 0$ fixed, there exists $C_u > 0$ such that for $t = \frac{1}{\sqrt{p}}$, $Z, Z' \in T_{x_0}X$, $x_0 \in X$, we have

$$\begin{aligned} \sup_{|\alpha|,|\alpha'|\leq m} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \Big(P_{0,t} - \sum_{r=0}^{k} t^r (J_{r,u} - F_{r,u}) \Big)(Z, Z') \right|_{\mathscr{C}^{m'}(X)} \\ & \leq C_u t^{k+1} (1 + |Z| + |Z'|)^N \exp(-\sqrt{C''\mu_0}|Z - Z'|), \end{aligned} \tag{4.110}$$

Set

$$P^{(r)} = J_{r,u} - F_{r,u}. \tag{4.111}$$

Then, $P^{(r)}$ does not depend on $u > 0$ by (4.110), as $P_{0,t}$ does not depend on u. Moreover, by taking the limit of (4.90) as $t \to 0$,

$$\begin{aligned} & \Big| F_{r,u}(Z, Z') \Big|_{\mathscr{C}^m(X)} \\ & \leq C(1 + |Z| + |Z'|)^N \exp\left(-\frac{1}{8}\mu_0 u - \sqrt{C''\mu_0}|Z - Z'| \right). \end{aligned} \tag{4.112}$$

Thus

$$J_{r,u}(Z, Z') = P^{(r)}(Z, Z') + F_{r,u}(Z, Z') = P^{(r)}(Z, Z') + \mathscr{O}(e^{-\frac{1}{8}\mu_0 u}), \tag{4.113}$$

uniformly on any compact set of $T_{x_0}X \times T_{x_0}X$.

Let $P(Z,Z')$ be the Bergman kernel of $L^0_{2,\mathbb{C}}$ in (4.105), i.e., the smooth kernel of the orthogonal projection from $L^2(\mathbb{R}^{2n},\mathbb{C})$ on $\operatorname{Ker} L^0_{2,\mathbb{C}}$. Then for $Z,Z'\in T_{x_0}X$,

(4.114)
$$P(Z,Z') = \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \exp\left(-\frac{1}{4}\left\langle |\mathcal{J}_{x_0}|(Z-Z'),(Z-Z')\right\rangle + \frac{1}{2}\left\langle \mathcal{J}_{x_0}Z,Z'\right\rangle\right).$$

Now, $e^{u\mathcal{J}_{x_0}} = \cosh(u|\mathcal{J}_{x_0}|)+\sinh(u|\mathcal{J}_{x_0}|)\frac{\mathcal{J}_{x_0}}{|\mathcal{J}_{x_0}|}$, thus $\frac{\mathcal{J}_{x_0}/2}{\sinh(u\mathcal{J}_{x_0})}e^{u\mathcal{J}_{x_0}} = \frac{1}{2}(|\mathcal{J}_{x_0}|+\mathcal{J}_{x_0})+\mathscr{O}(e^{-2u|\mathcal{J}_{x_0}|})$. From (4.106), and (4.107), we get as $u\to\infty$,

$$J_{0,u}(Z,Z') = e^{-uL^0_2}(Z,Z') = P(Z,Z')I_{\mathbb{C}\otimes E}+\mathscr{O}(e^{-\mu_0 u}), \tag{4.115}$$
$$P^{(0)}(Z,Z') = P(Z,Z')I_{\mathbb{C}\otimes E}.$$

uniformly on any compact set of $T_{x_0}X\times T_{x_0}X$. From (4.108), (4.113), and (4.115), we know that as $u\to\infty$,

$$J_{r,\beta,\beta'}(u) = J_{r,\beta,\beta'}(\infty)+\mathscr{O}(e^{-\frac{1}{8}\mu_0 u}). \tag{4.116}$$

and by (4.113), (4.115) and (4.116),

$$P^{(r)}(Z,Z') = J_{r,\infty}(Z,Z') = \sum_{\beta,\beta'} J_{r,\beta,\beta'}(\infty)Z^\beta Z'^{\beta'}P(Z,Z'). \tag{4.117}$$

Note that in (4.49), $\kappa(Z) = (\det g_{ij}(Z))^{1/2} = (\det(\theta^k_i\theta^k_j))^{1/2}$. By (4.37), for $Z,Z'\in T_{x_0}X$,

$$P^0_p(Z,Z') = p^n P_{0,t}(Z/t,Z'/t)\kappa^{-1}(Z'), \tag{4.118}$$
$$\exp\left(-\frac{u}{p}D^{X_0,2}_p\right)(Z,Z') = p^n e^{-uL^t_2}(Z/t,Z'/t)\kappa^{-1}(Z').$$

We now observe that, as a consequence of (4.110) and (4.118), we obtain the following important estimate.

Theorem 4.18. *For any $k,m,m'\in\mathbb{N}$, there exist $N\in\mathbb{N}, C>0$ such that for $\alpha,\alpha'\in\mathbb{N}^n$, $|\alpha|+|\alpha'|\le m$, $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|\le\varepsilon$, $x_0\in X$, $p\ge 1$,*

(4.119)
$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\left(\frac{1}{p^n}P^0_p(Z,Z') - \sum_{r=0}^k P^{(r)}(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-1}(Z')p^{-r/2}\right)\right|_{\mathscr{C}^{m'}(X)}$$
$$\le Cp^{-(k+1-m)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-\sqrt{C''\mu_0}\sqrt{p}|Z-Z'|).$$

By (4.26) and Theorem 4.18, we obtain the following full off-diagonal expansion for the Bergman kernel on X.

Theorem 4.18′. *With the notation in Theorem* 4.18*,*

(4.120)
$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\left(\frac{1}{p^n}P_p(Z,Z')-\sum_{r=0}^{k}P^{(r)}(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-1}(Z')p^{-r/2}\right)\right|_{\mathscr{C}^{m'}(X)}$$
$$\leq Cp^{-(k+1-m)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-\sqrt{C''\mu_0}\sqrt{p}|Z-Z'|)$$
$$+\mathscr{O}(p^{-\infty}).$$

The term $\mathscr{O}(p^{-\infty})$ means that for any $l,l_1\in\mathbb{N}$, there exists $C_{l,l_1}>0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1}p^{-l}$.

From Theorem 4.17, we know that $J_{r,u}(0,0)=0$ for r odd. Thus from (4.113), $P^{(r)}(0,0)=0$ for r odd. Thus from (4.120), for $Z=Z'=0$, $m=0$, we get

$$\left|\frac{1}{p^n}P_p(x_0,x_0)-\sum_{r=0}^{k}P^{(2r)}(0,0)p^{-r}\right|_{\mathscr{C}^{m'}(X)}\leq Cp^{-k-1}. \tag{4.121}$$

From (4.115),

$$P^{(0)}(0,0)=P(0,0)I_{\mathbb{C}\otimes E}=(\det\mathbf{J})^{1/2}I_{\mathbb{C}\otimes E}. \tag{4.122}$$

Moreover, from Theorems 4.6, 4.17, (4.104), we deduce the desired property on b_r in Theorem 1.1. To get the last part of Theorem 1.1, we notice that the constants in Theorems 4.11 and 4.15 will be uniform bounded under our condition, thus we can take $C_{k,l}$ in (1.6) independent of g^{TX}. Thus, we have proved Theorem 1.1.

From Proposition 4.3, we know that for any $u>0$ fixed, for any $l\in\mathbb{N}$, there exists $C>0$ such that for $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|\leq\varepsilon$, $x_0\in X$,

$$\left|\Big(\exp(-\frac{u}{p}D_p^2)-\exp\Big(-\frac{u}{p}D_p^{X_0,2}\Big)\Big)(Z,Z')\right|_{\mathscr{C}^{m'}(X)}\leq Cp^{-l}. \tag{4.123}$$

Thus, from Theorem 4.17, (4.102), (4.118), and (4.123), we get

(4.124)
$$\left|\frac{1}{p^n}\exp\Big(-\frac{u}{p}D_p^2\Big)(x_0,x_0)-\sum_{r=0}^{k}J_{2r,u}(0,0)p^{-r}\right|_{\mathscr{C}^{m'}(X)}\leq Cp^{-k-1}.$$

Hence, we have (1.4) and at x_0,

$$b_{r,u}=J_{2r,u}(0,0). \tag{4.125}$$

Now, from (4.106), (4.113), (4.121), and (4.125), we deduce Theorem 1.2.

From our proof of Theorems 1.1, and 1.2, we also obtain a method to compute the coefficients. Namely, we compute first the heat kernel expansion of $\exp(-\frac{u}{p}D_p^2)(x,x)$ when $p\to\infty$ by $\sum_{r=0}^{j} b_{r,u}(x)p^{n-r}$ (cf. (4.124)), then let $u\to\infty$, we get the corresponding coefficients of the expansion of $\frac{1}{p^n}P_p(x,x)$. As an example, we will calculate b_1 in the next section.

In practice, we choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$, such that

$$\mathscr{J}_{x_0} = \operatorname{diag}(a_1(x_0),\dots,a_n(x_0)) \in \operatorname{End}(T_{x_0}^{(1,0)}X), \tag{4.126}$$

with $0 < a_1(x_0) \le a_2(x_0) \le \cdots \le a_n(x_0)$, and let $\{w^j\}_{j=1}^n$ be its dual basis. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j=1,\dots,n$ forms an orthonormal basis of $T_{x_0}X$. In the coordinate induced by $\{e_i\}$ as above, all even function $g(\mathscr{J}_{x_0})$ of $\mathscr{J}_{x_0}$ is diagonal, and $g(\mathscr{J}_{x_0}) = g(|\mathscr{J}_{x_0}|)$.

5. Applications

This section is organized as follows. In Section 5.1, we calculate the coefficient b_1 in Theorem 1.1 when the manifold is Kähler. In Section 5.2, we extend Theorem 1.1 to the orbifold case. Again the finite propagation speed allows us to localize the problem which was also used in [**27**].

5.1. Kähler case. In this Section, we assume that (X,ω) is Kähler and $\mathbf{J}=J$, and the vector bundles E, L E, L are holomorphic on X with the holomorphic connections ∇^L, ∇^E. Then, $a_j(x) = 2\pi$ for $j\in\{1,\dots,n\}$ in (4.126). Note that for $\{w_j\}$ (resp. $\{e_j\}$) an orthonormal basis of $T^{(1,0)}X$ (resp TX), the scalar curvature r^X of (X, g^{TX}) is given by

$$r^X = -\sum_{jk}\left\langle R^{TX}(e_j,e_k)e_j, e_k\right\rangle = 2\sum_{jk}\left\langle R^{TX}(w_j,\overline{w}_j)w_k,\overline{w}_k\right\rangle. \tag{5.1}$$

Now, the Levi–Civita connection ∇^{TX} preserves $T^{(1,0)}X$ and $T^{(0,1)}X$, and $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ is the holomorphic Hermitian connection on $T^{(1,0)}X$. In this situation, the Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ is $\nabla^{\Lambda(T^{*(0,1)}X)}$, the natural connection induced by $\nabla^{T^{(1,0)}X}$. Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on $\Omega^{0,\bullet}(X, L^p\otimes E)$. Then, the operator D_p in (3.3) is $D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*})$. Note that D_p^2 preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^p\otimes E)$. Let $D_{p,i}^2 = D_p^2|_{\Omega^{0,i}(X,L^p\otimes E)}$, then for p large enough,

$$\operatorname{Ker} D_p = \operatorname{Ker} D_{p,0}^2 = H^0(X, L^p\otimes E). \tag{5.2}$$

By (5.2), $B_p(x)\in\operatorname{End}(E)$ and we only need to do the computation for $D_{p,0}^2$. In what follows, we compute everything on $\mathscr{C}^\infty(X, L^p\otimes E)$.

Especially, $\mathscr{Q}_r$ in (4.104) takes value in $\mathrm{End}(E)$. Now, we replace X by $\mathbb{R}^{2n} \simeq T_{x_0}X$ as in Section 4.2, and we use the notation therein. We denote by $(g^{ij}(Z))$ the inverse of the matrix $(g_{ij}(Z)) = (g^{TX}_{ij}(Z))$. Let $\Delta^{TX} = \sum_i \frac{\partial^2}{\partial Z_i^2}$ be the standard Euclidean Laplacian on $T_{x_0}X$ with respect to the metric $g^{T_{x_0}X}$. Then by (4.29), (4.35),

$$g_{ij}(Z) = \sum_k \theta^k_i \theta^k_j(Z) = \delta_{ij} + \frac{1}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle + \mathscr{O}(|Z|^3). \tag{5.3}$$

Theorem 5.1.

$$\mathscr{Q}_0 = -\Delta^{TX} + \pi^2|Z|^2 - 2\pi n + 2\sqrt{-1}\pi\nabla_{J\mathcal{R}}, \quad \mathscr{Q}_1 = 0, \tag{5.4}$$

$$\begin{aligned}\mathscr{Q}_2 = \sum_j \Bigg(&\frac{2}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)e_i, e_j\right\rangle \\ &- \frac{\sqrt{-1}\pi}{2}\left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_j\right\rangle - R^E_{x_0}(\mathcal{R}, e_j)\Bigg)\nabla_{e_j} \\ &- \frac{\sqrt{-1}}{2}\sum_j R^E_{x_0}(e_j, Je_j) \\ &+ \pi\sqrt{-1}R^E_{x_0}(\mathcal{R}, J\mathcal{R}) - \frac{\pi^2}{6}\left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, J\mathcal{R}\right\rangle \\ &+ \frac{1}{3}\sum_{ij}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle \nabla_{e_i}\nabla_{e_j}.\end{aligned}$$

Proof. Let Γ^l_{ij} be the connection form of ∇^{TX} with respect to the basis $\{e_i\}$, then $(\nabla^{TX}_{e_i} e_j)(Z) = \Gamma^l_{ij}(Z)e_l$. By (5.3),

$$\begin{aligned}\Gamma^l_{ij}(Z) &= \frac{1}{2}\sum_k g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(Z) \\ &= \frac{1}{3}\Big[\left\langle R^{TX}_{x_0}(\mathcal{R}, e_j)e_i, e_l\right\rangle_{x_0} + \left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)e_j, e_l\right\rangle_{x_0}\Big] \\ &\quad + \mathscr{O}(|Z|^2).\end{aligned} \tag{5.5}$$

Observe that J is parallel with respect to ∇^{TX}, thus $\langle J\widetilde{e}_i, \widetilde{e}_j\rangle_Z = \langle Je_i, e_j\rangle_{x_0}$. By (1.1), (4.29), and (4.35),

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}R^L(e_k, e_l)(Z) &= \sum_{ij}\theta^i_k(Z)\theta^j_l(Z)\left\langle J\widetilde{e}_i, \widetilde{e}_j\right\rangle_Z \\ &= \langle Je_k, e_l\rangle_{x_0} - \frac{1}{6}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_k)\mathcal{R}, Je_l\right\rangle_{x_0}\end{aligned} \tag{5.6}$$

$$+\frac{1}{6}\left\langle R^{TX}_{x_0}(\mathcal{R}, Je_k)\mathcal{R}, e_l\right\rangle_{x_0} + \mathscr{O}(|Z|^3).$$

By (4.37), (4.46) and (5.6), for $t=\frac{1}{\sqrt{p}}$, we get

$$\begin{aligned}\nabla_{t,e_i}|_Z &= tS_t^{-1}\nabla^{L^p\otimes E}_{e_i}S_t|_Z \\ &= \nabla_{e_i} + \frac{1}{t}\Gamma^L(e_i)(tZ) + t\Gamma^E(e_i)(tZ) \\ &= \nabla_{e_i} - \sqrt{-1}\pi\left\langle J\mathcal{R}, e_i\right\rangle - \frac{\sqrt{-1}\pi}{12}t^2\left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i\right\rangle \\ &\quad + \frac{t^2}{2}R^E_{x_0}(\mathcal{R}, e_i) + \mathscr{O}(t^3).\end{aligned} \tag{5.7}$$

By a direct calculation (4.60) or by Lichnerowicz formula in [**4**, Proposition 1.2], we know

$$\begin{aligned}D^2_{p,0} = -\sum_{ij} g^{ij}\left[\nabla^{L^p\otimes E}_{e_i}\nabla^{L^p\otimes E}_{e_j} - \Gamma^l_{ij}\nabla^{L^p\otimes E}_{e_l}\right] \\ - \frac{\sqrt{-1}}{2}\sum_i R^E(\widetilde{e}_i, J\widetilde{e}_i) - 2\pi np.\end{aligned} \tag{5.8}$$

Thus from (4.38), (5.3), (5.7), and (5.8),

(5.9)

$$\begin{aligned}\mathbf{D}^2_{t,0} &= S_t^{-1}t^2D^2_{p,0}S_t \\ &= -\sum_{ij} g^{ij}(tZ)\Big[\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^l_{ij}(t\cdot)\nabla_{t,e_l}\Big](Z) \\ &\quad - \frac{\sqrt{-1}}{2}t^2\sum_i R^E(\widetilde{e}_i, J\widetilde{e}_i)(tZ) - 2\pi n \\ &= -\sum_{ij}\left(\delta_{ij} - \frac{t^2}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle + \mathscr{O}(t^3)\right) \\ &\quad \cdot\Bigg\{\Bigg(\nabla_{e_i} - \sqrt{-1}\pi\left\langle J\mathcal{R}, e_i\right\rangle - \frac{\sqrt{-1}\pi}{12}t^2\left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_i\right\rangle \\ &\quad + \frac{t^2}{2}R^E_{x_0}(\mathcal{R}, e_i) + \mathscr{O}(t^3)\Bigg) \\ &\quad \cdot\Bigg(\nabla_{e_j} - \sqrt{-1}\pi\left\langle J\mathcal{R}, e_j\right\rangle - \frac{\sqrt{-1}\pi}{12}t^2\left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, e_j\right\rangle\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^2}{2}R^E_{x_0}(\mathcal{R},e_j)+\mathscr{O}(t^3)\Big)\\
&-t\Gamma^l_{ij}(tZ)\Big(\nabla_{e_l}-\sqrt{-1}\pi\left\langle J\mathcal{R},e_l\right\rangle+\mathscr{O}(t^2)\Big)\Big\}\\
&-\frac{\sqrt{-1}}{2}t^2\sum_i R^E_{x_0}(e_i,Je_i)-2\pi n+\mathscr{O}(t^3).
\end{aligned}
$$

Now since $e_{2j}=Je_{2j-1}$, we see that $\left\langle R^{TX}(\mathcal{R},e_{2j-1})\mathcal{R},Je_{2j-1}\right\rangle = -\left\langle R^{TX}(\mathcal{R},e_{2j})\mathcal{R},Je_{2j}\right\rangle$, and thus $\sum_j\left\langle R^{TX}(\mathcal{R},e_j)\mathcal{R},Je_j\right\rangle=0$. From (5.5), (5.9) and the fact that R^{TX} is a (1,1)-form, we derive (5.4). q.e.d.

Proof of Theorem 1.3. From (4.106), and (5.4),

$$
\begin{aligned}
e^{-uL^0_2}(Z,Z')=\frac{1}{(1-e^{-4\pi u})^n}\exp\Big(&-\frac{\pi(|Z|^2+|Z'|^2)}{2\tanh(2\pi u)}\\
&+\frac{\pi}{\sinh(2\pi u)}\left\langle e^{-2\sqrt{-1}\pi uJ}Z,Z'\right\rangle\Big).
\end{aligned}
\tag{5.10}
$$

By (4.107), (5.4), (5.10), $J_{1,u}(Z,Z')=0$, and

$$
\begin{aligned}
&J_{2,u}(0,0)\\
&=-\int_0^u du_1\int_{\mathbb{R}^{2n}}\frac{1}{(1-e^{-4\pi u_1})^n(1-e^{-4\pi(u-u_1)})^n}\\
&\qquad\cdot\exp\left(-\frac{\pi|Z|^2}{2\tanh(2\pi(u-u_1))}\right)\mathcal{Q}_2(Z)\exp\left(-\frac{\pi|Z|^2}{2\tanh(2\pi u_1)}\right).
\end{aligned}
\tag{5.11}
$$

By (5.4),

$$
\begin{aligned}
&\mathcal{Q}_2(Z)\exp\left(\frac{-\pi|Z|^2}{2\tanh(2\pi u_1)}\right)\\
&=\Big\{\pi\sqrt{-1}R^E_{x_0}(\mathcal{R},J\mathcal{R})-\frac{\pi^2}{6}\left\langle R^{TX}_{x_0}(\mathcal{R},J\mathcal{R})\mathcal{R},J\mathcal{R}\right\rangle\\
&\quad+\frac{\pi}{3\tanh(2\pi u_1)}\sum_i\left\langle R^{TX}_{x_0}(\mathcal{R},e_i)\mathcal{R},e_i\right\rangle\\
&\quad-\frac{\sqrt{-1}}{2}\sum_i R^E_{x_0}(e_i,Je_i)\Big\}\exp\left(\frac{-\pi|Z|^2}{2\tanh(2\pi u_1)}\right).
\end{aligned}
\tag{5.12}
$$

Now, $\int_{-\infty}^{+\infty} x^2 e^{-x^2/2}dx = \int_{-\infty}^{+\infty} e^{-x^2/2}dx = \sqrt{2\pi}$, and $\int_{-\infty}^{+\infty} x^4 e^{-x^2/2}dx = 3\sqrt{2\pi}$. Thus

$$\begin{aligned}
(5.13)\qquad & \int_{\mathbb{R}^{2n}} \left\langle R^{TX}_{x_0}(\mathcal{R}, J\mathcal{R})\mathcal{R}, J\mathcal{R}\right\rangle \exp\left(-\frac{|Z|^2}{2}\right) \\
&= (2\pi)^n \sum_{jk} \Big[\left\langle R^{TX}_{x_0}(e_j, Je_j)e_k, Je_k\right\rangle \\
&\qquad + \left\langle R^{TX}_{x_0}(e_j, Je_k)e_j, Je_k\right\rangle + \left\langle R^{TX}_{x_0}(e_j, Je_k)e_k, Je_j\right\rangle \Big] \\
&= -(2\pi)^n \times 4 r^X_{x_0}.
\end{aligned}$$

Set $c(u_1) = \frac{\sinh(2\pi(u-u_1))\sinh(2\pi u_1)}{\sinh(2\pi u)}$. Then from (5.11)–(5.13), we get

(5.14)

$$\begin{aligned}
& J_{2,u}(0,0) \\
&= -\int_0^u \frac{du_1}{(1-e^{-4\pi u})^n} \Bigg[\left(c(u_1) - \frac{1}{2}\right) \sqrt{-1} \sum_i R^E_{x_0}(e_i, Je_i) \\
&\quad + \frac{1}{3}\left(\frac{c(u_1)}{\tanh(2\pi u_1)} - 2c(u_1)^2\right) r^X_{x_0} \Bigg] \\
&= \frac{-1}{(1-e^{-4\pi u})^n} \Bigg\{ \left[\left(\frac{1}{\tanh(2\pi u)} - 1\right)\frac{u}{2} - \frac{1}{4\pi}\right] \sqrt{-1} \sum_i R^E_{x_0}(e_i, Je_i) \\
&\quad + \frac{1}{3}\left[\frac{u}{2} - \frac{u}{2\tanh^2(2\pi u)} - \frac{2}{\sinh^2(2\pi u)}\left(\frac{-3}{32\pi}\sinh(4\pi u) + \frac{u}{8}\right)\right] r^X_{x_0} \Bigg\}.
\end{aligned}$$

Thus, by (1.5), and (4.125),

$$(5.15)\qquad b_1 = \lim_{u\to\infty} J_{2,u}(0,0) = \frac{1}{4\pi}\Big[\sqrt{-1}\sum_i R^E(e_i, Je_i) + \frac{1}{2} r^X \,\mathrm{Id}_E\Big].$$

From Theorem 1.1 and (5.15), the proof of Theorem 1.3 is completed. q.e.d.

5.2. Orbifold case. Let (X,ω) be a compact symplectic orbifold of real dimension $2n$ with singular set X'. By definition, for any $x \in X$, there exists a small neighborhood $U_x \subset X$, a finite group G_x acting linearly on $\mathbb{R}^{2n}$, and $\widetilde{U}_x \subset \mathbb{R}^{2n}$ an G_x-open set such that $\widetilde{U}_x \overset{\tau_x}{\to} \widetilde{U}_x/G_x = U_x$ and $\{0\} = \tau_x^{-1}(x) \in \widetilde{U}_x$. We will use $\widetilde{z}$ to denote the point in $\widetilde{U}_x$ representing $z \in U_x$. Let $\Sigma X = \{(x, (h_x^j)) | x \in X, G_x \neq 1, (h_x^j)$ runs over the conjugacy classes in $G_x\}$. Then, ΣX has a natural orbifold structure defined by (cf. [**23**])

$$\left\{(Z_{G_x}(h_x^j)/K_x^j, \widetilde{U}_x^{h_x^j}) \to \widetilde{U}_x^{h_x^j}/Z_{G_x}(h_x^j)\right\}_{(x,U_x,j)}. \tag{5.16}$$

Here, $\widetilde{U}_x^{h_x^j}$ is the fixed point set of h_x^j over $\widetilde{U}_x$, $Z_{G_x}(h_x^j)$ is the centralizer of h_x^j in G_x, and K_x^j is the kernel of the representation $Z_{G_x}(h_x^j) \to$ Diffeo $(\widetilde{U}_x^{h_x^j})$. The number $|K_x^j|$ is locally constant on ΣX and we call it as the multiplicity m_i of each connected component X_i of $X \cup \Sigma X$.

An orbifold vector bundle E on an orbifold X means that for any $x \in X$, there exists $\widetilde{p}_{U_x} : \widetilde{E}_{U_x} \to \widetilde{U_x}$ a $G_{U_x}^E$-equivariant vector bundle and $(G_{U_x}^E, \widetilde{E}_{U_x})$ (resp. $(G_{U_x}^E/K_{U_x}, \widetilde{U}_x)$, $K_{U_x} = \mathrm{Ker}(G_{U_x}^E \to \mathrm{Diffeo}(\widetilde{U}_x)))$ is the orbifold structure of E (resp. X). Set $\widetilde{E}_{U_x}^{\mathrm{pr}}$ the K_{U_x}-invariant sub-bundle of $\widetilde{E}_{U_x}$ on $\widetilde{U}_x$, then $(G_{U_x}^E/K_{U_x}, \widetilde{E}_{U_x}^{\mathrm{pr}})$ defines an orbifold sub-bundle E^{pr} of E on X. We call E^{pr} the proper part of E. We say E is proper if $G_{U_x}^E = G_x$ for any $x \in X$.

Now, any structure on X or E should be locally G_x or $G_{U_x}^E$ equivariant.

Assume that there exists a proper orbifold Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi} R^L = \omega$ (Thus, there exist $k \in \mathbb{N}$ such that L^k is a line bundle in the usual sense). Let (E, h^E) be a proper orbifold Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Then the construction in Section 3 works well here. Especially, the spin^c Dirac operator D_p is well defined. In our situation, let $\{S_1^p, ..., S_{d_p}^p\}$ ($d_p = \dim \mathrm{Ker}\, D_p$) be any orthonormal basis of $\mathrm{Ker}\, D_p$ with respect to the inner product (3.2). We still have (4.3) for $B_p(x)$. In fact, on the local coordinate above, $\widetilde{S}_i^p(\widetilde{z})$ on $\widetilde{U}_x$ are G_x invariant, and

$$P_p(z, z') = \sum_{i=1}^{d_p} \widetilde{S}_i^p(\widetilde{z}) \otimes (\widetilde{S}_i^p(\widetilde{z'}))^*. \tag{5.17}$$

We note that if $Q : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F)$ is a pseudo-differential operator of order m ($m < -2n-k, k \in \mathbb{N}$), and E, F are proper orbifold vector bundles, then the operator Q has a $\mathscr{C}^k$-kernel. In fact, Q_{U_x} lifts to a pseudo-differential operator $\widetilde{Q}_{U_x}$ on $\widetilde{U}_x$ and for $\widetilde{Q}_{U_x}(\widetilde{z}, \widetilde{z'})$ the $\mathscr{C}^k$-kernel on $\widetilde{U}_x \times \widetilde{U}_x$ with respect to $dv_{\widetilde{U}_x}$, the kernel of the operator $Q_{U_x} : \mathscr{C}^\infty(U_x, E_{|U_x}) \to \mathscr{C}^\infty(U_x, F_{|U_x})$ is (cf. also [**27**, (2.2)])

$$Q_{U_x}(z, z') = \sum_{g \in G_x} (g, 1)\widetilde{Q}_{U_x}(g^{-1}\widetilde{z}, \widetilde{z'}), \quad (z, z') \in U_x \times U_x. \tag{5.18}$$

Indeed, for $s \in \mathscr{C}^\infty(U_x, E)$ with compact support, then s is a G_x-invariant section of $\widetilde{E}_{U_x}$ on $\widetilde{U}_x$, by definition,

$$
\begin{aligned}
(5.19) \qquad (Qs)(z) &= \int_{\widetilde{U}_x} \widetilde{Q}_{U_x}(\widetilde{z}, \widetilde{z}') s(\widetilde{z}') dv_{\widetilde{U}_x}(\widetilde{z}'),\\
&= \frac{1}{|G_x|} \sum_{g\in G_x} \int_{\widetilde{U}_x} (g,1)\widetilde{Q}_{U_x}(g^{-1}\widetilde{z}, \widetilde{z}') s(\widetilde{z}') dv_{\widetilde{U}_x}(\widetilde{z}'),\\
&= \int_{U_x} \sum_{g\in G_x} (g,1)\widetilde{Q}_{U_x}(g^{-1}\widetilde{z}, z') s(z') dv_{U_x}(z').
\end{aligned}
$$

Proof of Theorem 1.4. At first, we have the analogue of Propositions 4.1,

$$
(5.20) \qquad |P_p(x,x') - F(D_p)(x,x')|_{\mathscr{C}^m(X)} \le C_{l,m,\varepsilon} p^{-l}.
$$

To prove (5.20), we work on $\widetilde{U}_{x_i}$, and the Sobolev norm in (4.9) is summed on $\widetilde{U}_{x_i}$.

Note that on orbifold, the property of the finite propagation speed of solutions of hyperbolic equations still holds if we check the proof therein [**15**, Section 7.8], [**31**, Section 4.4] as pointed out in [**27**]. Thus, for $x, x' \in X$, if $d(x,x') \ge \varepsilon$, then $F(D_p)(x,x') = 0$, and given $x \in X$, $F(D_p)(x,\cdot)$ only depends on the restriction of D_p to $B^X(x,\varepsilon)$. Thus, the problem on the asymptotic expansion of $P_p(x,\cdot)$ is local.

Now, we replace X by $\mathbb{R}^{2n}/G_{x_0}$, and let $\widetilde{L}, \widetilde{E}$ be the G_{x_0}-equivariant vector bundles on $\widetilde{U}_{x_0}$ corresponding to L, E on $\widetilde{U}_{x_0}/G_{x_0}$. In particular, G_{x_0} acts linearly and effectively on $\mathbb{R}^{2n}$. We will add a superscript $\widetilde{\ }$ to indicate the corresponding objects on $\mathbb{R}^{2n}$.

Now, for $Z, Z' \in \mathbb{R}^{2n}/G_{x_0}$, $|Z|, |Z'| \le \varepsilon/2$ and $\widetilde{Z}, \widetilde{Z}' \in \mathbb{R}^{2n}$ represent Z, Z', then by (4.4), (4.26), and (5.18), for any $l, m \in \mathbb{N}$, there exists $C_{l,m,\varepsilon} > 0$ such that for $p \ge 1$,

$$
\begin{aligned}
(5.21) \qquad F(D_p)(Z,Z') &= \sum_{g\in G_x} (g,1) F(\widetilde{D}_p)(g^{-1}\widetilde{Z}, \widetilde{Z}'),\\
|F(\widetilde{D}_p)(\widetilde{Z}, \widetilde{Z}') &- \widetilde{P}^0_p(\widetilde{Z}, \widetilde{Z}')|_{\mathscr{C}^m} \le C_{l,m,\varepsilon} p^{-l}.
\end{aligned}
$$

Moreover, for $t = \frac{1}{\sqrt{p}}$,

$$
(5.22) \qquad \widetilde{P}^0_p(\widetilde{Z}, \widetilde{Z}') = \frac{1}{t^{2n}} \widetilde{P}_{0,t}(\widetilde{Z}/t, \widetilde{Z}'/t)\kappa^{-1}(Z').
$$

We will denote $P^{(r)}$ in (4.108) by $P^{(r)}_{x_0}$ to indicate the base point x_0. For $g \in G_{x_0}$, we denote by $\widetilde{Z} = \widetilde{Z}_{1,g} + \widetilde{Z}_{2,g}$ with $\widetilde{Z}_{1,g} \in T\widetilde{U}^g_{x_0}$, $\widetilde{Z}_{2,g} \in N_{g,x_0}$

(here, N_{g,x_0} is the normal bundle to $\widetilde{U}^g_{x_0}$ in $\widetilde{U}_{x_0}$). By (4.89), (4.102), as in (4.119), for $|\widetilde{Z}| \le \varepsilon/2$, α, α' with $|\alpha| \le m, |\alpha'| \le m'$,

$$\begin{aligned} &\Big| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha_{1,g}} \frac{\partial^{|\alpha'|}}{\partial \widetilde{Z}^{\alpha'}_{2,g}} \Big(\frac{1}{p^n} \widetilde{P}^0_p(g^{-1}\widetilde{Z}, \widetilde{Z}) \\ &\qquad - \sum_{r=0}^k t^r P^{(r)}_{\widetilde{Z}_{1,g}} (\sqrt{p} g^{-1} \widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g}) \kappa^{-1}_{\widetilde{Z}_{1,g}} (\widetilde{Z}_{2,g}) \Big) \Big| \\ &\le C t^{k-m'} (1 + \sqrt{p} |\widetilde{Z}_{2,g}|)^N \exp(-\sqrt{C''\mu_0} \sqrt{p} |\widetilde{Z}_{2,g}|). \end{aligned} \tag{5.23}$$

Especially, for $Z \in \mathbb{R}^{2n}/G_{x_0}$, $|\widetilde{Z}| \le \varepsilon/2$, as in (4.121),

$$\sup_{|\alpha| \le m} \left| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha} \left(\frac{1}{p^n} \widetilde{P}^0_p(\widetilde{Z}, \widetilde{Z}) - \sum_{r=0}^k p^{-r} P^{(2r)}_{\widetilde{Z}}(0,0) \right) \right| \le C p^{-k-1}. \tag{5.24}$$

Thus, from (5.20)–(5.24)[1] , we get for $|\widetilde{Z}| \le \varepsilon/2$,

$$\begin{aligned} &\sup_{|\alpha| \le m'} \Big| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha} \Big(\frac{1}{p^n} P_p(\widetilde{Z}, \widetilde{Z}) - \sum_{r=0}^k b_r(\widetilde{Z}) p^{-r} \\ &\quad - \sum_{r=0}^{2k} p^{-\frac{r}{2}} \sum_{1 \ne g \in G_{x_0}} (g,1) P^{(r)}_{\widetilde{Z}_{1,g}} (\sqrt{p} g^{-1} \widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g}) \kappa^{-1}_{\widetilde{Z}_{1,g}} (\widetilde{Z}_{2,g}) \Big) \Big| \\ &\le C \Big(p^{-k-1} + p^{-k+\frac{m'-1}{2}} \left(1 + \sqrt{p} d(Z, X')\right)^N \\ &\qquad \cdot \exp\left(-\sqrt{C''\mu_0} \sqrt{p} d(Z, X') \right) \Big). \end{aligned} \tag{5.25}$$

By (4.117), we get for α, α' with $|\alpha| \le m, |\alpha'| \le m'$,

$$\begin{aligned} &\left| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha_{1,g}} \frac{\partial^{|\alpha'|}}{\partial \widetilde{Z}^{\alpha'}_{2,g}} \sum_{r=0}^{2k} t^r P^{(r)}_{\widetilde{Z}_{1,g}} (g^{-1}\widetilde{Z}_{2,g}/t, \widetilde{Z}_{2,g}/t) \right| \\ &\le C t^{-m'} \left(1 + \left| \frac{\widetilde{Z}_{2,g}}{t} \right| \right)^N \exp\left(-\frac{C'}{t} |\widetilde{Z}_{2,g}| \right). \end{aligned} \tag{5.26}$$

For any compact set $K \subset X \setminus X'$, we get the uniform estimate (1.7) from (5.24) as in Section 4.5 as $G_x = \{1\}$. From (5.25), (5.26), we get (1.7) near the singular set X'.

[1]In the same way, by Theorem 4.18, (5.21), (5.22), we get the full off-diagonal expansion of the Bergman kernel on the orbifolds as in Theorem 4.18$'$.

By the argument in Section 5.1, we have established the last part of Theorem 1.4. q.e.d.

Note that if $x_0 \in X'$, then $|G_{x_0}| > 1$. Now, if in addition, L and E are usual vector bundles, i.e., G_{x_0} acts on both L_{x_0} and E_{x_0} as identity, then by (5.25),

$$\left|\frac{1}{p^n} P_p(x_0, x_0) - |G_{x_0}| b_0(x_0)\right| \le C p^{-1/2}. \tag{5.27}$$

Thus, we can never have an uniform asymptotic expansion on X if X' is not empty.

Remark 5.2. On $\widetilde{U}^g_{x_0}$, g acts on L by multiplication by $e^{i\theta}$, the action of g on $E_{\widetilde{U}^g_{x_0}}$ and on $\Lambda(T^{*(0,1)}X)$ is parallel with respect to the connections ∇^E and ∇^{Cliff}. We denote by $g|_{\Lambda\otimes E}$, $g|_E$ the action of g on $\Lambda(T^{*(0,1)}X) \otimes E$, E on $\widetilde{U}^g_{x_0}$. We define on $\widetilde{U}^g_{x_0}$

(5.28)
$$\begin{aligned}\psi_{r,q}(\widetilde{Z}_{1,g}) = \sum_{|\alpha|=q} \frac{1}{\alpha!} \left[\int_{N_{g,x_0}} g|_{\Lambda\otimes E} P^{(r)}_{\widetilde{Z}_{1,g}}(g^{-1}\widetilde{Z}_{2,g}, \widetilde{Z}_{2,g}) \widetilde{Z}^{\alpha}_{2,g} dv_N(\widetilde{Z}_{2,g})\right] \\ \cdot \left(\frac{\partial}{\partial \widetilde{Z}_{2,g}}\right)^{\alpha} (\kappa^{-1}_{\widetilde{Z}_{1,g}} \cdot).\end{aligned}$$

Then, $e^{i\theta p}\psi_{r,q}(\widetilde{Z}_{1,g})$ are a family of differential operators on $\widetilde{U}^g_{x_0}$ along the normal direction N_{g,x_0} with coefficients in $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)$, and they are well defined on $\widetilde{U}^g_{x_0}/Z_{G_{x_0}}(g)$ and on ΣX. By (4.117), (5.25), we know that in the sense of distributions,

(5.29)
$$\begin{aligned}&\frac{1}{p^n} B_p(x) \\ &= \sum_{r=0}^{k} p^{-r/2} \sum_{X_j \subset X\cup\Sigma X} \frac{1}{m_j} p^{-n+\dim X_j} e^{i\theta_j p} \delta_{X_j} \sum_{q\ge 0} p^{-\frac{q}{2}} \psi_{r,q} + \mathscr{O}(p^{-k}).\end{aligned}$$

Here, X_j runs over all the connected component of $X\cup\Sigma X$ and g acts on $L|_{X_j}$ as multiplication by $e^{i\theta_j}$, and m_j is the multiplicity of X_j defined in [**23**] (cf. also [**27**]).

Especially, if $\Sigma X = \{y_j\}$ is finite points, then $m_j = |G_{y_j}|$ and $g|_{\Lambda(T^{*(0,1)}X)\otimes E} \circ I_{\mathbb{C}\otimes E} = g|_E \circ I_{\mathbb{C}\otimes E}$. Moreover, as g commutes with

$\mathcal{J}_{x_0}$, from (4.114), for $Z = z + \overline{z}$,

(5.30)
$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} P(g^{-1}Z, Z)dZ \\
&= \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \exp\Big(-\frac{1}{4}||\mathcal{J}_{x_0}|^{\frac{1}{2}}(g^{-1}-1)Z|^2 + \frac{1}{2}\left\langle \mathcal{J}_{x_0} g^{-1}Z, Z\right\rangle \Big) dZ \\
&= \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \exp\Big(-\frac{1}{2}\left\langle |\mathcal{J}_{x_0}|Z, Z\right\rangle + \frac{1}{2}\left\langle (|\mathcal{J}_{x_0}| + \mathcal{J}_{x_0}) g^{-1}Z, Z\right\rangle \Big) dZ \\
&= \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \exp\Big(-\left\langle \mathcal{J}_{x_0} z, \overline{z}\right\rangle + \left\langle \mathcal{J}_{x_0} g^{-1}z, \overline{z}\right\rangle \Big) dZ \\
&= \frac{1}{\det_{\mathbb{C}}(1 - g^{-1}_{T^{(1,0)}X})}.
\end{aligned}$$

Thus from (4.117), (5.29), (5.30),

(5.31)
$$B_p(x) = \sum_{r=0}^{n} b_r(x) p^{n-r} + \sum_{y_j} \frac{e^{i\theta_j p} g|_E \circ I_{\mathbb{C}\otimes E}}{|G_{y_j}| \det_{\mathbb{C}}(1 - g^{-1}_{T^{(1,0)}X})} \delta_{y_j} + \mathscr{O}\left(\frac{1}{p}\right).$$

Remark 5.3. Now, assume that (X, ω) is a Kähler orbifold and $\mathbf{J} = J$, moreover L is an usual line bundle on X. Then, we can embed X into $\mathbb{P}(H^0(X, L^p)^*)$ by using the orbifold Kodaira embedding ϕ_p for p large enough (cf. [**1**, Section 7]). Let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbb{P}(H^0(X, L^p)^*)$ with canonical metric $h^{\mathcal{O}(1)}$. Then, $L^p = \phi_p^*\mathcal{O}(1)$ and $h^{L^p} = B_p(x)\phi_p^* h^{\mathcal{O}(1)}$. We can also interpret as following: Let $\{S_j\}_{j=1}^{d_p}$ be an orthonormal basis of $H^0(X, L^p)$ with respect to (3.2), then it induces an identification $H^0(X, L^p)^* \cong \mathbb{C}^{d_p}$; also, choose a local G_x-invariant holomorphic frame S_L (which is possible as G_x acts on L_x as identity) and write $S_j = f_j S_L^p$. Then $\phi_p : X \hookrightarrow \mathbb{P}(H^0(X, L^p)^*)$ is defined by $\phi_p(x) = [f_1(x), \ldots, f_{d_p}(x)]$. Let ω_{FS} be the Fubini–Study metric on $\mathbb{P}(H^0(X, L^p)^*)$. Then

$$\frac{1}{p}\phi_p^*\omega_{FS} = \frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log\left(\sum_{j=1}^{d_p}|f_j|^2\right) = \omega + \frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log B_p(x). \tag{5.32}$$

Note that $g \in G_{x_0}$ acts as identity on L_{x_0}, for $\widetilde{Z} = z + \overline{z}$, by (4.114),

$$\begin{aligned}
&(g,1)P_{\widetilde{Z}_{1,g}}(\sqrt{p}g^{-1}\widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g}) \\
&\quad = \exp\Big(-\frac{\pi}{2}p\left|(g^{-1}-1)\widetilde{Z}_{2,g}\right|^2 + \pi p\left\langle g^{-1}(z_{2,g} - \overline{z}_{2,g}), \widetilde{Z}_{2,g}\right\rangle \Big).
\end{aligned} \tag{5.33}$$

Set $\widetilde{b}_0(\widetilde{Z}) = 1 + \sum_{1\neq g\in G_{x_0}} (g,1)\, P_{\widetilde{Z}_{1,g}}(\sqrt{p}\, g^{-1}\, \widetilde{Z}_{2,g}, \sqrt{p}\widetilde{Z}_{2,g})\kappa_{\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})$. Then, $\widetilde{b}_0(\widetilde{Z})$ has a positive real part on $T_{x_0}X$. By (5.25), for $m \in \mathbb{N}$, taking $k \gg m$, then for p large enough, for $|Z| \leq \varepsilon/2$, under the norms $\mathscr{C}^m$,

(5.34)

$$\begin{aligned}
&\log\left(\frac{1}{p^n}B_p(Z)\right)\\
&= \log(\widetilde{b}_0(\widetilde{Z})) + \log\Bigg(1 + \sum_{r=1}^{k}\widetilde{b}_0(\widetilde{Z})^{-1}b_r(\widetilde{Z})p^{-r}\\
&\quad - \sum_{r=1}^{2k} p^{-\frac{r}{2}}\widetilde{b}_0(\widetilde{Z})^{-1}\sum_{1\neq g\in G_{x_0}}(g,1)P^{(r)}_{\widetilde{Z}_{1,g}}(\sqrt{p}g^{-1}\widetilde{Z}_{2,g},\sqrt{p}\widetilde{Z}_{2,g})\kappa_{\widetilde{Z}_{1,g}}(\widetilde{Z}_{2,g})\Bigg)\\
&\quad + \mathscr{O}(p^{-k+\frac{m}{2}}).
\end{aligned}$$

Thus, from (4.117), (5.32), (5.34), for any $l \in \mathbb{N}$, there exists $C_l > 0$ such that

$$\left|\frac{1}{p}\phi_p^*\omega_{FS}(x) - \omega(x)\right|_{\mathscr{C}^l} \leq C_l\left(\frac{1}{p} + p^{\frac{l}{2}}e^{-c\sqrt{p}d(x,X')}\right). \tag{5.35}$$

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