

## Characteristic Laplacian in Sub-Riemannian Geometry

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We study a Laplacian operator related to the characteristic cohomology of a smooth manifold endowed with a distribution. We prove that this Laplacian does not behave very well: it is not hypoelliptic in general and does not respect the bigrading on forms in a complex setting. We also discuss the consequences of these negative results for a conjecture of P. Griffiths, concerning the characteristic cohomology of period domains.

### 1 Introduction

Let  $X$  be a smooth manifold and denote by  $\Omega^\bullet(X)$  its differential graded algebra of smooth differential forms. Given a constant-rank distribution  $W$  on  $X$ , we consider the *Pfaffian system* associated to  $W$ , that is the graded differential ideal  $\mathcal{J}^\bullet$  generated by the smooth global sections of the annihilator of  $W$  in  $T^*X$ . Pfaffian systems constitute an important class of *exterior differential systems*, for which we refer the interested reader to [2].

For a differential map  $f$  from a smooth manifold  $Y$  to  $X$ , it is equivalent for the differential of  $f$  to have its values in the distribution  $W$  and for the pullback by  $f$  of any form in  $\mathcal{J}$  to vanish on  $Y$ ; we call such maps *solutions* of the Pfaffian system. Hence, it is reasonable to consider the complex  $\Omega^\bullet(X)/\mathcal{J}^\bullet$ , endowed with the differential induced by exterior differentiation on  $\Omega^\bullet(X)$ . This is well defined since  $\mathcal{J}$  is stable by exterior differentiation. We define the *characteristic cohomology* of  $(X, W)$  to be the cohomology

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of this complex and we denote it by  $H_{\mathcal{J}}^{\bullet}(X)$ . More generally, we will attach the adjective *characteristic* to the constructions related to the Pfaffian system. In [3, 4], this characteristic cohomology is intensively studied. Remark that if  $f: Y \rightarrow X$  is a solution of  $\mathcal{J}$ , then one has a map in cohomology

$$f^* : H_{\mathcal{J}}^{\bullet}(X) \rightarrow H^{\bullet}(Y).$$

In [12, Section III], the characteristic cohomology of period domains is the object of an interesting conjecture. Period domains  $D$  are homogeneous spaces  $G/V$ , where  $V$  is a compact subgroup of a Lie group  $G$ , encountered in the study of variations of Hodge structures. To every variation of Hodge structures over a complex manifold  $S$ , one can construct a holomorphic map from  $S$  to a quotient  $\Gamma \backslash D$ , where  $\Gamma$  is a discrete subgroup of  $G$ , and the well-known *Griffiths transversality condition* (a.k.a *infinitesimal period relation*; see, e.g., [6]) states that the differential of such a map has its values in some distribution  $W$  of  $\Gamma \backslash D$ , coming from a  $G$ -invariant distribution of  $D$ . The conjecture can be stated as follows.

**Conjecture 1.1.** Let  $\Gamma$  be a cocompact subgroup of  $G$ , acting freely on  $D = G/V$ . If the distribution  $W$  is bracket-generating and if  $m \leq m_0$ , where  $m_0$  is some integer determined by the Pfaffian system  $\mathcal{J}$  associated to  $W$ , then  $H_{\mathcal{J}}^m(\Gamma \backslash D)$  carries a natural pure real Hodge structure of weight  $m$ . □

A real pure Hodge structure of weight  $m$  is a real vector space  $E$  whose complexification  $E \otimes_{\mathbb{R}} \mathbb{C}$  carries a decomposition

$$E \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=m} E^{p,q},$$

where  $E^{p,q}$  are complex vector spaces satisfying  $E^{p,q} = \overline{E^{q,p}}$ . The prototype of real pure Hodge structure of weight  $m$  is the real cohomology in degree  $m$  of a compact Kähler manifold, as results from Hodge theory (especially the ellipticity of the Hodge Laplacian) and Kähler identities (see [13] or [15] for instance). One idea to study this conjecture is to develop an analog of Hodge theory in this *characteristic situation*. More precisely, following [11], one defines a Laplacian related to the Pfaffian system and one can try to prove that there is an isomorphism between its harmonic forms and the characteristic cohomology. In the complex setting, if the Laplacian respects the bigrading on forms, we get a Hodge structure on harmonic forms, hence on characteristic cohomology. However, we will see that the picture is not so bright and that Conjecture 1.1 certainly needs to be studied in another way.

In the first part, we construct this *characteristic Laplacian*. In [11, Section III. A], it is asserted that this characteristic Laplacian is hypoelliptic and that in this case we have an isomorphism of the characteristic cohomology with the space of harmonic forms. However, we explain why there seems to be no reason for the hypoellipticity of the characteristic Laplacian in general and, in Example 2.10, we give an explicit counterexample to the hypoellipticity in a complex setting.

**Proposition 1.2.** One can construct compact complex manifolds of dimension 3, endowed with a contact structure, for which the corresponding characteristic Laplacian is *not* hypoelliptic in degree 2.  $\square$

In particular, it seems difficult to understand the characteristic cohomology *via* harmonic forms. As remarked by one of the referees, this result would be much stronger if a Hodge-theoretic example was given. Look at Remark 2.13 for a discussion.

In the second part, we nevertheless study the characteristic Laplacian in more detail and answer the following question, asked in [11, Section III. A].

**Question 1.3.** Let  $(X, h)$  be a Hermitian manifold, endowed with a holomorphic distribution  $W$ . Let  $W_{\mathbb{R}}$  denote the underlying real distribution in  $TX$  and consider the Pfaffian system associated to  $W_{\mathbb{R}}$ . What are the necessary and sufficient conditions for the characteristic Laplacian to respect the bigrading on differential forms on  $X$ ?  $\square$

We give an unexpected answer to this question in Theorem 3.1.

**Theorem 1.4.** The characteristic Laplacian never respects the bigrading, in any positive degree, when the distribution  $W$  is not involutive (in particular, when it is bracket-generating, as in Conjecture 1.1).  $\square$

This is quite deceptive since it shows that there is nothing like a Kähler condition in the characteristic case. Indeed, in the classical case where the distribution  $W$  is the whole space  $TX$ , we show in Theorem 3.2:

**Theorem 1.5.** A complex Hermitian manifold is Kähler if and only if the Hodge Laplacian preserves the bigrading on differential forms on  $X$ .  $\square$

It seems that the necessity part was not written yet in the literature. This result and its proof are independent of the rest of the article.

In this paper, if  $E$  is a complex vector bundle on a manifold  $X$ , we will denote by  $E_{\mathbb{R}}$  its underlying real vector bundle. Moreover,  $\wedge$  and  $i$  are the exterior and interior products on  $\Omega^\bullet(X)$  and all distributions are assumed to be of constant rank.

If  $A: E \rightarrow E$  is an operator on some vector space endowed with an inner product, and if  $F \subset E$  is a vector subspace, we can define an operator  $A_F: F \rightarrow F$  by  $A_F = \pi_F \circ A$ , where  $\pi_F$  is the orthogonal projection from  $E$  onto  $F$ . We will often emphasize that this operator is defined only on  $F$  by writing  $A_F = \pi_F \circ A \circ \pi_F$ .

## 2 The Characteristic Laplacian

This section is organized as follows. In Section 2.1, we define the characteristic Laplacian associated with a distribution for Riemannian manifolds. In Section 2.2, we explain Taylor’s counterexample for the hypoellipticity of the characteristic Laplacian. In Section 2.3, we explain why the hypoellipticity does not seem to hold in general by computing its principal symbol. In Section 2.4, we give a counterexample for the hypoellipticity of the characteristic Laplacian in the complex setting which is the context of the original question.

### 2.1 Definitions and notations

Let  $(X, g^{TX})$  be a smooth compact Riemannian manifold, endowed with a (constant-rank) distribution  $W$ . We denote by  $F$  the annihilator of  $W$  in  $T^*X$ ; it is a vector subbundle of  $T^*X$ . We denote by  $\Omega^\bullet(X)$  the graded algebra of differential forms on  $X$  and we endow it with the natural metric  $g^{\Omega(X)}$  induced by  $g^{TX}$ . We consider

- $\mathcal{I}$  the algebraic ideal generated by the smooth sections  $I_X = \mathcal{C}^\infty(X, F)$ ;
- $\mathcal{J}$  the differential ideal generated by the smooth sections of  $F$  on  $X$ , that is the minimal algebraic ideal containing the smooth sections of  $F$  and stable by exterior differentiation.

Remark that  $\mathcal{J}$  is the algebraic ideal generated by  $I_X$  and  $dI_X$ . If  $(\theta_j)$  is a frame of  $F$ , the forms in  $\mathcal{I}$  can locally be written as

$$\sum_j \theta_j \wedge \phi_j,$$

where  $\phi_j$  are arbitrary forms on  $X$ , and those in  $\mathcal{J}$  are of the form

$$\sum_j \theta_j \wedge \phi_j + d\theta_j \wedge \psi_j,$$

where  $\phi_j, \psi_j$  are arbitrary forms on  $X$ .

Let  $\mathcal{Q}$  be the orthogonal complement of  $\mathcal{J}$  in  $\Omega(X)$ . Remark that  $\mathcal{Q}$  is naturally graded. We define a differential operator for forms in  $\mathcal{Q}$  by

$$d_{\mathcal{Q}} := \pi_{\mathcal{Q}} \circ d \circ \pi_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{Q}, \quad (2.1)$$

where  $\pi_{\mathcal{Q}}$  is the orthogonal projection from  $\Omega(X)$  onto  $\mathcal{Q}$ . Since  $\mathcal{J}$  is stable by  $d$ , we have

$$\pi_{\mathcal{Q}} \circ d \circ \pi_{\mathcal{Q}} = \pi_{\mathcal{Q}} \circ d : \Omega^{\bullet}(X) \rightarrow \Omega^{\bullet}(X). \quad (2.2)$$

As a direct consequence of (2.2), we know

$$d_{\mathcal{Q}}^2 = \pi_{\mathcal{Q}} \circ d \circ \pi_{\mathcal{Q}} \circ d \circ \pi_{\mathcal{Q}} = \pi_{\mathcal{Q}} \circ d^2 \circ \pi_{\mathcal{Q}} = 0. \quad (2.3)$$

Since  $\mathcal{Q}$  is the orthogonal complement of  $\mathcal{J}$  and  $\mathcal{J}$  is stable by  $d$ , we know that  $\mathcal{Q}$  is stable by the adjoint  $d^*$  of  $d$  and the restriction of  $d^*$  to  $\mathcal{Q}$  is  $d_{\mathcal{Q}}^*$ , the adjoint of  $d_{\mathcal{Q}}$  for the natural  $L^2$ -structure on  $\mathcal{Q}$ . Indeed, if  $\alpha$  (respectively,  $\beta$ ) is in  $\mathcal{Q}$  (respectively,  $\mathcal{J}$ ), then  $d\beta$  is in  $\mathcal{J}$  and this implies

$$(d^*\alpha, \beta) = (\alpha, d\beta) = 0.$$

Since this is true for any  $\beta$  in  $\mathcal{J}$ ,  $d^*\alpha$  is in  $\mathcal{Q}$ . Moreover, if  $\alpha$  and  $\beta$  are in  $\mathcal{Q}$ , then

$$(d^*\alpha, \beta) = (\alpha, d\beta) = (\alpha, d_{\mathcal{Q}}\beta) \quad (2.4)$$

and this shows that  $d_{\mathcal{Q}}^*\alpha = d^*\alpha$ .

**Definition 2.1.** The *characteristic Laplacian*  $\Delta_{\mathcal{Q}}$  on  $X$  with respect to  $W$  is the differential operator on  $\mathcal{Q}$

$$\Delta_{\mathcal{Q}} = d_{\mathcal{Q}}d_{\mathcal{Q}}^* + d_{\mathcal{Q}}^*d_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{Q}. \quad (2.5)$$

□

**Remark 2.2.** In sub-Riemmanian geometry (where the distribution  $W$  is involutive), one defines a sub-Laplacian on functions (see [16]). This sub-Laplacian is hypoelliptic and coincides with the characteristic Laplacian in degree 0. In Example 2.5, we will see that hypoellipticity can fail in positive degrees. □

Remember that we defined the characteristic cohomology of  $X$  (associated to the distribution  $W$ ) to be

$$H_{\mathcal{J}}^{\bullet}(X) := H^{\bullet}(\Omega^{\bullet}(X)/\mathcal{J}^{\bullet}, d), \quad (2.6)$$

with the differential induced by exterior differentiation on  $\Omega^{\bullet}(X)$ .

The characteristic cohomology of an exterior differential system was deeply studied in [3, 4]. More recently, some aspects of the characteristic cohomology of period

(or Mumford–Tate) domains have been treated in [17, 18]. In particular, it is explained in the last reference that, in low dimensions, characteristic and de Rham cohomologies coincide. Although written in a quite specific setting in [17], this is true in general, as will be explained in a forthcoming note of the first-named author [9].

By definition of  $\mathcal{Q}$  and  $d_{\mathcal{Q}}$ , this characteristic cohomology is naturally isomorphic to the cohomology of the complex  $(\mathcal{Q}^{\bullet}, d_{\mathcal{Q}})$ . An analog of Hodge theory would be the following conjecture.

**Conjecture 2.3.** We denote by  $\mathcal{H}_{\mathcal{Q}}^{\bullet}(X)$  the *characteristic harmonic forms*, that is the kernel of  $\Delta_{\mathcal{Q}}$ . Then

- $\mathcal{H}_{\mathcal{Q}}^{\bullet}(X)$  is of finite dimension;
- there is an orthogonal decomposition

$$\mathcal{Q}^{\bullet}(X) = \mathcal{H}_{\mathcal{Q}}^{\bullet}(X) \oplus d_{\mathcal{Q}}(\mathcal{Q}^{\bullet-1}(X)) \oplus d_{\mathcal{Q}}^*(\mathcal{Q}^{\bullet+1}(X)); \tag{2.7}$$

- the natural application  $\mathcal{H}_{\mathcal{Q}}^{\bullet}(X) \rightarrow H_{\mathcal{J}}^{\bullet}(X)$  is an isomorphism. □

In classical Hodge theory, one gets these results as consequences of the ellipticity of the Laplacian. In the next section, we show that the characteristic Laplacian is not even hypoelliptic in general. Hence we cannot really hope this conjecture to be true.

### 2.2 The question of hypoellipticity

We recall that if  $E$  and  $F$  are vector bundles over  $X$  and  $P : E \rightarrow F$  is a differential operator, then  $P$  is said to be *hypoelliptic* if the following condition is satisfied: for every local distribution  $u$  with values in  $E$ , if  $Pu$  is smooth on an open set  $U \subset X$ , then the restriction of  $u$  to  $U$  is smooth. Elliptic operators, like the usual Hodge Laplacian, are hypoelliptic. It is a natural question to ask whether  $\Delta_{\mathcal{Q}}$  is hypoelliptic since this would be the first step in order to prove Conjecture 2.3.

The best known sufficient condition for a second-order differential operator to be hypoelliptic is due to Hörmander ([14]).

**Theorem 2.4** (Sum of squares condition of hypoellipticity). Let  $P$  be a second-order differential operator from a vector bundle  $E$  to itself. Suppose that locally one can find smooth vector fields  $X_0, \dots, X_k$  and a smooth function  $c$  such that in a local frame of  $E$ ,

$$Pu = \left( \sum_{i=1}^k X_i^2 + X_0 + c \right) u$$

(in particular,  $P$  acts componentwise). Then  $P$  is hypoelliptic if and only if  $X_0, \dots, X_k$  generate  $TX$  by brackets.  $\square$

In [11], it is suggested that this theorem implies that the characteristic Laplacian is hypoelliptic when the distribution  $W$  is bracket-generating. We will first give a counterexample due to Taylor [19] and then compute the principal symbol of  $\Delta_Q$  in order to understand why the hypoellipticity certainly fails in general.

**Example 2.5.** A contact structure on a 3-manifold  $M$  is one of the simplest examples of Pfaffian systems. It is the datum of a 2-rank distribution  $W$  on  $M$  which is bracket-generating. For instance, one can take for  $W$  the kernel of the 1-form  $\theta = du - pdx$  in coordinates  $(x, u, p)$  in  $M = \mathbb{R}^3$ . A natural example in which  $M$  is compact is constructed as follows: we consider  $\mathcal{H}^3$  the 3D Heisenberg group, that is  $\mathbb{R}^3$  with coordinates  $(p, q, t)$  and group structure  $(p, q, t) \cdot (p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(pq' - p'q))$ . One checks that the 1-form  $\theta = dt - \frac{1}{2}qdp + \frac{1}{2}pdq$  is right-invariant and defines a contact structure on  $\mathcal{H}^3$ . Taking a cocompact discrete subgroup  $\Gamma$  of  $G$ ,  $M = G/\Gamma$  still carries the contact structure.

Locally, all contact structures on 3-manifolds are the same (however, the characteristic Laplacian also needs a Riemannian metric to be defined; see also Example 2.10 for this important issue). Let  $M$  be a 3D Riemannian manifold with a contact form  $\theta$  and the corresponding 2-rank distribution  $W$ . Let  $U$  be an open set in  $M$ . Then

- $\mathcal{J}^0(U) = 0$ ,
- $\mathcal{J}^1(U) = \mathcal{C}^\infty(U, \mathbb{R}\theta)$ ,
- $\mathcal{J}^2(U) = \Omega^2(U)$ ,  $\mathcal{J}^3(U) = \Omega^3(U)$ ,

and

- $\mathcal{Q}^0(U) = \mathcal{C}^\infty(U)$ ,
- $\mathcal{Q}^1(U) = \mathcal{C}^\infty(U, E)$ ,
- $\mathcal{Q}^2(U) = 0$ ,  $\mathcal{Q}^3(U) = 0$ ,

where  $E \rightarrow U$  is the real vector bundle of rank 2, which is the orthogonal of  $\mathbb{R}\theta$  in  $T^*\mathbb{R}^3$ . In degree 1, the characteristic Laplacian is well defined on forms with compact support. We have

$$\Delta_Q^1 = d_Q d_Q^* : \mathcal{C}_c^\infty(U, E) \rightarrow \mathcal{C}_c^\infty(U, E)$$

and one can consider it as a second-order differential operator on  $E$ . Let us denote by  $(X_1, X_2)$  a (smooth) orthonormal frame of  $W$  over  $\bar{U}$  and by  $(\alpha^1, \alpha^2)$  the dual frame of  $W^*$ .

Using the metric,  $W^*$  can be seen as a subbundle of  $T^*M$  and  $E$  can be identified with  $W^*$  (see also the following Section 2.3).

A form  $\mu \in L^2(U, E)$  can thus be written  $\mu = \mu_1\alpha^1 + \mu_2\alpha^2$ , with  $\mu_i \in L^2(U, \mathbb{R})$ , and  $d_{\mathcal{Q}}^*\mu$  is given by

$$d_{\mathcal{Q}}^*\mu = Y_1\mu_1 + Y_2\mu_2, \tag{2.8}$$

where  $Y_1, Y_2$  are first-order scalar differential operators on  $U$ .

In particular,

$$d_{\mathcal{Q}}^*(\mu_1\alpha^1) = Y_1\mu_1.$$

We claim that  $Y_1\mu_1$  can be smooth (even zero) without  $\mu_1$  being smooth. Indeed, as any first-order scalar differential operator,  $Y_1$  can be written  $Y_1 = X + f$ , with  $X$  a nonvanishing vector field and  $f$  a smooth function. Let  $(x_1, x_2, x_3)$  be some local system of centered coordinates, star-shaped in 0, such that  $X = \frac{\partial}{\partial x_1}$ ; then  $Y_1$  has the form

$$Y_1 = \frac{\partial}{\partial x_1} + f(x_1, x_2, x_3).$$

Consider the function

$$v(x_1, x_2, x_3) = \exp\left(\int_0^{x_1} f(t, x_2, x_3) dt\right) \mu_1(x_1, x_2, x_3),$$

which is smooth if and only if  $\mu_1$  is smooth. We compute that

$$\frac{\partial}{\partial x_1} v(x_1, x_2, x_3) = \exp\left(\int_0^{x_1} f(t, x_2, x_3) dt\right) Y_1\mu_1(x_1, x_2, x_3).$$

Choosing  $v$  independent of  $x_1$  but *not* smooth, one has  $Y_1\mu_1 = 0$ , proving the claim. □

**Remark 2.6.** In an analogous example occurring in the complex situation (Example 2.10), we will need to be more precise. In particular, formula (2.8) can be made explicit:

$$d_{\mathcal{Q}}^*\mu = -\operatorname{div}(\mu_1 X_1 + \mu_2 X_2), \tag{2.9}$$

where  $\operatorname{div}(X)$  is the divergence of the vector field  $X$ . □

In order to better understand why the hypoellipticity seems to fail, we will compute the principal symbol of  $\Delta_{\mathcal{Q}}$ . It will not show that  $\Delta_{\mathcal{Q}}$  is not hypoelliptic but it will at least show that the sum of squares condition cannot be applied, at least if one only considers the second-order terms.



2.3 The principal symbol of the characteristic Laplacian

Recall that we denote by  $F$  the annihilator of  $W$  in  $T^*X$ . Let  $N$  be the orthogonal complement of  $W$  in  $(TX, g^{TX})$ . Then, as a smooth vector bundle, we have

$$TX = W \oplus N, \quad \text{and} \quad T^*X = W^* \oplus N^*. \tag{2.10}$$

We can identify  $N^*$  and  $F$  as  $\mathcal{C}^\infty$  vector bundles, and

$$\mathcal{I} = \mathcal{C}^\infty(X, N^* \hat{\otimes} \Lambda(T^*X)). \tag{2.11}$$

The quotient  $\Omega(X)/\mathcal{I}$  can be identified with the orthogonal complement of  $\mathcal{I}$ , that is,

$$\Omega_W(X) := \mathcal{C}^\infty(X, \Lambda W^*). \tag{2.12}$$

In what follows, we do these identifications without further notice.

Since  $\mathcal{I} \subset \mathcal{J}$ ,  $\mathcal{Q} = \mathcal{J}^\perp$  can be viewed as a subspace of  $\Omega_W(X)$ . The orthogonal complement of  $\mathcal{Q}$  in  $\Omega_W(X)$  will be identified with  $\mathcal{J}/\mathcal{I}$ . We thus have the following decompositions:

$$\Omega(X) = \mathcal{I} \oplus \Omega_W(X) \tag{2.13}$$

and

$$\Omega_W(X) = \mathcal{J}/\mathcal{I} \oplus \mathcal{Q}. \tag{2.14}$$

All of these spaces are naturally graded. We define a map  $\varphi : F \rightarrow \Lambda^2(W^*)$  by: for  $\theta \in \mathcal{C}^\infty(X, F)$ ,  $v, w \in \mathcal{C}^\infty(X, W)$ , set

$$\varphi(\theta)(v, w) := (d\theta)(v, w) = -\theta([v, w]). \tag{2.15}$$

We check that, for any  $x \in X$ ,  $\varphi(\theta)(v, w)_x$  depends only on  $\theta_x$ ,  $v_x$ , and  $w_x$ .

The map  $\varphi : F \rightarrow \Lambda^2 W^*$  induces a map  $\varphi : F \hat{\otimes} \Lambda^k W^* \rightarrow \Lambda^{k+2} W^*$  for any  $k$ . We will assume that the rank of these maps is constant on  $X$  for any  $k$ . In particular,  $\varphi(F \hat{\otimes} \Lambda^k W^*)$  forms a vector subbundle of  $\Lambda^{k+2} W^*$  on  $X$  for any  $k$ . Set  $F_\varphi := \varphi(F \hat{\otimes} \Lambda W^*)$  and let  $F_{\varphi,\perp}$  be the orthogonal complement of  $F_\varphi$  in  $\Lambda W^*$  over  $X$ .

By construction, we thus have an orthogonal decomposition

$$\Lambda W^* = F_\varphi \oplus F_{\varphi,\perp} \tag{2.16}$$

and by (2.12) and (2.16), this decomposition induces (2.14), that is

$$\begin{aligned} \mathcal{J}/\mathcal{I} &= \mathcal{C}^\infty(X, F_\varphi), \\ \mathcal{Q} &= \mathcal{C}^\infty(X, F_{\varphi,\perp}). \end{aligned} \tag{2.17}$$

We denote by  $\pi_{F_\varphi}$  and  $\pi_{F_{\varphi,\perp}}$  the orthogonal projections from  $\Lambda W^*$  onto  $F_\varphi$  and  $F_{\varphi,\perp}$ . In order to make computations with the operators on  $\mathcal{Q}$ , we construct intermediate operators on  $\Lambda W^*$ . First, we define

$$d_W := \pi_W \circ d \circ \pi_W : \Lambda W^* \rightarrow \Lambda W^*, \tag{2.18}$$

where  $\pi_W$  is the projection from  $\Lambda(T^*X)$  on  $\Lambda W^*$  in the decomposition (2.13). Beware that there is no reason for  $d_W^2$  to be 0. The adjoint  $d_W^*$  of  $d_W$  satisfies

$$d_W^* = \pi_W \circ d^* \circ \pi_W. \tag{2.19}$$

From (2.1), (2.18), and (2.19), we have

$$\begin{aligned} d_{\mathcal{Q}} &= \pi_{\mathcal{Q}} \circ d_W \circ \pi_{\mathcal{Q}}, \\ d_{\mathcal{Q}}^* &= \pi_{\mathcal{Q}} \circ d_W^* \circ \pi_{\mathcal{Q}}. \end{aligned} \tag{2.20}$$

By (2.18) and (2.19),  $d_W$  and  $d_W^*$  are first-order differential operators on  $\Lambda W^*$  and their principal symbols are, for  $\xi \in T^*X$ ,

$$\begin{aligned} \sigma_1(d_W, \xi) &= \sqrt{-1} \xi_W \wedge, \\ \sigma_1(d_W^*, \xi) &= -\sqrt{-1} i_{\xi_W^*}, \end{aligned} \tag{2.21}$$

where  $\xi_W$  is the orthogonal projection of  $\xi$  on  $W^*$  and  $\xi_W^* \in W$  is the metric dual of  $\xi_W$ .

By (2.17) and (2.20),  $d_{\mathcal{Q}}$  and  $d_{\mathcal{Q}}^*$  are first-order differential operators on  $F_{\varphi,\perp}$  and their principal symbols are

$$\begin{aligned} \sigma_1(d_{\mathcal{Q}}, \xi) &= \sqrt{-1} \pi_{F_{\varphi,\perp}} \xi_W \wedge \pi_{F_{\varphi,\perp}}, \\ \sigma_1(d_{\mathcal{Q}}^*, \xi) &= -\sqrt{-1} \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \pi_{F_{\varphi,\perp}}. \end{aligned} \tag{2.22}$$

One also gets the adjoint formula of (2.2)

$$\pi_{\mathcal{Q}} \circ d^* \circ \pi_{\mathcal{Q}} = d^* \circ \pi_{\mathcal{Q}}. \tag{2.23}$$

Taking the principal symbols of (2.2) and (2.23), we have

$$\begin{aligned} \pi_{F_{\varphi,\perp}} \xi_W \wedge \pi_{F_{\varphi,\perp}} &= \pi_{F_{\varphi,\perp}} \xi_W \wedge, \\ \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \pi_{F_{\varphi,\perp}} &= i_{\xi_W^*} \pi_{F_{\varphi,\perp}}. \end{aligned} \tag{2.24}$$

**Proposition 2.7.** The characteristic Laplacian is a second-order differential operator on  $F_{\varphi,\perp}$  and its principal symbol is

$$\sigma_2(\Delta_{\mathcal{Q}}, \xi) = \pi_{F_{\varphi,\perp}}(|\xi_W|^2 - i_{\xi_W^*} \pi_{F_{\varphi}} \xi_W \wedge) \pi_{F_{\varphi,\perp}}. \tag{2.25}$$

□

**Remark 2.8.** The first term in (2.25) is the term suggested in [11] but the second term was forgotten. Because of this second term, one cannot apply Hörmander’s condition of hypoellipticity; see Example 2.9. □

**Proof of Proposition 2.7.** By (2.5), (2.22), and (2.24),  $\Delta_{\mathcal{Q}}$  is a second-order differential operator on  $F_{\varphi,\perp}$  and its principal symbol is

$$\begin{aligned} \sigma_2(\Delta_{\mathcal{Q}}, \xi) &= \pi_{F_{\varphi,\perp}} \xi_W \wedge \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \pi_{F_{\varphi,\perp}} + \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \pi_{F_{\varphi,\perp}} \xi_W \wedge \pi_{F_{\varphi,\perp}} \\ &= \pi_{F_{\varphi,\perp}} \xi_W \wedge i_{\xi_W^*} \pi_{F_{\varphi,\perp}} + \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \xi_W \wedge \pi_{F_{\varphi,\perp}} - \pi_{F_{\varphi,\perp}} i_{\xi_W^*} \pi_{F_{\varphi}} \xi_W \wedge \pi_{F_{\varphi,\perp}} \\ &= \pi_{F_{\varphi,\perp}} (|\xi_W|^2 - i_{\xi_W^*} \pi_{F_{\varphi}} \xi_W \wedge) \pi_{F_{\varphi,\perp}}. \end{aligned}$$

In the second equality, we use (2.24) and in the third, we use the identity

$$i_{\xi_W^*} \xi_W \wedge + \xi_W \wedge i_{\xi_W^*} = |\xi_W|^2.$$

The proof of Proposition 2.7 is completed. ■

**Example 2.9.** We review Example 2.5 and compute the principal symbol of the characteristic Laplacian in degree 1. With the identifications at the beginning of the paragraph,  $\mathcal{Q}^1$  is the space of sections of  $W^*$  and  $\mathcal{Q}^2$  is zero. Let  $\eta$  be in  $W^*$ . Since, in (2.25), only the projection  $\xi_W$  of  $\xi$  is involved, we can restrict the symbol to the  $\xi$  belonging to  $W^*$ . We have

$$\begin{aligned} \sigma_2(\Delta_{\mathcal{Q}}, \xi)\eta &= |\xi|^2 \eta - \pi_{F_{\varphi,\perp}} i_{\xi^*} \pi_{F_{\varphi}} (\xi \wedge \eta) \\ &= |\xi|^2 \eta - i_{\xi^*} (\xi \wedge \eta) \\ &= |\xi|^2 \eta - (|\xi|^2 \eta - \eta(\xi^*)\xi) \\ &= \eta(\xi^*)\xi. \end{aligned}$$

In the second equality, we use that  $F_{\varphi,\perp}$  (respectively,  $F_{\varphi}$ ) equals  $\Lambda W^*$  in degree 1 (respectively, 2).

If  $\xi = (\xi_1, \xi_2)^t$  and  $\eta = (\eta_1, \eta_2)^t$  in an orthonormal frame for  $W^*$ , we obtain

$$\sigma_2(\Delta_{\mathcal{Q}}, \xi) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = (\xi_1 \eta_1 + \xi_2 \eta_2) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1^2 \eta_1 + \xi_1 \xi_2 \eta_2 \\ \xi_1 \xi_2 \eta_1 + \xi_2^2 \eta_2 \end{pmatrix}.$$

Otherwise said, we have the equality

$$-\Delta_{\mathcal{Q}} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} X_1^2 \eta_1 + X_1 X_2 \eta_2 \\ X_1 X_2 \eta_1 + X_2^2 \eta_2 \end{pmatrix} + X_0 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

where  $(X_1, X_2)$  is a local frame of  $W$  and  $X_0$  is a first-order differential operator, which does not necessarily act componentwise. Since the second-order does not act componentwise, one cannot apply Hörmander’s condition of Theorem 2.4.  $\square$

### 2.4 The complex situation

In the remainder of the article, we will be interested in Pfaffian systems over a complex manifold. More precisely, let  $(X, J)$  be a compact complex manifold;  $J$  induces a splitting  $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$ , where  $T^{(1,0)}X$  and  $T^{(0,1)}X$  are the eigenbundles of  $J$  corresponding to the eigenvalues  $\sqrt{-1}$  and  $-\sqrt{-1}$ , respectively. Let  $T^{*(1,0)}X$  and  $T^{*(0,1)}X$  be the corresponding dual bundles.

We still denote by  $\Omega^k(X)$  the space of smooth  $k$ -forms on  $X$  with values in  $\mathbb{C}$ . Let

$$\Lambda^{p,q}(T^*X) = \Lambda^p(T^{*(1,0)}X) \otimes \Lambda^q(T^{*(0,1)}X), \quad \Omega^{p,q}(X) := \mathcal{C}^\infty(X, \Lambda^{p,q}(T^*X)). \tag{2.26}$$

Then  $\Omega^{p,q}(X)$  is the space of smooth  $(p, q)$ -forms on  $X$ , and  $\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$ .

Let  $\Theta$  be a real  $(1, 1)$ -form such that

$$g^{TX}(\cdot, \cdot) = \Theta(\cdot, J\cdot) \tag{2.27}$$

defines a Riemannian metric on  $TX$ . The triple  $(X, J, \Theta)$  is called a *complex Hermitian manifold*. If  $\Theta$  is a closed form, then the form  $\Theta$  is called a *Kähler form* on  $X$ .

We denote the holomorphic tangent bundle by  $T_hX$ . Let  $W \subset T_hX$  be a (constant-rank) holomorphic distribution. We consider the Pfaffian system associated to the distribution  $W_{\mathbb{R}}$ . Otherwise said, if we denote by  $F \subset T_h^*X$  the holomorphic annihilator of  $W$ , then the exterior differential system  $\mathcal{J}$  we consider is generated by  $F_{\mathbb{R}} \subset T^*X$ . Beware of the notations that differ from the real case. Remark that  $\mathcal{J}$  is not only  $d$ -stable: it is also  $\partial$  and  $\bar{\partial}$ -stable. Indeed, if  $I_X = \mathcal{C}^\infty(X, F)$  is the space of smooth sections of  $F$  on  $X$ , then  $\bar{I}_X = \mathcal{C}^\infty(X, \bar{F})$  and  $d$  acts on  $I_X$  (respectively,  $\bar{I}_X$ ) as  $\partial$  modulo  $I_X \cdot \Omega^{0,1}(X)$  (respectively,  $\bar{\partial}$  modulo  $\bar{I}_X \cdot \Omega^{1,0}(X)$ ).

Locally, if  $(\theta_j)$  is a holomorphic frame of  $F$ , the forms in  $\mathcal{I}$  can be written as

$$\sum_j \theta_j \wedge \psi_j + \bar{\theta}_j \wedge \phi_j,$$

where  $\psi_j, \phi_j$  are arbitrary smooth forms, and those in  $\mathcal{J}$  are of the form

$$\sum_j \theta_j \wedge \psi_j + \bar{\theta}_j \wedge \phi_j + d\theta_j \wedge \omega_j + d\bar{\theta}_j \wedge \chi_j,$$

where  $\psi_j, \phi_j, \omega_j, \chi_j$  are arbitrary smooth forms.

We still denote by  $\mathcal{Q}$  the orthogonal of  $\mathcal{J}$ . Besides the operator  $d_{\mathcal{Q}}$ , we also define  $\partial_{\mathcal{Q}}$  and  $\bar{\partial}_{\mathcal{Q}}$  by

$$\partial_{\mathcal{Q}} := \pi_{\mathcal{Q}} \circ \partial \circ \pi_{\mathcal{Q}}, \quad \bar{\partial}_{\mathcal{Q}} := \pi_{\mathcal{Q}} \circ \bar{\partial} \circ \pi_{\mathcal{Q}}. \tag{2.28}$$

Moreover, as in (2.4), the adjoints of  $\partial_{\mathcal{Q}}$  and  $\bar{\partial}_{\mathcal{Q}}$  (for the natural  $L^2$ -structure on  $\mathcal{Q}$ ) are the restrictions to  $\mathcal{Q}$  of  $\partial^*$ ,  $\bar{\partial}^*$  the adjoints of  $\partial$  and  $\bar{\partial}$ .

If we denote by  $N$  the orthogonal complement of  $W$  in  $(TX, g^{TX})$ , one obtains decompositions analogous to those in the previous subsection. In particular, denoting by  $\mathcal{I}$  the algebraic ideal generated by  $F_{\mathbb{R}}$ , one has the analogs of (2.11)–(2.14):

$$\begin{aligned} \mathcal{I} &= \mathcal{C}^\infty(X, N_{\mathbb{R}}^* \hat{\otimes} \Lambda(T^*X)), \\ \Omega_W(X) &:= \mathcal{C}^\infty(X, \Lambda W_{\mathbb{R}}^*), \\ \Omega(X) &= \mathcal{I} \oplus \Omega_W(X), \\ \Omega_W(X) &= \mathcal{J}/\mathcal{I} \oplus \mathcal{Q}. \end{aligned} \tag{2.29}$$

All of these spaces carry a natural bigrading. Denoting by  $\pi_W$  the orthogonal projection from  $\Lambda(T^*X)$  onto  $\Lambda W_{\mathbb{R}}^*$ , we define

$$\begin{aligned} \partial_W &:= \pi_W \circ \partial \circ \pi_W, & \bar{\partial}_W &:= \pi_W \circ \bar{\partial} \circ \pi_W, \\ \partial_W^* &= \pi_W \circ \partial^* \circ \pi_W, & \bar{\partial}_W^* &= \pi_W \circ \bar{\partial}^* \circ \pi_W. \end{aligned} \tag{2.30}$$

Then, we have, besides Equations (2.18), (2.19), and (2.20),

$$\begin{aligned} \partial_{\mathcal{Q}} &= \pi_{\mathcal{Q}} \circ \partial_W \circ \pi_{\mathcal{Q}}, & \bar{\partial}_{\mathcal{Q}} &= \pi_{\mathcal{Q}} \circ \bar{\partial}_W \circ \pi_{\mathcal{Q}}, \\ \partial_{\mathcal{Q}}^* &= \pi_{\mathcal{Q}} \circ \partial_W^* \circ \pi_{\mathcal{Q}}, & \bar{\partial}_{\mathcal{Q}}^* &= \pi_{\mathcal{Q}} \circ \bar{\partial}_W^* \circ \pi_{\mathcal{Q}}. \end{aligned} \tag{2.31}$$

We still define a map  $\varphi : F \rightarrow \Lambda^2 W^*$  as in (2.15). Since, in the definition (2.15), we can take  $\theta, v, w$  to be holomorphic sections, it proves that  $\varphi$  is a holomorphic map.

We still assume that the induced map  $\varphi : F \hat{\otimes} \Lambda W_{\mathbb{R}}^* \rightarrow \Lambda W_{\mathbb{R}}^*$  has constant rank. Set  $F_{\varphi} := \varphi(F \hat{\otimes} \Lambda W^*)$  and  $F_{\varphi, \perp}$  the orthogonal complement of  $F_{\varphi}$  in  $\Lambda W^*$  and let  $\pi_{F_{\varphi}}, \pi_{F_{\varphi, \perp}}$  be the orthogonal projections from  $\Lambda W^*$  onto  $F_{\varphi}, F_{\varphi, \perp}$ .

Then  $F_{\varphi} \hat{\otimes} \Lambda \bar{W}^* + \Lambda W^* \hat{\otimes} \bar{F}_{\varphi}$  and  $F_{\varphi, \perp} \hat{\otimes} \bar{F}_{\varphi, \perp}$  are vector subbundles of  $\Lambda W_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C}$  over  $X$ , and

$$\Lambda W_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = (F_{\varphi} \hat{\otimes} \Lambda \bar{W}^* + \Lambda W^* \hat{\otimes} \bar{F}_{\varphi}) \oplus (F_{\varphi, \perp} \hat{\otimes} \bar{F}_{\varphi, \perp}). \tag{2.32}$$

In this decomposition, we have, as in (2.17),

$$\begin{aligned} \mathcal{J}/\mathcal{I} &= \mathcal{C}^{\infty}(X, F_{\varphi} \hat{\otimes} \Lambda \bar{W}^* + \Lambda W^* \hat{\otimes} \bar{F}_{\varphi}), \\ \mathcal{Q} &= \mathcal{C}^{\infty}(X, F_{\varphi, \perp} \hat{\otimes} \bar{F}_{\varphi, \perp}). \end{aligned} \tag{2.33}$$

**Example 2.10.**

Example 2.5 can also be seen in the complex situation but an interesting phenomenon appears in degree 1. Consider the complex manifold  $M = \mathbb{C}^3$  with complex coordinates  $(x, u, p)$  and the holomorphic 1-form  $\theta = du - p dx$ . We denote by  $(\cdot, \cdot)$  a Hermitian metric on  $M$  and by  $|\cdot|$  the corresponding norm. Using the notations of this subsection,  $W$  is a holomorphic vector subbundle of  $T_h X$  of rank 2. Hence  $\Lambda^2 W^*$  is of rank 1 and we have  $F_{\varphi} = \Lambda^2 W^*$ . We thus obtain the following bidegree decomposition of  $\mathcal{Q}$  over an open set  $U$ :

- $\mathcal{Q}^{0,0}(U) = \mathcal{C}^{\infty}(U),$
- $\mathcal{Q}^{1,0}(U) = \mathcal{C}^{\infty}(U, W^*),$
- $\mathcal{Q}^{0,1}(U) = \mathcal{C}^{\infty}(U, \bar{W}^*),$
- $\mathcal{Q}^{1,1}(U) = \mathcal{C}^{\infty}(U, W^* \hat{\otimes} \bar{W}^*).$

Take  $(X_1, X_2)$  a holomorphic frame of  $W$  on  $\bar{U}$  and  $(\alpha^1, \alpha^2)$  its dual frame. We study the smoothness of harmonic forms for  $\Delta_{\mathcal{Q}}$  in degrees 0, 1, and 2.

For  $f \in L^2(U, \mathbb{C})$ , one has  $\Delta_{\mathcal{Q}} f = 0$  if and only if  $d_{\mathcal{Q}} f = 0$ . Using the frames, one obtains

$$d_{\mathcal{Q}} f = X_1(f) \alpha^1 + X_2(f) \alpha^2 + \bar{X}_1(f) \bar{\alpha}^1 + \bar{X}_2(f) \bar{\alpha}^2.$$

Since  $X_1, X_2, \bar{X}_1,$  and  $\bar{X}_2$  generate by brackets the tangent bundle of  $U$ , a function  $f$  such that  $d_{\mathcal{Q}} f = 0$  is in fact locally constant. Remark that the same argument works for any distribution  $W$  that is bracket-generating. This proves that the harmonic functions are smooth.

In degree 1, let  $\mu = \mu_1 \alpha^1 + \mu_2 \alpha^2$  be in  $L^2(U, W^*)$  (here  $\mu_1, \mu_2 \in L^2(U, \mathbb{C})$ ). Such a form is killed by  $\Delta_{\mathcal{Q}}^1$  if and only if it is killed by both  $d_{\mathcal{Q}}$  and  $d_{\mathcal{Q}}^*$ .

We denote by  $\operatorname{div}(Y)$  the divergence of a vector field  $Y$  and by  $dv_X$  the Riemannian volume form on  $(X, g^{TX})$ .

**Lemma 2.11.** The following identities hold:

$$d_{\mathcal{Q}}\mu = \bar{\partial}_{\mathcal{Q}}\mu = \sum_{i,j=1}^2 \bar{X}_j(\mu_i)\bar{\alpha}^j \wedge \alpha^i, \tag{2.34}$$

$$d_{\mathcal{Q}}^*\mu = -\operatorname{div}\left(\sum_{i=1}^2(\mu, \alpha^i)\bar{X}_i\right). \tag{2.35}$$

□

**Proof.** Since  $\mathcal{Q}^{2,0}(U) = 0$ , we know  $d_{\mathcal{Q}}\mu = \bar{\partial}_{\mathcal{Q}}\mu$ . The 2-form  $d\alpha^k$  satisfies

$$d\alpha^k(Y_1, Y_2) = Y_1(\alpha^k(Y_2)) - Y_2(\alpha^k(Y_1)) - \alpha^k([Y_1, Y_2]).$$

Since  $(\alpha^1, \alpha^2)$  is the dual basis of  $(X_1, X_2)$ , this simplifies to  $d\alpha^k(Y_1, Y_2) = -\alpha^k([Y_1, Y_2])$  when  $Y_i$  is either a  $X_i$  or a  $\bar{X}_j$ . Since the  $X_i$  are holomorphic, the brackets  $[X_i, \bar{X}_j]$  vanish and  $d\alpha^k(X_i, \bar{X}_j) = 0$ . This implies that

$$d_{\mathcal{Q}}\mu = \sum_{i=1}^2 \bar{\partial}_{\mathcal{Q}}\mu_i \alpha^i = \sum_{i,j=1}^2 \bar{X}_j(\mu_i)\bar{\alpha}^j \wedge \alpha^i.$$

For the second equality, one has, for every smooth function  $f$ ,

$$\begin{aligned} (d_{\mathcal{Q}}^*\mu, f) &= (\bar{\partial}_{\mathcal{Q}}^*\mu, f) \\ &= (\mu, \bar{\partial}_{\mathcal{Q}}f) \\ &= \sum_{i=1}^2 \int_M \overline{X_i(f)}(\mu, \alpha^i) dv_X \\ &= - \sum_{i=1}^2 \int_M \bar{f} \operatorname{div}((\mu, \alpha^i)\bar{X}_i) dv_X. \end{aligned}$$

Hence  $d_{\mathcal{Q}}^*\mu = -\operatorname{div}(\sum_{i=1}^2(\mu, \alpha^i)\bar{X}_i)$ . ■

Using Lemma 2.11, if  $\Delta_{\mathcal{Q}}^1\mu = 0$ , then  $\bar{X}_j(\mu_i) = 0$  for all  $i, j$  from 1 to 2. Since the  $\bar{X}_j$  generate by brackets the anti-holomorphic tangent bundle, this is equivalent to  $\mu_i$  being holomorphic in the weak sense. Since  $\bar{\partial}$  is an elliptic operator on functions, the  $\mu_i$  are holomorphic. This shows that the harmonic forms of bidegree  $(1, 0)$  (respectively,

(0, 1) are holomorphic (respectively, anti-holomorphic) sections of  $W^*$  (respectively,  $\bar{W}^*$ ). In particular, they are smooth, contrary to what happened in the real setting.

In degree 2, consider a form  $\nu = \sum_{i,j=1}^2 v_{ij} \bar{\alpha}^j \wedge \alpha^i$  in  $L^2(U, W^* \hat{\otimes} \bar{W}^*)$ . It is annihilated by  $\Delta_{\mathbb{Q}}^2$  if and only if  $d_{\mathbb{Q}}^* \nu = 0$ .

**Lemma 2.12.** The 2-form  $\nu$  is harmonic if and only if, for  $k = 1, 2$ ,

$$\begin{aligned} \operatorname{div} \left( \sum_{l=1}^2 (\nu, \bar{\alpha}^l \wedge \alpha^k) X_l \right) &= 0, \\ \operatorname{div} \left( \sum_{l=1}^2 (\nu, \bar{\alpha}^k \wedge \alpha^l) \bar{X}_l \right) &= 0. \end{aligned} \quad \square$$

**Proof.** We just show that the first equation is equivalent to the vanishing of  $\bar{d}_{\mathbb{Q}}^* \nu$ . For every (1, 0)-form  $\mu = \mu_1 \alpha^1 + \mu_2 \alpha^2$ , we obtain

$$\begin{aligned} (\bar{d}_{\mathbb{Q}}^* \nu, \mu) &= (\nu, \bar{d}_{\mathbb{Q}} \mu) \\ &= \left( \nu, \sum_{k,l=1}^2 \bar{X}_l (\mu_k) \bar{\alpha}^l \wedge \alpha^k \right) \text{ by Equation (2.34) in Lemma 2.11} \\ &= \sum_{k,l=1}^2 \int_M X_l (\bar{\mu}_k) (\nu, \bar{\alpha}^l \wedge \alpha^k) \, dv_X \\ &= - \sum_{k,l=1}^2 \int_M \bar{\mu}_k \operatorname{div}((\nu, \bar{\alpha}^l \wedge \alpha^k) X_l) \, dv_X. \end{aligned}$$

This is zero for all  $\mu$  if and only if  $\operatorname{div}(\sum_{l=1}^2 (\nu, \bar{\alpha}^l \wedge \alpha^k) X_l)$  is 0 for  $k = 1, 2$ . ■

It seems difficult to unravel these equations in general. We will only consider two different choices for the metric.

*Standard metric.* In the particular case where the metric on  $\mathbb{C}^3$  is the standard one, one can choose for  $X_1, X_2$  orthogonal holomorphic vectors with zero divergence (take  $X_1 = \frac{\partial}{\partial p}$  and  $X_2 = p \frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{x}}$ ). Moreover,  $X_1$  has norm 1. Then the equations of Lemma 2.12 become

$$\begin{aligned} \sum_l X_l (v_{lk} |\bar{\alpha}^l \wedge \alpha^k|^2) &= 0, \\ \sum_l \bar{X}_l (v_{kl} |\bar{\alpha}^k \wedge \alpha^l|^2) &= 0, \end{aligned}$$



where  $\nu = \nu_{lk}\bar{\alpha}^l \wedge \alpha^k$ . One can take  $\nu_{12} = \nu_{21} = \nu_{22} = 0$ . Then the equations are simply  $X_1\nu_{11} = \bar{X}_1\nu_{11} = 0$ . Since  $X_1$  is holomorphic,  $[X_1, \bar{X}_1] = 0$  and, by Frobenius theorem, one can choose  $\nu_{11}$  constant in the directions of  $X_1$  and  $\bar{X}_1$  but  $\nu_{11}$  not smooth. This shows the existence of nonsmooth harmonic 2-forms.

*Heisenberg metric.* Consider the case where we see  $\mathbb{C}^3$  as the complex Heisenberg group (see Example 2.5), endowed with a right-invariant Hermitian metric and with a right-invariant contact form. Choose a basis  $(X'_1, X'_2)$  of  $W$  at the identity  $e$  of  $\mathcal{H}^3$ . Consider the corresponding right-invariant vector fields on  $\mathcal{H}^3$ , denoted by  $\tilde{X}'_1$  and  $\tilde{X}'_2$ . Since these vector fields and the volume form are right-invariant, the divergences of  $\tilde{X}'_1$  and  $\tilde{X}'_2$  are constant. Hence, to certain linear combination  $X_1$  of  $X'_1$  and  $X'_2$  corresponds a right-invariant vector field with zero divergence. We can moreover assume that  $X_1$  is a unit vector and complete it to an orthonormal basis  $(X_1, X_2)$  of  $W_e$ . Thus, we get a holomorphic orthonormal frame  $(\tilde{X}_1, \tilde{X}_2)$  of  $W$  and  $\tilde{X}_1$  has zero divergence. Then, the same argument as above shows the existence of nonsmooth harmonic 2-forms.  $\square$

**Remark 2.13.** Still considering a contact system on a 3D complex manifold, there is another natural example to study. For more precisions, one can look at [5] or at the general reference [7] on period domains of (real) Hodge structures.

This example is one of the simplest situations encountered in Hodge theory. Let  $h$  be the nondegenerate Hermitian form of signature  $(2, 1)$  on the complex vector space  $\mathbb{C}^3$ , given by the diagonal matrix with diagonal entries  $1, -1, 1$ . We define a *complex Hodge structure* (of type  $(1, 1, 1)$ ) to be an  $h$ -orthogonal decomposition

$$\mathbb{C}^3 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

such that  $H^{i,2-i}$  is a complex line on which  $h$  is  $(-1)^i$  positive definite.

Writing  $(e_1, e_2, e_3)$  for the canonical basis of  $\mathbb{C}^3$ , then  $H^{2,0} = \mathbb{C}e_1$ ,  $H^{1,1} = \mathbb{C}e_2$ ,  $H^{0,2} = \mathbb{C}e_3$  is such a complex Hodge structure that we call the standard one. Around the standard one, the complex Hodge structures can be parameterized by three complex coordinates  $(x, u, p)$  in the following way: let  $F^2(x, u, p)$  be the line generated by the vector with coordinates  $(1, x, u)$  and let  $F^1(x, u, p)$  be the plane generated by  $F^2(x, u, p)$  and the vector with coordinates  $(0, 1, p)$ . Then, if  $x, u, p$  are sufficiently small,  $H^{2,0}(x, u, p) := F^2(x, u, p)$ ,  $H^{1,1} := F^2(x, u, p)^{\perp_{F^1(x,u,p)}}$ , and  $H^{0,2} := F^1(x, u, p)^{\perp_{\mathbb{C}^3}}$  define a complex Hodge structure, the orthogonals being taken with respect to  $h$ . In this way, the *period domain*  $\mathcal{D}$ , that is the set of all complex Hodge structures, is endowed with the structure of a complex manifold of dimension 3.

Moreover, the real Lie group  $U(\mathfrak{h}) \cong U(2, 1)$  naturally acts on  $\mathcal{D}$  and it can be shown that this action is transitive, with stabilizer  $V \cong U(1)^{\times 3}$  at the standard complex Hodge structure. Hence,  $\mathcal{D} \cong U(2, 1)/U(1)^{\times 3}$ . Since  $U(1)^{\times 3}$  is compact, one can endow  $\mathcal{D}$  with a  $U(2, 1)$ -invariant structure of Hermitian manifold.

Finally, we can also define a holomorphic distribution in the (holomorphic) tangent space of  $\mathcal{D}$ . Geometrically, this distribution gives constraints to the line  $H^{2,0}$ : more precisely, we want the so-called *infinitesimal period relation*  $dH^{2,0} \subset H^{2,0} \oplus H^{1,1} = F^1$  to be satisfied. By this, we mean that the derivative  $\dot{v}$  of a vector belonging to  $H^{2,0}$  has to be contained in  $F^1$ . In the coordinates  $(x, u, p)$ , this simply means that the values of the 1-form  $(0, dx, du)$  have to be in the vector space generated by  $(1, x, u)$  and  $(0, 1, p)$ , which is equivalent to the vanishing of the 1-form  $pdx - du$ . Hence, we find again a contact system, which is  $U(2, 1)$ -invariant.

In conclusion, we have defined a 3D complex manifold, with a Hermitian structure and a contact distribution. As before, we can ask whether the characteristic Laplacian is hypoelliptic in degree 2 in this situation. But the computations seem difficult to handle and we are not able to perform them at this moment. Our intuition is that there is no reason for the characteristic Laplacian to be hypoelliptic; but if it were, it would be very interesting to understand whether this is a general phenomenon coming from the Hodge-theoretic setting.  $\square$

### 3 Answer to Question 1.3

The aim of this section is to prove the following theorem, which is an answer to question 1.3.

**Theorem 3.1.** In the notations of Question 1.3 and Section 2, the characteristic Laplacian never respects the bigrading on  $\mathcal{Q}^\bullet$  when the distribution  $W$  is not involutive.  $\square$

This section is organized as follows. In Section 3.1, we show that a complex Hermitian manifold is Kähler if and only if the Hodge Laplacian preserves the bigrading on  $\Omega(X)$ . In Section 3.2, we establish a generalized sub-Kähler identity. In Section 3.3, we establish Theorem 3.1.

#### 3.1 The classical case

First, we study the case where the distribution  $W$  is the whole tangent space  $TX$ , which is interesting for itself. We thus have  $\mathcal{Q} = \Omega(X)$  and the characteristic Laplacian is the

usual Hodge Laplacian, which we simply denote by  $\Delta$ . Remark that Theorem 3.1 says nothing in this case.

It is well known that, for a Kähler manifold, its Hodge Laplacian preserves the bigrading of the differential forms (cf. [13, Section 0.7; 15, Corollary 1.4.13]). This implies the decomposition of the complex-valued de Rham cohomology in bidegree type for a compact Kähler manifold; this was in fact the initial interest of the authors for the general question 1.3. In [11, Section III. A], Green, Griffiths, and Kerr claimed that Chern [8] proved that, for Hermitian manifolds, if its Hodge Laplacian preserves the bigrading of the differential forms, then the Hermitian metric is Kähler. After communications with Professors Bryant and Griffiths, we realized that Chern did not claim this result in his paper [8], and it seems that one could not find a proof in the literature.

**Theorem 3.2.** The complex Hermitian manifold  $(X, J, \Theta)$  is Kähler if and only if  $\Delta$  preserves the bigrading on  $\Omega(X)$ , that is,  $\Delta$  sends  $(p, q)$ -forms to  $(p, q)$ -forms.  $\square$

We first introduce some notations from [15].

For any  $\mathbb{Z}_2$ -graded vector space  $V = V^+ \oplus V^-$ , the natural  $\mathbb{Z}_2$ -grading on  $\text{End}(V)$  is defined by

$$\text{End}(V)^+ = \text{End}(V^+) \oplus \text{End}(V^-), \quad \text{End}(V)^- = \text{Hom}(V^+, V^-) \oplus \text{Hom}(V^-, V^+),$$

and we define  $\text{deg } B = 0$  for  $B \in \text{End}(V)^+$ , and  $\text{deg } B = 1$  for  $B \in \text{End}(V)^-$ . For  $B, C \in \text{End}(V)$ , we define their supercommutator (or graded Lie bracket) by

$$[B, C] = BC - (-1)^{\text{deg } B \cdot \text{deg } C} CB. \tag{3.1}$$

Then, for  $B, B', C \in \text{End}(V)$ , the Jacobi identity holds:

$$\begin{aligned} &(-1)^{\text{deg } C \cdot \text{deg } B'} [B', [B, C]] + (-1)^{\text{deg } B' \cdot \text{deg } B} [B, [C, B']] \\ &+ (-1)^{\text{deg } B \cdot \text{deg } C} [C, [B', B]] = 0. \end{aligned} \tag{3.2}$$

We will apply the above notation for  $\Omega^\bullet(X)$  with natural  $\mathbb{Z}_2$ -grading induced by the parity of the degree (cf. [15, (1.3.31)]).

We define the Lefschetz operator  $L = \Theta \wedge$  on  $\Lambda^{\bullet, \bullet}(T^*X)$  and its adjoint  $\Lambda = i(\Theta)$  with respect to the Hermitian product  $\langle \cdot, \cdot \rangle_{\Lambda^{\bullet, \bullet}}$  induced by  $g^{TX}$ . For  $\{w_j\}_{j=1}^m$  a local orthonormal frame of  $T^{(1,0)}X$ , we have

$$L = \sqrt{-1} \sum_{j=1}^m w^j \wedge \bar{w}^j \wedge, \quad \Lambda = -\sqrt{-1} \sum_{j=1}^m i_{\bar{w}_j} i_{w_j}, \tag{3.3}$$

where  $\wedge$  and  $i$  denote the exterior and interior product, respectively. The *Hermitian torsion operator* is defined by

$$\mathcal{T} := [\Lambda, \partial\Theta] = [i(\Theta), \partial\Theta]. \tag{3.4}$$

**Proof of Theorem 3.2.** If  $(X, J, \Theta)$  is Kähler, then it is a classical result that  $\Delta$  preserves the bigrading on  $\Omega^\bullet(X)$ ; cf., for example, [15, Corollary 1.4.13] for a proof.

We assume now that  $\Delta$  preserves the bigrading on  $\Omega^\bullet(X)$ .

Let  $\bar{\square} := \partial\bar{\partial}^* + \bar{\partial}^*\partial$ ;  $\square := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$  be the usual  $\partial$ -Laplacian and  $\bar{\partial}$ -Laplacian. Then, as  $d = \partial + \bar{\partial}$  and  $d^2 = 0$ , we have (cf. [15, 1.4.50])

$$\Delta = [d, d^*] = [\partial + \bar{\partial}, \partial^* + \bar{\partial}^*] = \square + \bar{\square} + [\partial, \bar{\partial}^*] + [\bar{\partial}, \partial^*]. \tag{3.5}$$

As  $\square, \bar{\square}$  preserve the bigrading on  $\Omega^\bullet(X)$ , and  $[\partial, \bar{\partial}^*]: \Omega^{\bullet,\bullet}(X) \rightarrow \Omega^{\bullet+1,\bullet-1}(X)$ , we know that  $\Delta$  preserves the bigrading on  $\Omega^\bullet(X)$  if and only if

$$[\partial, \bar{\partial}^*] = 0. \tag{3.6}$$

By the generalized Kähler identities [15, (1.4.38d)] (cf. [10]) for  $E = \mathbb{C}$  therein, we obtain

$$[\Lambda, \partial] = \sqrt{-1}(\bar{\partial}^* + \bar{\mathcal{T}}^*). \tag{3.7}$$

From (3.7), we obtain

$$[\partial, \bar{\partial}^*] = -\sqrt{-1}[\partial, [\Lambda, \partial]] - [\partial, \bar{\mathcal{T}}^*]. \tag{3.8}$$

But by (3.2), we obtain

$$[\partial, [\Lambda, \partial]] = [\Lambda, [\partial, \partial]] + [\partial, [\partial, \Lambda]]. \tag{3.9}$$

As  $[\partial, \partial] = 2\partial^2 = 0$  and  $[\partial, \Lambda] = -[\Lambda, \partial]$ , we get from (3.9) that

$$[\partial, [\Lambda, \partial]] = 0. \tag{3.10}$$

From (3.8) and (3.10), we know that (3.6) is equivalent to

$$[\bar{\partial}^*, \mathcal{T}] = 0. \tag{3.11}$$

By [15, (1.4.9)], the operator  $\bar{\partial}^*$  has the form  $\bar{\partial}^* = -\sum_j i_{\bar{w}_j} \tilde{\nabla}_{w_j}^{TX} + 0$ -order terms; here  $\tilde{\nabla}^{TX}$  is a certain connection on  $\Lambda(T^*X)$ , thus  $[\bar{\partial}^*, \mathcal{T}]$  is a first-order differential operator, and

its principal symbol  $\sigma$  is as follows: for  $\xi \in T^*X$ ,

$$\sigma(\xi) = -\sqrt{-1} \sum_j (\xi, w_j) \cdot [i_{\bar{w}_j}, \mathcal{T}]. \tag{3.12}$$

By [15, Lemma 1.4.10, (1.2.48), (1.2.54)],

$$\mathcal{T} = -\frac{\sqrt{-1}}{2} \sum_{jkl} (\partial\Theta)(w_j, w_k, \bar{w}_l) [2w^k \wedge \bar{w}^l \wedge i_{\bar{w}_j} - 2\delta_{jl}w^k - w^j \wedge w^k \wedge i_{w_l}]. \tag{3.13}$$

From (3.13), we obtain

$$\begin{aligned} [i_{\bar{w}_i}, \mathcal{T}] &= \sqrt{-1} \sum_{jkl} (\partial\Theta)(w_j, w_k, \bar{w}_l) w^k \wedge [i_{\bar{w}_i}, \bar{w}^l] \wedge i_{\bar{w}_j} \\ &= \sqrt{-1} \sum_{jk} (\partial\Theta)(w_j, w_k, \bar{w}_i) w^k \wedge i_{\bar{w}_j}. \end{aligned} \tag{3.14}$$

By (3.12) and (3.14), Equation (3.11) implies that

$$\partial\Theta = 0. \tag{3.15}$$

Thus,  $\bar{\partial}\Theta = \partial\bar{\Theta} = 0$  and  $d\Theta = 0$ . This means that if  $\Delta$  preserves the bigrading on  $\Omega^\bullet(X)$ , then  $(X, J, \Theta)$  is Kähler. ■

**Remark 3.3.** After we sent our preliminary version to Professor Bryant, he sent us an easier proof which works also in the almost complex case. Here is the argument:

Let  $(X, J, \Theta)$  be an almost complex manifold with almost complex structure  $J$  and  $\Theta$  be a real  $(1, 1)$ -form as in (2.27). We suppose that the Hodge Laplacian  $\Delta$  preserves the bigrading. In fact, we may only suppose that  $\Delta$  sends  $(0, 1)$ -forms to  $(0, 1)$ -forms. In particular,  $\Delta$  commutes with  $J : TX \rightarrow TX$ . Using the following lemma, this implies that  $J$  is parallel with respect to the Levi-Civita connection  $\nabla^{TX}$  on  $(TX, g^{TX})$ . It is well known (cf. [15]) that this condition is equivalent to the metric being Kähler. □

**Lemma 3.4.** Let  $(X, g^{TX})$  be a Riemannian manifold and  $L \in \mathcal{C}^\infty(X, \text{End}(T^*X))$ . If  $L$  commutes with the Hodge Laplacian  $\Delta$  on 1-forms, then  $L$  is parallel with respect to the Levi-Civita connection  $\nabla^{TX}$ . □

**Proof.** Let  $\nabla^{\Lambda(T^*X)}$  (respectively,  $\nabla^{\text{End}(T^*X)}$ ) be the connection on  $\Lambda(T^*X)$  (respectively,  $\text{End}(T^*X)$ ) induced by the Levi-Civita connection  $\nabla^{TX}$  on  $(TX, g^{TX})$ .

For  $\nabla^F$  a connection on a vector bundle  $F$ , let  $\Delta^F$  be the Bochner Laplacian on  $F$  associated to  $\nabla^F$ . By definition, for  $\{e_j\}_j$  an orthonormal frame of  $(TX, g^{TX})$ , we have

$$\Delta^F = - \sum_j [(\nabla_{e_j}^F)^2 - \nabla_{\nabla_{e_j}^{TX} e_j}^F]. \tag{3.16}$$

As  $\nabla^{\Lambda(T^*X)}$  preserves the  $\mathbb{Z}$ -grading on  $\Lambda(T^*X)$ , we know that the Bochner Laplacian  $\Delta^{\Lambda(T^*X)}$  on  $\Lambda(T^*X)$  associated to  $\nabla^{\Lambda(T^*X)}$  also preserves the  $\mathbb{Z}$ -grading on  $\Omega(X)$ . One can relate  $\Delta$  and  $\Delta^{\Lambda(T^*X)}$  by the Weitzenböck formula (cf. [1, Section 3.6]). In particular, if  $\alpha \in \Omega^1(X)$ , one has the equality

$$\Delta\alpha = \Delta^{T^*X}\alpha + \text{Ric}\alpha, \tag{3.17}$$

where the Ricci curvature  $\text{Ric}$  is identified with a section of the bundle  $\text{End}(T^*X)$  by means of  $g^{TX}$ .

By (3.16) and (3.17), the principal symbol  $\sigma_2(\Delta)$  of  $\Delta$  is  $\sigma_2(\Delta)(\xi) = |\xi|^2 \text{Id}_{\Lambda(T^*X)}$  for  $\xi \in T^*X$ . Thus,  $\sigma_2(\Delta L - L\Delta) = 0$  and  $\Delta L - L\Delta$  is a first-order differential operator. We now compute the principal symbol  $\sigma_1(\Delta L - L\Delta)$  by computing  $\lim_{t \rightarrow \infty} t^{-1} e^{-itf} (\Delta L - L\Delta) e^{itf}$  when  $t \rightarrow +\infty$  for any  $f \in \mathcal{C}^\infty(X)$ . By (3.16) and (3.17), we know, for any  $s \in \mathcal{C}^\infty(X, T^*X)$ ,

$$\sigma_1(\Delta L - L\Delta)(df)s = \lim_{t \rightarrow \infty} t^{-1} e^{-itf} (\Delta L - L\Delta) e^{itf} s = -2i(\nabla_{e_j}^{\text{End}(T^*X)} L)e_j(f)s. \tag{3.18}$$

By assumption, one has

$$\Delta L - L\Delta = 0. \tag{3.19}$$

This implies  $\sigma_1(\Delta L - L\Delta) = 0$ . Thus from (3.18), we know (3.19) implies

$$\nabla^{\text{End}(T^*X)} L = 0. \tag{3.20}$$

The proof of Lemma 3.4 is completed. ■

**Remark 3.5.** In the first proof of Theorem 3.2, we use the generalized Kähler identity. When we began to clarify the situation of question 1.3, when there is a distribution, we computed an analog of the generalized Kähler identity in this case. This is the object of the following subsection, which is independent from Section 3.3. □

### 3.2 A generalized sub-Kähler identity

The Chern connection  $\nabla^{T_h X}$  on  $T_h X$  induces a connection on  $TX$  and on the bundle  $\Lambda^{\bullet, \bullet}(T^*X)$  ([15, §1.2.2]). This connection is denoted by  $\tilde{\nabla}^{TX}$ . In what follows, we identify

$T_h X$  with  $T^{(1,0)} X$  and thus we denote by  $\nabla^{T^{(1,0)} X}$  the connection  $\nabla^{T_h X}$  on  $T^{(1,0)} X$ . For  $v \in \mathcal{C}^\infty(X, T^{(0,1)} X)$ , we define  $\nabla^{T^{(0,1)} X} v = \nabla^{T^{(1,0)} X} \bar{v}$ . Then  $\tilde{\nabla}^{TX} = \nabla^{T^{(1,0)} X} \oplus \nabla^{T^{(0,1)} X}$ . Moreover, we denote by

$$T \in \Lambda^2(T^* X) \otimes TX$$

the torsion of  $\tilde{\nabla}^{TX}$ .

By identifying  $N$  in (2.10) to  $T_h X/W$ ,  $N$  gets a holomorphic structure from  $T_h X/W$ . Let  $\pi_N$  be the orthogonal projection from  $T_h X$  onto  $N$ . We denote by  $\langle \cdot, \cdot \rangle$  the  $\mathbb{C}$ -bilinear form on  $TX \otimes_{\mathbb{R}} \mathbb{C}$  induced by  $g^{TX}$ .

Let  $h^W, h^N$  be the Hermitian metrics on  $W, N$  induced by  $h^{T_h X}$ . Let  $\nabla^W, \nabla^N$  be the Chern connections on  $(W, h^W), (N, h^N)$ . Then we have

$$\nabla^W = \pi_W \nabla^{T_h X} \pi_W, \quad \nabla^N = \pi_N \nabla^{T_h X} \pi_N. \tag{3.21}$$

As  $W$  is a holomorphic subbundle, we know

$$A = \nabla^{T_h X''} - (\nabla^{W''} \oplus \nabla^{N''}) \in T^{*(0,1)} X \otimes \text{Hom}(N, W), \tag{3.22}$$

where  $\nabla''$  denotes the  $(0, 1)$ -part of a connection  $\nabla$ . The adjoint  $A^*$  of  $A$  takes values in  $T^{*(1,0)} X \otimes \text{Hom}(W, N)$ . Note that, for  $w \in W, v \in N, U \in TX \otimes_{\mathbb{R}} \mathbb{C}$ , we have

$$\langle A^*(U)w, \bar{v} \rangle = \langle w, \overline{A(\bar{U})(v)} \rangle. \tag{3.23}$$

Then, under the decomposition  $T_h X = W \oplus N$ , we have

$$\nabla^{T_h X} = \begin{pmatrix} \nabla^W & A \\ -A^* & \nabla^N \end{pmatrix}. \tag{3.24}$$

Let  $\tilde{\nabla}^W, \tilde{\nabla}^N$  be the connections on  $W_{\mathbb{R}}, N_{\mathbb{R}}$  induced by  $\nabla^W, \nabla^N$  as above or as in [15, (1.2.35)], respectively. Set

$$\oplus \nabla^{TX} = \tilde{\nabla}^W \oplus \tilde{\nabla}^N. \tag{3.25}$$

Let  $\tilde{\nabla}^W, \oplus \tilde{\nabla}^{TX}$  be the connections on  $\Lambda(W_{\mathbb{R}}^*), \Lambda(T^* X)$  induced by  $\nabla^W, \oplus \nabla^{TX}$  as in [15, Section 1.2.2], respectively.

Let  $\{w_j\}_{j=1}^m$  be an orthonormal frame of  $T^{(1,0)} X$  such that  $\{w_j\}_{j=1}^n$  is an orthonormal frame of  $W$ . Then, by [15, Lemma 1.4.4], we have

$$\partial = \sum_{j=1}^m w^j \wedge \tilde{\nabla}_{w_j}^{TX} + \frac{1}{2} \sum_{j,k,l=1}^m \langle T(w_j, w_k), \bar{w}_l \rangle w^j \wedge w^k \wedge \bar{w}_l, \tag{3.26}$$

and

$$\begin{aligned} \bar{\partial}^* &= -\sum_{j=1}^m i_{\bar{w}_j} \tilde{\nabla}_{w_j}^{TX} - \sum_{j,k=1}^m \langle T(w_j, w_k), \bar{w}_k \rangle i_{\bar{w}_j} \\ &\quad + \frac{1}{2} \sum_{j,k,l=1}^m \langle T(w_j, w_k), \bar{w}_l \rangle \bar{w}^l \wedge i_{\bar{w}_k} \wedge i_{\bar{w}_j}. \end{aligned} \tag{3.27}$$

Note that, for any  $1 \leq j, k \leq m, U \in TX$ , we have

$$\begin{aligned} \langle \tilde{\nabla}_U^{TX} w^k, w_j \rangle &= -(w^k, \tilde{\nabla}_U^{TX} w_j) = -\langle \bar{w}_k, \tilde{\nabla}_U^{TX} w_j \rangle = \langle \tilde{\nabla}_U^{TX} \bar{w}_k, w_j \rangle, \\ \langle \tilde{\nabla}_U^{TX} \bar{w}^k, \bar{w}_j \rangle &= \langle \tilde{\nabla}_U^{TX} w_k, \bar{w}_j \rangle. \end{aligned} \tag{3.28}$$

From (3.24), for  $1 \leq k \leq m, n+1 \leq \gamma \leq m$ , we obtain

$$\tilde{\nabla}_{w_k}^{TX} \bar{w}_\gamma = \oplus \nabla_{w_k}^{TX} \bar{w}_\gamma + \sum_{j=1}^n \langle \overline{A(\bar{w}_k)w_\gamma}, w_j \rangle \bar{w}_j. \tag{3.29}$$

From (3.24), (3.28), and (3.29), we know that, on  $\Lambda^{\bullet\bullet}(T^*X)$ ,

$$\begin{aligned} \tilde{\nabla}_{w_k}^{TX} &= \oplus \tilde{\nabla}_{w_k}^{TX} + \sum_{\beta=n+1}^m \sum_{j=1}^n [-\langle A^*(w_k)w_j, \bar{w}_\beta \rangle \bar{w}^\beta \wedge i_{\bar{w}_j} + \langle \overline{A(\bar{w}_k)w_\beta}, w_j \rangle w^j \wedge i_{w_\beta}] \\ &= \oplus \tilde{\nabla}_{w_k}^{TX} + \sum_{\beta=n+1}^m \sum_{j=1}^n \langle \overline{A(\bar{w}_k)w_\beta}, w_j \rangle (-\bar{w}^\beta \wedge i_{\bar{w}_j} + w^j \wedge i_{w_\beta}). \end{aligned} \tag{3.30}$$

By (2.30), (3.25), (3.26), and (3.30), we know that

$$\partial_W = \sum_{j=1}^n w^j \wedge \left( \tilde{\nabla}_{w_j}^W + \frac{1}{2} \sum_{k,l=1}^n \langle T(w_j, w_k), \bar{w}_l \rangle w^k \wedge i_{w_l} \right). \tag{3.31}$$

From (3.30), we obtain, for  $n+1 \leq \alpha \leq m$ ,

$$\pi_W i_{\bar{w}_\alpha} \tilde{\nabla}_{w_\alpha}^{TX} \pi_W = -\sum_{j=1}^n \langle w_j, \overline{A(\bar{w}_\alpha)w_\alpha} \rangle i_{\bar{w}_j}. \tag{3.32}$$

From (2.30), (3.27), (3.30), and (3.32), we obtain

$$\begin{aligned} \bar{\partial}_W^* &= \sum_{j=1}^n \left( -i_{\bar{w}_j} \tilde{\nabla}_{w_j}^W - \sum_{k=1}^m \langle T(w_j, w_k), \bar{w}_k \rangle i_{\bar{w}_j} \right) \\ &\quad + \frac{1}{2} \sum_{j,k,l=1}^n \langle T(w_j, w_k), \bar{w}_l \rangle \bar{w}^l \wedge i_{\bar{w}_k} \wedge i_{\bar{w}_j} + \sum_{\alpha=n+1}^m \sum_{j=1}^n \langle w_j, \overline{A(\bar{w}_\alpha)w_\alpha} \rangle i_{\bar{w}_j} \end{aligned}$$



$$\begin{aligned}
 &= \sum_{j=1}^n i_{\bar{w}_j} \left\{ -\tilde{\nabla}_{w_j}^W + \frac{1}{2} \sum_{k,l=1}^n \langle T(w_j, w_k), \bar{w}_l \rangle i_{\bar{w}_k} \wedge \bar{w}^l \right. \\
 &\quad \left. + \sum_{\alpha=n+1}^m (\langle w_j, \overline{A(\bar{w}_\alpha)w_\alpha} \rangle - \langle T(w_j, w_\alpha), \bar{w}_\alpha \rangle) \right\}. \tag{3.33}
 \end{aligned}$$

We also generalize the definition of the operators  $L = \Theta \wedge$  and  $\Lambda$  its adjoint, by defining  $\Theta_W \in \Lambda^{1,1}(W_{\mathbb{R}}^*)$  as the restriction to  $\Lambda^{1,1}(W_{\mathbb{R}}^*)$  of  $\Theta$ . We thus get operators  $L_W$  and  $\Lambda_W$  on  $\Lambda^{\bullet,\bullet}(W_{\mathbb{R}}^*)$ . By (3.3), we have

$$L_W = \sqrt{-1} \sum_{j=1}^n w^j \wedge \bar{w}^j \wedge, \quad \Lambda_W = -\sqrt{-1} \sum_{j=1}^n i_{\bar{w}_j} i_{w_j}. \tag{3.34}$$

The Hermitian torsion operator is defined as in (3.4) and [15, (1.4.34)] by

$$\mathcal{T}_W := [\Lambda_W, \partial_W \Theta_W]. \tag{3.35}$$

We have the analog of [15, Theorem 1.4.11].

**Proposition 3.6** (Generalized sub-Kähler identity).

$$\begin{aligned}
 [\Lambda_W, \partial_W] &= \sqrt{-1} (\bar{\partial}_W^* + \tilde{T}_W^*) \\
 &\quad - \sqrt{-1} \sum_{\alpha=n+1}^m \sum_{j=1}^n (\langle w_j, \overline{A(\bar{w}_\alpha)w_\alpha} \rangle - \langle T(w_j, w_\alpha), \bar{w}_\alpha \rangle) i_{\bar{w}_j} \pi_W. \tag{3.36}
 \end{aligned}$$

□

**Proof.** Set  $\pi_{W,\perp} = \text{Id} - \pi_W$ . By (3.7), we have

$$\pi_W [\Lambda, \partial] \pi_W = \sqrt{-1} (\bar{\partial}_W^* + \pi_W \tilde{T}^* \pi_W). \tag{3.37}$$

Note that

$$\pi_W \Lambda \pi_W = \Lambda \pi_W. \tag{3.38}$$

From (3.38), we know

$$\begin{aligned}
 \pi_W [\Lambda, \partial] \pi_W &= \pi_W \Lambda \pi_W \partial \pi_W + \pi_W \Lambda \pi_{W,\perp} \partial \pi_W - \pi_W \partial \pi_W \Lambda \pi_W \\
 &= [\Lambda_W, \partial_W] + \pi_W \Lambda \pi_{W,\perp} \partial \pi_W. \tag{3.39}
 \end{aligned}$$

By (3.26) and (3.30), we have

$$\begin{aligned} \pi_W \Lambda \pi_{W,\perp} \partial \pi_W &= -\sqrt{-1} \pi_W \sum_{\gamma=n+1}^m i_{\bar{w}_\gamma} i_{w_\gamma} \pi_{W,\perp} \partial \pi_W \\ &= \sqrt{-1} \sum_{\alpha=n+1}^m \sum_{j=1}^n \langle w_j, \overline{A(\bar{w}_\alpha) w_\alpha} \rangle i_{\bar{w}_j} \pi_W. \end{aligned} \tag{3.40}$$

By (3.3), (3.26), and (3.30), we have  $\partial_W \Theta_W = \pi_W \partial \Theta \pi_W$ . Thus, similarly to (3.39), we have

$$\pi_W [\Lambda, \partial \Theta] \pi_W = [\Lambda_W, \partial_W \Theta_W] + \pi_W \Lambda \pi_{W,\perp} \partial \Theta \pi_W. \tag{3.41}$$

By [15, (1.2.48), (1.2.54)], we have

$$\partial \Theta = \frac{\sqrt{-1}}{2} \sum_{i,j,k=1}^m \langle T(w_i, w_j), \bar{w}_k \rangle w^i \wedge w^j \wedge \bar{w}^k. \tag{3.42}$$

From (3.3) and (3.42), we know

$$\pi_W \Lambda \pi_{W,\perp} \partial \Theta \pi_W = - \sum_{\alpha=n+1}^m \sum_{j=1}^n \langle T(w_\alpha, w_j), \bar{w}_\alpha \rangle w^j \pi_W. \tag{3.43}$$

Taking the adjoint of (3.41), from (3.35) and (3.43), we know

$$\pi_W T^* \pi_W = T_W^* - \sum_{\alpha=n+1}^m \sum_{j=1}^n \langle T(\bar{w}_\alpha, \bar{w}_j), w_\alpha \rangle i_{w_j} \pi_W. \tag{3.44}$$

Thus

$$\pi_W \bar{T}^* \pi_W = \bar{T}_W^* - \sum_{\alpha=n+1}^m \sum_{j=1}^n \langle T(w_\alpha, w_j), \bar{w}_\alpha \rangle i_{\bar{w}_j} \pi_W. \tag{3.45}$$

Finally, from (3.37), (3.39), (3.40), and (3.45), we obtain

$$\begin{aligned} [\Lambda_W, \partial_W] &= \sqrt{-1} (\bar{\partial}_W^* + \bar{T}_W^*) \\ &\quad - \sqrt{-1} \sum_{\alpha=n+1}^m \sum_{j=1}^n (\langle w_j, \overline{A(\bar{w}_\alpha) w_\alpha} \rangle + \langle T(w_\alpha, w_j), \bar{w}_\alpha \rangle) i_{\bar{w}_j} \pi_W. \end{aligned} \tag{3.46}$$

From (3.46), we get (3.36). ■

**Remark 3.7.** Because of the presence of the double sum in (3.36), we do not know if there is a nice geometric interpretation for the vanishing of  $\mathcal{T}_W$ , except in the case where  $W = TX$ , when the vanishing is equivalent to the metric being Kähler, as explained in Section 3.1. □

3.3 The proof of Theorem 3.1

Remember the construction of the holomorphic map  $\varphi : F \hat{\otimes} \Lambda^\bullet W^* \rightarrow \Lambda^{\bullet+2} W^*$  in Section 2.4. There, we assumed for simplicity that this map has constant rank. For the purpose of Theorem 3.1, one can easily reduce to this case. Indeed, there exists an analytic subset  $V$  of  $X$  such that  $\varphi : F \hat{\otimes} \Lambda^k W^* \rightarrow \Lambda^{k+2} W^*$  has maximum rank on  $X \setminus V$  for any  $k$ . In particular,  $\varphi(F \hat{\otimes} \Lambda^k W^*)$  forms a vector subbundle of  $\Lambda^{k+2} W^*$  on  $X \setminus V$  for any  $k$ . We can define the vector bundles  $F_\varphi$  and  $F_{\varphi,\perp}$  on  $X \setminus V$  as before. On  $X \setminus V$ , we have the decompositions (2.32) and (2.33) for forms with compact support; in particular

$$\mathcal{Q} \cap \Omega_c^\bullet(X \setminus V) = \mathcal{C}_c^\infty(X \setminus V, F_{\varphi,\perp} \hat{\otimes} \bar{F}_{\varphi,\perp}). \tag{3.47}$$

As  $X \setminus V$  is an open connected dense subset of  $X$ , the characteristic Laplacian  $\Delta_{\mathcal{Q}}$  preserves the bigrading on  $\mathcal{Q}$  if and only if it preserves the bigrading on  $\mathcal{Q} \cap \Omega_c^\bullet(X \setminus V)$ . Thus, we can work on  $X \setminus V$  instead of  $X$ .

From the above discussion, in the rest, we will assume that  $\varphi(F \hat{\otimes} \Lambda^k W^*)$  forms a vector subbundle of  $\Lambda^{k+2} W^*$  on  $X$  for any  $k$ . Then we can use the formalism developed in Section 2.4.

By (2.1), (2.28), as in (3.5), we have

$$\Delta_{\mathcal{Q}} = [\partial_{\mathcal{Q}} + \bar{\partial}_{\mathcal{Q}}, \partial_{\mathcal{Q}}^* + \bar{\partial}_{\mathcal{Q}}^*] = \square_{\mathcal{Q}} + \bar{\square}_{\mathcal{Q}} + [\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*] + [\bar{\partial}_{\mathcal{Q}}, \partial_{\mathcal{Q}}^*]. \tag{3.48}$$

As  $\square_{\mathcal{Q}}, \bar{\square}_{\mathcal{Q}}$  preserve the bigrading on  $\mathcal{Q}$ , and  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*] : \mathcal{Q}^{\bullet,\bullet} \rightarrow \mathcal{Q}^{\bullet+1,\bullet-1}$ , we know that  $\Delta_{\mathcal{Q}}$  preserves the bigrading on  $\mathcal{Q}$  if and only if

$$[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*] = 0. \tag{3.49}$$

We would like to understand the operator  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]$ .

For  $f \in \mathcal{C}^\infty(X)$ , by (2.30), we have

$$\partial_W f = \sum_{j=1}^n w_j(f) w^j \in W^*, \tag{3.50}$$

where  $\{w_j\}_{j=1}^m$  is an orthonormal frame of  $T^{(1,0)} X$  such that  $\{w_j\}_{j=1}^n$  is an orthonormal frame of  $W$ . For  $\xi \in T_{\mathbb{R}}^* X$ , let  $\xi^* \in T_{\mathbb{R}} X$  be the metric dual of  $\xi$ . In particular, if  $\xi \in W^*$ , then  $\xi^* \in \bar{W}$ .

Since  $\mathcal{J}$  is stable by  $d, \partial, \bar{\partial}$ , as in (2.2), we have, as maps on  $\Omega(X)$ ,

$$\begin{aligned} \pi_{\mathcal{Q}} \circ \partial \circ \pi_{\mathcal{Q}} &= \pi_{\mathcal{Q}} \circ \partial, \\ \pi_{\mathcal{Q}} \circ \bar{\partial} \circ \pi_{\mathcal{Q}} &= \pi_{\mathcal{Q}} \circ \bar{\partial}, \quad \pi_{\mathcal{Q}} \circ \bar{\partial}^* \circ \pi_{\mathcal{Q}} = \bar{\partial}^* \circ \pi_{\mathcal{Q}}. \end{aligned} \tag{3.51}$$

Let  $h^{F_\varphi}, h^{F_{\varphi,\perp}}$  be, respectively, the Hermitian metrics on  $F_\varphi, F_{\varphi,\perp}$  induced by  $h^{\Lambda W^*}$  on  $\Lambda W^*$ , which is induced by  $h^W$ . We recall that  $F_\varphi$  is a holomorphic vector subbundle of  $\Lambda W^*$ . As in (3.21), let  $\nabla^{F_\varphi}, \nabla^{F_{\varphi,\perp}}$  be the Chern connections on  $(F_\varphi, h^{F_\varphi}), (F_{\varphi,\perp}, h^{F_{\varphi,\perp}})$ . Let  $\nabla^{\Lambda W^*}$  be the connection on  $\Lambda W^*$  induced by  $\nabla^W$ ; then  $\nabla^{\Lambda W^*}$  is the Chern connection on  $(\Lambda W^*, h^{\Lambda W^*})$ . Set

$$B = \nabla^{\Lambda W^*} - (\nabla^{F_\varphi} \oplus \nabla^{F_{\varphi,\perp}}) \in T^{*(0,1)}X \otimes \text{Hom}(F_{\varphi,\perp}, F_\varphi). \tag{3.52}$$

The adjoint  $B^*$  of  $B$  takes values in  $T^{*(1,0)}X \otimes \text{Hom}(F_\varphi, F_{\varphi,\perp})$ . Then, under the decomposition  $\Lambda W^* = F_\varphi \oplus F_{\varphi,\perp}$ , we have

$$\nabla^{\Lambda W^*} = \begin{pmatrix} \nabla^{F_\varphi} & B \\ -B^* & \nabla^{F_{\varphi,\perp}} \end{pmatrix}. \tag{3.53}$$

We also denote by  $\pi_Q$  the orthogonal projection from  $\Lambda W^* \hat{\otimes} \Lambda \bar{W}^*$  onto  $F_{\varphi,\perp} \hat{\otimes} \bar{F}_{\varphi,\perp}$ , and  $\pi_Q^\perp = \text{Id}_{\Lambda W^* \hat{\otimes} \Lambda \bar{W}^*} - \pi_Q$ . Then

$$\begin{aligned} \pi_Q &= \pi_{F_{\varphi,\perp}} \otimes \bar{\pi}_{F_{\varphi,\perp}}, \\ \pi_Q^\perp &= \pi_{F_\varphi} \otimes \bar{\pi}_{F_{\varphi,\perp}} + \pi_{F_{\varphi,\perp}} \otimes \bar{\pi}_{F_\varphi} + \pi_{F_\varphi} \otimes \bar{\pi}_{F_\varphi}. \end{aligned} \tag{3.54}$$

**Lemma 3.8.** The operator  $[\partial_Q, \bar{\partial}_Q^*]$  is a first-order differential operator acting on  $F_{\varphi,\perp} \hat{\otimes} \bar{F}_{\varphi,\perp}$ . Its principal symbol, evaluated on  $\xi \in T^*X$ , is  $\sqrt{-1}$  times

$$\begin{aligned} \pi_Q \sum_{j,k=1}^n i_{\bar{w}_j} w^k \wedge [(\xi_W, T(w_j, w_k)) - (\xi, \pi_N[w_j, w_k])] \pi_Q \\ + \sum_{j=1}^n (B^*(w_j) \pi_{F_\varphi} \xi_W) \hat{\otimes} i_{\bar{w}_j} - \pi_Q \sum_{j=1}^n w^j \hat{\otimes} i_{(\xi_W)^*} \overline{B(\bar{w}_j)}, \end{aligned} \tag{3.55}$$

where  $\xi_W$  is the orthogonal projection of  $\xi$  on  $W^*$  □

**Remark 3.9.** Theorem 3.1 is an easy corollary of (3.55). Indeed, if we take  $\xi$  a holomorphic 1-form that is orthogonal to  $W^*$ , the principal symbol is

$$-\sqrt{-1} \pi_Q \sum_{j,k=1}^n i_{\bar{w}_j} w^k \wedge (\xi, \pi_N[w_j, w_k]) \pi_Q.$$

Evaluating at  $\bar{w}^j$ , which belongs to  $F_{\varphi,\perp}$ , one obtains

$$\sqrt{-1} \sum_{k=1}^n (\xi, \pi_N[w_j, w_k]) w^k.$$

This term vanishes for any  $j$  and  $\xi$  if and only if the distribution is involutive, which shows Theorem 3.1.  $\square$

**Proof of Lemma 3.8.** Note that, for  $\xi \in W^*$ ,  $\psi \in F_{\varphi, \perp} \hat{\otimes} \bar{F}_{\varphi, \perp}$ ; by (2.16), (2.24), and (3.54), we know that  $\pi_{\mathcal{Q}}^{\perp}(\xi \wedge \psi) \in F_{\varphi} \hat{\otimes} \bar{F}_{\varphi, \perp}$ . Thus, by (2.24),  $i_{\xi^*} \pi_{\mathcal{Q}}^{\perp}(\xi \wedge \psi) \in F_{\varphi} \hat{\otimes} \bar{F}_{\varphi, \perp}$ , as  $\xi^* \in \bar{W}$ , and this implies

$$\pi_{\mathcal{Q}} i_{\xi^*} \pi_{\mathcal{Q}}^{\perp} \xi \wedge \pi_{\mathcal{Q}} = 0 \quad \text{for any } \xi \in W^*. \tag{3.56}$$

We compute now the principal symbol of  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]$  by computing the asymptotics of  $e^{-itf}[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]e^{itf}$  when  $t \rightarrow +\infty$  for  $f \in \mathcal{C}^{\infty}(X)$ . By (2.31), (3.31), and (3.33), we have first

$$\begin{aligned} e^{-itf} \partial_{\mathcal{Q}} e^{itf} &= it \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}} + \partial_{\mathcal{Q}}, \\ e^{-itf} \bar{\partial}_{\mathcal{Q}}^* e^{itf} &= -it \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}} + \bar{\partial}_{\mathcal{Q}}^*. \end{aligned} \tag{3.57}$$

Thus, from (2.24), (3.57), the principal symbol of  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]$  as a second-order differential operator is  $\lim_{t \rightarrow \infty} t^{-2} e^{-itf}[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]e^{itf}$ , that is

$$\begin{aligned} [i \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}}, -i \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}}] &= \pi_{\mathcal{Q}} \partial_W f \wedge i_{(\partial_W f)^*} \pi_{\mathcal{Q}} + \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}} \\ &= -\pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}} = 0; \end{aligned} \tag{3.58}$$

here we use (3.56) in the last equality. Equation (3.58) means that  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]$  is a first-order differential operator.

By (3.57), the principal symbol of  $[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]$  as a first-order differential operator is  $\lim_{t \rightarrow \infty} t^{-1} e^{-itf}[\partial_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*]e^{itf}$ , that is

$$i[\pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*] - i[\partial_{\mathcal{Q}}, \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}}]. \tag{3.59}$$

By (2.24) and (2.31), we obtain

$$\begin{aligned} [\pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}}, \bar{\partial}_{\mathcal{Q}}^*] &= \pi_{\mathcal{Q}} \partial_W f \wedge \bar{\partial}_W^* \pi_{\mathcal{Q}} + \pi_{\mathcal{Q}} \bar{\partial}_W^* \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}} \\ &= \pi_{\mathcal{Q}} \bar{\partial}_W^*(\partial_W f) \pi_{\mathcal{Q}} - \pi_{\mathcal{Q}} \bar{\partial}_W^* \pi_{\mathcal{Q}} \partial_W f \wedge \pi_{\mathcal{Q}}. \end{aligned} \tag{3.60}$$

Again by (2.24) and (2.31), we obtain

$$\begin{aligned} -[\partial_{\mathcal{Q}}, \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}}] &= -\pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}} \partial_W \pi_{\mathcal{Q}} - \pi_{\mathcal{Q}} \partial_W i_{(\partial_W f)^*} \pi_{\mathcal{Q}} \\ &= -\pi_{\mathcal{Q}} \partial_W (i_{(\partial_W f)^*}) \wedge \pi_{\mathcal{Q}} + \pi_{\mathcal{Q}} i_{(\partial_W f)^*} \pi_{\mathcal{Q}} \partial_W \pi_{\mathcal{Q}}. \end{aligned} \tag{3.61}$$

By (3.33) and (3.50), we obtain

$$\begin{aligned} \bar{\partial}_W^*(\partial_W f) &= [\bar{\partial}_W^*, \partial_W f] = - \left[ \sum_{j=1}^n i_{\bar{w}_j} \tilde{\nabla}_{w_j}^W, \partial_W f \right] \\ &= - \sum_{j,k=1}^n i_{\bar{w}_j} [w_j(w_k(f))w^k + w_k(f)\tilde{\nabla}_{w_j}^W w^k] \\ &= - \sum_{j,k=1}^n i_{\bar{w}_j} w^k [w_j(w_k(f)) - (\partial_W f, \nabla_{w_j}^W w_k)]. \end{aligned} \tag{3.62}$$

By (3.31) and (3.50), we obtain

$$\begin{aligned} -\partial_W(i_{(\partial_W f)^*}) &= -[\partial_W, i_{(\partial_W f)^*}] = - \left[ \sum_{k=1}^n w^k \tilde{\nabla}_{w_k}^W, i_{(\partial_W f)^*} \right] \\ &= - \sum_{j,k=1}^n w^k \wedge [w_k(w_j(f))i_{\bar{w}_j} + w_j(f)i_{\nabla_{w_k}^W \bar{w}_j}] \\ &= \sum_{j,k=1}^n i_{\bar{w}_j} w^k [w_k(w_j(f)) - (\partial_W f, \nabla_{w_k}^W w_j)]. \end{aligned} \tag{3.63}$$

By (3.24), (3.62), and (3.63), we obtain

$$\begin{aligned} \bar{\partial}_W^*(\partial_W f) - \partial_W(i_{(\partial_W f)^*}) &= \sum_{j,k=1}^n i_{\bar{w}_j} w^k [(\partial f, -[w_j, w_k]) + (\partial_W f, \nabla_{w_j}^W w_k - \nabla_{w_k}^W w_j)] \\ &= \sum_{j,k=1}^n i_{\bar{w}_j} w^k [(\partial_W f, T(w_j, w_k)) - (\partial f, \pi_N[w_j, w_k])]. \end{aligned} \tag{3.64}$$

From (3.59)–(3.64), we know that the principal symbol of the first-order differential operator  $[\partial_Q, \bar{\partial}_Q^*]$  is  $\sqrt{-1}$  times

$$\begin{aligned} &\pi_Q \sum_{j,k=1}^n i_{\bar{w}_j} w^k [(\partial_W f, T(w_j, w_k)) - (\partial f, \pi_N[w_j, w_k])] \pi_Q \\ &- \pi_Q \bar{\partial}_W^* \pi_Q^\perp \partial_W f \wedge \pi_Q + \pi_Q i_{(\partial_W f)^*} \pi_Q^\perp \partial_W \pi_Q. \end{aligned} \tag{3.65}$$

Now, by (3.54),  $\pi_{\mathbb{Q}}^{\perp} \partial_W f \wedge \pi_{\mathbb{Q}} \subset F_{\varphi} \otimes \bar{F}_{\varphi, \perp}$ . By (2.24), (3.33), and (3.53), we know that

$$\begin{aligned} \pi_{\mathbb{Q}} \bar{\partial}_W^* \pi_{\mathbb{Q}}^{\perp} \partial_W f \wedge \pi_{\mathbb{Q}} &= \pi_{\mathbb{Q}} \sum_{j=1}^n (-i_{\bar{w}_j} \tilde{\nabla}_{w_j}^W) \pi_{\mathbb{Q}}^{\perp} \partial_W f \wedge \pi_{\mathbb{Q}} \\ &= \pi_{\mathbb{Q}} \sum_{j=1}^n (i_{\bar{w}_j} B^*(w_j) \otimes 1) \pi_{\mathbb{Q}}^{\perp} \partial_W f \wedge \pi_{\mathbb{Q}} = - \sum_{j=1}^n (B^*(w_j) \pi_{F_{\varphi}} \partial_W f \wedge) \hat{\otimes} i_{\bar{w}_j}. \end{aligned} \tag{3.66}$$

Let  $P$  be the orthogonal projection from  $\Lambda^{**}(W_{\mathbb{R}}^*)$  onto  $F_{\varphi, \perp} \otimes \bar{F}_{\varphi}$ . Note that  $\pi_{\mathbb{Q}}^{\perp} \partial_W \pi_{\mathbb{Q}} \subset F_{\varphi} \otimes \Lambda \bar{W}^* \oplus F_{\varphi, \perp} \otimes \bar{F}_{\varphi}$ , as  $i_{(\partial_W f)^*} F_{\varphi} \otimes \Lambda \bar{W}^* \subset F_{\varphi} \otimes \Lambda \bar{W}^*$ ; from (3.31), (3.53), and (3.54), we have also

$$\begin{aligned} \pi_{\mathbb{Q}} i_{(\partial_W f)^*} \pi_{\mathbb{Q}}^{\perp} \partial_W \pi_{\mathbb{Q}} &= \pi_{\mathbb{Q}} i_{(\partial_W f)^*} P \partial_W \pi_{\mathbb{Q}} \\ &= \pi_{\mathbb{Q}} i_{(\partial_W f)^*} P \sum_{j=1}^n w^j \wedge \tilde{\nabla}_{w_j}^W \pi_{\mathbb{Q}} = - \pi_{\mathbb{Q}} \sum_{j=1}^n w^j \hat{\otimes} i_{(\partial_W f)^*} \overline{B(\bar{w}_j)}. \end{aligned} \tag{3.67}$$

The proof of Lemma 3.8 is completed. ■

We finally give an explicit computation of this phenomenon, in the setting of Example 2.10, for the choice of the standard metric on  $\mathbb{C}^3$ , described after Lemma 2.12.

We start with a form  $\mu = \mu_1 \alpha^1 + \mu_2 \alpha^2$  in  $\mathcal{C}^{\infty}(U, W^*) = \mathcal{Q}^{1,0}(U)$ .

**Lemma 3.10.** The 1-form  $[\bar{\partial}_Q, \partial_Q^*] \mu$  is given by

$$[\bar{\partial}_Q, \partial_Q^*] \mu = \sum_{i,j=1}^2 \left( \frac{\bar{X}_j \cdot ((\bar{X}_i \cdot \mu_j)(\alpha^j \wedge \bar{\alpha}^i, \alpha^j \wedge \bar{\alpha}^i))}{(\alpha^i, \alpha^i)} - \bar{X}_i \cdot (\bar{X}_j \cdot (\mu_j(\alpha^j, \alpha^j))) \right) \bar{\alpha}^i. \tag{3.68}$$

□

**Proof.** By Equation (2.34), we have

$$\bar{\partial}_Q \mu = \sum_{i,j=1}^2 (\bar{X}_j \cdot \mu_i) \bar{\alpha}^j \wedge \alpha^i =: \sum_{i,j=1}^2 \beta_{ij} \bar{\alpha}^j \wedge \alpha^i = \beta. \tag{3.69}$$

Let us compute  $\partial_Q^* \beta$ . Let  $v = v_1 \bar{\alpha}^1 + v_2 \bar{\alpha}^2$  be in  $\mathcal{C}^{\infty}(U, \bar{W}^*) = \mathcal{Q}^{0,1}(U)$ . Then, as in (2.34),

$$\begin{aligned} (\partial_Q^* \beta, v) &= (\beta, \partial_Q v) \\ &= \left( \beta, \sum_{i,j=1}^2 (X_j \cdot v_i) \alpha^j \wedge \bar{\alpha}^i \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j=1}^2 \int_M (\bar{X}_j \cdot \bar{v}_i)(\beta, \alpha^j \wedge \bar{\alpha}^i) \, dv_X \\
 &= - \sum_{i,j=1}^2 \int_M \bar{v}_i \bar{X}_j \cdot (\beta, \alpha^j \wedge \bar{\alpha}^i) \, dv_X.
 \end{aligned} \tag{3.70}$$

In the last equality, we use that the  $\bar{X}_j$  have zero divergence, as explained after Lemma 2.12. This shows that  $\partial_Q^* \beta = \gamma_1 \bar{\alpha}^1 + \gamma_2 \bar{\alpha}^2$ , where  $\gamma_i$  satisfies

$$\begin{aligned}
 \gamma_i &= - \frac{\sum_{j=1}^2 \bar{X}_j \cdot (\beta, \alpha^j \wedge \bar{\alpha}^i)}{(\bar{\alpha}^i, \bar{\alpha}^i)} \\
 &= \sum_{j=1}^2 \frac{\bar{X}_j \cdot (\beta_{ji}(\alpha^j \wedge \bar{\alpha}^i, \alpha^j \wedge \bar{\alpha}^i))}{(\alpha^i, \alpha^i)}.
 \end{aligned} \tag{3.71}$$

Here, we use that the  $\alpha_i$  are orthogonal, since we consider the standard metric. Hence,

$$\partial_Q^* \bar{\partial}_Q \mu = \sum_{i,j=1}^2 \frac{\bar{X}_j \cdot ((\bar{X}_i \cdot \mu_j)(\alpha^j \wedge \bar{\alpha}^i, \alpha^j \wedge \bar{\alpha}^i))}{(\alpha^i, \alpha^i)} \bar{\alpha}^i. \tag{3.72}$$

On the other hand, since the  $X_i$  have zero divergence and by Equation (2.35), one has

$$\partial_Q^* \mu = - \sum_{j=1}^2 \bar{X}_j \cdot (\mu_j(\alpha^j, \alpha^j)). \tag{3.73}$$

Hence,

$$\bar{\partial}_Q \partial_Q^* \mu = - \sum_{i,j=1}^2 \bar{X}_i \cdot (\bar{X}_j \cdot (\mu_j(\alpha^j, \alpha^j))) \bar{\alpha}^i. \tag{3.74}$$

■

In order to simplify the notations, we write  $N_i := (\alpha^i, \alpha^i)$  and  $N_{ij} := (\alpha^j \wedge \bar{\alpha}^i, \alpha^j \wedge \bar{\alpha}^i)$ . In particular,  $N_{ij} = N_i \cdot N_j$ . Then,

$$\begin{aligned}
 [\bar{\partial}_Q, \partial_Q^*] \mu &= \sum_{i,j=1}^2 \left( \frac{\bar{X}_j \cdot ((\bar{X}_i \cdot \mu_j) N_{ij})}{N_i} - \bar{X}_i \cdot \bar{X}_j \cdot (\mu_j N_j) \right) \bar{\alpha}^i \\
 &= \sum_{i,j=1}^2 \left( [\bar{X}_j, \bar{X}_i] \cdot \mu_j \times N_j + (\bar{X}_i \cdot \mu_j) \left( \frac{\bar{X}_j \cdot N_{ji}}{N_i} - \bar{X}_j \cdot N_j \right) \right. \\
 &\quad \left. - (\bar{X}_j \cdot \mu_j)(\bar{X}_i \cdot N_j) - \mu_j \bar{X}_i \cdot \bar{X}_j \cdot N_j \right) \bar{\alpha}^i.
 \end{aligned} \tag{3.75}$$



Now, we take  $\mu_1 = 0$  and consider the term before  $\bar{\alpha}^1$ . We consider a function  $\mu_2$  that, at some fixed point  $z_0 \in U$  satisfies:  $\mu_2(z_0) = 0$ ,  $(\bar{X}_1 \cdot \mu_2)(z_0) = (\bar{X}_2 \cdot \mu_2)(z_0) = 0$  and  $([\bar{X}_2, \bar{X}_1] \cdot \mu_2)(z_0) \neq 0$ . Such a function exists since the bracket  $[\bar{X}_1, \bar{X}_2]$  is not contained in the vector subspace generated by  $\bar{X}_1$  and  $\bar{X}_2$ .

Then  $[\bar{\partial}_Q, \bar{\partial}_Q^*]\mu$  does not vanish at  $z_0$ , which proves that in this particular example the characteristic Laplacian cannot respect the bigrading.

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