SCIENCE CHINA
Mathematics

• ARTICLES •

June 2017 Vol. 60 No. 6: 1047–1056
doi: 10.1007/s11425-016-9047-6

Donaldson's Q-operators for symplectic manifolds

In memory of Professor LU QiKeng (1927–2015)

LU Wen[1], MA Xiaonan[2,*] & MARINESCU George[3]

[1]*School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan* 430074*, China;*
[2]*Institut de Mathématiques de Jussieu-Paris Rive Gauche, UFR de Mathématiques, Université Paris Diderot-Paris* 7*, Paris* 75205*, France;*
[3]*Mathematisches Institut, Univerisität zu Köln, Köln* 50939*, Germany*
Email: wlu@hust.edu.cn, xiaonan.ma@imj-prg.fr, gmarines@math.uni-koeln.de

Received December 8, 2016; accepted February 24, 2017; published online March 21, 2017

Abstract We prove an estimate for Donaldson's Q-operator on a prequantized compact symplectic manifold. This estimate is an ingredient in the recent result of Keller and Lejmi (2017) about a symplectic generalization of Donaldson's lower bound for the L^2-norm of the Hermitian scalar curvature.

Keywords Q-operator, quantization, symplectic manifold

MSC(2010) 53D50

Citation: Lu W, Ma X, Marinescu G. Donaldson's Q-operators for symplectic manifolds. Sci China Math, 2017, 60: 1047–1056, doi: 10.1007/s11425-016-9047-6

1 Introduction

The Q-operator is an integral operator whose kernel is the square norm of the Bergman kernel of a positive line bundle (see (1.8) and (1.9)). It was introduced by Donaldson [5] in order to find explicit numerical approximations of Kähler-Einstein metrics on projective manifolds, and have attracted much attention recently (see [1, 6, 8–10, 16]).

Using the full asymptotic expansion of the Bergman kernel [2], Liu and Ma [10, Theorem 0.1] verified a statement of Donaldson [5, Subsection 4.2] about the relation of the asymptotics of Q_{K_p} to the heat kernel. Such statement was needed for the convergence of the approximation procedure in [5]. In [6], Liu and Ma improved the statement to a $\mathscr{C}^m$-estimate for Q_{K_p} on Kähler manifolds, as a crucial step towards the result of [6] about the convergence of the balancing flow to the Calabi flow. This is a parabolic analogue of Donaldson's theorem relating balanced embeddings to metrics with constant scalar curvature [3]. Besides, such results also turn out to be important in Cao and Keller's work [1] on Calabi's problem.

The purpose of this paper is to extend the $\mathscr{C}^m$-estimates of the operators Q_{K_p} to the case of symplectic manifolds. This result, together with [11], plays an important role in the recent work of Keller and Lejmi [8] about a lower bound for the L^2-norm of the Hermitian scalar curvature. Such a lower bound was obtained in the Kähler case by Donaldson [4]. Our proof is based on the asymptotic expansion of the (generalized) Bergman kernel, which in our case is the kernel of the spectral projection on lower

*Corresponding author

lying eigenstates of the normalized Bochner Laplacian. We refer the readers to the monograph [14] (see also [12, 15]) for more information on the Bergman kernel on symplectic manifolds.

Let us describe our result in detail. Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Let (L,h^L) be an Hermitian line bundle on X, and let ∇^L be an Hermitian connection on (L,h^L) with curvature $R^L=(\nabla^L)^2$. Let (E,h^E) be an auxiliary Hermitian vector bundle with Hermitian connection ∇^E. We assume throughout the paper that (L,h^L) satisfies the pre-quantization condition

$$\frac{\sqrt{-1}}{2\pi}R^L=\omega. \tag{1.1}$$

We choose an almost complex structure J on TX (i.e., $J\in\mathrm{End}(TX)$ and $J^2=-1$) such that ω is J-invariant and $\omega(\cdot,J\cdot)>0$. The almost complex structure J induces a splitting

$$TX\otimes_{\mathbb{R}}\mathbb{C}=T^{(1,0)}X\oplus T^{(0,1)}X,$$

where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $g^{TX}(\cdot,\cdot):=\omega(\cdot,J\cdot)$ be the Riemannian metric on TX induced by ω and J. The Riemannian volume form dv_X of (X,g^{TX}) has the form $dv_X=\omega^n/n!$. We denote by $L^p:=L^{\otimes p}$ the tensor powers of L for $p\in\mathbb{N}$ and by

$$h^{L^p}:=(h^L)^{\otimes p},\quad h^{L^p\otimes E}=h^{L^p}\otimes h^E,$$

the induced Hermitian metrics on L^p and $L^p\otimes E$, respectively. The L^2-Hermitian product on the space $\mathscr{C}^\infty(X,L^p\otimes E)$ of smooth sections of $L^p\otimes E$ on X is given by

$$\langle s_1,s_2\rangle=\int_X\langle s_1(x),s_2(x)\rangle_{h^{L^p\otimes E}}dv_X(x). \tag{1.2}$$

Let ∇^{TX} be the Levi-Civita connection on (X,g^{TX}), and let $\nabla^{L^p\otimes E}$ be the connection on $L^p\otimes E$ induced by ∇^L and ∇^E. Let $\{e_k\}$ be a local orthonormal frame of (TX,g^{TX}). The Bochner Laplacian acting on $\mathscr{C}^\infty(X,L^p\otimes E)$ is given by

$$\Delta^{L^p\otimes E}=-\sum_k[(\nabla^{L^p\otimes E}_{e_k})^2-\nabla^{L^p\otimes E}_{\nabla^{TX}_{e_k}e_k}]. \tag{1.3}$$

Let $\Phi\in\mathscr{C}^\infty(X,\mathrm{End}(E))$ be Hermitian (i.e., self-adjoint with respect to h^E). The renormalized Bochner Laplacian is defined by

$$\Delta_{p,\Phi}=\Delta^{L^p\otimes E}-2\pi np+\Phi. \tag{1.4}$$

By [7] and [13, Corollary 1.2], there exists $C_L>0$ independent of p such that

$$\mathrm{Spec}(\Delta_{p,\Phi})\subset[-C_L,C_L]\cup[4\pi p-C_L,+\infty), \tag{1.5}$$

where $\mathrm{Spec}(A)$ denotes the spectrum of the operator A. Since $\Delta_{p,\Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let

$$\mathcal{H}_p:=\bigoplus_{\lambda\in[-C_L,C_L]}\mathrm{Ker}(\Delta_{p,\Phi}-\lambda)\subset\mathscr{C}^\infty(X,L^p\otimes E) \tag{1.6}$$

be the direct sum of eigenspaces of $\Delta_{p,\Phi}$ corresponding to the eigenvalues lying in $[-C_L,C_L]$. In mathematical physics terms, the operator $\Delta_{p,\Phi}$ is a semiclassical Schrödinger operator and the space $\mathcal{H}_p$ is the space of its bound states as $p\to\infty$. By [14, Theorem 8.3.1],

$$\dim\mathcal{H}_p=\int_X\mathrm{Td}(T^{(1,0)}X)\mathrm{ch}(L^p\otimes E), \tag{1.7}$$

where $\mathrm{Td}(\cdot)$ and $\mathrm{ch}(\cdot)$ denote the Todd class and the Chern character of the corresponding complex vector bundle. The formula (1.7) agrees with the Riemann-Roch-Hirzebruch theorem and Kodaira vanishing theorem in the Kähler case. The space $\mathcal{H}_p$ proves to be an appropriate replacement for the space of holomorphic sections $H^0(X, L^p \otimes E)$ from the Kähler case.

Let $P_{\mathcal{H}_p}$ be the orthogonal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $\mathcal{H}_p$. The kernel $P_{\mathcal{H}_p}(x, x')$ of $P_{\mathcal{H}_p}$ with respect to $dv_X(x')$ is called a generalized Bergman kernel [15]. Note that

$$P_{\mathcal{H}_p}(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'}.$$

Set

$$\mathrm{Vol}(X, dv_X) = \int_X dv_X.$$

Following Donaldson [5, Section 4], we set

$$K_p(x, x') = |P_{\mathcal{H}_p}(x, x')|^2, \quad R_p := (\dim \mathcal{H}_p)/\mathrm{Vol}(X, dv_X). \tag{1.8}$$

Let K_p and Q_{K_p} be the integral operators associated to K_p which is defined by for $f \in \mathscr{C}^\infty(X)$,

$$(K_p f)(x) = \int_X K_p(x, y) f(y) dv_X(y), \quad Q_{K_p} = \frac{1}{R_p} K_p f. \tag{1.9}$$

The operator Q_{K_p} has been studied by Donaldson [5], Liu and Ma [6, Appendix; 10], and Ma and Marinescu [16, Section 6] in the case of Kähler manifolds.

The main result of this paper is as follows. For Kähler manifolds it was obtained by Liu and Ma [6, Appendix; 10].

Theorem 1.1. *For any integer $m \geqslant 0$, there exists a constant $C > 0$ such that for any $f \in \mathscr{C}^\infty(X)$,*

$$\|Q_{K_p}(f) - f\|_{\mathscr{C}^m(X)} \leqslant \frac{C}{p} \|f\|_{\mathscr{C}^{m+2}(X)}. \tag{1.10}$$

Moreover, (1.10) *is uniform in the following sense. Consider Q_{K_p} as a function of the parameters $g^{TX}, h^L, \nabla^L, h^E, \nabla^E$ and Φ, i.e.,*

$$Q_{K_p} = Q_{K_p}(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, \Phi).$$

Let $\mathcal{M}$ be a subset of the infinite dimensional manifold $\mathscr{D}$ of all compatible tuples $g^{TX}, h^L, \nabla^L, h^E, \nabla^E$ and Φ. Assume that

(i) *the covariant derivatives in the direction X of order $\ell \leqslant 2n + m + 6$ of elements of $\mathcal{M}$ form a set of tensors on $X \times \mathcal{M}$ which is bounded in the $\mathscr{C}^0$-norm calculated in the direction of $\mathcal{M}$;*

(ii) *the projection of $\mathcal{M}$ on the space of Riemannian metrics is bounded below in the $\mathscr{C}^0$-norm.*

Then there exists $C = C_m(\mathcal{M})$ such that (1.10) *holds for all tuples of parameters from $\mathcal{M}$. Moreover, the $\mathscr{C}^m$-norm in* (1.10) *can be taken on $X \times \mathcal{M}$.*

The organization of this paper is as follows. In Section 2, we establish the asymptotic expansion of the generalized Bergman kernel which extends [14, Subsection 8.3]. In Section 3, we prove Theorem 1.1.

2 Asymptotic expansion of the generalized Bergman kernel

In this section, we assume that g^{TX} is an arbitrary J-invariant Riemannian metric on X. Let $\Delta^{L^p \otimes E}$ be the Bochner Laplacian acting on $\mathscr{C}^\infty(X, L^p \otimes E)$ associated with g^{TX} and $\nabla^{L^p \otimes E}$. Let $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ be Hermitian.

Let dv_X be the Riemannian volume form on (X, g^{TX}). Now the Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E)$ is induced by h^L, h^E and dv_X.

We identify the two form R^L with the Hermitian matrix $\dot{R}^L \in \mathrm{End}(T^{(1,0)}X)$ such that for $W, Y \in T^{(1,0)}X$,

$$R^L(W, \overline{Y}) = \langle \dot{R}^L W, \overline{Y} \rangle. \tag{2.1}$$

Set

$$\tau = \operatorname{Tr}|_{T^{(1,0)}X} \dot{R}^L, \quad \mu_0 = \inf_{u \in T_x^{(1,0)}X,\, x \in X} R_x^L(u, \overline{u})/|u|_{g^{TX}}^2 > 0. \tag{2.2}$$

Note that if $g^{TX} = \omega(\cdot, J\cdot)$, then $\tau = 2\pi n$ and $\mu_0 = 2\pi$.

Then the renormalized Bochner Laplacian is defined as

$$\Delta_{p,\Phi} = \Delta^{L^p \otimes E} - \tau p + \Phi. \tag{2.3}$$

By the same references as those in Section 1, there exists $C_L > 0$ independent of p such that

$$\operatorname{Spec}(\Delta_{p,\Phi}) \subset [-C_L, C_L] \cup [2\mu_0 p - C_L, +\infty). \tag{2.4}$$

Thus $\mathcal{H}_p$ in (1.6) is still well-defined and (1.7) holds.

Let $P_{\mathcal{H}_p}(x, x')$ be the smooth kernel of the orthogonal projection $P_{\mathcal{H}_p}$ from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $\mathcal{H}_p$ with respect to $dv_X(x')$. In this section, we study the asymptotics of $P_{\mathcal{H}_p}(x, x')$ as $p \to \infty$.

Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in (0, a^X/4)$. Let $d(x, y)$ denote the Riemannian distance from x to y on (X, g^{TX}). By [14, Proposition 8.3.5] and the argument after [14, Proposition 8.3.5], we get for any $l, m \in \mathbb{N}$ and $0 < \theta < 1$, there exists $C > 0$ such that

$$|P_{\mathcal{H}_p}(x, x')|_{\mathscr{C}^m(X \times X)} \leqslant C p^{-l}, \quad \text{if} \quad d(x, x') > \varepsilon p^{-\frac{\theta}{2}}. \tag{2.5}$$

Now we still need to understand the asymptotics of $P_{\mathcal{H}_p}(x, x')$ for $d(x, x') \leqslant \varepsilon p^{-\frac{\theta}{2}}$.

We recall first the procedure of [15, Subsection 1.2] and [14, Subsection 8.3].

Denote by $B^X(x, \varepsilon)$ and $B^{T_xX}(0, \varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. We identify $B^{T_xX}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ by using the exponential map of (X, g^{TX}).

We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0, \varepsilon)$, we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve

$$\gamma_Z : [0, 1] \ni u \to \exp_{x_0}^X(uZ).$$

Then under our identification, $P_{\mathcal{H}_p}(Z, Z')$ is a function on $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon$. We denote it by $P_{\mathcal{H}_p, x_0}(Z, Z')$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $P_{\mathcal{H}_p, x_0}(Z, Z')$ as a smooth function over $TX \times_X TX$ by identifying a section

$$s \in \mathscr{C}^\infty(TX \times_X TX, \pi^*(\operatorname{End}(E)))$$

with the family $(s_x)_{x \in X}$, where $s_x = s|_{\pi^{-1}(x)}$.

Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{\mathrm{e}^i\}_i$ be its dual basis. For $\varepsilon > 0$ small enough, we extend the geometric objects from $B^{T_{x_0}X}(0, \varepsilon)$ to $\mathbb{R}^{2n} \simeq T_{x_0}X$ where the identification is given by

$$(Z_1, \ldots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_i Z_i e_i \in T_{x_0}X, \tag{2.6}$$

such that $\Delta_{p,\Phi}$ is the restriction of a renormalized Bochner-Laplacian on $\mathbb{R}^{2n}$ associated with an Hermitian line bundle with positive curvature. In this way, we replace X by $\mathbb{R}^{2n}$.

At first, we denote by L_0 and E_0 the trivial bundles with fiber L_{x_0} and E_{x_0} on $X_0 = \mathbb{R}^{2n}$. We still denote by ∇^L, ∇^E and h^L, etc. the connections and metrics on L_0 and E_0 on $B^{T_{x_0}X}(0, 4\varepsilon)$ induced by the above identification. Then h^L and h^E are identified to the constant metrics $h^{L_0} = h^{L_{x_0}}$ and $h^{E_0} = h^{E_{x_0}}$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(v) = 1 \quad \text{if} \quad |v| < 2, \quad \rho(v) = 0 \quad \text{if} \quad |v| > 4. \tag{2.7}$$

Let $\varphi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the map defined by $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. Then $\Phi_0 = \Phi \circ \varphi_\varepsilon$ is a smooth function on X_0. Let $g^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(Z))$ be the metric on X_0. Set $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$. Then ∇^{E_0} is the extension of ∇^E on $B^{T_{x_0}X}(0,\varepsilon)$. Denote by $\mathcal{R} = \sum_i Z_i e_i = Z$ the radial vector field on $\mathbb{R}^{2n}$. We define the Hermitian connection ∇^{L_0} on (L^0, h^{L_0}) by

$$\nabla^{L_0}|_Z = \varphi_\varepsilon^* \nabla^L + \frac{1}{2}(1 - \rho^2(|Z|/\varepsilon)) R^L_{x_0}(\mathcal{R}, \cdot). \tag{2.8}$$

Let R^{L_0} denote the curvature of ∇^{L_0} and $\{e_i\}_i$ be an orthonormal frame of (TX_0, g^{TX_0}). Let J_0 be an almost complex structure on X_0 compatible with g^{TX_0} and $\frac{\sqrt{-1}}{2\pi}R^{L_0}$ such that $J_0 = J$ on $B^{T_{x_0}X}(0, 2\varepsilon)$ and $J_0 = J_{x_0}$ outside $B^{T_{x_0}X}(0, 4\varepsilon)$. Set (see (2.2))

$$\tau_0 = \frac{\sqrt{-1}}{2} \sum_i R^{L_0}(e_i, J_0 e_i). \tag{2.9}$$

Let

$$\Delta^{X_0}_{p,\Phi_0} = \Delta^{L_0^p \otimes E_0} - p\tau_0 + \Phi_0$$

be the renormalized Bochner-Laplacian on X_0 associated to the above data as in (1.4). By [15, (1.23)], there exists $C_{L_0} > 0$ such that

$$\mathrm{Spec}(\Delta^{X_0}_{p,\Phi_0}) \subset [-C_{L_0}, C_{L_0}] \cup [\mu_0 p - C_{L_0}, +\infty). \tag{2.10}$$

Let S_L be a unit vector of L_0. Using S_L and the above discussion, we get an isometry $L_0^p \simeq \mathbb{C}$. Let $P_{0,\mathcal{H}_p}$ be the spectral projection of $\Delta^{X_0}_{p,\Phi_0}$ from $\mathscr{C}^\infty(X_0, L_0^p \otimes E_0) \simeq \mathscr{C}^\infty(X_0, E_0)$ corresponding to the interval $[-C_{L_0}, C_{L_0}]$, and let $P_{0,\mathcal{H}_p}(x, x')$ be the smooth kernel of $P_{0,\mathcal{H}_p}$ with respect to the volume form $dv_{X_0}(x')$. By [15, Proposition 1.3] (for $q = 0$ therein), for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x, x' \in B^{T_{x_0}X}(0, \varepsilon)$, we have

$$|(P_{0,\mathcal{H}_p} - P_{\mathcal{H}_p})(x, x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m} p^{-l}, \tag{2.11}$$

where the $\mathscr{C}^m$-norm is induced by $\nabla^{TX}, \nabla^L, \nabla^E, h^L, h^E$ and g^{TX}.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z), \tag{2.12}$$

with $\kappa(0) = 1$. Denote by ∇_U the ordinary differentiation operation on $T_{x_0}X$ in the direction U. Denote by $t = \frac{1}{\sqrt{p}}$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_0)$ and $Z \in \mathbb{R}^{2n}$, set

$$\begin{aligned} &(S_t s)(Z) = s(Z/t), \quad \nabla_t = t S_t^{-1} \kappa^{\frac{1}{2}} \nabla^{L_0} \kappa^{-\frac{1}{2}} S_t, \\ &\mathscr{L}_t = S_t^{-1} \kappa^{\frac{1}{2}} t^2 \Delta^{X_0}_{p,\Phi_0} \kappa^{-\frac{1}{2}} S_t. \end{aligned} \tag{2.13}$$

It follows from (2.10) and (2.13) that for t small enough (see [15, (1.43)]),

$$\mathrm{Spec}(\mathscr{L}_t) \subset [-C_{L_0} t^2, C_{L_0} t^2] \cup \left[\frac{1}{2}\mu_0, +\infty\right). \tag{2.14}$$

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 radius $\frac{1}{4}\mu_0$. By (2.14), there exists $t_0 > 0$ such that the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta$ and $t \in (0, t_0]$.

We denote by $\langle \cdot, \cdot \rangle_{0,L^2}$ and $\|\cdot\|_{0,L^2}$ the scalar product and the L^2-norm on $\mathscr{C}^\infty(X_0, E_0)$ induced by g^{TX_0} as in (1.2). For $s \in C^\infty(X_0, E_0)$, set

$$\begin{aligned} &\|s\|^2_{t,0} = \|s\|^2_0 = \int_{\mathbb{R}^{2n}} |s(Z)|^2_{h^{E_0}} dv_{TX}(Z), \\ &\|s\|^2_{t,m} = \sum_{l=1}^m \sum_{i_1,\ldots,i_l=1}^{2n} \|\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_l}} s\|^2_{t,0}. \end{aligned} \tag{2.15}$$

We denote by $\langle\cdot,\cdot\rangle$ the inner product on $C^\infty(X_0,E_0)$ corresponding to $\|\cdot\|_{t,0}$. Let H_t^m be the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. Let H_t^{-1} be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on H_t^{-1} defined by

$$\|s\|_{t,-1}=\sup_{0\neq s'\in H_t^1}|\langle s,s'\rangle_{t,0}|/\|s'\|_{t,1}.$$

If $A\in\mathscr{L}(H^m,H^{m'})$, then we denote by $\|A\|_t^{m,m'}$ the norm of A with respect to the norms $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$.

Let $\mathcal{P}_{0,t}$ be the orthogonal projection from $(\mathscr{C}^\infty(X_0,E_0),\|\cdot\|_0)$ onto the space of the direct sum of eigenspaces of $\mathscr{L}_t$ corresponding to the eigenvalues lying in $[-C_{L_0}t^2,C_{L_0}t^2]$. Let $\mathcal{P}_{0,t}(Z,Z')=\mathcal{P}_{0,t,x_0}(Z,Z')$ (with $Z,Z'\in X_0$) be the smooth kernel of $\mathcal{P}_{0,t}$ with respect to $dv_{TX}(Z')$. Denote by $\mathscr{C}^m(X)$ the $\mathscr{C}^m$-norm for the parameter $x_0\in X$. By [14, (4.2.9)], we have the following extension of [15, Theorem 1.10] (for $q=0$).

Claim. *For any $r,m',m\in\mathbb{N}$, there exists $C>0$ such that for $t\in(0,t_0]$ and $Z,Z'\in T_{x_0}X$,*

$$\sup_{|\alpha|+|\alpha'|\leqslant m'}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}(Z,Z')\right|_{\mathscr{C}^m(X)}\leqslant C(1+|Z|+|Z'|)^{M_{r,m',m}}\tag{2.16}$$

with

$$M_{r,m',m}=2n+2+2r+m'+2m.\tag{2.17}$$

We will sketch the proof of the claim. The readers are referred to [2], [14, Chapter 4] and [15, Section 1] for more details. In fact, by (2.14), for any $k\in\mathbb{N}^*$ (see [15, (1.55)]),

$$\mathcal{P}_{0,t}=\frac{1}{2\pi\sqrt{-1}}\int_\delta\lambda^{k-1}(\lambda-\mathscr{L}_t)^{-k}d\lambda.\tag{2.18}$$

For $m\in\mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\{\nabla_{t,e_{i_1}}\cdots\nabla_{t,e_{i_j}}\}_{j\leqslant m}$. By [15, (1.58)],

$$\left\|Q\mathcal{P}_{0,t}Q'\right\|_t^{0,0}\leqslant C_m,\quad\text{for}\quad Q,Q'\in\mathcal{Q}^m.\tag{2.19}$$

Let $\|\cdot\|_m$ be the usual Sobolev norm on $C^\infty(\mathbb{R}^n,E_0)$ induced by h^{E_0} and the volume form $dv_{TX}(Z)$. By [14, (4.29)], there exists $C>0$ such that for $s\in C^\infty(X_0,E_0)$ with $\mathrm{supp}(s)\subset B^{T_{x_0}X}(0,q)$, $m\geqslant0$,

$$\frac{1}{C}(1+q)^{-m}\|s\|_{t,m}\leqslant\|s\|_m\leqslant C(1+q)^m\|s\|_{t,m}.\tag{2.20}$$

Now (2.19) and (2.20) together with Sobolev inequalities imply that for $Q,Q'\in\mathcal{Q}^m$,

$$\sup_{|Z|,|Z'|\leqslant q}|Q_ZQ'_{Z'}\mathcal{P}_{0,t}(Z,Z')|\leqslant C(1+q)^{2n+2}.\tag{2.21}$$

Combining [15, (1.35)] and (2.21) yields (2.16) for $r=m'=0$. To obtain (2.16) for $r\geqslant1$ and $m'=0$, note that by (2.18),

$$\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}=\frac{1}{2\pi\sqrt{-1}}\int_\delta\lambda^{k-1}\frac{\partial^r}{\partial t^r}(\lambda-\mathscr{L}_t)^{-k}d\lambda.\tag{2.22}$$

For $k,r\in\mathbb{N}^*$, let

$$I_{k,r}=\left\{(\boldsymbol{k},\boldsymbol{r})=(k_i,r_i)\,\middle|\,\sum_{i=0}^jk_i=k+j,\sum_{i=1}^jr_i=r,k_i+r_i\in\mathbb{N}^*\right\}.\tag{2.23}$$

Then there exist $a_{\boldsymbol{r}}^{\boldsymbol{k}}\in\mathbb{R}$ such that

$$A_{\boldsymbol{r}}^{\boldsymbol{k}}(\lambda,t)=(\lambda-\mathscr{L}_t)^{-k_0}\frac{\partial^{r_1}\mathscr{L}_t}{\partial t^{r_1}}(\lambda-\mathscr{L}_t)^{-k_1}\cdots\frac{\partial^{r_j}\mathscr{L}_t}{\partial t^{r_j}}(\lambda-\mathscr{L}_t)^{-k_j},$$

$$\frac{\partial^r}{\partial t^r}(\lambda - \mathscr{L}_t)^{-k} = \sum_{(\boldsymbol{k},\boldsymbol{r})\in I_{k,r}} a_{\boldsymbol{r}}^{\boldsymbol{k}} A_{\boldsymbol{r}}^{\boldsymbol{k}}(\lambda, t). \tag{2.24}$$

We can now carry on nearly word by word the corresponding part of the proof of [15, Theorem 1.10] to finish the proof of (2.16). We finish the proof of the claim.

Set (see [14, (4.1.65)])

$$\begin{aligned} \mathscr{F}_r &= \frac{1}{2\pi\sqrt{-1}r!}\int_\delta \lambda^{k-1} \sum_{(\boldsymbol{k},\boldsymbol{r})\in I_{k,r}} a_{\boldsymbol{r}}^{\boldsymbol{k}} A_{\boldsymbol{r}}^{\boldsymbol{k}}(\lambda, 0)d\lambda, \\ \mathscr{F}_{r,t} &= \frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t} - \mathscr{F}_r. \end{aligned} \tag{2.25}$$

Let $\mathscr{F}_r(Z,Z')$ $(Z,Z' \in T_{x_0}X)$ be the smooth kernel of $\mathscr{F}$ with respect to $dv_{TX}(Z')$. Then $\mathscr{F}_r(Z,Z') \in \mathscr{C}^\infty(TX\times_X TX, \pi^*\mathrm{End}(E))$. By the proof of (2.16), we observe that $\mathscr{F}_r$ verifies the similar inequalities to (2.16), i.e., to replace the factor $\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,r}$ in (2.16) by $\mathscr{F}_r$. Using this observation, (2.16) and (2.25), we obtain the extension of [15, Theorem 1.12]. There exists $C>0$ such that for $t\in(0,t_0]$ and $Z,Z'\in T_{x_0}X$,

$$|\mathscr{F}_{r,t}(Z,Z')| \leqslant Ct^{1/(2n+1)}(1+|Z|+|Z'|)^{2n+2}. \tag{2.26}$$

By (2.25) and (2.26), we have (see [15, (1.78)])

$$\frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{0,t}\,|_{t=0} = \mathscr{F}_r. \tag{2.27}$$

By (2.16), (2.27) and the Taylor expansion

$$G(t) - \sum_{r=0}^k \frac{1}{r!}\frac{\partial^r G}{\partial t^r}(0)t^r = \frac{1}{k!}\int_0^t (t-s)^k \frac{\partial^{k+1}G}{\partial s^{k+1}}(s)ds, \tag{2.28}$$

we obtain the extension of [15, Theorem 1.13]. For any $k,m,m'\in\mathbb{N}$, there exists $C>0$ such that for $t\in(0,t_0]$, $Z,Z'\in T_{x_0}X$ and for $|\alpha|+|\alpha'|\leqslant m'$,

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\left(\mathcal{P}_{0,t} - \sum_{r=0}^k \mathscr{F}_r t^r\right)(Z,Z')\right|_{\mathscr{C}^m(X)} \leqslant Ct^{k+1}(1+|Z|+|Z'|)^{M_{k+1,m',m}}. \tag{2.29}$$

By (2.12) and (2.13), for $Z,Z'\in\mathbb{R}^{2n}$ (see [15, (1.112)]),

$$P_{0,\mathcal{H}_p}(Z,Z') = t^{-2n}\kappa^{-\frac{1}{2}}(Z)\mathcal{P}_{0,t}(Z/t,Z'/t)\kappa^{-\frac{1}{2}}(Z'). \tag{2.30}$$

Combining (2.11), (2.29) and (2.30), we obtain

$$\begin{aligned} &\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\left(\frac{1}{p^n}P_{\mathcal{H}_p,x_0}(Z,Z') - \sum_{r=0}^k \mathscr{F}_r(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^{-\frac{k-m'+1}{2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z'|)^{M_{k+1,m',m}}. \end{aligned} \tag{2.31}$$

Now we fix k_0, m' and m. Take

$$k = k_0+m'+2 \quad \text{and} \quad \theta = 1/(2M_{k+1,m',m}). \tag{2.32}$$

Then for $|\alpha|+|\alpha'|\leqslant m'$ and $|Z|,|Z'|<p^{-\frac{1}{2}+\theta}$, we have

$$\begin{aligned} &\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\left(\frac{1}{p^n}P_{\mathcal{H}_p,x_0}(Z,Z') - \sum_{r=0}^k \mathscr{F}_r(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^m(X)} \\ &\leqslant Cp^{-\frac{k_0}{2}-1}. \end{aligned} \tag{2.33}$$

To sum up, we have finished the proof of the following result.

Theorem 2.1. *For any $k_0, m', m \in \mathbb{N}$, there exists $C > 0$ such that for $|\alpha| + |\alpha'| \leqslant m'$ and $|Z|, |Z'| < p^{-\frac{1}{2}+\theta}$ with*

$$\theta = \frac{1}{2(2n + 8 + 2k_0 + 3m' + 2m)}, \tag{2.34}$$

we have

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}}\left(\frac{1}{p^n} P_{\mathcal{H}_p,x_0}(Z, Z') - \sum_{r=0}^{k} \mathscr{F}_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^m(X)} \leqslant Cp^{-\frac{k_0}{2}-1}, \tag{2.35}$$

where $k = k_0 + m' + 2$.

We choose $\{w_j\}_{j=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that

$$\dot{R}_{x_0}^L = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{End}(T_{x_0}^{(1,0)}X). \tag{2.36}$$

Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$, form an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ induced by $\{e_i\}$ as in (2.6) and in what follows we also introduce the complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Set

$$\mathscr{P}(Z, Z') = \prod_{j=1}^{n} \frac{a_j}{2\pi} \exp\left[-\frac{1}{4}\sum_{j=1}^{n} a_j(|z_j|^2 + |z_j'|^2 - 2z_j\overline{z}_j')\right]. \tag{2.37}$$

By [15, Theorem 1.18], there exist $J_r(Z, Z')$ polynomials in Z and Z' with the same parity as r and degree $\leqslant 3r$ such that

$$\mathscr{F}_r(Z, Z') = J_r(Z, Z')\mathscr{P}(Z, Z'), \quad J_0(Z, Z') = 1. \tag{2.38}$$

3 Proof of Theorem 1.1

Now $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$, thus $a_j = 2\pi$ in (2.37).

Recall that the classical heat kernel on $\mathbb{C}^n$ is $\mathrm{e}^{-u\Delta}(Z, Z') = (4\pi u)^{-n}\mathrm{e}^{-\frac{1}{4u}|Z-Z'|^2}$. Then

$$|\mathscr{P}(Z, Z')|^2 = \mathrm{e}^{-\pi|Z-Z'|^2} = \mathrm{e}^{-\frac{\Delta}{4\pi}}(Z, Z'). \tag{3.1}$$

Note that $|P_{\mathcal{H}_p,x_0}(Z, Z')|^2 = P_{\mathcal{H}_p,x_0}(Z, Z')\overline{P_{\mathcal{H}_p,x_0}(Z, Z')}$. By (1.8), (2.35), (2.38) and (3.1), there exist polynomials $J_r'(Z, Z')$ in Z and Z' such that for $|Z|, |Z'| < p^{-\frac{1}{2}+\theta}$ with θ in (2.34),

$$\left|\frac{1}{p^{2n}} K_{p,x_0}(Z, Z') - \left(1 + \sum_{r=1}^{k} p^{-\frac{r}{2}} J_r'(\sqrt{p}Z, \sqrt{p}Z')\right)\mathrm{e}^{-\pi p|Z-Z'|^2}\right|_{\mathscr{C}^m(X)} \leqslant Cp^{-\frac{k_0}{2}-1}, \tag{3.2}$$

with

$$J_1'(0, Z') = (J_1 + \overline{J_1})(0, Z'). \tag{3.3}$$

For a function $f \in \mathscr{C}^\infty(X)$, we denote by $f_{x_0}(Z)$ the function f in normal coordinates Z around a point $x_0 \in X$. We have thus a family (f_{x_0}) of functions indexed by the parameter $x_0 \in X$. Combining (1.8), (2.5) with θ in (2.34), and (3.2), we obtain

$$\begin{aligned}&\left|\frac{1}{p^n} K_p f - p^n \int_{|Z'| \leqslant \varepsilon p^{-\theta/2}} \left(1 + \sum_{r=1}^{k} p^{-\frac{r}{2}} J_r'(0, \sqrt{p}Z')\right)\mathrm{e}^{-\pi p|Z'|^2} f_{x_0}(Z') dv_X(Z')\right|_{\mathscr{C}^m(X)} \\ &\quad \leqslant Cp^{-\frac{k_0}{2}-1}|f|_{\mathscr{C}^m(X)}.\end{aligned} \tag{3.4}$$

By using Taylor expansion of $f_{x_0}(Z')$ at 0, we obtain

$$\begin{aligned} &\left|p^n\int_{|Z'|\leqslant \varepsilon p^{-\theta/2}} J_r'(0,\sqrt{p}Z')\mathrm{e}^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\right|_{\mathscr{C}^m(X)}\leqslant C|f|_{\mathscr{C}^m(X)},\\ &\left|p^n\int_{|Z'|\leqslant \varepsilon p^{-\theta/2}} \mathrm{e}^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')-f(x_0)\right|_{\mathscr{C}^m(X)}\leqslant \frac{C}{p}|f|_{\mathscr{C}^{m+2}(X)}. \end{aligned} \tag{3.5}$$

Finally, by [15, Theorem 1.18] and [15, (1.97), (1.98) and (1.111)], we obtain

$$\begin{aligned} &\int_{Z'\in\mathbb{C}^n}\overline{J_1}(0,Z')|\mathscr{P}|^2(0,Z')dZ'\\ &\quad=\int_{Z'\in\mathbb{C}^n}\mathscr{P}(0,Z')J_1(Z',0)\mathscr{P}(Z',0)dZ'\\ &\quad=(\mathscr{P}J_1\mathscr{P})(0,0)=0. \end{aligned} \tag{3.6}$$

Combining Taylor expansion of $f_{x_0}(Z')$ at 0, and (3.6) yields

$$\left|p^n\int_{|Z'|\leqslant \varepsilon p^{-\theta/2}} p^{-1/2}J_1'(0,\sqrt{p}Z')\mathrm{e}^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\right|_{\mathscr{C}^m(X)}\leqslant \frac{C}{p}|f|_{\mathscr{C}^{m+2}(X)}. \tag{3.7}$$

Combining (3.4) for $k_0=0$, (3.5) and (3.7) yields

$$\left|\frac{1}{p^n}K_pf-f\right|_{\mathscr{C}^m(X)}\leqslant \frac{C}{p}|f|_{\mathscr{C}^{m+2}(X)}. \tag{3.8}$$

Then the desired $\mathscr{C}^m$-estimate (1.10) follows from (1.9) and (3.8). The proof of the uniformity assertion from Theorem 1.1 is modeled on [14, Subsection 4.1.7] and [15, Subsection 1.5]. First, we notice that in the proof of (2.16), we only use the derivatives of the coefficients of $\mathscr{L}_t$ with order $\leqslant 2n+m+m'+r+2$. Thus, by (2.28), the constants in (2.16) and (2.26) ((2.29) and (2.31), respectively) are bounded, if with respect to a fixed metric g_0^{TX}, the $\mathscr{C}^{2n+m+m'+r+3}$ ($\mathscr{C}^{2n+m+m'+k+4}$, respectively)-norms on X of the data $g^{TX},h^L,\nabla^L,h^E,\nabla^E$ and Φ are bounded and g^{TX} is bounded below. Note $k=k_0+m'+2$ in (2.35). Then the constants in (2.35) ((3.2), (3.4) and (3.8), respectively) are bounded if with respect to a fixed metric g_0^{TX}, the $\mathscr{C}^{2n+m+2m'+k_0+6}$ ($\mathscr{C}^{2n+m+k_0+6}$, $\mathscr{C}^{2n+m+k_0+6}$ and $\mathscr{C}^{2n+m+6}$, respectively)-norm on X of the data $g^{TX},h^L,\nabla^L,h^E,\nabla^E$ and Φ are bounded and g^{TX} is bounded below. Moreover, taking derivatives with respect to the parameters we obtain a similar equation to (2.22) (see [15, (1.65)]). Thus the $\mathscr{C}^m$-norm in (3.8) can also include the parameters of the $\mathscr{C}^m$-norm if the $\mathscr{C}^m$-norms (with respect to the parameter $x_0\in X$) of derivatives of the above data with order $\leqslant 2n+6$ are bounded. Thus we can take C in (1.10) independent of g^{TX}. The proof of Theorem 1.1 is completed.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11401232 and 11528103), Agence nationale de la recherche (Grant No. ANR-14-CE25-0012-01), funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative and Deutsche Forschungsgemeinschaft Funded Project Sonderforschungsbereich Transregio 191.

References

1 Cao H, Keller J. On the Calabi problem: A finite-dimensional approach. J Eur Math Soc (JEMS), 2013, 15: 1033–1065
2 Dai X, Liu K, Ma X. On the asymptotic expansion of Bergman kernel. J Differential Geom, 2006, 72: 1–41
3 Donaldson S K. Scalar curvature and projective embeddings. I. J Differential Geom, 2001, 59: 479–522
4 Donaldson S K. Lower bounds on the Calabi functional. J Differential Geom, 2005, 70: 453–472
5 Donaldson S K. Some numerical results in complex differential geometry. Pure Appl Math Q, 2009, 5: 571–618
6 Fine J. Calabi flow and projective embeddings, with an appendix by Kefeng Liu and Xiaonan Ma. J Differential Geom, 2010, 84: 489–523
7 Guillemin V, Uribe A. The Laplace operator on the n-th tensor power of a line bundle: Eigenvalues which are bounded uniformly in n. Asymptot Anal, 1988, 1: 105–113

8 Keller J, Lejmi M. On the lower bounds of the L^2-norm of the Hermitian scalar curvature. ArXiv:1702.01810, 2017
9 Keller J, Seyyedali R. Quantization of Donaldson's heat flow over projective manifolds. Math Z, 2016, 282: 839–866
10 Liu K, Ma X. A remark on 'Some numerical results in complex differential geometry'. Math Res Lett, 2007, 14: 165–171
11 Lu W, Ma X, Marinescu G. Optimal convergence speed of Bergman metrics on symplectic manifolds. ArXiv:1702.00974, 2017
12 Ma X. Geometric quantization on Kähler and symplectic manifolds. In: Proceedings of the International Congress of Mathematicians, vol. 2. New Delhi: Hindustan Book Agency, 2010, 785–810
13 Ma X, Marinescu G. The Spin^c Dirac operator on high tensor powers of a line bundle. Math Z, 2002, 240: 651–664
14 Ma X, Marinescu G. Holomorphic Morse Inequalities and Bergman Kernels. Boston: Birkhäuser, 2007
15 Ma X, Marinescu G. Generalized Bergman kernels on symplectic manifolds. Adv Math, 2008, 217: 1756–1815
16 Ma X, Marinescu G. Berezin-Toeplitz quantization on Kähler manifolds. J Reine Angew Math, 2012, 662: 1–58