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ON ELLIPTIC GENERA AND FOLIATIONS

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Abstract. We prove several vanishing theorems for a class of generalized elliptic genera on foliated manifolds, by using classical equivariant index theory. The main techniques are the use of the Jacobi theta-functions and the construction of a new class of elliptic operators associated to foliations.

1. Introduction

The main purpose of this paper is to prove some vanishing theorems of characteristic numbers for foliated manifolds with group actions. Such type of results are usually proved by using index theorems or fixed point theorems for foliations (cf. [HL1]). In this paper we take a rather different route. Instead of using the heavy machinery about index theory for foliations as developed by various people, we use certain new elliptic operators particularly designed to study integrable subbundles with spin struture, the so-called sub-Dirac operator (cf. [LiuZ]). With the help of such operators, we are able to prove our theorems for foliated manifolds by using the classical index theory. Compare with [HL2].

More precisely, let (M, F) be a transversally oriented compact foliated manifold such that the integrable bundle F is spin and carries a fixed spin structure. Assume that there is an effective S^3-action on M which preserves the leaves induced by F and also the spin structure on F. Then a special case of our result shows that the Witten genus ([W, (17)]) of M vanishes if the first Pontryagin class of F verifies $p_1(F) = 0$. Note that here we do not assume that the manifold M is spin, so that the Witten genus under consideration is not a priori an integer.

When the foliation happens to be a fibration, then the above vanishing result is a direct consequence of the family vanishing theorem proved in [LiuMa1]. Thus our results here generalize the corresponding vanishing results in [LiuMa1] to foliations.

On the other hand, we will also prove certain vanishing theorems for spin manifolds with split tangent bundles, by using the similar technique. The elliptic genera we derive in this situation can be viewed as interpolations between the various classical elliptic genera. They are actually the mixture of the two

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universal elliptic genera. It is interesting to note that, under some mild conditions, we get quite general vanishing theorems. Similar theorems can be proved for loop group representations.

This paper is organized as follows. In Section 2, we introduce the sub-Dirac operator. In Section 3, we state our main vanishing theorem for elliptic genus on a foliated manifold, which will be proved in Section 4 by combining the construction in Section 2 with Jacobi-theta functions. In Section 5, we prove several vanishing theorems for certain twisted elliptic genera associated to spin manifolds with split tangent bundle. In Section 6, we point out other generalizations and state a conjecture concerning the vanishing of the Witten genus of a foliation with spin leaves of positive Ricci curvature.

2. Sub-Dirac operator

Let M be an even dimensional smooth compact oriented manifold. Let F be a sub-bundle of the tangent vector bundle TM of M. Let g^{TM} be a Riemannian metric on TM. Let $F^{\perp}$ be the orthogonal complement to F in TM. Then one has the orthogonal splittings

$$\begin{aligned} TM &= F \oplus F^{\perp}, \\ g^{TM} &= g^{F} \oplus g^{F^{\perp}}. \end{aligned} \tag{2.1}$$

Moreover, one has the obvious identification that

$$TM/F \simeq F^{\perp}. \tag{2.2}$$

From now on we make the special assumption that F is even dimensional, oriented, spin and carries a fixed spin structure. Then $F^{\perp}$ carries an induced orientation. Set $2p = \dim F$ and $2r = \dim F^{\perp}$.

Let $S(F)$ be the bundle of spinors associated to (F, g^F). For any $X \in F$, denote by $c(X)$ the Clifford action of X on $S(F)$. We have the splitting

$$S(F) = S_+(F) \oplus S_-(F) \tag{2.3}$$

and $c(X)$ exchanges $S_{\pm}(F)$.

Let $\Lambda(F^{\perp,*})$ be the exterior algebra bundle of $F^{\perp}$. Then $\Lambda(F^{\perp,*})$ carries a canonically induced metric $g^{\Lambda(F^{\perp,*})}$ from $g^{F^{\perp}}$. By using $g^{F^{\perp}}$, one has the canonical identification $F^{\perp} \simeq F^{\perp,*}$. For any $U \in F^{\perp}$, let $U^* \in F^{\perp,*}$ be the corresponding dual of U with respect to $g^{F^{\perp}}$. Now for $U \in F^{\perp}$, set

$$c(U) = U^* \wedge -i_U, \tag{2.4}$$

where $U^*\wedge$ and i_U are the exterior and inner multiplications by U^* and U on $\Lambda(F^{\perp,*})$ respectively. One has the following obvious identities,

$$c(U)c(V) + c(V)c(U) = -2\langle U, V\rangle_{g^{F^{\perp}}} \tag{2.5}$$

for $U,\ V \in F^{\perp}$.

Let $h_1, \cdots, h_{2r}$ be an oriented local orthonormal basis of $F^{\perp}$. Set

$$\tau\left(F^{\perp}, g^{F^{\perp}}\right) = (\sqrt{-1})^r c(h_1)\cdots c(h_{2r}). \tag{2.6}$$

Then clearly

$$\tau\left(F^{\perp}, g^{F^{\perp}}\right)^2 = \mathrm{Id}_{\Lambda(F^{\perp,*})}. \tag{2.7}$$

Denote

$$\Lambda_{\pm}\left(F^{\perp,*}\right) = \left\{h \in \Lambda\left(F^{\perp,*}\right) :\ \tau\left(F^{\perp}, g^{F^{\perp}}\right) h = \pm h\right\}. \tag{2.8}$$

Then $\Lambda_{\pm}(F^{\perp,*})$ are sub-bundles of $\Lambda(F^{\perp,*})$. Also, one verifies that for any $h \in F^{\perp}$, $c(h)$ exchanges $\Lambda_{\pm}(F^{\perp,*})$.

We will view both vector bundles

$$S(F) = S_+(F) \oplus S_-(F) \tag{2.9}$$

and

$$\Lambda\left(F^{\perp,*}\right) = \Lambda_+\left(F^{\perp,*}\right) \oplus \Lambda_-\left(F^{\perp,*}\right) \tag{2.10}$$

as super-vector bundles. Their $\mathbf{Z}_2$-graded tensor product is given by

$$\begin{aligned} S(F)\widehat{\otimes}\Lambda\left(F^{\perp,*}\right) = &\left[S_+(F)\otimes\Lambda_+\left(F^{\perp,*}\right) \oplus S_-(F)\otimes\Lambda_-\left(F^{\perp,*}\right)\right] \\ &\bigoplus \left[S_+(F)\otimes\Lambda_-\left(F^{\perp,*}\right) \oplus S_-(F)\otimes\Lambda_+\left(F^{\perp,*}\right)\right]. \end{aligned} \tag{2.11}$$

For $X \in F$, $U \in F^{\perp}$, the operators $c(X)$, $c(U)$ extend naturally to

$$S(F)\widehat{\otimes}\Lambda(F^{\perp,*}).$$

Let ∇^{TM} be the Levi-Civita connection associated to g^{TM}. Let ∇^F, $\nabla^{F^{\perp}}$ be the resitriction of ∇^{TM} on ∇^F, $\nabla^{F^{\perp}}$ respectively. Then ∇^F, $\nabla^{F^{\perp}}$ lift to $S(F)$ and $\Lambda(F^{\perp,*})$ naturally, and preserve the splittings (2.9) and (2.10). We write them as

$$\nabla^{S(F)} = \nabla^{S_+(F)} \oplus \nabla^{S_-(F)}, \quad \nabla^{\Lambda(F^{\perp,*})} = \nabla^{\Lambda_+(F^{\perp,*})} \oplus \nabla^{\Lambda_-(F^{\perp,*})}.$$

Then $S(F)\widehat{\otimes}\Lambda(F^{\perp,*})$ carries the induced tensor product connection

$$\nabla^{S(F)\otimes\Lambda(F^{\perp,*})} = \nabla^{S(F)} \otimes \mathrm{Id}_{\Lambda(F^{\perp,*})} + \mathrm{Id}_{S(F)} \otimes \nabla^{\Lambda(F^{\perp,*})}.$$

And similarly for $S_{\pm}(F)\widehat{\otimes}\Lambda_{\pm}(F^{\perp,*})$.

For any vector bundle E over M, by an integral polynomial of E we will mean a vector bundle $\varphi(E)$ which is a polynomial in the exterior and symmetric powers of E with integral coefficients.

Let $\psi(F)$ (resp. $\varphi(F^{\perp})$) be an integral polynomial of F (resp. $F^{\perp}$), then $\psi(F)$ (resp. $\varphi(F^{\perp})$) carries a naturally induced metric $g^{\psi(F)}$ (resp. $g^{\varphi(F^{\perp})}$) from g^F (resp. $g^{F^{\perp}}$) and also a naturally induced Hermitian connection $\nabla^{\psi(F)}$ (resp. $\nabla^{\varphi(F^{\perp})}$) induced from ∇^F (resp. $\nabla^{F^{\perp}}$).

Our main concern will be on the $\mathbf{Z}_2$-graded vector bundle

$$\left(S(F)\widehat{\otimes}\Lambda\left(F^{\perp,*}\right)\right) \otimes \psi(F) \otimes \varphi\left(F^{\perp}\right)$$

which is

$$\left[S_+(F)\otimes\Lambda_+\left(F^{\perp,*}\right)\otimes\psi(F)\otimes\varphi\left(F^{\perp}\right)\oplus S_-(F)\otimes\Lambda_-\left(F^{\perp,*}\right)\otimes\psi(F)\otimes\varphi\left(F^{\perp}\right)\right]$$

$$\bigoplus\left[S_+(F)\otimes\Lambda_-\left(F^{\perp,*}\right)\otimes\psi(F)\otimes\varphi\left(F^{\perp}\right)\oplus S_+(F)\otimes\Lambda_-\left(F^{\perp,*}\right)\otimes\psi(F)\otimes\varphi\left(F^{\perp}\right)\right].$$

The Clifford actions $c(X)$, $c(U)$ for $X\in F$, $U\in F^{\perp}$ extend further to these bundles by acting as identity on $\psi(F)\otimes\varphi(F^{\perp})$.

We can also form the tensor product metric on the new bundles as well as the tensor product connection on $(S(F)\widehat{\otimes}\Lambda(F^{\perp,*}))\otimes\psi(F)\otimes\varphi(F^{\perp})$ given by

$$\nabla^{(S(F)\otimes\Lambda(F^{\perp,*}))\otimes\psi(F)\otimes\varphi(F^{\perp})}=\nabla^{S(F)\otimes\Lambda(F^{\perp,*})}\otimes\mathrm{Id}_{\psi(F)\otimes\varphi(F^{\perp})}+\mathrm{Id}_{S(F)\hat{\otimes}\Lambda(F^{\perp,*})}\otimes\nabla^{\psi(F)\otimes\varphi(F^{\perp})},$$

where $\nabla^{\psi(F)\otimes\varphi(F^{\perp})}$ is the tensor product connection on $\psi(F)\otimes\varphi(F^{\perp})$ obtained from $\nabla^{\psi(F)}$ and $\nabla^{\varphi(F^{\perp})}$, as well as on the $\pm$ subbundles.

Now let $\{f_i\}_{i=1}^{2p}$ be an oriented orthonormal basis of F. Recall that $\{h_s\}_{s=1}^{2r}$ is an oriented orthonormal basis of $F^{\perp}$. The follwoing elliptic operator is introduced mainly for the reason that the vector bundle $F^{\perp}$ might well be non-spin.

Definition 2.1. *Let $D_{F,\psi(F)\otimes\varphi(F^{\perp})}$ be the operator which maps $\Gamma(S(F)\widehat{\otimes}\Lambda(F^{\perp,*})\otimes\psi(F)\otimes\varphi(F^{\perp}))$ to itself defined by*

$$D_{F,\psi(F)\otimes\varphi(F^{\perp})}=\sum_{i=1}^{2p}c(f_i)\nabla_{f_i}^{(S(F)\otimes\Lambda(F^{\perp,*}))\otimes\psi(F)\otimes\varphi(F^{\perp})}+\sum_{s=1}^{2r}c(h_s)\nabla_{h_s}^{(S(F)\otimes\Lambda(F^{\perp,*}))\otimes\psi(F)\otimes\varphi(F^{\perp})}.$$

Let $D_{F,\psi(F)\otimes\varphi(F^{\perp}),+}$ be the restriction of $D_{F,\psi(F)\otimes\varphi(F^{\perp})}$ to the even part of $(S(F)\widehat{\otimes}\Lambda(F^{\perp,*}))\otimes\psi(F)\otimes\varphi(F^{\perp})$.

Let $\widehat{A}(x)$, $L(x)$ be the functions of x defined by

$$\widehat{A}(x)=\frac{x/2}{\sinh(x/2)},\quad L(x)=\frac{x}{\tanh(x/2)}.$$

Let $\widehat{A}(F)$, $L(F^{\perp})$ be the corresponding characteristic classes of F, $F^{\perp}$.

The following result follows easily from the Atiyah-Singer index theorem [AS].

Theorem 2.2. *The following index formula holds,*

$$\operatorname{ind}\left(D_{F,\psi(F)\otimes\varphi(F^{\perp}),+}\right)=\left\langle\widehat{A}(F)\mathrm{ch}(\psi(F))L\left(F^{\perp}\right)\mathrm{ch}\left(\varphi\left(F^{\perp}\right)\right),[M]\right\rangle.$$

Now assume that M admits an S^1-action which preserves g^{TM}, as well as the spin structure on F. Then it also preserves the splittings in (2.1). Furthermore, an equivariant version of the above index formula still holds.

More precisely, let $\{N\}$ be the set of connected components of the fixed point set of this circle action. When restricted to the fixed point set, we have the equivariant decompositions

$$F|_N = F_0 \oplus (\oplus_j E_j), \quad F^\perp = F_0^\perp \oplus (\oplus_j L_j) \tag{2.12}$$

such that the generator $e^{2\pi it} \in S^1$ acts trivially on the real vector bundles F_0 and $F_0^\perp$, and acts on the complex vector bundles E_j and L_j by multiplications by $e^{2\pi i m_j t}$ and $e^{2\pi i n_j t}$ respectively. Let $\{2\pi i x_j^k\}$ be the Chern roots of E_j and $\{2\pi i z_j^k\}$ be the Chern roots of L_j. Note that in our notation, if E_j is a complex line bundle and R^{E_j} is the curvature of a connection on E_j, then $\frac{-R^{E_j}}{2\pi i} = 2\pi i x_j$.

By (2.12), F_0, $F_0^\perp$ are naturally oriented. We fix the orientation on N induced by the orientations on F_0, $F_0^\perp$.

The following result follows easily from the equivariant index theorem of Atiyah, Bott, Segal and Singer (cf. [AS]).

Theorem 2.3. *The following equivariant index formula for the Lefschetz number $L(g)$ of the generator $g = e^{2\pi it} \in S^1$ associated to the elliptic operator $D_{F,\psi(F)\otimes\varphi(F^\perp),+}$ holds,*

$$L(g) = \sum_N \left\langle \widehat{A}(F_0) L\left(F_0^\perp\right) A(F,t) L\left(F^\perp, t\right) \mathrm{ch}_g\left(\psi(F|_N)\right) \mathrm{ch}_g\left(\varphi\left(F^\perp|_N\right)\right), [N] \right\rangle,$$

where

$$A(F,t) = \prod_{j,k} \frac{1}{2\sinh(\pi i(x_j^k + m_j t))}, \quad L\left(F^\perp, t\right) = \prod_{j,k} \frac{1}{\tanh(\pi i(z_j^k + n_j t))}$$

and ch_g *denotes the equivariant Chern character, for examples,*

$$\mathrm{ch}_g(E_j) = \sum_k e^{2\pi i(x_j^k + m_j t)}, \quad \mathrm{ch}_g(L_j) = \sum_k e^{2\pi i(z_j^k + n_j t)}$$

3. Elliptic genus for foliations

For any vector bundle E, let us denote the two operations in K-theory, the total symmetric and exterior power operations, by

$$\begin{aligned} \mathrm{Sym}_q(E) &= 1 + qE + q^2 \mathrm{Sym}^2(E) + \cdots, \\ \Lambda_q(E) &= 1 + qE + q^2 \Lambda^2(E) + \cdots, \end{aligned} \tag{3.1}$$

where q is a parameter. Recall the relations:

$$\mathrm{Sym}_q(E_1 - E_2) = \mathrm{Sym}_q(E_1) \cdot \Lambda_{-q}(E_2); \quad \Lambda_q(E_1 - E_2) = \Lambda_q(E_1) \cdot \mathrm{Sym}_{-q}(E_2).$$

In what follows, we will take $\psi(F)$ to be the Witten element [W]

$$\Psi_q(F) = \otimes_{j=1}^\infty \mathrm{Sym}_{q^j}(F - \dim F) \tag{3.2}$$

with q a parameter.

We now further assume in this section that F is a nontrivial integrable subbundle of TM. Then F induces a foliation on M. We make the basic assumption that the S^1-action on M preserves the leaves induced by F.

Recall that the equivariant cohomology group $H^*_{S^1}(M)$ is defined to be the usual cohomology group of the space $ES^1 \times_{S^1} M$, where ES^1 denotes the universal principal S^1-bundle over the classifying space BS^1. Here we take cohomology with rational coefficients. The projection

$$\pi : \; ES^1 \times_{S^1} M \to BS^1 \tag{3.3}$$

induces a map

$$\pi^* : \; H^*_{S^1}(\text{pt.}) \to H^*_{S^1}(M) \tag{3.4}$$

which makes $H^*_{S^1}(M)$ a module over $H^*_{S^1}(\text{pt.}) \simeq \mathbf{Q}[[u]]$ with u a generator of degree 2.

Let $p_1(F)_{S^1}$ be the equivariant first Pontryagin class of F. We can now state the main result of this section as follows.

Theorem 3.1. *If M is connected and the S^1-action on M is nontrivial and $p_1(F)_{S^1} = n \cdot \pi^* u^2$ for some integer n, then the equivariant index of $D_{F,\Psi_q(F)\otimes\varphi(F^\perp),+}$ is 0. As a consequence, for any Pontryagin class $p(TM/F)$ of TM/F, we have*

$$\left\langle \widehat{A}(F)\mathrm{ch}\left(\Psi_q(F)\right)p(TM/F), [M] \right\rangle = 0. \tag{3.5}$$

In particular, the Witten genus $[W]$ of M, which is defined by

$$\left\langle \widehat{A}(TM)\mathrm{ch}\left(\Psi_q(TM)\right), [M] \right\rangle, \tag{3.6}$$

vanishes.

If the S^1-action is induced from an effective S^3-action which also preserves the foliation and the spin structure on F, then one can show that $p_1(F)_{S^1} = n \cdot \pi^* u^2$ is equivalent to the condition that $p_1(F) = 0$. This gives us the following

Corollary 3.2. *Assume that M is connected and that there is an effective S^3-action that preserves the foliation and the spin structure on F, and that $p_1(F) = 0$, then the equivariant index of $D_{F,\Psi_q(F)\otimes\varphi(F^\perp),+}$ is 0. In particular, the vanishing formula* (3.5) *holds and the Witten genus given by* (3.6) *vanishes.*

Theorem 3.1 and Corollary 3.2 will be proved in the next section.

4. Proof of Theorem 3.1

Let us first recall the defintion of the Jacobi-theta functions as in [Ch].

For $v \in \mathbf{C}$, $\tau \in \mathbf{H} = \{\tau \in \mathbf{C}, \mathrm{Im}\tau > 0\}$, $q = e^{2\pi i \tau}$, let $\theta(v,\tau)$ denote the classical Jacobi theta-function

$$\theta(v,\tau) = c(q)q^{1/8} 2\sin(\pi v) \prod_{n=1}^{\infty} \left(1 - q^n e^{2\pi i v}\right)\left(1 - q^n e^{-2\pi i v}\right) \tag{4.1}$$

where $c(q) = \prod_{n=1}^{\infty}(1-q^n)$. Set

$$\theta'(0,\tau) = \left.\frac{\partial\theta(v,\tau)}{\partial v}\right|_{v=0}. \tag{4.2}$$

Since any Pontryagin class $p(F^\perp)$ of $F^\perp$ can be written as a linear combination with rational coefficients of classes of the form $L(F^\perp)\mathrm{ch}(\varphi(F^\perp))$, we can and we will assume first that $p(F^\perp)$ is of homogeneous degree $2l$ and that $\varphi(F^\perp)$ verifies that $L(F^\perp)\mathrm{ch}(\varphi(F^\perp))$ is a nonzero rational multiple of $p(F^\perp)$.

Let $g = e^{2\pi it} \in S^1$ be a generator of the S^1-action. Let $\{N\}$ denote the set of connected components of the fixed point set of the S^1-action. Since the S^1-action preserves the leaves induced by F, according to Lemma 3.3 in [HL2], it induces a trivial action on $F^\perp|_N$. Assume that the bundle $F|_N$ has the decomposition

$$F|_N = F_0 \oplus (\oplus_j E_j), \tag{4.3}$$

where each E_j is a complex vector bundle on which $e^{2\pi it}$ acts by $e^{2\pi i m_j t}$, while the S^1 acts trivially on the real vector bundle F_0.

Let $\{2\pi i x_j^k\}$ denote the Chern roots of E_j, and let $\{\pm 2\pi i y_j\}$ denote the Chern roots of $F_0 \otimes_{\mathbf{R}} \mathbf{C}$. By Theorem 2.3 one deduces easily that the Lefschetz number $L(g)$ associated to the operator $D_{F,\Psi_q(F)\otimes\varphi(F^\perp),+}$ is given by

$$\begin{aligned}&H(t,\tau) = \\ &(2\pi i)^{-p}\sum_N \left\langle H(F_0,\tau)\prod_{j,k}\left(\frac{\theta'(0,\tau)}{\theta(x_j^k + m_j t,\tau)}\right) L\left(F^\perp|_N\right)\mathrm{ch}\left(\varphi\left(F^\perp|_N\right)\right), [N]\right\rangle,\end{aligned} \tag{4.4}$$

where the term $H(F_0,\tau)$ denotes the characteristic class

$$H(F_0,\tau) = \prod_j \left(2\pi i y_j \frac{\theta'(0,\tau)}{\theta(y_j,\tau)}\right). \tag{4.5}$$

Considered as function of (t,τ), we can obviously extend $H(t,\tau)$ to meromorphic function on $\mathbf{C}\times\mathbf{H}$. Note that this function is holomorphic in τ.

Recall that $2p = \dim F$ and that $L(F^\perp)\mathrm{ch}(\varphi(F^\perp))$ is of homogeneous degree $2l$. As the S^1 action on M induces a trivial action on $F^\perp|_N$, we know that $\dim N = 2r + \dim F_0$.

Lemma 4.1. *The following formulas hold under the modular transformations,*

$$H\left(\frac{t}{\tau}, -\frac{1}{\tau}\right) = \tau^{p+r-l} e^{-\pi i n t^2/\tau} H(t,\tau), \qquad H(t,\tau+1) = H(t,\tau). \tag{4.6}$$

Proof. One deduces easily that the condition on $p_1(F)_{S^1}$ implies that

$$\sum_{j,k}\left(2\pi i x_j^k + m_j u\right)^2 + \sum_j (2\pi i y_j)^2 = n\cdot u^2,$$

which in turn implies that

$$\sum_{j,k}\left(x_j^k\right)^2 + \sum_j y_j^2 = 0, \quad \sum_{j,k} m_j\, x_j^k = 0, \quad \sum_j (\dim_{\mathbf{C}} E_j)\, m_j^2 = n. \tag{4.7}$$

By [Ch], we have the following transformation formulas

$$\theta\left(\frac{t}{\tau},-\frac{1}{\tau}\right)=\frac{1}{i}\sqrt{\frac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta(t,\tau),\qquad \theta(t,\tau+1)=e^{\frac{\pi i}{4}}\theta(t,\tau). \tag{4.8}$$

By (4.7) and (4.8), we get

$$\begin{aligned}H\left(\frac{t}{\tau},-\frac{1}{\tau}\right)&=\frac{1}{(2\pi i)^p}\\ &\cdot\sum_N\left\langle H\left(F_0,-\frac{1}{\tau}\right)\prod_{j,k}\left(\frac{\theta'\left(0,-\frac{1}{\tau}\right)}{\theta\left(x_j^k+m_j\frac{t}{\tau},-\frac{1}{\tau}\right)}\right)L\left(F^{\perp}\right)\mathrm{ch}\left(\varphi\left(F^{\perp}\right)\right),[N]\right\rangle=\\ &(2\pi i)^{-p}\tau^p e^{-\pi i n t^2/\tau}\sum_N\left\langle\prod_j\left(2\pi i y_j\frac{\theta'(0,\tau)}{\theta(\tau y_j,\tau)}\right)\right.\\ &\left.\prod_{j,k}\left(\frac{\theta'(0,\tau)}{\theta(\tau x_j^k+m_j t,\tau)}\right)L\left(F^{\perp}\right)\mathrm{ch}\left(\varphi\left(F^{\perp}\right)\right),[N]\right\rangle.\end{aligned} \tag{4.9}$$

By comparing the $(\frac{1}{2}\dim F_0+r)=\frac{1}{2}\dim N$ homogeneous terms of the polynomials in x's and y's and the Chern roots of $F^{\perp}$, on both sides, we get the following equation:

$$\begin{aligned}&\tau^{-\dim F_0/2}\\ &\cdot\left\langle\prod_j\left(2\pi i\tau y_j\frac{\theta'(0,\tau)}{\theta(\tau y_j,\tau)}\right)\prod_{j,k}\left(\frac{\theta'(0,\tau)}{\theta(\tau x_j^k+m_j t,\tau)}\right)L\left(F^{\perp}\right)\mathrm{ch}\left(\varphi\left(F^{\perp}\right)\right),[N]\right\rangle\\ &=\tau^{r-l}\left\langle H(F_0,\tau)\prod_{j,k}\left(\frac{\theta'(0,\tau)}{\theta(x_j^k+m_j t,\tau)}\right)L\left(F^{\perp}|_N\right)\mathrm{ch}\left(\varphi\left(F^{\perp}|_N\right)\right),[N]\right\rangle.\end{aligned} \tag{4.10}$$

By (4.9), (4.10), we get the first identity of (4.6). By using (4.8), the second identity can also be verified easily. □

Lemma 4.2. *For $a,\ b\in 2\mathbf{Z}$, the following identity holds,*

$$H(t+a\tau+b,\tau)=e^{\pi i n(a^2\tau+2at)}H(t,\tau). \tag{4.11}$$

Proof. By [Ch], for $a,b\in 2\mathbf{Z}$, we have the transformation formula for the theta-function,

$$\theta(t+a\tau+b,\tau)=e^{-\pi i(a^2\tau+2at)}\theta(t,\tau). \tag{4.12}$$

By using (4.7) and (4.12), we obtain immediately the wanted identity. □

Let $\mathbf{R}$ denote the real number field.

Lemma 4.3. *The function $H(t,\tau)$ is holomorphic for $(t,\tau)\in\mathbf{R}\times\mathbf{H}$.*

Remark 4.4. Lemma 4.3 is the place where the spin condition on F comes in, which guarantees that the function $H(t,\tau)$ is defined as the equivariant index of an elliptic operator, which is a virtual character of the S^1 representation and therefore is holomorphic for $(t,\tau) \in \mathbf{R} \times \mathbf{H}$.

Proof of Lemma 4.3. Let $z = e^{2\pi i t}$, $K = \max_{j,N}|m_j|$. Denote by $D_K \subset \mathbf{C}^2$ the domain

$$|q|^{1/K} < |z| < |q|^{-1/K}, \quad 0 < |q| < 1.$$

Let f_N be the contribution of the fixed component N in the function H. Then by (4.1), (4.4), in D_K, f_N has expansion of the form

$$\prod_j (1 - z^{m_j})^{-p(p+r)} \sum_{n=0}^{\infty} b_{N,n}(z) q^n,$$

where $\Sigma_{n=0}^{\infty} b_{N,n}(z) q^n$ is a holomorphic function of $(z,q) \in D_K$, and $b_{N,n}(z)$ are polynomial functions of z. So as a meromorphic function, in D_K, H has an expansion of the form

$$\sum_{n=0}^{\infty} b_n(z) q^n$$

with each $b_n(z)$ a rational function of z, which can only have poles on the unit circle $\{z : |z| = 1\}$.

Now if we multiply the function H by a function of the form

$$f(z) = \prod_N \prod_j (1 - z^{m_j})^{p(p+r)},$$

where N runs over the connected components of the fixed point set, we get a holomorphic function which has convergent power series expansion of the form

$$\sum_{n=0}^{\infty} c_n(z) q^n$$

with $\{c_n(z)\}$ polynomial functions of z in D_K.

By comparing the above two expansions, we get

$$c_n(z) = f(z) b_n(z).$$

On the other hand, we can expand the element $\Psi_q(F) \otimes \varphi(F^{\perp})$ into formal power series of the form $\Sigma_{n=0}^{\infty} R_n q^n$ with $R_n \in K(M)$. So, for $t \in [0,1] \setminus \mathbf{Q}$, $z = e^{2\pi i t}$, by applying the equivariant index formula to each term we get a formal power series of q for H:

$$\sum_{n=0}^{\infty} \left(\sum_{m=-N(n)}^{N(n)} a_{m,n} z^m \right) q^n$$

with $a_{m,n} \in \mathbf{C}$ and $N(n)$ some positive integer depending on n.

By comparing the above formulas we get for $t \in [0,1] \setminus \mathbf{Q}$, $z = e^{2\pi i t}$,

$$b_n(z) = \sum_{m=-N(n)}^{N(n)} a_{m,n} z^m.$$

Since both sides are analytic functions of z, this equality holds for any $z \in \mathbf{C}$.

By using the Weierstrass preparation theorem, we then deduce that

$$\sum_{n=0}^{\infty} b_n(z) q^n = \frac{1}{f(z)} \sum_{n=0}^{\infty} c_n(z) q^n$$

is holomorphic on (z,q) in D_K which clearly contains the set $\{(t,q) : t \in \mathbf{R}, q \in \mathbf{H}\}$. □

We recall that a (meromorphic) Jacobi form of index m and weight k over $L \rtimes \Gamma$, where L is an integral lattice in the complex plane $\mathbf{C}$ preserved by the modular subgroup $\Gamma \subset SL(2,\mathbf{Z})$, is a (meromorphic) function $F(t,\tau)$ on $\mathbf{C} \times \mathbf{H}$ such that

$$\begin{aligned} F\left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^k e^{2\pi i m (ct^2/(c\tau+d))} F(t,\tau), \\ F\left(t+\lambda\tau+\mu, \tau\right) &= e^{-2\pi i m(\lambda^2\tau + 2\lambda t)} F(t,\tau), \end{aligned} \tag{4.13}$$

where $(\lambda,\mu) \in L$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If F is holomorphic on $\mathbf{C} \times \mathbf{H}$, we say that F is a holomorphic Jacobi form [EZ].

Lemma 4.5. *The function $H(t,\tau)$ is a holomorphic Jacobi form of weight $p+r-l$ and index $-n/2$ over $(2\mathbf{Z})^2 \rtimes SL(2,\mathbf{Z})$.*

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$, we define its modular transformation on $\mathbf{C} \times \mathbf{H}$ by

$$\gamma(t,\tau) = \left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right).$$

Recall the two generators of $SL(2,\mathbf{Z})$ are $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which act on $\mathbf{C} \times \mathbf{H}$ in the following way:

$$S(t,\tau) = \left(\frac{t}{\tau}, -\frac{1}{\tau}\right), \qquad T(t,\tau) = (t, \tau+1).$$

So Lemmas 4.1-4.3 imply that $H(t,\tau)$ is a (meromorphic) Jacobi form of weight $p+r-l$ and index $-n/2$ over $(2\mathbf{Z})^2 \rtimes SL(2,\mathbf{Z})$. We now show that it actually is holomorphic.

From (4.1) and (4.4) we know that the possible poles of H in $\mathbf{C} \times \mathbf{H}$ are of the form

$$t = \frac{h}{s}(c\tau+d), \tag{4.14}$$

where $h,\ s,\ c,\ d$ are integers with $(c,d)=1$ or $c=1,\ d=0$.

We can always find integers $a,\ b$ such that $ad-bc=1$. Then the matrix

$$\gamma=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\in SL(2,\mathbf{Z}) \tag{4.15}$$

induces an action

$$H(\gamma(t,\tau))=H\left(\frac{t}{-c\tau+a},\frac{d\tau-b}{-c\tau+a}\right). \tag{4.16}$$

Now, if $t=\frac{h}{s}(c\tau+d)$ is a polar divisor of $H(t,\tau)$, then one polar divisor of $H(\gamma(t,\tau))$ is given by

$$\frac{t}{-c\tau+a}=\frac{h}{s}\left(c\frac{d\tau-b}{-c\tau+a}+d\right), \tag{4.17}$$

which exactly gives $t=h/s$. But by Lemma 4.1, up to a factor that is holomorphic in $(t,\tau)\in\mathbf{C}\times\mathbf{H}$, $H(\gamma(t,\tau))$ is still equal to $H(t,\tau)$ which is holomorphic for $t\in\mathbf{R}$. This implies that $H(t,\tau)$ has no poles in $\mathbf{C}\times\mathbf{H}$. □

Proof of Theorem 3.1. Since by (4.7), $n=\sum_j(\dim_{\mathbf{C}}E_j)m_j^2$, we have $n\geq 0$.

(i) If $n=0$, then since the S^1-action is nontrivial, it has no fixed point on M. Thus all the Lefschetz number $L(g)$ vanishes by the fixed point formula.

(ii) If $n>0$, then Lemma 4.5 shows that $H(t,\tau)$ is a holomorphic Jacobi form of negative index. By [EZ, Theorem 1.2], $H(t,\tau)$ must be zero.

By (i), (ii) and our choice of $\varphi(F^\perp)$, one gets (3.5) easily. Formula (3.6) then follows from (3.5) and the multiplicativity of the Witten elements:

$$\Psi_q(TM)=\Psi_q(F)\cdot\Psi_q\left(F^\perp\right). \tag{4.18}$$

Now for a general $\varphi(F^\perp)$, we write

$$L\left(F^\perp\right)\mathrm{ch}\left(\varphi\left(F^\perp\right)\right)=\sum_i\omega_i\left(F^\perp\right)\quad\text{with}\quad\omega_i\left(F^\perp\right)\in H^i(M;\mathbf{Q}). \tag{4.19}$$

Then for each $\omega_i(F^\perp)$, one can find an integral polynomial $\varphi_i(F^\perp)$ such that

$$L\left(F^\perp\right)\mathrm{ch}\left(\varphi_i\left(F^\perp\right)\right)=n_i\cdot\omega_i\left(F^\perp\right) \tag{4.20}$$

for some nonzero integer n_i. One then verifies that the equivariant index of $D_{F,\Psi_q(F)\otimes\varphi(F^\perp)}$ can be expressed as a linear combination with rational coefficients of the equivariant indices of $D_{F,\Psi_q(F)\otimes\varphi_i(F^\perp)}$'s, which have been proved to be zero.

The proof of Theorem 3.1 is completed. □

Proof of Corollary 3.2.. We can either use the simple exact sequence for the S^3-equivariant cohomology groups,

$$H^4(BS^3)\to H^4_{S^3}(M)\to H^4(M) \tag{4.21}$$

which follows from the spectral sequence for the fibration

$$ES^3\times_{S^3}M\to BS^3$$

by using the fact that BS^3 is 3-connected. Here ES^3 is the universal S^3-principal bundle over the classifying space BS^3 of S^3.

Alternatively, one may prove this by using the following simple observation. In fact, at least formally, we may write $p_1(F)_{S^1}$ as

$$p_1(F)_{S^1} = p_1(F) + Au + n \cdot \pi^* u^2 \tag{4.22}$$

with A a two form on M.

If the S^1-action is induced from an S^3-action, then $p_1(F)_{S^1}$ should be invariant under the Weyl group action $u \to -u$, which implies $A = 0$.

This means under the condition of Corollary 3.2, there exists $n \in \mathbf{Z}$, such that $p_1(F)_{S^1} = n\pi^* u^2$. By Theorem 3.1, we get Corollary 3.2. □

By using a special case of the above arguments, one gets the following result in which the condition on $p_1(F)$ is no longer needed. Compare with [HL2, Prop. 3.2]. It generalizes the classical Atiyah-Hirzebruch vanishing theorem [AH] to the foliated manifolds.

Theorem 4.6. *If an S^1 acts nontrivially on a compact connected foliation (M, F) and preserves the leaves induced by F as well as the spin structure on F, then $\langle \widehat{A}(TM), [M] \rangle = 0$.*

5. Manifolds with split tangent bundle

In this section, we no longer assume that F is integrable. We assume instead that M itself is a spin manifold and the S^1-action preserves the spin structures on TM and F. Then it also preserves the induced spin structure on $TM/F \simeq F^\perp$. Consequently, the S^1-action on the restriction of $F^\perp$ to the fixed point set of the S^1-action on M need not be trivial.

Let us introduce elements

$$\begin{aligned}
R_q\left(F^\perp\right) &= \otimes_{j=1}^\infty \mathrm{Sym}_{q^j}\left(F^\perp - \dim F^\perp\right) \\
&\qquad \otimes \left(\otimes_{m=1}^\infty \Lambda_{q^m} \cdot \left(F^\perp - \dim F^\perp\right)\right), \\
R'_q\left(F^\perp\right) &= \otimes_{j=1}^\infty \mathrm{Sym}_{q^j}\left(F^\perp - \dim F^\perp\right) \\
&\qquad \otimes \left(\otimes_{m=1}^\infty \Lambda_{q^{m-1/2}}\left(F^\perp - \dim F^\perp\right)\right), \\
R''_q\left(F^\perp\right) &= \otimes_{j=1}^\infty \mathrm{Sym}_{q^j}\left(F^\perp - \dim F^\perp\right) \\
&\qquad \otimes \left(\otimes_{m=1}^\infty \Lambda_{-q^{m-1/2}}\left(F^\perp - \dim F^\perp\right)\right).
\end{aligned} \tag{5.1}$$

Recall that since the S^1-action preserves g^{TM}, it preserves the orthogonal splitting

$$TM = F \oplus F^\perp. \tag{5.2}$$

Let D denote the canonical Dirac operator on M associated to g^{TM}. We also consider the twisted Dirac operators

$$D_{\Psi_q(F) \otimes R'_q(F^\perp)} = D \otimes \Psi_q(F) \otimes R'_q\left(F^\perp\right)$$

and

$$D_{\Psi_q(F)\otimes R_q''(F^\perp)} = D \otimes \Psi_q(F) \otimes R_q''\left(F^\perp\right).$$

Under the above assumptions and notations, the main result of this section can be stated as follows.

Theorem 5.1. *If $p_1(F)_{S^1} = n \cdot \pi^* u^2$ for some integer $n \neq 0$, then the equivariant indices of $D_{F,\Psi_q(F)\otimes R_q(F^\perp),+}$, $D_{\Psi_q(F)\otimes R_q'(F^\perp),+}$ and $D_{\Psi_q(F)\otimes R_q''(F^\perp),+}$ vanish. In particular, the following three formulas hold,*

$$\begin{aligned}
&\left\langle \widehat{A}(F) L\left(F^\perp\right) \mathrm{ch}\left(\Psi_q(F)\right) \mathrm{ch}\left(R_q\left(F^\perp\right)\right), [M]\right\rangle = 0,\\
&\left\langle \widehat{A}(TM) \mathrm{ch}\left(\Psi_q(F)\right) \mathrm{ch}\left(R_q'\left(F^\perp\right)\right), [M]\right\rangle = 0,\\
&\left\langle \widehat{A}(TM) \mathrm{ch}\left(\Psi_q(F)\right) \mathrm{ch}\left(R_q''\left(F^\perp\right)\right), [M]\right\rangle = 0.
\end{aligned}$$

Proof of Theorem 5.1. Let

$$F|_N = F_0 \oplus \left(\oplus_j E_j\right), \quad F^\perp\big|_N = F_0^\perp \oplus \left(\oplus_j L_j\right) \tag{5.4}$$

be the corresponding equivariant decomopositions of F and $F^\perp$, when restricted to the connected component N of the fixed point set of the S^1-action on M. Assume the generator $g = e^{2\pi i t} \in S^1$ acts on E_j by multiplication by $e^{2\pi i m_j t}$ and on L_j by multiplication by $e^{2\pi i n_j t}$.

Let $\{2\pi i x_j^k\}$ denote the Chern roots of E_j and $\{2\pi i z_j^k\}$ denote the Chern roots of L_j. We also denote by $\{\pm 2\pi i y_j\}$ and $\{\pm 2\pi i w_j\}$ the Chern roots of $F_0 \otimes_{\mathbf{R}} \mathbf{C}$ and $F_0^\perp \otimes_{\mathbf{R}} \mathbf{C}$ respectively.

Let $\theta_1(v,\tau)$, $\theta_2(v,\tau)$ and $\theta_3(v,\tau)$ be the three theta functions (cf. [Ch]):

$$\begin{aligned}
\theta_3(v,\tau) &= c(q) \prod_{n=1}^\infty \left(1 + q^{n-1/2} e^{2\pi i v}\right) \prod_{n=1}^\infty \left(1 + q^{n-1/2} e^{-2\pi i v}\right),\\
\theta_2(v,\tau) &= c(q) \prod_{n=1}^\infty \left(1 - q^{n-1/2} e^{2\pi i v}\right) \prod_{n=1}^\infty \left(1 - q^{n-1/2} e^{-2\pi i v}\right),\\
\theta_1(v,\tau) &= c(q) q^{1/8} 2\cos(\pi v) \prod_{n=1}^\infty \left(1 + q^n e^{2\pi i v}\right) \prod_{n=1}^\infty \left(1 + q^n e^{-2\pi i v}\right).
\end{aligned}$$

Let us write

$$\begin{aligned}
G_0(\tau) &= \widehat{A}(F_0) L\left(F_0^\perp\right) \mathrm{ch}\left(\Psi_q\left(F_0\right)\right) \mathrm{ch}\left(R_q\left(F_0^\perp\right)\right),\\
G_0'(\tau) &= \widehat{A}\left(TN\right) \mathrm{ch}\left(\Psi_q\left(F_0\right)\right) \mathrm{ch}\left(R_q'\left(F_0^\perp\right)\right),\\
G_0''(\tau) &= \widehat{A}\left(TN\right) \mathrm{ch}\left(\Psi_q\left(F_0\right)\right) \mathrm{ch}\left(R_q''\left(F_0^\perp\right)\right).
\end{aligned}$$

By applying the equivariant index formula (2.21), we get three functions,

$$G(t,\tau) = \sum_N (2\pi i)^{-(p+r-\frac{\dim N}{2})} \cdot \left\langle G_0(\tau) \prod_{j,k} \frac{\theta'(0,\tau)}{\theta(x_j^k + m_j t,\tau)} \prod_{j,k} \frac{\theta_1(z_j^k + n_j t,\tau)\theta'(0,\tau)}{\theta(z_j^k + n_j t,\tau)\theta_1(0,\tau)}, [N] \right\rangle,$$

$$G'(t,\tau) = \sum_N (2\pi i)^{-(p+r-\frac{\dim N}{2})} \cdot \left\langle G_0'(\tau) \prod_{j,k} \frac{\theta'(0,\tau)}{\theta(x_j^k + m_j t,\tau)} \prod_{j,k} \frac{\theta_2(z_j^k + n_j t,\tau)\theta'(0,\tau)}{\theta(z_j^k + n_j t,\tau)\theta_2(0,\tau)}, [N] \right\rangle,$$

$$G''(t,\tau) = \sum_N (2\pi i)^{-(p+r-\frac{\dim N}{2})} \cdot \left\langle G_0''(\tau) \prod_{j,k} \frac{\theta'(0,\tau)}{\theta(x_j^k + m_j t,\tau)} \prod_{j,k} \frac{\theta_3(z_j^k + n_j t,\tau)\theta'(0,\tau)}{\theta(z_j^k + n_j t,\tau)\theta_3(0,\tau)}, [N] \right\rangle$$

corresponding to the equivariant indices of the three elliptic operators $D_{F,\Psi_q(F)\otimes R(F^\perp),+}$, $D_{\Psi_q(F)\otimes R'(F^\perp),+}$ and $D_{\Psi_q(F)\otimes R''(F^\perp),+}$ respectively.

Now recall the definitions of the following three modular subgroups:

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) : c \equiv 0 \pmod 2 \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) : b \equiv 0 \pmod 2 \right\},$$

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}.$$

By using the modular transformation formula of the theta-functions [Ch], we can immediately prove the following result by proceeding as in the proofs of Lemmas 4.1 and 4.2.

Lemma 5.2. If $p_1(F)_{S^1} = n \cdot \pi^* u^2$, then $G(t,\tau)$ is a Jacobi form over $(2\mathbf{Z}) \rtimes \Gamma_0(2)$, $G'(t,\tau)$ is a Jacobi form over $(2\mathbf{Z}) \rtimes \Gamma^0(2)$ and $G''(t,\tau)$ is a Jacobi form over $(2\mathbf{Z}) \rtimes \Gamma_\theta$. *All of them are of index $-n/2$ and weight $p+r$.*

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$, let us use the notation

$$H(\gamma(t,\tau))|_{m,k} = (c\tau+d)^{-k} e^{-2\pi i m c t^2/(c\tau+d)} H\left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) \tag{5.7}$$

to denote the action of γ on a Jacobi form H of index m and weight k.

Recall that $2p = \dim F$ and $2r = \dim F^{\perp}$. The following lemma can be proved easily by proceeding as the proof of Lemma 4.3.

Lemma 5.3. *For any $\gamma \in SL(2, \mathbf{Z})$, let $F(t, \tau)$ be one of the functions $G(t, \tau)$, $G'(t, \tau)$ and $G''(t, \tau)$. Then $F(\gamma(t, \tau))|_{\frac{-n}{2}, p+r}$ is holomorphic for $(t, \tau) \in \mathbf{R} \times \mathbf{H}$.*

Again this is the place where the index theory comes in to cancel part of the poles of these functions. Here the spin conditions on F, TM are crucially needed.

Now, by using the same argument as in the proof of Lemma 4.5, we get

Lemma 5.4. *For a (meromorphic) Jacobi form $H(t, \tau)$ of index m and weight k over $L \rtimes \Gamma$, assume that H may only have polar divisors of the form $t = (c\tau + d)/l$ in $\mathbf{C} \times \mathbf{H}$ for some integers c, d and $l \neq 0$. If $H(\gamma(t, \tau))|_{m,k}$ is holomorphic for $t \in \mathbf{R}$, $\tau \in \mathbf{H}$ for every $\gamma \in SL(2, \mathbf{Z})$, then $H(t, \tau)$ is holomorphic for any $t \in \mathbf{C}$ and $\tau \in \mathbf{H}$.*

From Lemmas 5.3 and 5.4 one sees, as in the proof of Theorem 3.1, that the G's are holomorphic Jacobi forms of index $-n/2$, and therefore must be zero. (Here we have used the fact that $n > 0$.)

The proof of Theorem 5.1 is completed. □

Remark 5.5. If $n = 0$, then we get the rigidity properties in Theorem 5.1 instead of the vanishing results.

6. Concluding remarks

Motivitate by Corollary 3.2, we find it is interesting and reasonable to make the following conjecture which may be viewed as a foliation analogue of a conjecture of Hoehn and Stolz [S].

We consider an oriented compact foliation M which is foliated by a spin integrable subbundle F of TM. Let g^F be a metric on F.

Conjecture 6.1. *If $\frac{1}{2}p_1(F) = 0$, and the Ricci curvature of g^F along each leaf is positive, then the Witten genus of M, $\langle \widehat{A}(TM)\mathrm{ch}(\Psi_q(TM)), [M]\rangle$, vanishes.*

As have been remarked in the introduction, we may as well take $\psi(F)$ or $\varphi(F^{\perp})$ in Section 2 as the elements in $K(M)$ induced from loop group representations. Then the modularity of the characters of the loop group representations can be used to prove vanishing theorems for the correponding twisted sub-Dirac operators. On the other hand, the construction of the sub-elliptic operators is very flexible. For example if the the integrable subbundle of the foliation has almost complex structure or Spin^c-structure, then we can construct sub $\bar{\partial}$-operator or Spin^c sub-Dirac operator correspondingly. If there exists a compact Lie group action on M preserving the leaves, then the rigidity and vanishing theorems can be proved for the equivariant indices of such operators which generalize the corresponding rigidity and vanishing results for the usual elliptic genera. See [Liu2] or [LiuMa2] for some details about these.

In concluding, we may also replace the signature operator in the normal direction by other elliptic operators like the de Rham type operator from which we can derive the vanishing of characteristic numbers like

$$\left\langle \widehat{A}(TM)\ \mathrm{ch}\left(\Psi_q(F)\right) e\left(F^{\perp}\right), [M] \right\rangle,$$

where $e(F^{\perp})$ denotes the Euler class of $F^{\perp}$.

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